Elliptic Operators on S^2 whose Weyl Spectral Asymptotics Have Small Remainders

Michael Taylor *

Abstract

We show that if Δ is the Laplace-Beltrami operator on the unit 2-sphere S^2, and

$$L = \Delta + Z^2,$$

where Z is the real vector field that generates 2π-periodic rotation about the x_3-axis, then the spectral counting function of L satisfies

$$\mathcal{N}(L, R) = \frac{\pi}{4} R^2 + O(R^{1-\alpha}),$$

with $\alpha = 1/3$. Going further, we analyze the spectral counting functions of

$$\Delta + aZ^2,$$

in the entire elliptic range $a \in (-1, \infty)$, and show that, for all such nonzero a,

$$\mathcal{N}(\Delta + aZ^2, R) = K_a R^2 + O(R^{1-\alpha}),$$

with K_a given in §4. The proofs involve a variety of lattice point estimates, established in §5.

*MSC-2020 Math Subject Classification: 35P20, 42B05, 43A85. *Key words.* Spectrum, joint spectrum, Laplace operator, elliptic operator, lattice point estimates

1 Introduction

As is well known, the Laplace operator Δ on the unit sphere $S^2 \subset \mathbb{R}^3$, with its standard metric, exhibits spectral clustering. In fact,

$$\operatorname{Spec}(-\Delta) = \{k^2 + k : k \in \mathbb{Z}^+\}, \tag{1.1}$$

and, if

$$V_k = \{u \in C^\infty(S^2) : -\Delta u = (k^2 + k)u\}, \tag{1.2}$$

then

$$\dim V_k = 2k + 1. \tag{1.3}$$

Going further, we can let Z denote the real vector field generating rotation of S^2 about the x_3-axis, with period 2π, and set $X = -iZ$, so X is a self-adjoint operator, commuting with Δ, and then V_k has an orthonormal basis

$$Y_k^\ell, \quad -k \le \ell \le k, \quad XY_k^\ell = \ell Y_k^\ell. \tag{1.4}$$

There are many places where such basic results on spherical harmonics can be found. We mention in particular [9], Chapter 8, §4, and [10], Chapter 7, §4.

One neat way to record the information in (1.1)–(1.4) is to bring in the operators

$$\Lambda = \Big(-\Delta + \frac{1}{4}\Big)^{1/2}, \quad \Lambda_0 = \Lambda - \frac{1}{2}. \tag{1.5}$$

Then

$$u \in V_k \Longrightarrow \Lambda_0 v = kv. \tag{1.6}$$

Hence the *joint spectrum* of Λ_0 and X is given by

$$\operatorname{Spec}(\Lambda_0, X) = \{(k, \ell) \in \mathbb{Z} \times \mathbb{Z} : k \ge 0,\ |\ell| \le k\}, \tag{1.7}$$

and each joint eigenspace is one dimensional, spanned by Y_k^ℓ, satisfying

$$\Lambda_0 Y_k^\ell = kY_k^\ell, \quad XY_k^\ell = \ell Y_k^\ell. \tag{1.8}$$

See Figure 1.1.

For the Laplace operator Δ_M on a compact Riemannian manifold M, one defines the spectral counting function

$$\mathcal{N}(\Delta_M, R) = \sum_{\lambda \le R} \dim V_\lambda(\Delta_M), \tag{1.9}$$

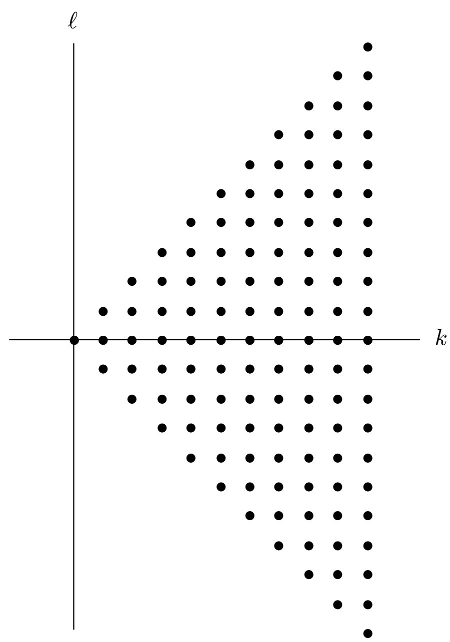

Figure 1.1: $\mathrm{Spec}(\Lambda_0, X)$ on S^2

where

$$V_\lambda(\Delta_M) = \{u \in C^\infty(M) : \Delta_M u = -\lambda^2 u\}. \tag{1.10}$$

In general,

$$\mathcal{N}(\Delta_M, R) = C_n \operatorname{Vol}(M) R^n + O(R^{n-1}). \tag{1.11}$$

See [6]. Spectral clustering for a sphere gives a case where the remainder estimate cannot be improved. Conditions have been found that do lead to improved remainder estimates. In [4] it is shown that one can take $o(R^{n-1})$ in case the geodesic flow on M has "not too many" periodic orbits. In [1] it is shown that if either $\dim M = 2$ and there are no conjugate points, or $\dim M \geq 3$ and the sectional curvatures are all ≤ 0, then one can take

$$O(R^{n-1}/\log R), \tag{1.12}$$

as a remainder estimate in (1.11). Recently [3] has obtained an improved remainder estimate for a much broader class of Riemannian manifolds. Among the results obtained there is the result that the remainder estimate (1.12) holds whenever M is a Cartesian product, $M = X \times Y$ of two Riemannian manifolds, with the product metric.

In another vein, [7] obtain remainder estimates of the form

$$O(R^{n-1-\alpha}), \tag{1.13}$$

for products of spheres,

$$M = S^{d_1} \times \cdots \times S^{d_k}, \quad n = d_1 + \cdots + d_k, \tag{1.14}$$

with $\alpha > 0$. The authors express $\mathcal{N}(\Delta_M, R)$ as a weighted lattice count.

Here we produce some second-order differential operators on S^2 whose spectral counting functions have improved Weyl asymptotics of the form (1.13). To start, we examine spectral asymptotics of the operators

$$L_0 = -\Lambda_0^2 + Z^2, \quad L = -\Lambda^2 + Z^2. \tag{1.15}$$

Recall that Z is a real vector field and that $Z^2 = -X^2$.

Parallel to (1.9), we define the spectral counting function

$$\mathcal{N}(L_0, R) = \sum_{\lambda \leq R} \dim V_\lambda(L_0), \tag{1.16}$$

where

$$V_\lambda(L_0) = \{u \in C^\infty(S^2) : L_0 u = -\lambda^2 u\}, \tag{1.17}$$

and similarly define $\mathcal{N}(L, R)$. In §2 we see that

$$\mathcal{N}(L_0, R) = \frac{\pi}{4}R^2 + \frac{\sqrt{2}}{2}R + O(R^{1-\alpha}), \tag{1.18}$$

where $\alpha = 1/3$.

Now L_0 is not a differential operator. Indeed, since

$$\begin{aligned} \Lambda_0^2 &= \Lambda^2 - \Lambda + \frac{1}{4} \\ &= -\Delta - \Lambda + \frac{1}{2}, \end{aligned} \tag{1.19}$$

we have

$$L_0 = \Delta + Z^2 + \Lambda - \frac{1}{2}, \tag{1.20}$$

so L_0 is an elliptic pseudodifferential operator, but not a differential operator. On the other hand,

$$L = \Delta + Z^2 - \frac{1}{4}, \tag{1.21}$$

which is an elliptic differential operator. In §3 we see that, with α as above,

$$\mathcal{N}(L, R) = \frac{\pi}{4}R^2 + O(R^{1-\alpha}). \tag{1.22}$$

Note that (1.18) has two terms in its expansion, but this is not unusual for a pseudodifferential modification of an elliptic differential operator. For another class of examples, suppose M is a compact n-dimensional Riemannian manifold whose Laplace operator Δ_M has the property

$$\mathcal{N}(\Delta_M, R) = CR^n + o(R^{n-1}). \tag{1.23}$$

If we set $A = \sqrt{-\Delta_M}$, then

$$\mathcal{N}(-(A+1)^2, R) = \mathcal{N}(\Delta_M, R+1), \tag{1.24}$$

hence

$$\mathcal{N}(\Delta_M - 2A - 1, R) = CR^n + nCR^{n-1} + o(R^{n-1}). \tag{1.25}$$

The result (1.18) enters here because it is relatively easy to prove and it provides a stepping stone to (1.22), via a curious cancellation effect, which we discuss in §3. The main significance of (1.18) lies in exactly what that second term is.

Going further, we examine the operators

$$L_{0a} = -\Lambda_0^2 + aZ^2, \quad L_a = -\Lambda^2 + aZ^2, \tag{1.26}$$

with $a \in (0,\infty) \cup (-1,0)$. Again, L_a is a differential operator,

$$L_a = \Delta + aZ^2 + \frac{1}{4}. \tag{1.27}$$

Parallel to (1.18) and (1.22), we show that

$$\mathcal{N}(L_{0a}, R) = K_a R^2 + (1+a)^{-1/2} R + O(R^{1-\alpha}), \tag{1.28}$$

and from there establish the following.

Theorem 1.1 *For all nonzero $a \in (-1,\infty)$,*

$$\mathcal{N}(L_a, R) = K_a R^2 + O(R^{1-\alpha}), \tag{1.29}$$

for $\alpha = 1/3$. Here

$$K_a = \text{Area}\, \mathcal{O}_a^+(1), \tag{1.30}$$

where $\mathcal{O}_a^+(R)$ is a family of planar domains defined by (4.2).

The proof of (1.28) is somewhat parallel to that of (1.18), but more effort is required to carry it out, since the domain $\mathcal{O}_a(R)$, defined by (4.4), has some corners, and, for $a \in (1,\infty) \cup (-1,0)$ is not convex. The deduction of (1.29) from (1.28) (and of (1.22) from (1.18)) also involves a novel lattice point estimate. These results are treated in §5.

Remark. It is elementary that, if (1.29) holds, so does its analogue with L_a replaced by $L_a - 1/4 = \Delta + aZ^2$.

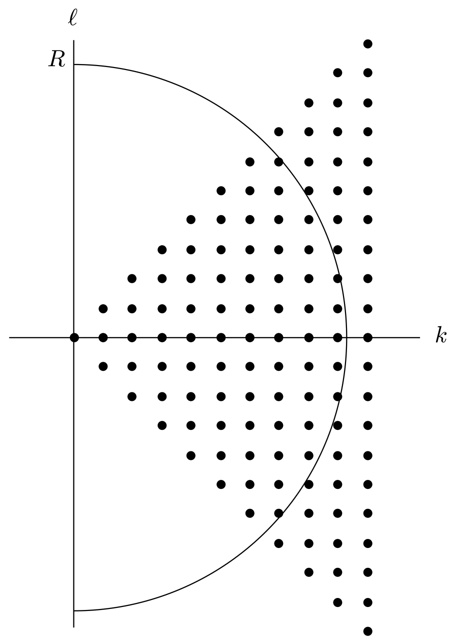

Figure 2.1: Counting Spec$(-L_0)$

2 Spectral asymptotics of L_0

Our goal here is to obtain an asymptotic evaluation of $\mathcal{N}(L_0, R)$, defined by (1.16). Referring to (1.7)–(1.15), and to Figure 2.1, we see that

$$\mathcal{N}(L_0, R) = \#\{(k, \ell) \in \mathbb{Z} \times \mathbb{Z} : k \geq 0, |\ell| \leq k, k^2 + \ell^2 \leq R^2\}. \tag{2.1}$$

By symmetry, we have

$$4\mathcal{N}(L_0, R) = \mathcal{D}(R) + 4\mathcal{L}(R) - 1, \tag{2.2}$$

where

$$\mathcal{D}(R) = \#\{(k, \ell) \in \mathbb{Z} \times \mathbb{Z} : k^2 + \ell^2 \leq R^2\}, \tag{2.3}$$

and

$$\begin{aligned} \mathcal{L}(R) &= \#\{(k, \ell) \in \mathbb{Z} \times \mathbb{Z} : k \geq 0, \ell = k, k^2 + \ell^2 \leq R^2\} \\ &= \#\{k \in \mathbb{Z}^+ : 2k^2 \leq R^2\}. \end{aligned} \tag{2.4}$$

Clearly

$$\mathcal{L}(R) = \frac{\sqrt{2}}{2} R + O(1). \tag{2.5}$$

Meanwhile, the evaluation of $\mathcal{D}(R)$ is the classical *circle problem*. For this, one has

$$\mathcal{D}(R) = \pi R^2 + O(R^{1-\alpha}). \tag{2.6}$$

There are numerous partial results on this, but the optimal value of α has not been established. It is known one can take $\alpha > 1/3$, but one cannot take $\alpha = 1/2$. See [5], p. 450, for a discussion, and [2] for recent progress.

Putting together (2.2)–(2.6) gives

$$\mathcal{N}(L_0, R) = \frac{\pi}{4} R^2 + \frac{\sqrt{2}}{2} R + O(R^{1-\alpha}), \tag{2.7}$$

as advertised in the Introduction.

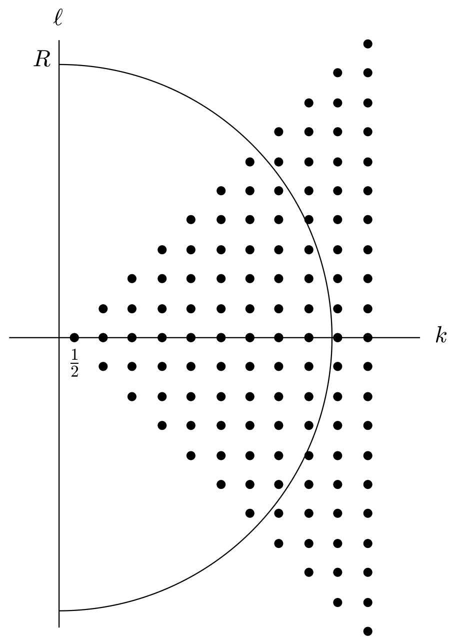

Figure 3.1: Counting Spec($-L$)

3 Spectral asymptotics of L

Our goal here is to establish (1.22), i.e.,

$$\mathcal{N}(L,R)=\frac{\pi}{4}R^2+O(R^{1-\alpha}), \tag{3.1}$$

for $\alpha=1/3$. Having established (1.18) in §2, we are left with the task of showing that

$$\mathcal{N}(L_0,R)-\mathcal{N}(L,R)=\frac{\sqrt{2}}{2}R+O(R^{1-\alpha}). \tag{3.2}$$

As suggested by Figures 3.1–3.2, we can approach this as follows. Set

$$\begin{aligned}E_R=\{(x,y)\in\mathbb{R}^2 : x\ge 0,\ |y|\le x,\\ x^2+y^2\le R^2,\ (x+\tfrac{1}{2})^2+y^2>R^2\}.\end{aligned} \tag{3.3}$$

Then

$$\mathcal{N}(L_0,R)-\mathcal{N}(L,R)=\#(E_R\cap\mathbb{Z}^2). \tag{3.4}$$

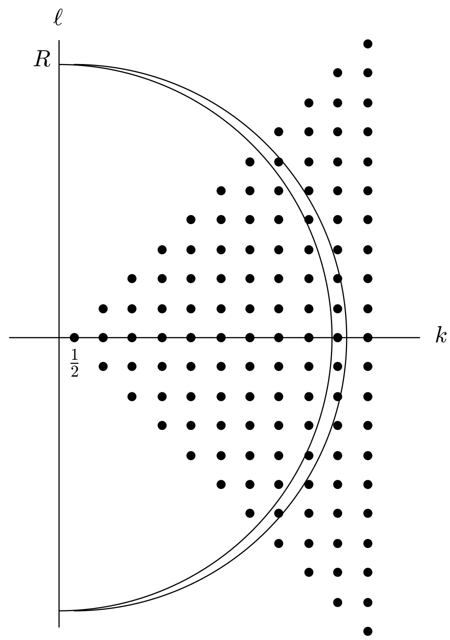

Figure 3.2: A look at $\mathcal{N}(L_0, R) - \mathcal{N}(L, R)$

Note that

$$\text{Area}\, E_R = \frac{\sqrt{2}}{2} R + O(1). \tag{3.5}$$

Hence the desired estimate (3.2) is a consequence of the following, which will be proved in §5.

Proposition 3.1 *As* $R \to \infty$,

$$\#(E_R \cap \mathbb{Z}^2) = \text{Area}\, E_R + O(R^{1-\alpha}). \tag{3.6}$$

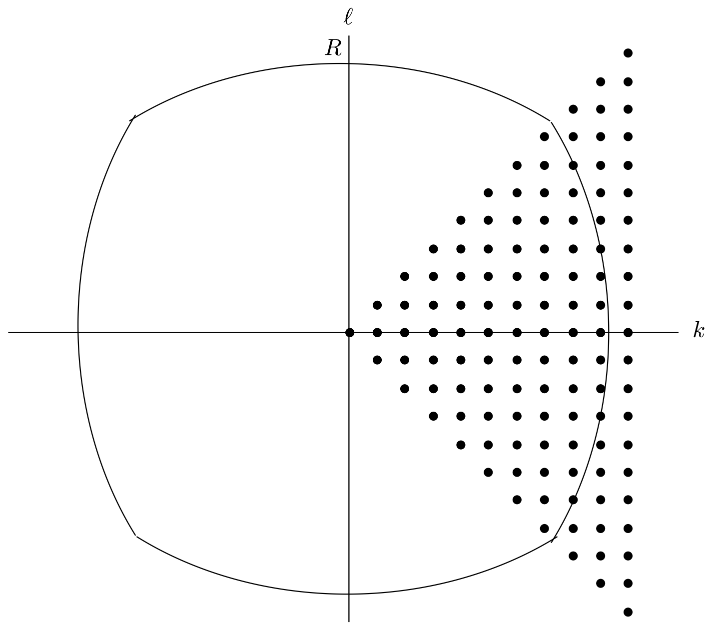

Figure 4.1: Counting Spec$(-L_{0a})$, $0 < a < 1$

4 Spectral asymptotics of L_{0a} and of L_a

Here we extend our study of the operators L_0 and L, defined by (1.8), to that of

$$L_{0a} = -\Lambda_0^2 + aZ^2, \quad L_a = -\Lambda^2 + aZ^2, \tag{4.1}$$

with $a \in (0, \infty) \cup (-1, 0)$. In (1.8), the operators have this form with $a = 1$. To start, if we set

$$\mathcal{O}_a^+(R) = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ |y| \leq x,\ x^2 + ay^2 \leq R^2\}, \tag{4.2}$$

then

$$\mathcal{N}(L_{0a}, R) = \#\big(\mathcal{O}_a^+(R) \cap \mathbb{Z}^2\big). \tag{4.3}$$

We seek a formula parallel to (2.2). That formula exploited symmetry of the circle centered at 0. In the current setting, we need to impose symmetry. We set

$$\mathcal{O}_a(R) = \bigcup_{m=0}^{3} J^m \mathcal{O}_a^+(R), \tag{4.4}$$

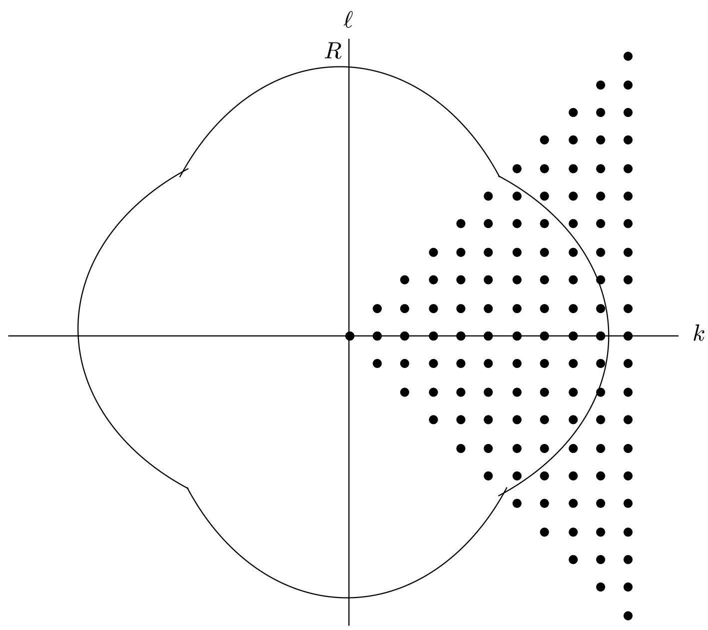

Figure 4.2: Counting Spec$(-L_{0a})$, $a > 1$

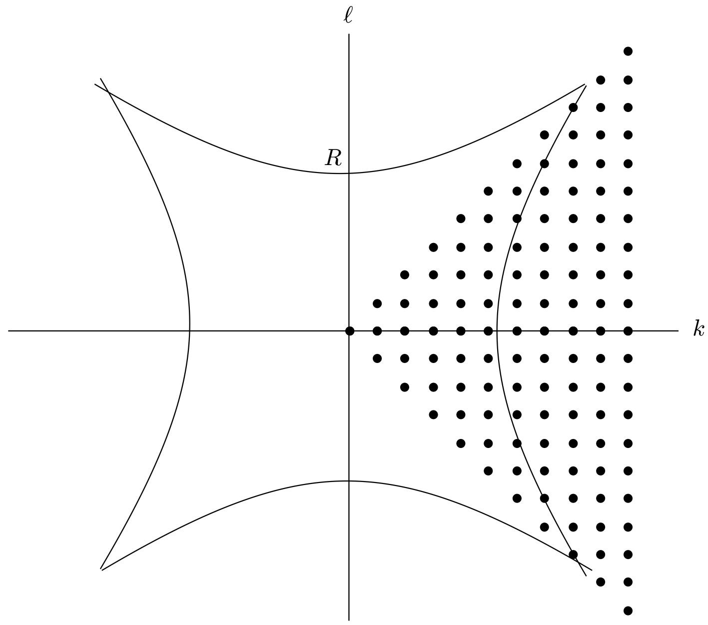

Figure 4.3: Counting Spec$(-L_{0a})$, $-1 < a < 0$

where $J \in M(2,\mathbb{R})$ is counterclockwise rotation by 90°. See Figures 4.1–4.3, depicting cases where $0 < a < 1$, $a > 1$, and $-1 < a < 0$, respectively. Then, parallel to (2.2)–(2.4), we have

$$4\mathcal{N}(L_{0a}, R) = \mathcal{D}_a(R) + 4\mathcal{L}_a(R) - 1, \tag{4.5}$$

where

$$\mathcal{D}_a(R) = \#\big(\mathcal{O}_a(R) \cap \mathbb{Z}^2\big), \tag{4.6}$$

and

$$\mathcal{L}_a(R) = \#\{k \in \mathbb{Z}^+ : (1+a)k^2 \le R^2\}. \tag{4.7}$$

Clearly

$$\mathcal{L}_a(R) = (1+a)^{-1/2} R + O(1). \tag{4.8}$$

If $a = 1$, $\mathcal{O}_a(R)$ is a disc, and estimates on $\mathcal{D}_a(R)$ are classical, as discussed in §2. As indicated in Figures 4.1–4.3, $\mathcal{O}_a(R)$ has four corners if $a \ne 1$. Also, this domain is convex if $a \in (0,1)$, but not if $a > 1$ or $a \in (-1,0)$. We will show in §5 that, whenever $a \in (0,\infty) \cup (-1,0)$,

$$\mathcal{D}_a(R) = \operatorname{Area} \mathcal{O}_a(R) + O(R^{1-\alpha}), \tag{4.9}$$

for $\alpha = 1/3$. This leads to the following result.

Proposition 4.1 *If $a \in (0,\infty) \cup (-1,0)$, then*

$$\mathcal{N}(L_{0a}, R) = K_a R^2 + (1+a)^{-1/2} R + O(R^{1-\alpha}), \tag{4.10}$$

where

$$K_a = \operatorname{Area} \mathcal{O}_a^+(1). \tag{4.11}$$

To proceed, parallel to (3.3)–(3.4), we have

$$\mathcal{N}(L_{0a}, R) - \mathcal{N}(L_a, R) = \#(E_{a,R} \cap \mathbb{Z}^2), \tag{4.12}$$

where

$$\begin{aligned} E_{a,R} = \{(x,y) \in \mathbb{R}^2 : x \ge 0,\ |y| \le x, \\ x^2 + ay^2 \le R^2,\ (x+\tfrac{1}{2})^2 + ay^2 > R^2\}. \end{aligned} \tag{4.13}$$

In §5 we will show that

$$\#(E_{a,R} \cap \mathbb{Z}^2) = \operatorname{Area} E_{a,R} + O(R^{1-\alpha}). \tag{4.14}$$

Meanwhile, parallel to (3.5), we have

$$\operatorname{Area} E_{a,R} = (1+a)^{-1/2} R + O(1). \tag{4.15}$$

Therefore, given Proposition 4.1 and (4.12), we have Theorem 1.1.

5 Lattice point estimates

Our goal here is to prove the lattice point estimates (3.6), (4.9), and (4.14). As a tool for this, we will establish a generalization of Proposition 3.2 of [8]. We describe the setup for this result. As done in [8], we work in n dimensions, and specialize to $n = 2$ later. Let $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ denote the n-dimensional torus. We take

$$\nu \in \mathbb{N}, \quad h = \frac{2\pi}{\nu}, \tag{5.1}$$

and aim to estimate how well the mean value

$$Mu = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} u(x)\, dx \tag{5.2}$$

of a given bounded, Borel function u on $\mathbb{T}^n$ is approximated by

$$\sigma_h u(x) = \nu^{-n} \sum_{\ell \in (\mathbb{Z}/\nu)^n} u(x + h\ell), \tag{5.3}$$

in terms of an estimate of the sup norm of

$$\rho_h u = \sigma_h u - Mu. \tag{5.4}$$

A key identity for the analysis of (5.4) is

$$\rho_h u(x) = \sum_{j \in \mathbb{Z}^n \setminus 0} \hat{u}(\nu j) e^{-i\nu j \cdot x}, \tag{5.5}$$

where $\hat{u}(k)$ are the Fourier coefficients,

$$\hat{u}(k) = (2\pi)^{-n} \int\limits_{\mathbb{T}^n} u(x) e^{-ik\cdot x}\, dx, \quad k \in \mathbb{Z}^n. \tag{5.6}$$

In light of this, the following scale of distributions makes a natural appearance. Given $u \in \mathcal{D}'(\mathbb{T}^n)$, we say

$$u \in F^r(\mathbb{T}^n) \Longleftrightarrow |\hat{u}(k)| \le C\langle k\rangle^{-r}, \quad \|u\|_{F^r} = \sup_k \langle k\rangle^r |\hat{u}(k)|, \tag{5.7}$$

where $\langle k\rangle = (1 + |k|^2)^{1/2}$. One simple consequence of (5.5) is the estimate

$$\|\rho_h u\|_{L^\infty} \le Ch^r \|u\|_{F^r}, \quad \text{if} \ \ r > n, \tag{5.8}$$

but this will not suffice for our needs, which involve functions (or distributions) in $F^r(\mathbb{T}^n)$ with $r < n$.

We will use the following strategy to estimate $\|\rho_h u\|_{L^\infty}$ when we have $u \in F^r(\mathbb{T}^n)$ and we also have some further information on u. Pick

$$\varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi(0) = 1, \quad \text{and set } J_\varepsilon = \varphi(\varepsilon D). \tag{5.9}$$

We estimate $\|\rho_h J_\varepsilon u\|_{L^\infty}$ in terms of $\|u\|_{F^r}$, and then we apply another method to estimate $\|\rho_h(u - J_\varepsilon u)\|_{L^\infty}$. We take $\varepsilon = h^{1+\gamma}$ for some $\gamma > 0$, and find an optimal value of γ, where the two sorts of estimates have the same order of magnitude. To start, one has

$$\rho_h J_\varepsilon u(x) = \sum_{j \neq 0} \hat{u}(\nu j)\varphi(\nu\varepsilon j)e^{i\nu j\cdot x}. \tag{5.10}$$

This leads to the following estimate, established in (3.3) of [8].

Lemma 5.1 *If $r \in (0, n)$, then*

$$\begin{aligned} \|\rho_h J_\varepsilon u\|_{L^\infty} &\le C_1 \varepsilon^r \Big(\frac{h}{\varepsilon}\Big)^n \|u\|_{F^r} \\ &= C_1 h^{r-(n-r)\gamma}\|u\|_{F^r}, \end{aligned} \tag{5.11}$$

for all $u \in F^r(\mathbb{T}^n)$, the second formula holding provided $\varepsilon = h^{1+\gamma}$.

We move to an analysis of $\rho_h(u - J_\varepsilon u)$, in the following situation. Choose φ satisfying (5.9) and also

$$\varphi(\xi) = \hat{\psi}(\xi), \quad \operatorname{supp}\psi \subset \{x : |x| \le 1\}, \quad \psi \ge 0. \tag{5.12}$$

Let $\Omega \subset \mathbb{T}^n$ be open, and assume

$$\begin{gathered} u \text{ vanishes outside } \overline{\Omega} \text{ and is Lipschitz on } \overline{\Omega}, \text{ with seminorm} \\ \operatorname{Lip}_{\overline{\Omega}}(u) = \sup_{x,y\in\overline{\Omega}} \frac{|u(x) - u(y)|}{|x - y|}. \end{gathered} \tag{5.13}$$

Then

$$|u(x) - J_\varepsilon u(x)| \le \varepsilon \operatorname{Lip}_{\overline{\Omega}}(u), \quad x \in \mathbb{T}^n \setminus \Sigma_\varepsilon, \tag{5.14}$$

where

$$\Sigma_\varepsilon = \{x \in \mathbb{T}^n : \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}. \tag{5.15}$$

Write

$$u - J_\varepsilon u = (\chi_{\mathbb{T}^n\setminus\Sigma_\varepsilon} + \chi_{\Sigma_\varepsilon})(u - J_\varepsilon u) = v_\varepsilon + w_\varepsilon, \tag{5.16}$$

so

$$|v_\varepsilon(x)| \le \varepsilon \operatorname{Lip}_{\overline{\Omega}}(u), \quad \forall\, x \in \mathbb{T}^n,$$
$$\operatorname{supp} w_\varepsilon \subset \Sigma_\varepsilon, \quad \|w_\varepsilon\|_{L^\infty} \le 2\|u\|_{L^\infty(\Sigma_{2\varepsilon})}. \tag{5.17}$$

Then $\rho_h(u - J_\varepsilon u) = \rho_h v_\varepsilon + \rho_h w_\varepsilon$, and clearly

$$|\rho_h v_\varepsilon(x)| \le 2\varepsilon \operatorname{Lip}_{\overline{\Omega}}(u). \tag{5.18}$$

The estimate (5.18) can sometimes be improved by noting that $v_\varepsilon(x) = 0$ for $\operatorname{dist}(x, \overline{\Omega}) \ge \varepsilon$, and hence

$$\operatorname{supp} v_\varepsilon \subset \Omega. \tag{5.19}$$

We return to this point below.

For now we turn to an estimate on $\rho_h w_\varepsilon$, given the information on w_ε in (5.17). Let us abstract the situation, and suppose w is a bounded Borel function on $\mathbb{T}^n$, satisfying

$$\operatorname{supp} w \subset K, \quad \sup |w| = \|w\|_{L^\infty}, \tag{5.20}$$

with $K \subset \mathbb{T}^n$ compact. We want an estimate on $\|\rho_h w\|_{L^\infty}$ that takes careful account of metric properties of K. As (5.17) suggests, we expect to encounter situations where K is rather "thin" and do not expect cancellations between Mw and $\sigma_h w$, so our task here is actually to estimate these quantities separately. The first is easy:

$$|Mw| \le (2\pi)^{-n} (\operatorname{Vol} K) \|w\|_{L^\infty}. \tag{5.21}$$

To estimate $\sigma_h w$, we bring in the following quantities. First, we set

$$N_h(K, x) = \#\{\lambda \in \Lambda_h : x + \lambda \in K\}, \quad \text{where}$$
$$\Lambda_h = \{h\ell \in \mathbb{T}^n : \ell \in (\mathbb{Z}/\nu)^n\}. \tag{5.22}$$

Recall from (5.1) that $h = 2\pi/\nu$. Then we set

$$N_h(K) = \sup_x N_h(K, x), \quad V_h(K) = h^n N_h(K). \tag{5.23}$$

Let us note that

$$\begin{aligned} N_h(K,x) &= \sum_{\lambda \in \Lambda_h} \chi_{K-\lambda}(x) \\ \Longrightarrow (2\pi)^{-n} \int_{\mathbb{T}^n} N_h(K,x)\, dx &= (2\pi)^{-n} \sum_{\lambda \in \Lambda_h} \operatorname{Vol} K = h^{-n} \operatorname{Vol} K \\ \Longrightarrow V_h(K) &\ge \operatorname{Vol} K. \end{aligned} \tag{5.24}$$

It follows directly from the definitions of $N_h(K,x)$ and of $\sigma_h w(x)$ that

$$\|\sigma_h w\|_{L^\infty} \le (2\pi)^{-n} V_h(K)\, \|w\|_{L^\infty}, \tag{5.25}$$

hence

$$\|\rho_h w\|_{L^\infty} \le (2\pi)^{-n} \Big\{ \operatorname{Vol} K + V_h(K) \Big\} \|w\|_{L^\infty}. \tag{5.26}$$

Returning to the source of our interest in the estimate (5.26), given (5.20), we apply this to w_ε, defined in (5.16) and described in (5.17). Then (5.26) yields

$$\|\rho_h w_\varepsilon\|_{L^\infty} \le 2(2\pi)^{-n} \Big\{ \operatorname{Vol} \Sigma_\varepsilon + V_h(\Sigma_\varepsilon) \Big\} \|u\|_{L^\infty(\Sigma_{2\varepsilon})}, \tag{5.27}$$

under the hypothesis (5.13) on u, and with Σ_ε given by (5.15). In combination with the estimate (5.18) on $\rho_h v_\varepsilon$, we have

$$\begin{aligned} &\|\rho_h(u - J_\varepsilon u)\|_{L^\infty} \\ &\le 2\varepsilon \operatorname{Lip}_{\overline{\Omega}}(u) + 2(2\pi)^{-n} \Big\{ \operatorname{Vol} \Sigma_\varepsilon + V_h(\Sigma_\varepsilon) \Big\} \|u\|_{L^\infty(\Sigma_{2\varepsilon})}. \end{aligned} \tag{5.28}$$

This is to be combined with the estimate (5.11) on $\rho_h J_\varepsilon u$, under the further hypothesis that $u \in F^r(\mathbb{T}^n)$, for some $r \in (0, n)$.

As we will see, the occasion also arises to let Ω depend on h and/or ε, and then it is useful to sharpen (5.18) to the estimate

$$\|\rho_h v_\varepsilon\|_{L^\infty} \le 2(2\pi)^{-n} \Big\{ \operatorname{Vol} \overline{\Omega} + V_h(\overline{\Omega}) \Big\} \varepsilon \operatorname{Lip}_{\overline{\Omega}}(u), \tag{5.29}$$

and to make the corresponding adjustment on the right side of (5.28).

At this point we bring together various estimates established above, and record the following result, which generalizes Proposition 3.2 of [8].

Proposition 5.2 *Let $\Omega \subset \mathbb{T}^n$ be open, and assume u is a bounded function on $\mathbb{T}^n$ satisfying* (5.13). *Also assume $u \in F^r(\mathbb{T}^n)$, for some $r \in (0, n)$. Then, for $\varepsilon, h \in (0, 1]$,*

$$\begin{aligned} \|\rho_h u\|_{L^\infty} \le\ & 2(2\pi)^{-n} \Big\{ \operatorname{Vol} \overline{\Omega} + V_h(\overline{\Omega}) \Big\} \varepsilon \operatorname{Lip}_{\overline{\Omega}}(u) \\ & + 2(2\pi)^{-n} \Big\{ \operatorname{Vol} \Sigma_\varepsilon + V_h(\Sigma_\varepsilon) \Big\} \|u\|_{L^\infty(\Sigma_{2\varepsilon})} \\ & + C_1 \varepsilon^r \Big(\frac{h}{\varepsilon} \Big)^n \|u\|_{F^r}. \end{aligned} \tag{5.30}$$

In this statement, recall that Σ_ε is given by (5.15), and that ultimately we will take $\varepsilon = h^{1+\gamma}$, for some $\gamma > 0$.

Now it seems that, in the pursuit of one lattice point problem, we have acquired two others. However, we need only upper bounds on $V_h(\overline{\Omega})$ and $V_h(\Sigma_\varepsilon)$. When Ω is fat, as it will be in the proof of (4.9), we use the universal bound

$$V_h(\overline{\Omega}) \le (2\pi)^n. \tag{5.31}$$

When Ω is thin, as in the proof of (3.6) and (4.14), we can take further steps, as indicated in the discussion of Σ_ε below, but in the case at hand, $\mathrm{Lip}_{\overline{\Omega}}(u) = 0$, and we do not need to do so.

As for Σ_ε, the ε-neighborhood of $\partial\Omega$, this is thin, if $\partial\Omega$ is sufficiently smooth. We turn to an estimate of $V_h(\Sigma_\varepsilon)$. We assume $\partial\Omega$ is rectifiable, and bring in the surface measure

$$\mu = \mathcal{H}^{n-1} \rfloor \partial\Omega. \tag{5.32}$$

We also assume $\partial\Omega$ is lower Ahlfors regular, satisfying

$$x \in \partial\Omega \Longrightarrow \mu(B_{\varepsilon/4}(x)) \ge b(\partial\Omega)\varepsilon^{n-1}, \tag{5.33}$$

for $\varepsilon \in (0, 1]$. Now take φ and ψ as in (5.9) and (5.12), and arrange in addition that

$$\psi(x) \ge A, \quad \text{for } \ |x| \le \frac{1}{2}. \tag{5.34}$$

One can take $A = 1/2\,\mathrm{Vol}(B_{1/2})$, where $B_{1/2}$ is a ball in $\mathbb{R}^n$ of radius 1/2. As before, take $J_\varepsilon = \varphi(\varepsilon D)$, so

$$J_\varepsilon \mu = \psi_\varepsilon * \mu, \quad \psi_\varepsilon(x) = \varepsilon^{-n}\psi(\varepsilon^{-1}x). \tag{5.35}$$

The hypothesis (5.33) implies that

$$\begin{aligned} x \in \Sigma_{\varepsilon/4} &\Longrightarrow \mu(B_{\varepsilon/2}(x)) \ge b(\partial\Omega)\varepsilon^{n-1} \\ &\Longrightarrow J_\varepsilon\mu(x) \ge A b(\partial\Omega)\varepsilon^{-1}. \end{aligned} \tag{5.36}$$

Hence

$$\begin{aligned} N_h(\Sigma_{\varepsilon/4}, x) &\le \frac{\varepsilon}{A b(\partial\Omega)} \sum_{\lambda \in \Lambda_h} J_\varepsilon\mu(x+\lambda) \\ &= \frac{\varepsilon \nu^n}{A b(\partial\Omega)} \sigma_h J_\varepsilon \mu(x). \end{aligned} \tag{5.37}$$

Now

$$\sigma_h J_\varepsilon\mu(x) = M J_\varepsilon\mu + \rho_h J_\varepsilon\mu(x), \quad M J_\varepsilon \mu = \mathrm{Area}\,\partial\Omega. \tag{5.38}$$

Recalling (5.22)–(5.23), we obtain

$$V_h(\Sigma_{\varepsilon/4}) \le \frac{\varepsilon}{Ab(\partial\Omega)}(2\pi)^n\Big\{\text{Area}\,\partial\Omega + \|\rho_h J_\varepsilon \mu\|_{L^\infty}\Big\}. \tag{5.39}$$

We now apply Lemma 5.1, with u replaced by μ, assumed to belong to $F^{r-1}(\mathbb{T}^n)$, obtaining

$$\|\rho_h J_\varepsilon \mu\|_{L^\infty} \le C_1 \varepsilon^{r-1}\Big(\frac{h}{\varepsilon}\Big)^n \|\mu\|_{F^{r-1}}. \tag{5.40}$$

Replacing $\varepsilon/4$ by ε, we have the following.

Proposition 5.3 *Assume $\partial\Omega$ is rectifiable, with surface measure μ satisfying* (5.33) *and*

$$\mu \in F^{r-1}(\mathbb{T}^n), \quad r-1 \in (0,n). \tag{5.41}$$

Then

$$V_h(\Sigma_\varepsilon) \le \frac{4(2\pi)^n}{Ab(\partial\Omega)}\Big\{\varepsilon\,\text{Area}\,\partial\Omega + C_2\varepsilon^r\Big(\frac{h}{\varepsilon}\Big)^n\|\mu\|_{F^{r-1}}\Big\}, \tag{5.42}$$

where $C_2 = 4^{r-1-n}C_1$.

By (5.24), we can also take (5.42) as an upper bound for $\text{Vol}\,\Sigma_\varepsilon$. Furthermore, $\text{Area}\,\partial\Omega = \hat{\mu}(0) \le \|\mu\|_{F^{r-1}}$. Plugging these estimates into (5.30) gives an estimate of the form

$$\|\rho_h u\|_{L^\infty} \le \mathfrak{C}_1\varepsilon + \mathfrak{C}_2\varepsilon^r\Big(\frac{h}{\varepsilon}\Big)^n, \tag{5.43}$$

where $\mathfrak{C}_j$ depend on a number of quantities associated to u, Ω, and μ. Note that

$$\begin{aligned}\varepsilon = \varepsilon^r\Big(\frac{h}{\varepsilon}\Big)^n &\Longleftrightarrow \varepsilon^{n+1-r} = h^n \\ &\Longleftrightarrow \varepsilon = h^{1+\gamma}, \quad \text{with} \quad \gamma = \frac{r-1}{n-(r-1)}.\end{aligned} \tag{5.44}$$

Using this value for ε in (5.43) then gives the following.

Proposition 5.4 *There is a constant $C = C(n,r)$ with the following significance. Let $\Omega \subset \mathbb{T}^n$ be open, u satisfy* (5.13), *$\partial\Omega$ satisfy* (5.32)–(5.33), *and assume*

$$u \in F^r(\mathbb{T}^n), \quad \mu \in F^{r-1}(\mathbb{T}^n), \quad \textit{for some} \quad r \in (1,n). \tag{5.45}$$

Then

$$\|\rho_h u\|_{L^\infty} \\ \le Ch^{1+\gamma}\Big\{V_h(\overline{\Omega})\,\mathrm{Lip}_{\overline{\Omega}}(u) + b(\partial\Omega)^{-1}\|\mu\|_{F^{r-1}}\|u\|_{L^\infty} + \|u\|_{F^r}\Big\}, \tag{5.46}$$

with

$$\gamma = \frac{r-1}{n-(r-1)}. \tag{5.47}$$

Example. If $\partial\Omega$ is smooth and has nowhere vanishing Gauss curvature, and if $u \in C^\infty(\overline{\Omega})$, classical stationary-phase arguments yield (5.45) with

$$r = \frac{n+1}{2}, \quad \text{hence} \;\; \gamma = \frac{n-1}{n+1}, \tag{5.48}$$

and then (5.46) leads to the classical estimate

$$\|\rho_h u\|_{L^\infty} \le Ch^{2n/(n+1)}, \tag{5.49}$$

which applies to (2.6) (with $\alpha = 1/3$).

Proof of (4.9)**.** Our strategy for this task will be to show that Proposition 5.4 is applicable, with

$$\overline{\Omega} = \mathcal{O}_a(1), \quad u = \chi_{\overline{\Omega}}, \tag{5.50}$$

and $\mathcal{O}_a(R)$ defined by (4.4) ($a \in (0,\infty) \cup (-1,0)$). For $a = 1$, this is the unit disk, but otherwise, as noted in §4, the region has four corners, and, for $a > 1$ or $a \in (-1,0)$, it is not convex. Nevertheless, we will show that

$$u = \chi_{\overline{\Omega}} \in F^{3/2}(\mathbb{T}^2), \quad \mu \in F^{1/2}(\mathbb{T}^2), \tag{5.51}$$

where μ is arclength measure on $\partial\Omega$. Once we have this, (5.46) gives

$$\|\rho_h u\|_{L^\infty} \le Ch^{4/3}, \tag{5.52}$$

which upon scaling, gives (4.9), with $\alpha = 1/3$.

We start with the analysis of μ, and use the fact that $\partial\Omega$ is contained in the union of two smooth curves with nowhere vanishing curvature, namely

$$\gamma_0 = \{(x,y) \in \mathbb{R}^2 : x^2 + ay^2 = 1\}, \\ \gamma_1 = \{(x,y) \in \mathbb{R}^2 : ax^2 + y^2 = 1\}, \tag{5.53}$$

which are ellipses if $a > 0$ and hyperbolas if $-1 < a < 0$. We can chop μ into four pieces, $\mu = \mu_{00} + \mu_{01} + \mu_{10} + \mu_{11}$, with the property that μ_{00} and μ_{01}

are compactly supported on γ_0, and μ_{10} and μ_{11} are compactly supported on γ_1. Each such measure is arclength measure, on the particular arc which is its support. We have

$$\mu_{ij} = \chi_{ij}\mu_i, \tag{5.54}$$

where μ_i is arclength measure on γ_i and χ_{ij} is the characteristic function of the interval supporting μ_{ij}. The result on μ stated in (5.51) will follow once we establish that

$$\mu_{ij} \in F^{1/2}(\mathbb{R}^2). \tag{5.55}$$

It is convenient to extend the setting a bit. Let $\gamma \subset \mathbb{R}^2$ be a smooth curve with nowhere vanishing curvature, and let $f : \gamma \to \mathbb{R}$ be a piecewise smooth function with compact support and a finite number of simple jumps. Let σ denote arc-length measure on γ, and set $\lambda = f\sigma$. We assert that $\lambda \in F^{1/2}(\mathbb{R}^2)$. Using a partition of unity, we see it suffices to assume f has only one jump, say at $p \in \gamma$, and that f is supported on a small neighborhood of p. Write

$$\hat{\lambda}(\xi) = \int\limits_{\gamma} e^{i|\xi|\psi(x,\zeta)} f(x)\, d\sigma(x), \tag{5.56}$$

with $\xi = |\xi|\zeta,\ \psi(x,\zeta) = x \cdot \zeta$. It is elementary to check that, under the hypotheses made above, $\hat{\lambda}(\xi) = O(|\xi|^{-1})$ outside a small conic neighborhood of $\pm N(p)$, where, for $y \in \gamma$, $N(y)$ denotes the outward-pointing unit normal. The remaining task is to estimate $|\hat{\lambda}(\xi)|$ for $\xi/|\xi|$ in a small neighborhood of $\pm N(p)$. Since $\hat{\lambda}(-\xi) = \overline{\hat{\lambda}(\xi)}$, we can concentrate on $\xi = |\xi|\zeta$, with ζ close to $N(p)$.

For this, we can bring in a family of coordinate charts $\chi_\zeta : I \to \mathcal{O} \subset \gamma$, where I is an interval in $\mathbb{R}$ and $\mathcal{O}$ is a neighborhood of $\operatorname{supp} f$, satisfying

$$N(\chi_\zeta(0)) = \zeta, \tag{5.57}$$

and

$$\psi(\chi_\zeta(x), \zeta) = \zeta \cdot N^{-1}(\zeta) - \frac{|x|^2}{4}. \tag{5.58}$$

Hence, for $\xi = |\xi|\zeta$,

$$\hat{\lambda}(\xi) = \int\limits_{I} e^{i|\xi|(\zeta\cdot N^{-1}(\zeta) - |x|^2/4)} f(\chi_\zeta(x)) J_\zeta(x)\, dx. \tag{5.59}$$

We set

$$f_\zeta(x) = f(\chi_\zeta(x)) J_\zeta(x), \quad t = |\xi|^{-1}, \tag{5.60}$$

and conclude that

$$\hat{\lambda}(\xi) = e^{i|\xi|\zeta\cdot N^{-1}(\zeta)}(4\pi it)^{1/2} e^{it\partial_x^2} f_\zeta(0), \tag{5.61}$$

where $e^{it\partial_x^2}$ is the solution operator to the 1D Schrödinger equation. Here f_ζ is a compactly supported function on $\mathbb{R}$, piecewise smooth with one simple jump, at $\chi_\zeta^{-1}(p)$. A straightforward argument involving the Fresnel integral (cf. [11], Proposition 4.1) yields

$$\sup_{x,t} \left| e^{it\partial_x^2} f_\zeta(x) \right| \leq C\|f_\zeta\|_{\mathrm{BV}(\mathbb{R})}. \tag{5.62}$$

This then gives

$$|\hat{\lambda}(\xi)| \leq C|\xi|^{-1/2}, \tag{5.63}$$

hence $\lambda \in F^{1/2}(\mathbb{R}^2)$. In particular, we have (5.55), so in (5.51) we have $\mu \in F^{1/2}(\mathbb{T}^2)$.

We next check that $\chi_{\overline{\Omega}} \in F^{3/2}(\mathbb{T}^2)$ in (5.51). This is equivalent to the assertion that

$$\partial_j \chi_{\overline{\Omega}} \in F^{1/2}(\mathbb{R}^2), \quad \text{for} \quad j = 1, 2. \tag{5.64}$$

In turn,

$$\partial_j \chi_{\overline{\Omega}} = -(N \cdot e_j)\mu, \tag{5.65}$$

with μ as in (5.51), and N the unit outward pointing normal to $\partial\Omega$. In turn, it is readily verified that a decomposition and analysis parallel to (5.54) applies to this situation, yielding (5.64), and completing the proof of (4.9).

Proof of (3.6) **and** (4.14)**.** We concentrate on (4.14), which is more general. It is convenient to bring in an argument parallel to that involving (4.4), and set

$$\mathcal{E}_{a,R} = \bigcup_{m=0}^{3} J^m E_{a,R}. \tag{5.66}$$

Then $\operatorname{Area} \mathcal{E}_{a,R} = 4 \operatorname{Area} E_{a,R}$, and, parallel to (4.5),

$$\#(\mathcal{E}_{a,R} \cap \mathbb{Z}^2) = 4 \cdot \#(E_{a,R} \cap \mathbb{Z}^2) + O(1), \tag{5.67}$$

so (4.14) is equivalent to the assertion that

$$\#(\mathcal{E}_{a,R} \cap \mathbb{Z}^2) = \operatorname{Area} \mathcal{E}_{a,R} + O(R^{1-\alpha}). \tag{5.68}$$

We now form a family of domains Ω to which to apply Proposition 5.4. For $R > 1$, take $\nu \in \mathbb{N}$ such that

$$|\nu - 2\pi R| < 1, \tag{5.69}$$

as usual, take $h = 2\pi/\nu$, and set

$$\Omega_R^b = h\mathcal{E}_{a,R}, \quad hR = 1 + O(R^{-1}). \tag{5.70}$$

Hence

$$\#(\mathcal{E}_{a,R} \cap \mathbb{Z}^2) = \#(\Omega_R^b \cap \Lambda_h). \tag{5.71}$$

Each set Ω_R^b is an annular region, containing its outer boundary but not its inner boundary. We denote the interior of Ω_R^b by Ω_R, and its closure by $\overline{\Omega}_R$. Upon addressing the minor difference between Ω_R^b and $\overline{\Omega}_R$ (as we do below), we have that (5.68) is equivalent to the estimate

$$\|\rho_h u_R\|_{L^\infty} \le C h^{1+\alpha}, \quad u_R = \chi_{\overline{\Omega}_R}. \tag{5.72}$$

To deduce this from (5.46), with $\Omega = \Omega_R$, with $\Omega = \Omega_R,\ u = u_R$, note that $\mathrm{Lip}_{\overline{\Omega}_R}(u_R) = 0$, so what we need to show (parallel to (5.51)) is that

$$\begin{aligned} u_R \in F^{3/2}(\mathbb{T}^2), &\quad \|u_R\|_{F^{3/2}} \le C, \\ \mu_R \in F^{1/2}(\mathbb{T}^2), &\quad \|\mu_R\|_{F^{1/2}} \le C, \end{aligned} \tag{5.73}$$

where μ_R is arc-length measure on $\partial\Omega_R^b$. First, μ_R is the sum of arc length measures on the two connected components of $\partial\Omega_R$, and each of these is amenable to an analysis similar to that done on arc length measure μ in (5.51). Next, $\partial_j u_R$ has an analysis parallel to that described in (5.64)–(5.65). Consequently, Proposition 5.4 yields (5.72). To relate this to (5.71), it suffices to note that an analogue of Proposition 5.3 holds. Hence we have the estimates (3.6) and (4.14).

The proof of Theorem 1.1 is complete.

References

[1] P. Berard, On the wave equation on a compact Riemannian manifold without conjugate points, Math. Zeit. 155 (1977), 249–276.

[2] J. Bourgain and N. Watt, Mean square of zeta function, circle problem, and divisor problem revisited, arXiv:1709.04340, 2017.

[3] Y. Canzani and J. Galkowski, Weyl remainder estimates: an application of geodesic beams, Preprint, 2020.

[4] J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975), 49–79.

[5] D. Hejhal, The Selberg trace formula, and the Riemann zeta function, Duke Math. J. 43 (1976), 441–482.

[6] L. Hörmander, The spectral function of an elliptic operator, Acta Math. 121 (1968), 193–218.

[7] A. Iosevich and E. Weyman, Weyl law improvement for products of spheres, Preprint, 2019.

[8] J. Rauch and M. Taylor, Quadrature estimates for multidimensional integrals, Houston J. Math. 36 (2010), 727–749.

[9] M. Taylor, Partial Differential Equations, Springer, New York, 1996 (2nd ed. 2011).

[10] M. Taylor, Introduction to Analysis in Several Variables – Advanced Calculus, AMS, 2020.

[11] M. Taylor, Short time behavior of solutions to nonlinear Schrödinger equations, Comm. PDE 31 (2006), 945–957.