

## Calculating Natural Logarithms

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The integral formula

$$(1) \quad \log(1+x) = \int_0^x \frac{dy}{1+y}$$

leads to the power series

$$(2) \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

convenient for calculating  $\log a$  when  $|a-1|$  is small. However, we desire to compute  $\log a$  when this quantity is not so small, for example,  $a = 2$ . Setting  $x = 1$  in (2) yields

$$(3) \quad \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which actually does converge, but very slowly. One would not want to use (3) to compute  $\log 2$  to 12 decimal places. A better formula arises from replacing  $x$  by  $-x$  and using

$$(4) \quad \log \frac{1}{a} = -\log a.$$

We get

$$(5) \quad \lambda(x) := \log \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

(the first identity defining  $\lambda(x)$ ), also convergent for  $|x| < 1$ , and

$$(6) \quad \log 2 = \lambda\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{1}{2}\right)^k.$$

This has better convergence properties; 40 terms yield 12 digits of accuracy. Similarly,

$$(7) \quad \log 5 = \lambda\left(\frac{4}{5}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{4}{5}\right)^k,$$

but this converges a bit more slowly than (6).

Here we use the identity

$$(8) \quad \log ab = \log a + \log b$$

to show how we can get power series for  $\log n$ , for various values of  $n$ , that converge much faster than (6) (not to mention (7)). The key is to simultaneously produce power series for  $\log n$  for several different values of  $n$ . For example,

$$(9) \quad \begin{aligned} \log 3 &= \log 2 + \log \frac{3}{2} = \log 2 + \lambda\left(\frac{1}{3}\right), \\ \log 4 &= \log 3 + \log \frac{4}{3} = \log 3 + \lambda\left(\frac{1}{4}\right), \end{aligned}$$

gives the pair of equations

$$(10) \quad \begin{aligned} \log 2 - \log 3 &= -\lambda\left(\frac{1}{3}\right), \\ 2 \log 2 - \log 3 &= \lambda\left(\frac{1}{4}\right), \end{aligned}$$

hence we get

$$(11) \quad \begin{aligned} \log 2 &= \lambda\left(\frac{1}{3}\right) + \lambda\left(\frac{1}{4}\right), \\ \log 3 &= 2\lambda\left(\frac{1}{3}\right) + \lambda\left(\frac{1}{4}\right), \end{aligned}$$

and the power series  $\lambda(1/3)$  and  $\lambda(1/4)$  converge faster than (6), at least moderately so. We can do better, starting with

$$(12) \quad 2^3 = 8, \quad 3^2 = 9 \implies 2 \log 3 = 3 \log 2 + \lambda\left(\frac{1}{9}\right).$$

We can combine this with

$$(13) \quad \begin{aligned} 2^4 = 16, \quad 3 \cdot 5 = 15 &\implies 4 \log 2 = \log 3 + \log 5 + \lambda\left(\frac{1}{16}\right), \\ 2 \cdot 5 = 10, \quad 3^2 = 9 &\implies \log 2 + \log 5 = 2 \log 3 + \lambda\left(\frac{1}{10}\right). \end{aligned}$$

Together, (12)–(13) provide 3 linear equations for  $\log 2$ ,  $\log 3$ , and  $\log 5$ , which are readily solved to yield

$$(14) \quad \begin{aligned} \log 2 &= 3\lambda\left(\frac{1}{9}\right) + 2\lambda\left(\frac{1}{10}\right) + 2\lambda\left(\frac{1}{16}\right), \\ \log 3 &= 5\lambda\left(\frac{1}{9}\right) + 3\lambda\left(\frac{1}{10}\right) + 3\lambda\left(\frac{1}{16}\right), \\ \log 5 &= 2 \log 3 - \log 2 + \lambda\left(\frac{1}{10}\right). \end{aligned}$$

Other identities involving the logs of 2, 3, and 5, arise as follows:

$$(15) \quad \begin{aligned} 5^2 = 25, \quad 8 \cdot 3 = 24 &\implies 2 \log 5 = 3 \log 2 + \log 3 + \lambda\left(\frac{1}{25}\right), \\ 3^3 = 27, \quad 5^2 = 25 &\implies 3 \log 3 = 2 \log 5 + \lambda\left(\frac{2}{27}\right), \\ 3^4 = 81, \quad 16 \cdot 5 = 80 &\implies 4 \log 3 = 4 \log 2 + \log 5 + \lambda\left(\frac{1}{81}\right), \\ 2^7 = 128, \quad 5^3 = 125 &\implies 7 \log 2 = 3 \log 5 + \lambda\left(\frac{3}{128}\right). \end{aligned}$$

For an improvement on the formulas in (14), we can use the first, third, and fourth identities in (15) to produce three linear equations for  $\log 2$ ,  $\log 3$ , and  $\log 5$ , and solve this system, to get

$$(15A) \quad \begin{aligned} \log 2 &= 12\lambda\left(\frac{1}{25}\right) + 3\lambda\left(\frac{1}{81}\right) + 7\lambda\left(\frac{3}{128}\right), \\ \log 5 &= \frac{7}{3}\log 2 - \frac{1}{3}\lambda\left(\frac{3}{128}\right), \\ \log 3 &= \frac{2}{3}\log 5 + \frac{1}{3}\left[\lambda\left(\frac{1}{25}\right) + \lambda\left(\frac{1}{81}\right) + \lambda\left(\frac{3}{128}\right)\right]. \end{aligned}$$

We move along to the computation of  $\log 7$ , using

$$(16) \quad 3 \cdot 7 = 21, \quad 4 \cdot 5 = 20 \implies \log 3 + \log 7 = 2\log 2 + \log 5 + \lambda\left(\frac{1}{21}\right),$$

or alternatives, such as

$$(17) \quad \begin{aligned} 7^2 = 49, \quad 16 \cdot 3 = 48 &\implies 2\log 7 = 4\log 2 + \log 3 + \lambda\left(\frac{1}{49}\right), \\ 2 \cdot 5^2 = 50, \quad 7^2 = 49 &\implies 2\log 5 + \log 2 = 2\log 7 + \lambda\left(\frac{1}{50}\right). \end{aligned}$$

Next we can tackle  $\log 11$ , using

$$(18) \quad \log 11 = \log 2 + \log 5 + \lambda\left(\frac{1}{11}\right),$$

or, alternatively,

$$(19) \quad \begin{aligned} 2 \cdot 11 = 22, \quad 3 \cdot 7 = 21 &\implies \log 11 + \log 2 = \log 3 + \log 7 + \lambda\left(\frac{1}{22}\right), \\ 3 \cdot 11 = 33, \quad 2^5 = 32 &\implies \log 3 + \log 11 = 5\log 2 + \lambda\left(\frac{1}{33}\right), \\ 11^2 = 121, \quad 3 \cdot 5 \cdot 8 = 120 &\implies 2\log 11 = 3\log 2 + \log 3 + \log 5 + \lambda\left(\frac{1}{121}\right). \end{aligned}$$

To calculate  $\log 13$ , we can use

$$(20) \quad \log 13 = 2\log 2 + \log 3 + \lambda\left(\frac{1}{13}\right),$$

or alternatively,

$$(21) \quad \begin{aligned} 3^3 = 27, \quad 2 \cdot 13 = 26 &\implies 3\log 3 = \log 2 + \log 13 + \lambda\left(\frac{1}{27}\right), \\ 8 \cdot 5 = 40, \quad 3 \cdot 13 = 39 &\implies 3\log 2 + \log 5 = \log 3 + \log 13 + \lambda\left(\frac{1}{40}\right), \\ 5 \cdot 13 = 65, \quad 2^6 = 64 &\implies \log 5 + \log 13 = 6\log 2 + \lambda\left(\frac{1}{65}\right). \end{aligned}$$

Moving to  $\log 17$ , we can use

$$(22) \quad \log 17 = 4\log 2 + \lambda\left(\frac{1}{17}\right),$$

or alternatively

$$(23) \quad \begin{aligned} 5 \cdot 7 = 35, \quad 2 \cdot 17 = 34 &\implies \log 5 + \log 7 = \log 2 + \log 17 + \lambda\left(\frac{1}{35}\right), \\ 3 \cdot 17 = 51, \quad 2 \cdot 25 = 50 &\implies \log 3 + \log 17 = \log 2 + 2\log 5 + \lambda\left(\frac{1}{51}\right), \\ 2 \cdot 5 \cdot 17 = 170, \quad 13^2 = 169 &\implies \log 2 + \log 5 + \log 17 = 2\log 13 + \lambda\left(\frac{1}{170}\right). \end{aligned}$$

We can proceed further, using

$$(24) \quad \begin{aligned} \log 19 &= \log 18 + \lambda\left(\frac{1}{19}\right) = 2 \log 3 + \log 2 + \lambda\left(\frac{1}{19}\right), \\ \log 23 &= \log 24 - \lambda\left(\frac{1}{24}\right) = 3 \log 2 + \log 3 - \lambda\left(\frac{1}{24}\right), \end{aligned}$$

and so forth.

Having logs of primes, we can evaluate  $\log n$  when  $n$  is a product of primes, via (8). For example,

$$(25) \quad \begin{aligned} \log 10 &= \log 2 + \log 5 \\ &= 2 \log 3 + \lambda\left(\frac{1}{10}\right) \\ &= 10\lambda\left(\frac{1}{9}\right) + 7\lambda\left(\frac{1}{10}\right) + 6\lambda\left(\frac{1}{16}\right), \end{aligned}$$

the latter two identities by (14). Alternatively, we can use the last identity in (15) to write

$$(26) \quad \log 10 = \frac{10}{3} \log 2 - \frac{1}{3} \lambda\left(\frac{3}{128}\right).$$

Then from (15A) we get

$$(27) \quad \log 10 = 40\lambda\left(\frac{1}{25}\right) + 10\lambda\left(\frac{1}{81}\right) + 23\lambda\left(\frac{3}{128}\right).$$

## Log Table

Here is a table of  $\log n$ , for  $n \in \{2, 3, 4, \dots, 21\}$ , to 15 digits after the decimal point.

$\log 2 = 0.693147180559945$	$\log 3 = 1.098612288668110$
$\log 4 = 1.386294361119891$	$\log 5 = 1.609437912428055$
$\log 6 = 1.791759469228055$	$\log 7 = 1.945910149055313$
$\log 8 = 2.079441541679836$	$\log 9 = 2.197224577336220$
$\log 10 = 2.302585092994046$	$\log 11 = 2.397895272798371$
$\log 12 = 2.484906649788000$	$\log 13 = 2.564949357461537$
$\log 14 = 2.639057329615258$	$\log 15 = 2.708050201102210$
$\log 16 = 2.772588722239781$	$\log 17 = 2.833213344056216$
$\log 18 = 2.890371757896165$	$\log 19 = 2.944438979166440$
$\log 20 = 2.995732273553991$	$\log 21 = 3.044522437723423$

An evaluation of  $\lambda(y)$ , at the values used in (15A) and (27), obtained by summing the first 11 terms in (5), yields

$$(28) \quad \begin{aligned} \lambda\left(\frac{1}{25}\right) &= 0.04082199452025513, \\ \lambda\left(\frac{1}{81}\right) &= 0.01242251999855715, \\ \lambda\left(\frac{3}{128}\right) &= 0.02371652661731604. \end{aligned}$$

One can check that using these values in (15A) and (27) yields  $\log 2$  and  $\log 10$  to full accuracy.