Product Manifolds with Small Weyl Remainders

Michael Taylor [*]

Abstract

We say a compact, ℓ-dimensional Riemannian manifold Y has spectral asymptotics with algebraically small Weyl remainder if the spctral counting function satisfies

$$\mathcal{N}(\Delta_Y, R) = C(Y)R^\ell + O(R^{\ell-1-\alpha}),$$

for some $\alpha > 0$. We show that if this holds, then each compact Cartesian product $M = X \times Y$ also has spectral asymptotics with algebraically small Weyl remainder.

Contents

[*]MSC-2020 Math Subject Classification: 35P20, 42B05, 43A85. *Key words.* Laplace operator, spectrum, joint spectrum, spectral counting function, Weyl asymptotics, wave trace

1 Introduction

Let M be a compact, m-dimensional Riemannian manifold, with Laplace operator Δ_M. Then $L^2(M)$ has an orthonormal basis of eigenfunctions u_j,

$$\Delta_M u_j = -\lambda_j^2 u_j, \quad 0 = \lambda_1 \le \cdots \le \lambda_j \nearrow +\infty. \tag{1.1}$$

We define the spectral counting function

$$\mathcal{N}(\Delta_M, R) = \#\{j \in \mathbb{N} : \lambda_j \le R\}. \tag{1.2}$$

It is classical that

$$\mathcal{N}(\Delta_M, R) = C(M)R^m + O(R^{m-1}). \tag{1.3}$$

See [5]. The unit sphere S^m is an example for which the remainder $O(R^{m-1})$ cannot be improved. There are a number of results that do yield improved Weyl remainder estimates. In [4] it is shown that one can take $o(R^{m-1})$ in case the geodesic flow on M has "not too many" periodic orbits. The paper [1] shows that under certain geometrical conditions, one can take

$$O(R^{m-1}/\log R) \tag{1.4}$$

as a remainder estimate in (1.3). Recently [3] obtained such an improved remainder estimes for a much broader class of Riemannian manifolds. Among the results obtained there is that (1.4) holds whenever M is a Cartesian product, $M = X \times Y$, of two Riemannian manifolds, with the product metric.

Now it is classically known that one can obtain remainder estimates of the form

$$O(R^{m-1-\alpha}), \tag{1.5}$$

with $\alpha > 0$, when $M = \mathbb{T}^m$ is a flat torus. (Just how big one can take α is an unsolved problem; see [2].) Recently [6] have obtained remainder estimates of the form (1.5) whenever M is a product of spheres,

$$M = S^{d_1} \times \cdots \times S^{d_k}, \quad m = d_1 + \cdots + d_k. \tag{1.6}$$

Also, [9] has a remainder estimate of the form (1.5) for $\mathcal{N}(L_a, R)$, when $M = S^2$ and $L_a = X_1^2 + X_2^2 + aX_3^2$, $a \in (0,1) \cup (1,\infty)$, where X_j generates 2π-periodic rotation about the x_j-axis in $\mathbb{R}^3$. The classical analysis of $M = \mathbb{T}^m$, uses lattice point estimates, and the analyses in [6] and [9] use certain weighted lattice point estimates, [9] bringing in results from [7].

In this paper we look further at Cartesian products,

$$M = X \times Y, \quad \dim X = n,\ \dim Y = \ell,\ m = n + \ell, \tag{1.7}$$

and establish the following.

Theorem 1.1 *Assume X and Y in* (1.7) *are compact Riemannian manifolds. Assume that, for some $\alpha \in (0,1)$,*

$$\mathcal{N}(\Delta_Y, \rho) = C(Y)\rho^\ell + R_Y(\rho), \quad R_Y(\rho) = O(\rho^{\ell-1-\alpha}), \tag{1.8}$$

Then

$$\mathcal{N}(\Delta_M, R) = C(M)R^m + O(R^{m-1-\alpha}). \tag{1.9}$$

The proof divides into two pieces. First, if $\{v_j\}$ is an orthonormal basis of $L^2(X)$ such that

$$\Delta_X v_j = -\mu_j^2 v_j, \quad \mu_j \nearrow +\infty, \tag{1.10}$$

then

$$\mathcal{N}(\Delta_M, R) = \sum_{\mu_j \le R} \mathcal{N}(\Delta_Y, \sqrt{R^2 - \mu_j^2}), \tag{1.11}$$

and we show in §2 that if the hypothesis (1.8) holds, then

$$\mathcal{N}(\Delta_M, R) = C(Y) \sum_{\mu_j \le R} (R^2 - \mu_j^2)^{\ell/2} + O(R^{m-1-\alpha}). \tag{1.12}$$

Now the sum over $\mu_j \le R$ on the right side of (1.12) is equal to

$$R^\ell \sum_{\mu_j \le R} \Bigl(1 - \Bigl(\frac{\mu_j}{R}\Bigr)^2\Bigr)^{\ell/2} = R^\ell \operatorname{Tr} \varphi_R(\Lambda), \tag{1.13}$$

with

$$\Lambda = \sqrt{-\Delta_X}, \quad \varphi_R(\lambda) = \varphi(\lambda/R), \quad \varphi(\lambda) = (1-\lambda^2)_+^{\ell/2}. \tag{1.14}$$

We show in §3 that, for φ as in (1.14), $\ell > 0$,

$$\operatorname{Tr} \varphi_R(\Lambda) = C_X(\varphi)R^n + O(R^{n-1-\gamma}), \tag{1.15}$$

with $\gamma = \ell/2$, and thereby obtain (1.9).

2 Reduction to study of $\operatorname{Tr}\varphi_R(\Lambda)$

Here we show that the hypothesis (1.8) leads to (1.12). To start, we complement the orthonormal basis $\{v_j\}$ of $L^2(X)$, satisfying (1.10), by the orthonormal basis $\{w_i\}$ of $L^2(Y)$, satisfying

$$\Delta_Y w_i = -\nu_i^2 w_i, \quad \nu_i \nearrow +\infty. \tag{2.1}$$

Thus we have an orthonormal basis $\{u_{ji}\}$ of $L^2(M)$:

$$u_{ji} = v_j w_i, \quad \Delta_M u_{ji} = -(\mu_j^2 + \nu_i^2) u_{ji}, \tag{2.2}$$

giving the spectrum of Δ_M in terms of the joint spectrum of (Δ_X, Δ_Y). We see that

$$\begin{aligned} \mathcal{N}(\Delta_M, R) &= \#\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu_j^2 + \nu_i^2 \le R^2\} \\ &= \#\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu_j \le R,\ \nu_i \le (R^2 - \mu_j^2)^{1/2}\} \\ &= \sum_{\mu_j \le R} \mathcal{N}(\Delta_Y, \sqrt{R^2 - \mu_j^2}). \end{aligned} \tag{2.3}$$

Now we bring in the hypothesis (1.8), i.e.,

$$\mathcal{N}(\Delta_Y, \rho) = C(Y)\rho^\ell + R_Y(\rho), \quad R_Y(\rho) = O(\rho^{\ell-1-\alpha}). \tag{2.4}$$

We obtain

$$\mathcal{N}(\Delta_M, R) = \sum_{\mu_j \le R} \Big\{ C(Y)(R^2 - \mu_j^2)^{\ell/2} + R_Y(\sqrt{R^2 - \mu_j^2}) \Big\}, \tag{2.5}$$

and

$$\begin{aligned} \sum_{\mu_j \le R} \Big| R_Y(\sqrt{R^2 - \mu_j^2}) \Big| &\le C \sum_{\mu_j \le R} (R^2 - \mu_j^2)^{(\ell-1-\alpha)/2} \\ &\le C R^{\ell-1-\alpha} \cdot \mathcal{N}(\Delta_X, R) \\ &\le C_2 R^{\ell-1-\alpha} \cdot R^n, \end{aligned} \tag{2.6}$$

in view of the analogue of (1.3) for X. Hence

$$(2.6) \le C_2 R^{m-1-\alpha}, \tag{2.7}$$

and (2.5) yields

$$\begin{aligned} \mathcal{N}(\Delta_M, R) = C(Y) \sum_{\mu_j \le R} (R^2 - \mu_j^2)^{\ell/2} + \mathcal{R}_{XY}(R), \\ |\mathcal{R}_{XY}(R)| \le C_2 R^{m-1-\alpha}, \end{aligned} \tag{2.8}$$

as asserted in (1.12). As noted in §1, we can write (2.8) as

$$\mathcal{N}(\Delta_M, R) = C(Y) R^\ell \ \mathrm{Tr}\, \varphi_R(\Lambda) + \mathcal{R}_{XY}(R), \tag{2.9}$$

with $\varphi_R(\Lambda)$ given by (1.14).

3 Analysis of $\operatorname{Tr} \varphi_R(\Lambda)$

We follow the results of §2 with an analysis of $\operatorname{Tr} \varphi_R(\Lambda)$, with $\varphi_R(\lambda) = \varphi(\lambda/R)$, for a variety of functions φ, including

$$\varphi(\lambda) = (1-\lambda^2)_+^{\ell/2}, \tag{3.1}$$

particularly for $\ell \in \mathbb{N}$, but we will also consider more general cases. We look at the following class of bounded, Borel functions φ. We assume that, for each $k \in \mathbb{N}$,

$$|\varphi(\lambda)| \le C_k \langle \lambda \rangle^{-k}, \tag{3.2}$$

and we also assume that each $\lambda \in \mathbb{R}$ is a Lebesgue point of φ. Note that (3.2) implies

$$\hat{\varphi} \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}), \quad \hat{\varphi}^{(k)} \in L^\infty(\mathbb{R}), \quad \forall\, k \ge 0. \tag{3.3}$$

A major tool will involve wave equation techniques, coming from

$$\psi(\Lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}(t) e^{it\Lambda}\, dt, \tag{3.4}$$

provided $\hat{\psi} \in L^1(\mathbb{R})$, which leads to the notion of the trace of $U(t) = e^{it\Lambda}$, as a tempered distribution, given by

$$\langle \hat{\psi}, \operatorname{Tr} U \rangle = \sqrt{2\pi}\, \operatorname{Tr} \psi(\Lambda), \quad \psi \in \mathcal{S}(\mathbb{R}). \tag{3.5}$$

See Appendix A for a general discussion of this approach. To apply this to $\varphi_R(\Lambda)$, it will be convenient to decompose $\varphi_R(\lambda)$ into two pieces, making use of

$$\beta \in C_0^\infty((-T_0, T_0)), \;\text{ such that } \beta(\lambda) = 1 \text{ for } |\lambda| \le T_0/2, \tag{3.6}$$

where $T_0 > 0$ will be specified below. We write

$$\varphi_R = \varphi_R^\# + \varphi_R^b, \quad \hat{\varphi}_R^\# = \beta \hat{\varphi}_R. \tag{3.7}$$

Hence

$$\varphi_R(\Lambda) = \varphi_R^\#(\Lambda) + \varphi_R^b(\Lambda), \tag{3.8}$$

with

$$\varphi_R^\#(\Lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t)\beta(t) e^{it\Lambda}\, dt, \tag{3.9}$$

an identity that fits into the mold (3.4). Thus, as in (3.5), we have

$$\operatorname{Tr} \varphi_R^{\#}(\Lambda) = \frac{1}{\sqrt{2\pi}} \langle \hat{\varphi}_R, v \rangle, \tag{3.10}$$

where

$$v = \beta \operatorname{Tr} U \in \mathcal{E}'(\mathbb{R}), \tag{3.11}$$

so $\hat{v} \in C^\infty(\mathbb{R})$ and is polynomially bounded, and we have

$$\operatorname{Tr} \varphi_R^{\#}(\Lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_R(\lambda) \hat{v}(\lambda) \, d\lambda. \tag{3.12}$$

To apply this, we make use of the fact that there exists $T_0 > 0$ such that

$$\operatorname{Tr} U\big|_{(-T_0,T_0)} \in C^\infty((-T_0, T_0) \setminus 0) \tag{3.13}$$

(cf. (A.22)), and use the seminal results of [5] and [4] (cf. (A.27)–(A.36)) that, if we take such T_0 in (3.6), then $\hat{v} \in S^{n-1}(\mathbb{R})$, satisfying

$$\hat{v}(\lambda) \sim (2\pi)^{-n} \sum_{k \geq 0} c_k \lambda^{n-1-k}, \quad \lambda \to +\infty, \tag{3.14}$$

(and $\hat{v}(\lambda)$ is rapidly decreasing as $\lambda \to -\infty$), with

$$c_0 = \operatorname{Vol} S^*X, \quad c_k = 0 \text{ for } k \text{ odd}. \tag{3.15}$$

In connection with this, note that

$$\begin{aligned} \int_0^\infty \varphi_R(\lambda) \lambda^{n-1-k} \, d\lambda &= \int_0^\infty \varphi\Big(\frac{\lambda}{R}\Big) \lambda^{n-1-k} \, d\lambda \\ &= R^{n-k} \int_0^\infty \varphi(\lambda) \lambda^{n-1-k} \, d\lambda, \end{aligned} \tag{3.16}$$

provided

$$0 \leq k \leq n-1. \tag{3.17}$$

This yields the following.

Proposition 3.1 *As $R \to \infty$,*

$$\begin{aligned} \operatorname{Tr} \varphi_R^{\#}(\Lambda) - C_X(\varphi) R^n &= O(R^{n-2}), \quad \text{if } n \geq 3, \\ & O(\log R), \quad \text{if } n = 2, \end{aligned} \tag{3.18}$$

where

$$C_X(\varphi) = (2\pi)^{-n} \operatorname{Vol} S^*X \int_0^\infty \varphi(\lambda) \lambda^{n-1} \, d\lambda. \tag{3.19}$$

We turn to an estimate of $\operatorname{Tr} \varphi_R^b(\Lambda)$, where, by (3.7), φ_R^b is given by

$$\hat{\varphi}_R^b = (1-\beta)\hat{\varphi}_R, \quad \text{i.e.,} \quad \varphi_R^b = \varphi_R - \tilde{\beta} * \varphi_R. \tag{3.20}$$

We attack this using the Hörmander estimate (A.37), which is equivalent to

$$\operatorname{Tr} \chi_I(\Lambda - \mu) \le C\langle \mu \rangle^{n-1}, \quad I = [-1, 1]. \tag{3.21}$$

From (3.21) we have:

Proposition 3.2 *Assume that σ_R is a family of positive measures on $\mathbb{R}$ such that*

$$|\varphi_R^b(\lambda)| \le \int \chi_I(\lambda - \mu)\, d\sigma_R(\mu), \quad \textit{for} \ \ \lambda \ge 0. \tag{3.22}$$

Then

$$\begin{aligned} |\operatorname{Tr} \varphi_R^b(\Lambda)| &\le \int \operatorname{Tr} \chi_I(\Lambda - \mu)\, d\sigma_R(\mu) \\ &\le C \int \langle \mu \rangle^{n-1}\, d\sigma_R(\mu). \end{aligned} \tag{3.23}$$

This leads to the following key estimate, which, together with (3.19), establishes the result

$$\operatorname{Tr} \varphi_R(\Lambda) = C_X(\varphi) R^n + O(R^{n-1-\gamma}), \tag{3.24}$$

as advertised in (1.15). As noted there, (1.12)–(1.15) imply (1.9) and hence prove Theorem 1.1.

Proposition 3.3 *Assume that*

$$\varphi(\lambda) = (1 - \lambda^2)_+^\gamma, \quad \gamma \ge 0. \tag{3.25}$$

Then

$$|\operatorname{Tr} \varphi_R^b(\Lambda)| \le C R^{n-1-\gamma}. \tag{3.26}$$

Note that taking $\gamma = 0$ in (3.25) recovers, in the setting of X, the classical result (1.3). In this case,

$$\varphi_R(\lambda) = \chi_{[-R,R]}(\lambda), \tag{3.27}$$

and estimating $\varphi_R^b(\lambda)$ is an exercise. For $\gamma > 0$, the exercise is perhaps a little less elementary, and we describe one route to the desired estimate. In the following analysis, we take $\gamma > 0$.

Note that

$$\varphi_R^b(\lambda) = \psi_\varepsilon\Big(\frac{\lambda}{R}\Big), \tag{3.28}$$

where

$$\psi_\varepsilon(\lambda) = \varphi(\lambda) - \tilde{\beta}_\varepsilon * \varphi(\lambda), \quad \tilde{\beta}_\varepsilon(\lambda) = R\tilde{\beta}(R\lambda), \quad \varepsilon = \frac{1}{R}. \tag{3.29}$$

Equivalently,

$$\hat{\psi}_\varepsilon(t) = \big(1 - \beta(\varepsilon t)\big)\hat{\varphi}(t). \tag{3.30}$$

We claim that, outside layers of thickness $\sim \varepsilon$ about $\lambda = \pm 1$, $\psi_\varepsilon(\lambda)$ decays rapidly to 0 as $\varepsilon \to 0$, and has an amplitude $\sim \varepsilon^\gamma$ on these layers. To do this analysis, it is convenient to use a partition of unity and split φ into two pieces, each with just one singularity, one at $\lambda = 1$ and one at $\lambda = -1$. Then translate to move the singularity to $\lambda = 0$. Having so modified φ, to a function we denote ψ, we want to analyze $\psi_\varepsilon(\lambda)$, given by

$$\hat{\psi}_\varepsilon(t) = \big(1 - \beta(\varepsilon t)\big)\hat{\psi}(t), \tag{3.31}$$

where $\psi \in \mathcal{E}'(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus 0)$ satisfies

$$\hat{\psi} \in S^{-1-\gamma}(\mathbb{R}), \quad \hat{\psi}(\pm t) \sim \sum_{k \geq 0} a_k^\pm t^{-1-\gamma-k}, \quad t \to +\infty. \tag{3.32}$$

We write

$$\hat{\psi}(t) \sim \sum_{k \geq 0} \hat{\psi}_k(t), \tag{3.33}$$

with $\hat{\psi}_k(t)$ homogeneous of degree $-1-\gamma-k$ on $\mathbb{R} \setminus 0$. Then the principal part of $\psi_\varepsilon(\lambda)$ is given by

$$\varepsilon^\gamma \Psi_0(\varepsilon^{-1}\lambda), \tag{3.34}$$

where $\widehat{\Psi}_0 \in S^{-1-\gamma}(\mathbb{R})$ is given by $(1 - \beta(t))\hat{\psi}_0(t)$, i.e.,

$$\widehat{\Psi}_0(t) = a_0^\pm \big(1 - \beta(t)\big)|t|^{-1-\gamma}, \tag{3.35}$$

for $\pm t > 0$. Note that $\widehat{\Psi}_0(t)$ vanishes near $t = 0$. It follows that Ψ_0 is continuous on $\mathbb{R}$ and smooth on $\mathbb{R} \setminus 0$, with a conormal singularity of the same sort as $|\lambda|^\gamma$ at $\lambda = 0$, and that $\Psi_0(\lambda)$ is rapidly decreasing, together with all its derivatives, as $|\lambda| \to \infty$.

Further terms in the expansion (3.33) yield contributions to $\psi_\varepsilon(\lambda)$ of the form

$$\varepsilon^{\gamma+k}\Psi_k(\varepsilon^{-1}\lambda). \tag{3.36}$$

At some point, one can cut off the sum (3.33) and estimate the contribution of the remainder in an elementary fashion. The behavior of (3.34) then leads to an estimate of $\varphi_R^b(\lambda)$, rapidly decreasing as $R \to \infty$ outside layers of thickness ~ 1 about $\lambda = \pm R$, allowing one to apply Proposition 3.2 and observe the conclusion (3.22) in Proposition 3.3.

A Generalities on the wave trace

Here we take X and Λ as in §1 (cf. (1.14)), though Λ could be a more general ellipic, self-adjoint element of $OPS^1(X)$. If $\varphi : \mathbb{R} \to \mathbb{C}$ is a Borel function, we define $\varphi(\Lambda)$ via the spectral theorem. Combining Sobolev space regularity and the Sobolev embedding theorem, we have

$$\begin{aligned} a > \frac{n}{2},\ |\varphi(\lambda)| \le C\langle\lambda\rangle^{-a} &\Longrightarrow \varphi(\Lambda) : L^2(X) \to H^a(X) \subset C(X) \\ &\Longrightarrow \varphi(\Lambda) \text{ is Hilbert-Schmidt,} \end{aligned} \tag{A.1}$$

hence

$$b > n,\ |\varphi(\lambda)| \le C\langle\lambda\rangle^{-b} \Longrightarrow \varphi(\Lambda) \text{ is trace class.} \tag{A.2}$$

Recall $n = \dim X$. We use the trace norm

$$\|A\|_{\mathrm{Tr}} = \mathrm{Tr}(A^*A)^{1/2}. \tag{A.3}$$

One convenient formula for $\varphi(\Lambda)$ arises from combining the spectral theorem and the Fourier inversion formula:

$$\varphi(\Lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{it\Lambda}\, dt, \tag{A.4}$$

assuming

$$\hat{\varphi} \in L^1(\mathbb{R}). \tag{A.5}$$

In particular, (A.4) works for $\varphi \in \mathcal{S}(\mathbb{R})$, and serves to define the wave trace as a tempered distribution,

$$\mathrm{Tr}\, U \in \mathcal{S}'(\mathbb{R}), \quad U(t) = e^{it\Lambda}, \tag{A.6}$$

by

$$\langle \hat{\varphi}, \mathrm{Tr}\, U \rangle = \sqrt{2\pi}\ \mathrm{Tr}\, \varphi(\Lambda), \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{A.7}$$

We want to use these constructions to study $\mathrm{Tr}\, \varphi(\Lambda)$ in cases where $\varphi \notin \mathcal{S}(\mathbb{R})$, such as

$$\varphi(\lambda) = (1 - \lambda^2)_+^{\ell/2}, \tag{A.8}$$

with $\ell > 0$, introduced in (1.14), and its dilates $\varphi_R(\lambda) = \varphi(\lambda/R)$. For now, we will work with functions φ satisfying the following: each $\lambda \in \mathbb{R}$ is a Lebesgue point of φ, and

$$|\varphi(\lambda)| \le C_k \langle\lambda\rangle^{-k}, \quad \forall\, k \in \mathbb{N}. \tag{A.9}$$

This implies

$$\hat{\varphi} \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}), \quad \hat{\varphi}^{(k)} \in L^\infty(\mathbb{R}), \quad \forall\, k \geq 0. \tag{A.10}$$

It does not imply (A.5), though the functions (A.8) do satisfy (A.5) for $\ell > 0$. But we also want to handle the limiting case $\ell = 0$. To proceed, we fix $T_0 \in (0, \infty)$, take

$$\beta \in C_0^\infty((-T_0, T_0)), \quad \beta(t) = 1 \text{ for } |t| \leq T_0/2, \tag{A.11}$$

and write

$$\varphi = \varphi^\# + \varphi^b, \quad \varphi^\# = \tilde{\beta} * \varphi, \ \text{ i.e., } \ \hat{\varphi}^\#(t) = \beta(t)\hat{\varphi}(t). \tag{A.12}$$

Then $\varphi^\# \in \mathcal{S}(\mathbb{R})$, and

$$\varphi(\Lambda) = \varphi^\#(\Lambda) + \varphi^b(\Lambda), \tag{A.13}$$

with

$$\varphi^\#(\Lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t)\beta(t)e^{it\Lambda}\, dt. \tag{A.14}$$

For judiciously chosen β we use (A.10) and results on the wave trace to analyze $\operatorname{Tr} \varphi^\#(\Lambda)$. We use other methods to estimate the trace of

$$\varphi^b(\Lambda), \quad \varphi^b(\lambda) = \varphi(\lambda) - \tilde{\beta} * \varphi(\lambda). \tag{A.15}$$

Let us set

$$v = \beta \ \operatorname{Tr} U \in \mathcal{E}'(\mathbb{R}), \tag{A.16}$$

so $\hat{v} \in C^\infty(\mathbb{R})$, and is polynomially bounded. We therefore have the identity

$$\begin{aligned} \operatorname{Tr} \varphi^\#(\Lambda) &= \frac{1}{\sqrt{2\pi}} \langle \hat{\varphi}, v \rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\lambda)\hat{v}(\lambda)\, d\lambda. \end{aligned} \tag{A.17}$$

Remark. We can readily obtain a definite estimate on $|\hat{v}(\lambda)|$, as follows. First,

$$\begin{aligned} \hat{v}(\lambda) &= \frac{1}{\sqrt{2\pi}} \langle \beta e^{-i\lambda t}, \operatorname{Tr} U \rangle \\ &= \operatorname{Tr} \tilde{\beta}_\lambda(\Lambda), \end{aligned} \tag{A.18}$$

where

$$\tilde{\beta}_\lambda(\mu) = \tilde{\beta}(\mu - \lambda). \tag{A.19}$$

From (A.2) we have

$$\|\psi(\Lambda)\|_{\mathrm{Tr}} \le C_b \sup_\lambda \langle\lambda\rangle^b |\psi(\lambda)|, \quad \forall\, b > n, \tag{A.20}$$

which leads to

$$|\hat{v}(\lambda)| \le C_\delta \langle\lambda\rangle^{n+\delta}, \quad \forall\, \delta > 0. \tag{A.21}$$

If T_0 in (A.11) is chosen appropriately, methods of microlocal analysis yield much more precise information on $\hat{v}(\lambda)$. (For one thing, we can replace $n+\delta$ by $n-1$ in (A.21).) To start, results on propagation of singularities show that if you pick $T_0 > 0$ so small that, for each $x \in X$, the orbits of H_{p_1} in $T^*X \setminus 0$, starting in $T^*_x X \setminus 0$ (p_1 denoting the principal symbol of Λ) do not pass over x for $0 < |t| < T_0$, then

$$\operatorname{Tr} U\big|_{(-T_0,T_0)} \in C^\infty((-T_0,T_0) \setminus 0). \tag{A.22}$$

A more refined result (cf. [4]) is that if $T \in \operatorname{sing\,supp} \operatorname{Tr} U$, then either $T = 0$ or the flow generated by H_{p_1} has an orbit of period T. The seminal analysis of [5] says that, perhaps on a smaller interval $(-T_1, T_1)$, $e^{it\Lambda}$ is equal, modulo a smoothing operator, to $Q(t)$, given by

$$Q(t)f(x) = (2\pi)^{-n} \iint e^{i\varphi(x,y,\xi)+itp(y,\xi)} q(t,x,y,\xi) f(y)\, d\xi\, dy, \tag{A.23}$$

where $\varphi(x,y,\xi)$ is real valued and homogeneous of degree 1 in ξ, and satisfies

$$\varphi(x,y,\xi) = \langle x-y, \xi\rangle + O(|x-y|^2|\xi|), \tag{A.24}$$

and q is a classical symbol of order 0. This leads to a precise analysis of

$$\hat{v}(\lambda) = \sum_j \tilde{\beta}(\lambda - \mu_j), \tag{A.25}$$

with μ_j as in (1.10), whose neatest form is perhaps Proposition 2.1 and (2.16) of [4]. The conclusion is that whenever β satisfies (A.11), with T_0 as in (A.22), and v is given by (A.16), then

$$\hat{v} \in S^{n-1}(\mathbb{R}), \tag{A.26}$$

with

$$\hat{v}(\lambda) \sim (2\pi)^{-n} \sum_{k\ge 0} c_k \lambda^{n-1-k}, \ \text{ as } \ \lambda \to +\infty, \tag{A.27}$$

where

$$c_0 = \operatorname{Vol} S^*X, \tag{A.28}$$

where $S^*X = \{(x,\xi) \in T^*X : p_1(x,\xi) = 1\}$. If Λ^2 is a differential operator (e.g., $\Lambda^2 = -\Delta_X$), then

$$c_k = 0, \quad \text{for } k \text{ odd}, \tag{A.29}$$

and also

$$c_{n+2\ell} = 0, \quad \text{for} \ \ \ell \in \mathbb{N}. \tag{A.30}$$

Of course, as is clear from (A.25),

$$\hat{v}(\lambda) \text{ is rapidly decreasing as } \ \lambda \to -\infty. \tag{A.31}$$

Remark. By the use of (A.23), the results (A.26)–(A.30) are first established under the hypothesis that (A.11) holds with T_0 replaced by a smaller quantity $T_1 \in (0, T_0)$. The extension to the more general class described by (A.11), with T_0 as in (A.22), is straightforward.

Here is another straightforward extension of (A.26)–(A.27). Replace β in (A.11) by $\hat{\rho}$, where

$$\hat{\rho} \in C_0^\infty((-T_0, T_0)). \tag{A.32}$$

The difference is that we do not assume $\hat{\rho} = 1$ for small $|t|$. We still have

$$v_\rho = \hat{\rho} \, \operatorname{Tr} U \Longrightarrow \hat{v}_\rho \in S^{n-1}(\mathbb{R}), \tag{A.33}$$

hence

$$\sum_j \rho(\lambda - \mu_j) \sim \sum_{k \ge 0} c_k' \lambda^{n-1-k}, \quad \text{as} \ \ \lambda \to +\infty. \tag{A.34}$$

Proof. Pick β as in (A.11) with $\beta(t) = 1$ on $\operatorname{supp} \hat{\rho}$. Then $v_\rho = \hat{\rho} v$, so

$$\hat{v}_\rho = \rho * \hat{v}. \tag{A.35}$$

□

This extension is significant because we can pick $\hat{\rho}$, satisfying (A.32), such that

$$\rho \ge 0 \text{ on } \mathbb{R}, \quad \rho(\lambda) \ge 1 \text{ for } |\lambda| \le 1, \tag{A.36}$$

and then (A.34) yields the important Hörmander estimate

$$\#\{j : |\lambda - \mu_j| \le 1\} \le C \langle \lambda \rangle^{n-1}. \tag{A.37}$$

References

[1] P. Berard, On the wave equation on a compact Riemannian manifold without conjugate points, Math. Zeit. 155 (1977), 249–276.

[2] B. Berndt, S. Kim, and A. Zaharescu, The circle problem of Gauss and the divisor problem – still unsolved, Amer. Math. Monthly 125 (2018), 99–114.

[3] Y. Canzani and J. Galkowski, Weyl remainder estimates: an application of geodesic beams, arXiv:2010.03969, 2020.

[4] J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975), 49–79.

[5] L. Hörmander, The spectral function of an elliptic operator, Acta Math. 121 (1968), 193–218.

[6] A. Iosevich and E. Weyman, Weyl law improvement for products of spheres, arXiv:1909.11844v1, IMRN, to appear.

[7] J. Rauch and M. Taylor, Quadrature estimates for multidimensional integrals, Houston J. Math. 36 (2010), 727–749.

[8] M. Taylor, Partial Differential Equations, Springer, New York, 1996 (2nd ed. 2011).

[9] M. Taylor, Elliptic operators on S^2 whose Weyl spectral asymptotics have small remainders, Preprint, 2021.

Michael E. Taylor
Mathematics Dept., UNC, Chapel Hill NC, 27599
email: met@math.unc.edu