Howl for ODE

Michael Taylor

Over the decades I have been teaching, I have come to the conclusion that there are serious shortcomings in the structure of the elementary differential equations courses, as they are taught in most American universities. This has led me to produce a text, *Introduction to Differential Equations*, Amer. Math. Soc., Providence RI, 2010 (2nd ed., 2021). I'm not the first to see there is a problem. An earlier critique was given by

G.-C. Rota, Ten lessons I wish I had learned before I started teaching differential equations, https://web.williams.edu/Mathematics/lg5/Rota.pdf

Rota's essay makes a number of points that agree with mine, though we do not agree on everything. Perhaps our main difference is that I opted not to despair, and undertook an effort to produce a text whose goal was to reform the presentation of introductory ODE.

My work on what was to become my ODE text began with a set of linear algebra notes (which ultimately became Chapter 2). The intention was to provide a brief but complete, self-contained treatment of results on linear algebra needed for the study of systems of ODE, as a supplement to the texts typically chosen for our ODE course, whose coverage of this topic struck me as not really sufficient.

As time went on, it dawned on me that what I needed to produce was more than a linear algebra supplement to the general run of basic ODE texts. Indeed, what was needed was a reconception of what the subject of differential equations is, and a serious consideration of what fundamental changes this reconception tells us we should make to produce a better course, and a better text.

My understanding that such a reconception is necessary arose in the course of taking a hard look at how typical ODE texts organize their material. Here is a synopsis of a typical presentation of a year-long beginning ODE course, in a text of moderate length (say, 460 pages), followed by a description of how my reaction to such presentations has motivated me to produce an alternative treatment.

1. Single first order equations, $1 - 100$,
2. Single linear equations of order ≥ 2, $101 - 280$,
3. Linear systems, $281 - 350$,
4. Nonlinear systems, $351 - 460$.

To start, the amount of space devoted to single, first-order differential equations, indicated above, is way out of proportion to the actual significance of this topic. To be sure, there are some useful points one wants to touch on in this area, such as studying the ODE $dx/dt = ax$ to achieve a solid understanding of what the exponential function is. The rest of the course will be making constant use of the exponential function, and its cousins (thanks to Euler's formula), the trigonometric functions. Also separation of variables is a valuable tool, and its value will grow in subsequent studies, such as the use of conservation of energy to take a class of second-order ODEs and produce first-order equations that are separable. But to run on for 100 pages entails devoting way too much time to minor tricks, and to applications that are nowhere near as significant as those arising once one is acquainted with a bit more ODE. Worse, this bloated treatment eats seriously into the time students have for the study of more important aspects of ODE and gives them a poor impression of what the subject of ODE really is.

Item #2, single equations of order ≥ 2, is meatier than item #1, but still, given that the treatment runs about 180 pages, it tends to disappoint. For one thing, standard treatments tend to confine themselves to linear equations. Our colleagues in physics and other sciences are not amused that such basic equations as arise from Newton's law $F = ma$ for motion in 1D (treating, for example, the motion of a pendulum) are not covered early on in introductory ODE courses. Such a topic should arise on p. 25 of the right ODE textbook, not wait till page 350+.

The use of power series to treat linear ODEs with analytic coefficients is a natural topic that arises in item #2. A common disappointment here is that one sees how to obtain formulas for the coefficients of the power series, but the text commonly fails to discuss convergence of the resulting power series. Now, what is the point of producing such formulas if one does not understand why they work? Here's a little secret that students (and maybe also instructors) should find good to know: these convergence proofs are not hard. It is the case that a presentation of such a proof is more compact when carried out in the setting of first-order systems. Actually, this speaks to the desirability of transitioning from single equations to systems a bit earlier, leading to a shortening of the amount of text devoted to single equations, which is certainly a point I want to emphasize.

Item #3, linear systems, tends to take up about 70 pages. Unlike items #1–2, at least this part is not too long, but standard treatments of this topic also have shortcomings, whose primary nature was described in the second paragraph to this note. It concerns the use of linear algebra in the treatment of linear systems of ODE. It is not uncommon for a text to devote about 20 pages to a presentation of background in linear algebra, but somehow these presentations omit crucial explanations of how key results in linear algebra work. One might, for example, see a handful of statements about the determinant of a square matrix, without a real definition, followed by assertions about eigenvalues, eigenvectors, and generalized eigenvectors, with no explanation of why these results are true. Moving along, one sees the matrix exponential e^{tA}, defined by the standard infinite series, accompanied by the assertion that the series converges and can be differentiated term by term, without arguments to support these claims. The underlying message to the student is clear: these results are too hard for you to grasp, so just believe them.

In fact, it is possible to present accessible demonstrations of these results, and students deserve to have them. It is for that reason that I wrote Chapter 2 of my text, and used this material in my Chapter 3.

Finally we get to item #4, nonlinear systems. This is actually the heart of a well presented ODE course, the part that reflects upon all the knowledge gleaned from items #1–3, and takes off from there. A balanced course should have close to half of its resources committed to this central theme. Instead, one sees standard texts devoting less than a quarter of their space to this, and those topics that do make it into the text seem scattershot. Lacking are foundational results on existence and uniqueness of solutions to $n \times n$ systems, and, of equal importance, results on dependence of solutions on initial data and other parameters. Results that bring ODE past the 19th century proceed from there to relate solutions to such systems to flows generated by vector fields, which provides a very convenient setting in which to appreciate the behavior of solutions to such systems. Chapter 4 of my text gives the student insight into such matters that is lacking in many standard introductory ODE texts.

My Text

I have organized my own *Introduction to Differential Equations* into four chapters, as follows;

Chapter 1 treats single differential equations, linear and nonlinear. The first section provides a self-contained development of exponential functions e^{at}, as solutions of the differential equation $dx/dt = ax$. We use a power series approach and allow a to be complex. We use this to provide a self-contained treatment of the trigonometric functions. Sections 2 and 3 give further basic results on first-order equations, such as the method of separation of variables. Then we quickly move to second-order equations. We emphasize equations arising from Newton's laws, applied to 1D motion, showing how conservation of energy leads to a reduction to a first-order separable equation. Then we bring in friction and see how matters become more complicated. We are motivated to study linearizations of these equations. Sections 8–18 deal with linear ODE, mostly of order 2, (with higher-order discussed in §17). Sections 9–14 deal with constant-coefficient equations, where the use of exponential functions (real or complex) are the key tools. Other tools include a further development of power series, in §15, applied to Bessel's equation in §16; and the Laplace transform, introduced in §18.

While single differential equations is the place to start, the theory of differential equations is and always has been (from the time of Newton's work on the equations of planetary motion) mainly about *systems* of equations. We prepare for this study in Chapter 2 and pursue it in Chapters 3 and 4.

Chapter 2 is devoted to linear algebra. This includes definitions of vector spaces and linear transformations, the notion of basis and dimension of a vector space, and representations of a linear transformation by a matrix, in terms of a choice of basis. We have a treatment of deteriminants of square matrices, followed by a discussion of eigenvalues and eigenvectors of a linear transformation, and then of generalized eigenvectors. We also discuss several special classes of linear transformations, particularly nilpotent transformations, and also self-adjoint, skew-adjoint, unitary, and orthogonal transformations.

In Chapter 3 we apply the material of Chapter 2 to the study of linear systems of differential equations. The first section is devoted to the matrix exponential, extending results that began in Chapter 1. Section 2 provides a complementary approach to trigonometric functions, and extends the scope to the sine and cosine of matrices. We proceed to various topics on linear systems, first constant-coefficient and homogeneous, then non-homogeneous, then variable coefficient. An integral formula, Duhamel's formula, is seen to provide an elegant replacement for the method of variation of parameters. Several sections are devoted to applications to electrical circuits, spring systems, and the Frenet-Serret equations for space curves. We end the chapter with a treatment of power series methods for systems with analytic coefficients and a treatment of systems with regular singular points. We show how this material applies to single equations of second order, such as the Bessel equation considered in Chapter 1. An appendix looks at the matrix Laplace transform, and shows how its use compares with that of Duhamel's formula.

Chapter 4 crowns the text, with a treatment of nonlinear systems of differential equations. To start, we prove results on existence and uniqueness of solutions, and dependence on initial conditions and other parameters. We then discuss geometrical aspects, interpreting solving the initial value problem as constructing the flow generated by a vector field. We bring in the phase portrait and discuss the nature of critical points of a vector field. We resume the discussion from Chapter 1 of equations arising from applying Newton's laws, this time to the interaction of several bodies in multi-dimensional space. We produce Newton's solution to the planetary motion problem. We discuss variational methods, pioneered by Euler and Lagrange, and show how they expand one's ability to derive and analyze equations of physics. We also discuss some nonlinear systems that arise in mathematical biology, such as predator-prey equations and competing species equations. We devote a section to difference schemes for numerical approximations to nonlinear systems of ODE, with special attention to the Runge-Kutta scheme. The last section discusses how, in dimension 3 and higher, flows can have chaotic behavior. Chapter 4 ends with a set of appendices, one reviewing multivariable differential calculus, and others touching on complementary topics regarding nonlinear systems, including behavior of critical points, equations of geodesics on surfaces, and ODE systems arising in the study of rigid body motion, and their connections with geodesics.