Supplementary Material for the Text,
Introduction to Analysis in One Variable
Pure and Applied Undergraduate Texts, #47

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Contents

Introduction

These notes were produced in the course of teaching the first semester of the undergraduate Analysis sequence at UNC, as a supplement to the text I used, *Introduction to Analysis in One Variable.* There are several categories of items. Some beef up homework problems given in the text. Others present a different way to prove some specific result. There are a couple of typos corrected.

One bit of new material concerns an extension of the fundamental theorem of calculus,

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Our extension of the textbook result, presented in the supplement to Section 4.2, concerns cases where F is not differentiable on the whole interval $I = (a, b)$, but rather differentiable on $I \setminus K$, where $\mathrm{cont}^+ K = 0$.

Chapter 1. Numbers

§1.1. Another exercise: labeling the elements of $\mathbb{N}$.

9. Complementing the definition of 1 as $s(0)$, we have

$$\begin{aligned} 2 &= s(1), & 3 &= s(2), & 4 &= s(3), \\ 5 &= s(4), & 6 &= s(5), & 7 &= s(6), \\ 8 &= s(7), & 9 &= s(8), & 10 &= s(9). \end{aligned}$$

From here, one proceeds to express larger integers in decimal notation, such as

$$20, \quad 25, \quad 251, \quad 2516, \quad 25163,$$

and so on. In general, with $a_j \in D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the string

$$a_n a_{n-1} \cdots a_1 a_0$$

represents the number

$$a_0 + 10a_1 + \cdots + 10^n a_n.$$

For example,

$$25 = 2 \cdot 10 + 5, \quad 2516 = 2 \cdot 10^3 + 5 \cdot 10^2 + 1 \cdot 10 + 6.$$

Establish the following.

Proposition. For each $x \in \mathbb{N}$, there exist unique $n \in \widetilde{\mathbb{N}}$ and $a_k \in D$, $0 \le k \le n$, $a_n \neq 0$, such that

$$x = \sum_{k=0}^{n} a_k \cdot 10^k.$$

Hint. To start, show that there exists a unique $n \in \widetilde{\mathbb{N}}$ such that

$$10^n \le x < 10^{n+1}.$$

If $n = 0$, $x \in D$. If $n > 0$, show that there exists a unique $a_n \in D$ (necessarily nonzero) such that

$$a_n \cdot 10^n \le x \quad \text{and} \quad x - a_n \cdot 10^n < 10^n.$$

Proceed inductively to treat $y = x - a_n \cdot 10^n$.

§1.6. Another approach to the proof of Proposition 1.5.6.

In §1.5 it is proved that $|a| < 1 \Rightarrow a^k \to 0$, or equivalently

$$0 < r < 1 \Longrightarrow r^k \to 0, \tag{1}$$

as $k \to \infty$. Here we provide an alternative approach to the proof of (1), using Proposition 1.6.11.

Second proof of (1). Note that $r^{k+1} = r \cdot r^k < r^k$, so (r^k) is a monotone sequence, satisfying $0 < r^k < 1$, hence a bounded monotone sequence. Proposition 1.6.11 implies this sequence converges,

$$r^k \longrightarrow s, \tag{2}$$

as $k \to \infty$. We have $s \in [0, 1)$. Now (2) implies

$$r^{k+1} = r \cdot r^k \longrightarrow rs, \tag{3}$$

hence $s = rs$, so $(1 - r)s = 0$, hence $s = 0$, as asserted.

§1.9. Dense vs. discrete subgroups of $\mathbb{R}$ (exercises).

8. Let $G \subset \mathbb{R}$ be a subgroup (cf. (1.3.10)). Assume $G \neq \{0\}$.
(a) Show that either $\{0\}$ is an isolated point of G or G is dense in $\mathbb{R}$.
Hint. If $x_\nu \in G,\ x_\nu \neq 0,\ |x_\nu| = \varepsilon_\nu \to 0$, then $\{kx_\nu : k \in \mathbb{Z}\}$ is "ε_ν-dense" in $\mathbb{R}$.
(b) If $\{0\}$ is isolated, set $\alpha = \inf\{x \in G : x > 0\}$, so $\alpha > 0$. Show that $\alpha \in G$.
Hint. If $\alpha \notin G$, there exist distinct $\alpha_\nu \in G$ such that $\alpha_\nu \to \alpha$, and then $\{\alpha_\nu - \alpha_\mu\}$ has 0 as a limit point.
(c) In the setting of part (b), show that

$$G = \{k\alpha : k \in \mathbb{Z}\}.$$

9. Suppose $\xi \in \mathbb{R}$ is irrational. Show that

$$\{j + k\xi : j, k \in \mathbb{Z}\} \text{ is dense in } \mathbb{R}.$$

§2.3. Another proof that (2.3.4) $\Rightarrow$ (2.3.2).

Section 2.3 discusses the equivalence of several properties of a metric space X, including

(2.3.2) Each infinite set $S \subset X$ has an accumulation point,

and

(2.3.4) Every open cover $\{U_\alpha\}$ of X has a finite subcover.

Each of these properties expresses *compactness* of X. Here we present an alternative proof that (2.3.4) $\Rightarrow$ (2.3.2).

So assume (2.3.4) holds and let $S \subset X$. If $p \in X$ is not an accumulation point of S, then there is an open set $\mathcal{O}_p \ni p$ that contains at most one point of S. If S has no accumulation points, then $\{\mathcal{O}_p : p \in X\}$ is an open cover of X, so by (2.3.4) it has a finite subcover

$$\{\mathcal{O}_{p_j} : 1 \le j \le K\}.$$

Thus S must be finite.

Chapter 3. Functions

§3.1. Alternative proof of Proposition 3.1.5.

Proposition 3.1.5. *Let X be a compact metric space. Assume $f : X \to Y$ is continuous, one-to-one, and onto. Then the inverse $g : Y \to X$ is continuous. That is, f is a homeomorphism.*

Proof. Assume $f(x_k) = y_k \to y$. We need to show that $x_k \to g(y)$. If not, we can pass to a subsequence and arrange that $d(x_k, g(y)) \geq \alpha > 0$ for all k. Then X compact $\Rightarrow$ some further subsequence $x_k \to x$. Then f continuous $\Rightarrow f(x_k) \to f(x)$, hence $f(x) = y$, so $x = g(y)$. Contradiction.

§4.1. Alternative approach to the inverse function theorem.

We use the mean value theorem to produce a criterion for constructing the inverse of a function. Let

$$f : [a, b] \longrightarrow \mathbb{R}, \quad f(a) = \alpha, \quad f(b) = \beta. \tag{4.1.17}$$

Assume f is continuous on $[a, b]$, differentiable on (a, b), and

$$0 < \gamma_0 \le f'(x) \le \gamma_1 < \infty, \quad \forall\, x \in (a, b). \tag{4.1.18}$$

We can apply Theorem 4.1.2 to f, restricted to the interval $[x_1, x_2] \subset [a, b]$, to get

$$\gamma_0 \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \gamma_1, \quad \text{if } a \le x_1 < x_2 \le b, \tag{4.1.19}$$

or

$$\gamma_0(x_2 - x_1) \le f(x_2) - f(x_1) \le \gamma_1(x_2 - x_1). \tag{4.1.20}$$

It follows that

$$f : [a, b] \longrightarrow [\alpha.\beta] \text{ is one-to-one.} \tag{4.1.21}$$

The intermediate value theorem implies $f : [a, b] \longrightarrow [\alpha, \beta]$ is onto. Consequently f has an inverse

$$g : [\alpha, \beta] \longrightarrow [a, b], \quad g(f(x)) = x, \quad f(g(y)) = y, \tag{4.1.22}$$

and (4.1.19) implies

$$\frac{1}{\gamma_1} \le \frac{g(y_2) - g(y_1)}{y_2 - y_1} \le \frac{1}{\gamma_0}, \quad \text{if } \alpha \le y_1 < y_2 \le \beta. \tag{4.1.23}$$

The following result is known as the *inverse function theorem.*

Theorem 4.1.3. *If f is continuous on $[a, b]$ and differentiable on (a, b), and (4.1.17)–(4.1.18) hold, then its inverse $g : [\alpha, \beta] \to [a, b]$ is differentiable on (α, β), and*

$$g'(y) = \frac{1}{f'(x)}, \quad \textit{for } y = f(x) \in (\alpha, \beta). \tag{4.1.24}$$

The same conclusion holds if in place of (4.1.18) we have

$$-\gamma_1 \le f'(x) \le -\gamma_0 < 0, \quad \forall\, x \in (a, b), \tag{4.1.25}$$

except that then $\beta < \alpha$.

Proof. Fix $y \in (\alpha, \beta)$, and let $x = g(y)$, so $y = f(x)$. To say that f is differentiable at x is to say

$$\lim_{\xi \to x} \frac{f(x) - f(\xi)}{x - \xi} = f'(x). \tag{4.1.26}$$

Now take $\eta = f(\xi)$, so $\xi = g(\eta)$, and note from (4.1.19) that

$$\xi \to x \Longleftrightarrow \eta \to y. \tag{4.1.27}$$

Hence, by (4.1.18)–(4.1.19) and (4.1.23), we have

$$\lim_{\eta \to y} \frac{g(y) - g(\eta)}{y - \eta} = \frac{1}{f'(x)}, \tag{4.1.28}$$

which proves (4.1.24).

Remark. If one knew that g were differentiable, as well as f, then the identity (4.1.24) would follow by differentiating $g(f(x)) = x$, and applying the chain rule. However, an additional argument, such as given above, is necessary to guarantee that g is differentiable.

Comments on (4.1.33) and (4.1.34).

We start with the observation that, for $j, k, \ell, m \in \mathbb{Z},\ y > 0$,

$$y^{km} = (y^k)^m, \quad \text{and} \ \ y^{j+k} = y^j y^k.$$

Regarding (4.1.33), we have, for $x > 0,\ k, n \neq 0$,

$$\begin{aligned}(x^{1/kn})^{km} = (x^{1/n})^m &\Longleftrightarrow (x^{1/kn})^k = x^{1/n} \\ &\Longleftrightarrow (x^{1/kn})^{kn} = x.\end{aligned}$$

Regarding (4.1.34), we have, for $x > 0,\ r = m/n,\ s = j/k$,

$$\begin{aligned}x^{r+s} &= x^{(km+jn)/nk} = (x^{1/nk})^{km+jn} \\ &= (x^{1/nk})^{km}(x^{1/nk})^{jn} = x^{m/n}\, x^{j/k}.\end{aligned}$$

New exercise: alternative computation of $dx^{1/n}/dx$.

11. Use the formula

$$(*) \qquad z^n - w^n = (z-w)(z^{n-1} + z^{n-2}w + \cdots + w^{n-1}),$$

with

$$z = (x+h)^{1/n}, \quad w = x^{1/n}.$$

Plug these values in, divide by h, and use the continuity of $f(x) = x^{1/n}$ (cf. (4.1.42)) to take the limit as $h \to 0$, thereby obtaining

$$\frac{d}{dx}\, x^{1/n} = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n} x^{(1/n)-1},$$

by a means different from (4.1.36). Note how the case $n = 2$ is worked out in (4.1.37)–(4.1.41).

12. In the setting of Exercise 11 above, use (*) with

$$z = x + h, \quad w = x$$

to get another proof that

$$\frac{d}{dx} x^n = nx^{n-1},$$

for $n \in \mathbb{N}$.

§4.2A. One-sided derivatives, symmetric derivatives, and the fundamental theorem of calculus

If $I = (a, b)$ and $f : I \to \mathbb{R}$, we say f is right-differentiable at $x \in I$, with right derivative $D_r f(x)$, provided

$$\lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h} = D_r f(x). \tag{2A.1}$$

Similarly,

$$\lim_{h \searrow 0} \frac{f(x) - f(x-h)}{h} = D_\ell f(x) \tag{2A.2}$$

defines left differentiability.

For example, if we set

$$\begin{aligned} f(x) = 1, &\quad \text{for } x \geq 0, \\ 0, &\quad \text{for } x < 0, \end{aligned} \tag{2A.3}$$

then

$$D_r f \equiv 0, \quad \text{and} \quad D_\ell f(x) = 0 \text{ for } x \neq 0, \tag{2A.4}$$

but f is not left-differentiable at $x = 0$. This illustrates the fact that to get useful results from one-sided differentiability, we want to impose the condition that f be continuous. We propose to prove the following.

Proposition 2A.1. *Assume $f : I \to \mathbb{R}$ is continuous and right differentiable. Then*

$$D_r f = g \in C(I) \Longrightarrow f \in C^1(I), \ \textit{and } f' = g. \tag{2A.5}$$

(A similar conclusion holds if f is left-differentiable.)

To start the proof, we apply the fundamental theorem of calculus to produce $\varphi \in C^1(I)$ such that $\varphi' = g$. Then

$$f - \varphi \in C(I), \quad \text{and} \quad D_r(f - \varphi) \equiv 0. \tag{2A.6}$$

The conclusion (2A.5) follows from the assertion that, if (2A.6) holds, then $f - \varphi$ is constant. This in turn is a consequence of part (c) of the following.

Lemma 2A.2. *Assume $f \in C(I)$ is right differentiable at each $x \in I$. Then*

(a) $D_r f > 0$ *on* $I \Longrightarrow f \nearrow$,

(b) $D_r f < 0$ *on* $I \Longrightarrow f \searrow$,

(c) $D_r f \equiv 0$ *on* $I \Longrightarrow f$ *is constant.*

Proof. We start with part (a). If f is not $\nearrow$, then there exist $x_0 < x_1 \in (a, b)$ such that $f(x_1) < f(x_0)$. Now $f\big|_{[x_0,x_1]}$ has a maximum, say at $\xi \in [x_0, x_1]$. This maximum is not at x_0, since $D_r f(x_0) > 0$. It is not at x_1, since $f(x_1) < f(x_0)$. Hence f achieves a maximum at $\xi \in (x_0, x_0)$. But f maximal at $\xi \Rightarrow D_r f(\xi) \le 0$. Contradiction.

The proof of (b) is similar.

Now for part (c). If $D_r f \equiv 0$, set $f_\varepsilon(x) = x + \varepsilon x$. For $\varepsilon > 0$, part (a) implies $f_\varepsilon \nearrow$, hence (letting $\varepsilon \searrow 0$) $f \nearrow$. Also, part (b) implies $f_\varepsilon \searrow$ for $\varepsilon < 0$, so (letting $\varepsilon \nearrow 0$) $f \searrow$. We have part (c).

We turn to the symmetric derivative. We say a function $f : I \to \mathbb{R}$ is s-differentiable at $x \in I$, with s-derivative $D_s f(x)$, provided

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = D_s f(x). \tag{2A.7}$$

For example, if $f(x) = |x|$,

$$D_s f(x) = -1 \text{ for } x < 0, \quad 0 \text{ for } x = 0, \quad 1 \text{ for } x > 0. \tag{2A.8}$$

The following result parallels Proposition 2A.1.

Proposition 2A.3. *Assume $f : I \to \mathbb{R}$ is continuous and s-differentiable. Then*

$$D_s f = g \in C(I) \Longrightarrow f \in C^1(I) \text{ and } f' = g. \tag{2A.9}$$

As in Proposition 2A.1, we pick $\varphi \in C^1(I)$ such that $\varphi' = g$ and analyze $D_s(f - \varphi)$, obtaining (2A.9) as a consequence of the following.

Lemma 2A.4. *Assume $f \in C(I)$ is s-differentiable at each $x \in I$. Then*

(a) $D_s f > 0$ *on* $I \Rightarrow f \nearrow$,

(b) $D_s f < 0$ *on* $I \Rightarrow f \searrow$,

(c) $D_s f \equiv 0$ *on* $I \Rightarrow f$ *is constant.*

As in Lemma 2A.2, the proof of part (b) is similar to that of part (a), and part (c) follows from parts (a) and (b).

Proof of part (a). Pick $\alpha \in (a, b)$. We aim to show that

$$f(x) \geq f(\alpha), \quad \forall\, x \in (\alpha, b). \tag{2A.10}$$

Pick $\delta > 0$. Then pick $\alpha_1 > \alpha$ such that

$$x \in [\alpha, \alpha_1] \Longrightarrow |f(x) - f(\alpha)| \leq \delta. \tag{2A.11}$$

Now set

$$A = \min_{x \in [\alpha, \alpha_1]} f(x), \tag{2A.12}$$

and note that $|A - f(\alpha)| \leq \delta$. Set

$$S_A = \{x \in [\alpha_1, b) : f(x) < A\}, \tag{2A.13}$$

which is open in $[\alpha_1, b)$. Note that

$$D_s f(\alpha_1) > 0 \Longrightarrow f(x) > A \text{ for } \alpha_1 < x \leq \alpha_1 + \varepsilon_1, \tag{2A.14}$$

with $\varepsilon_1 > 0$ sufficiently small. Hence

$$S_A \text{ is disjoint from } \ [\alpha_1, \alpha_1 + \varepsilon_1]. \tag{2A.15}$$

If $S_A \neq \emptyset$, set

$$x_1 = \inf\, S_A, \quad \text{so} \ \ x_1 \in (a_1 + \varepsilon_1, b). \tag{2A.16}$$

We have

$$f(x_1) = A, \tag{2A.17}$$

and $f(x) \geq A$ for $x < x_1$. Hence

$$D_s f(x_1) > 0 \Longrightarrow f(x_1 + \varepsilon) > f(x_1), \text{ for sufficiently small } \varepsilon > 0. \tag{2A.18}$$

But this says

$$x_1 + \varepsilon \notin S_A \tag{2A.19}$$

for such ε. Hence in fact $S_A = \emptyset$, and we have

$$x \in [\alpha, b) \Longrightarrow f(x) \geq A, \tag{2A.20}$$

hence $f(x) \geq f(\alpha) - \delta$, for all $\delta > 0$, and this gives (2A.10), hence $f \nearrow$, as asserted.

§4.2B. Another extension of the fundamental theorem of calculus: F not differentiable everywhere.

One drawback to Proposition 4.2.10 is that it requires the function F to be differentiable on the entire interval (a, b). Here is a natural extension.

Proposition 2B.1. *With $I = [a, b]$, assume $F \in C(I)$ satisfies a Lipschitz condition*

$$|F(x) - F(y)| \le L|x - y|, \quad \forall\, x, y \in I, \tag{2B.1}$$

for some $L < \infty$. Let $g \in \mathcal{R}(I)$. Assume $K \subset I$ is compact, $\mathrm{cont}^+ K = 0$, *and*

$$F \ \text{ is differentiable on } \ \mathcal{O} = I \setminus K, \quad F' = g \ \text{ on } \ \mathcal{O}. \tag{2B.2}$$

Then

$$\int_a^b g(t)\, dt = F(b) - F(a). \tag{2B.3}$$

Proof. Assume $|g| \le M$ on I. Pick $\varepsilon > 0$. Cover K by a finite number of open intervals, $J_1, \dots, J_n$, with disjoint closures, such that $\sum_k \ell(J_k) \le \varepsilon$. Let

$$U = I \setminus \bigcup_k J_k, \tag{2B.4}$$

so $U \subset \mathcal{O}$. The set U is a union of a finite number of closed intervals $I_k = [\alpha_k, \beta_k]$. For each I_k, we have

$$\int\limits_{I_k} g(t)\, dt = F(\beta_k) - F(\alpha_k), \tag{2B.5}$$

by Proposition 4.2.10. Meanwhine,

$$\Bigl| \int_a^b g(t)\, dt - \sum_k \int\limits_{I_k} g(t)\, dt \Bigr| \le M\varepsilon. \tag{2B.6}$$

Now if $J \subset I$ is an interval, with endpoints α, β, write

$$\Delta F(J) = F(\beta) - F(\alpha). \tag{2B.7}$$

We have

$$F(b) - F(a) = \sum_k \Delta F(I_k) + \sum_k \Delta F(J_k). \tag{2B.8}$$

By (2B.5),

$$\sum_k \int_{I_k} g(t)\,dt = \sum_k \Delta F(I_k). \tag{2B.9}$$

Furthermore,

$$\Bigl|\sum_k \Delta F(J_k)\Bigr| \le L \sum_k \ell(J_k) \le L\varepsilon. \tag{2B.10}$$

Hence

$$\Bigl|\int_a^b g(t)\,dt - \bigl\{F(b) - F(a)\bigr\}\Bigr| \le M\varepsilon + L\varepsilon. \tag{2B.11}$$

Taking $\varepsilon \searrow 0$ gives (2B.3).

Remark 1. In the setting of Proposition 2B.1, we have, for each $x \in I$,

$$\int_a^x g(t)\,dt = F(x) - F(a).$$

We say F is an antiderivative of g.

Remark 2. As mentioned in the text, the Lebesgue theory yields substantially stronger versions of the fundamental theorem of calculus. However, Proposition 2B.1 applies to many more interesting cases than does Proposition 4.2.10.

Example 1. Take $I = [-1, 1]$,

$$F(t) = |t|, \quad g(t) = \operatorname{sgn} t.$$

Example 2. Take $I = [0, 1]$, $\mathcal{K} \subset I$ the Cantor middle third set, and take

$$F(t) = \operatorname{dist}(t, \mathcal{K}).$$

Then Proposition 2B.1 applies with $K = \mathcal{K} \cup \{p_k\}$, where p_k are the midpoints of the intervals I_k that make up $I \setminus \mathcal{K}$, and $g(t) = +1$ on the left half of each such interval, and -1 on the right half.

See comments on §4.6 for further results.

4.3. Solution to Exercise 8.

The following exercise deals with a numerical evaluation of $\sqrt{2}$.

8. Note that

$$\sqrt{2} = 2\sqrt{1 - \frac{1}{2}}.$$

Expand the right side in a power series, using (4.3.28)–(4.3.29). How many terms suffice to approximate $\sqrt{2}$ to 12 digits?

Solution. Using (4.3.28)–(4.3.29) with $r = -1/2$ gives

$$\sqrt{1-t} = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k, \quad |t| < 1, \tag{1}$$

with

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = -\frac{1}{2} \cdot \frac{1}{2}, \quad a_k = -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \Bigl(k - \frac{3}{2}\Bigr).$$

By (4.3.30), we have a convenient iteration,

$$\frac{a_k}{k!} = \alpha_k, \quad \alpha_0 = 1, \quad \alpha_{k+1} = \frac{k - \frac{1}{2}}{k+1} \alpha_k. \tag{2}$$

We next want to estimate the remainder

$$\sqrt{1-t} = \sum_{k=0}^{n} \alpha_k t^k + R_n(t), \quad \text{with } \ t = \frac{1}{2}. \tag{3}$$

As seen in (4.3.50), the Lagrange remainder formula does not provide an adequate estimate. Instead, the Cauchy remainder formula gives, by (4.3.54),

$$|t| < 1 \Longrightarrow |R_n(t)| \le |b_n| \cdot |t|^{n+1}, \tag{4}$$

in this case, where, by (4.3.45), $b_n = a_{n+1}/n!$, in particular

$$b_0 = -\frac{1}{2}, \quad b_1 = -\frac{1}{4}.$$

By (4.3.46),

$$\frac{b_{n+1}}{b_n} = \frac{n + \frac{1}{2}}{n+1},$$

so $n \ge 1 \Rightarrow |b_n| \le 1/4$. We have

$$|R_n(\tfrac{1}{2})| \le \frac{1}{4} \cdot 2^{-n-1} = 2^{-n-3},$$

so the error E_n in evaluating $\sqrt{2}$ satisfies

$$E_n \le 2^{-n-2}.$$

Since $2^{10} = 1024$, we have

$$E_{40} \le \frac{1}{4} \cdot 10^{-12}, \quad E_{38} \le 10^{-12}.$$

REMARK 1. Further computation reveals

$$|b_{32}| \le \frac{1}{20},$$

and hence

$$E_{36} \le 10^{-12}.$$

REMARK 2. Here is an alternative approach, which yields a sharper result. We know that (1) holds, with coefficients α_k whose absolute values, by (2), are monotonically decreasing. Hence, in (3), we can take

$$R_n(t) = \sum_{k=n+1}^{\infty} \alpha_k t^k, \quad |t| < 1,$$

and deduce that

$$|R_n(t)| \le |\alpha_{n+1}| \frac{|t|^{n+1}}{1 - |t|},$$

and in particular

$$|R_n(\tfrac{1}{2})| \le |\alpha_{n+1}| 2^{-n}.$$

Now $|\alpha_{n+1}| = |b_n|/(n+1)$, and an estimate on $|b_{31}|$ gives

$$|\alpha_{32}| \le \frac{1}{600},$$

hence

$$E_{32} \le \frac{1}{600} \cdot \frac{1}{2} \cdot 2^{-30} \le 10^{-12}.$$

REMARK 3. Exercises 9 and 11 deal with some much more rapid approximations to $\sqrt{2}$, some using power series, some using another method.

§4.4. Remark on the calculation of $C'(t)$

In (4.4.33)–(4.4.39) we examine $C(t) = (\cos t, \sin t)$. We apply d/dt to the identity $C(t) \cdot C(t) \equiv 1$ and use the fact that also $\|C'(t)\| \equiv 1$ to deduce in (4.4.37)–(4.4.38) that, for each $t \in \mathbb{R}$, either

$$C'(t) = (\sin t, -\cos t), \tag{1}$$

or

$$C'(t) = (-\sin t, \cos t). \tag{2}$$

We then argue that actually only (2) can hold.

To add a detail to that argument, note that since both sides of (1) and (2) are continuous in t, the set $A \subset \mathbb{R}$ of t for which (1) holds and the set $B \subset \mathbb{R}$ of t such that (2) holds are both closed. Of course, A and B are disjoint. Since $\mathbb{R}$ is connected, one of them must be empty. Now, by (4.4.33), we have $0 \in B$, so in fact $A = \emptyset$, and (2) holds for all $t \in \mathbb{R}$.

Typo in (4.4.54).

In (4.4.54), change $x(t) = \rho(t) \sin t$ to $x(t) = \rho(t) \cos t$, as in (4.4.51).

NOTE. The point of (4.4.54) is to derive the formula (4.4.56) for arc length in polar coordinates. See the neater formulation in this supplement, concerning §4.5, exercise 57.

§4.5. Extensions of exercises 7, 45, and 56, and addition of exercise 57.

7. Show that
$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$
Use (4.4.27)–(4.4.31) to obtain a rapidly convergent infinite series for π.
Hint. Show that $\sin \pi/6 = 1/2$. Use Exercise 2 and the identity $e^{\pi i/6} = e^{\pi i/2} e^{-\pi i/3}$.
Note that a_k in (4.4.29)–(4.4.31) satisfies $a_{k+1} = (k+1/2)a_k$. Deduce that

$$\pi = \sum_{k=0}^{\infty} \frac{b_k}{2k+1}, \quad b_0 = 3, \quad b_{k+1} = \frac{1}{4}\frac{2k+1}{2k+2} b_k. \tag{4.5.48}$$

Note that $b_k \le 3 \cdot 4^{-k}$. Deduce that

$$\mathrm{pi}(n) = \sum_{k=0}^{n} \frac{b_k}{2k+1} \Longrightarrow 0 < \pi - \mathrm{pi}(n) < \frac{1}{n+1} 2^{-2n-1}. \tag{4.5.49}$$

In particular,

$$\pi - \mathrm{pi}(3) < \frac{1}{500}, \quad \pi - \mathrm{pi}(5) < 10^{-4}, \quad \pi - \mathrm{pi}(20) < 10^{-13}. \tag{4.5.50}$$

Compute pi(3) and pi(5) by hand, and show that

$$\pi \approx 3.14, \quad \text{and then} \quad \pi \approx 3.1416.$$

Use a calculator or computer to evaluate pi(20), and verify that

$$\pi \approx 3.1415926535897 \cdots .$$

Revised version of Exercise
45. Evaluate

$$I(x) = \int_0^x \frac{dy}{1+y^2}$$

in two ways.

(a) Set $y = \tan t$ to obtain

$$I(x) = \tan^{-1} x.$$

(b) Set $y = \sinh u$ to obtain

$$I(x) = \int_0^{\sinh^{-1} x} \frac{du}{\cosh u}.$$

Comparing these results, obtain

$$\int_0^x \frac{du}{\cosh u} = \tan^{-1}(\sinh x).$$

45A. Evaluate

$$I(x) = \int_0^x \sqrt{1+y^2}\, dy$$

in two ways.

(a) Set $y = \sinh u$ ($z = \sinh^{-1} x$) to obtain

$$\begin{aligned} I(x) &= \int_0^z \cosh^2 u\, du \\ &= \frac{1}{4}\int_0^z (e^{2u} + 2 + e^{-2u})\, du \\ &= \frac{1}{4}\sinh 2z + \frac{1}{2}z. \end{aligned}$$

Use $\sinh 2z = 2\sinh z \cosh z$ to obtain

$$I(x) = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1} x.$$

(b) Set $y = \tan t$ to obatin

$$I(x) = \int_0^{\tan^{-1} x} \sec^3 t\, dt.$$

Compare these results, to obtain

$$\int_0^t \sec^3 s\, ds = I(\tan t) = \frac{1}{2}\sec t \tan t + \frac{1}{2}\sinh^{-1} \tan t.$$

Compare results of Exercises 13–14.

56. In the implementation of the inequality

$$n! > e^{(n+1/2)\log(n+1/2)} e^{-(n+1/2)} e^{-(3/2)\log(3/2)} e^{3/2}$$
$$= \left(\frac{n}{e}\right)^n \left[e\left(\frac{2}{3}\right)^{3/2}\right] \sqrt{n}\, e^{(n+1/2)\log(1+1/2n)},$$

we never did make use of the estimate

$$\log(1+\delta) > \delta - \tfrac{1}{2}\delta^2,$$

valid for $0 < \delta < 1$. Actually, we might prefer

$$(1+\delta)\log(1+\delta) > \delta.$$

Bring this in, and show that

$$n! > \left(\frac{n}{e}\right)^n \left(\frac{2e}{3}\right)^{3/2} \sqrt{n}.$$

We have

$$\left(\frac{2e}{3}\right)^{3/2} < \sqrt{2\pi} < e.$$

Compute each of these three quantities to 5 digits of accuracy.

Back to arc length

57. Looking at the analysis of a curve γ given in polar coordinates by $r = \rho(\theta)$, as in (4.4.50)–(4.4.51), show that you can replace (4.4.54)–(4.4.55) by

$$\begin{aligned}\gamma(t) = \rho(t)e^{it} &\Rightarrow \gamma'(t) = [\rho'(t) + i\rho(t)]e^{it} \\ &\Rightarrow |\gamma'(t)|^2 = \rho'(t)^2 + \rho(t)^2,\end{aligned}$$

and rederive the arc length formula (4.4.56).

§4.6A. Antiderivatives of unbounded integrable functions.

In §4.6 we extend the space $\mathcal{R}(I)$ of Riemann integrable functions (necessarily bounded) on an interval $I = [a,b]$ to a class of unbounded integrable functions, denoted $\mathcal{R}^{\#}(I)$, defined in (4.6.3)–(4.6.12). Here we provide results that allow one to extend the fundamental theorem of calculus to deal with integrands in $\mathcal{R}^{\#}(I)$.

Lemma 6A.1. *Take $I = [a,b]$, $g \in \mathcal{R}^{\#}(I)$. Then*

$$F(x) = \int_a^x g(t)\,dt \Longrightarrow F \in C(I).$$

Proof. Exercise.

Proposition 6A.2. *Let $g, g_k \in \mathcal{R}^{\#}(I)$, and set*

$$F_k(x) = \int_a^x g_k(t)\,dt, \quad F(x) = \int_a^x g(t)\,dt.$$

Then

$$\|g - g_k\|_{L^1(I)} \to 0 \Longrightarrow F_k \to F, \text{ uniformly on } I.$$

Proof. One has

$$|F_k(x) - F(x)| \le \|g_k - g\|_{L^1(I)}, \quad \forall\, x \in I.$$

Corollary 6A.3. *Let $g, g_k \in \mathcal{R}^{\#}(I)$. Set*

$$F_k(x) = \int_a^x g_k(t)\,dt.$$

Then

$$\|g - g_k\|_{L^1(I)} \to 0, \quad F_k(x) \to F(x),\ \forall\, x \in I \Longrightarrow \int_a^x g(t)\,dt = F(x).$$

Example. Take $I = [-1,1]$, $r \in (0,1)$,

$$g(t) = |t|^{-r}, \quad F(t) = \frac{1}{1-r}|t|^{1-r}(\operatorname{sgn} t).$$

Then

$$\int_{-1}^x g(t)\,dt = F(x) - F(-1).$$

New exercises: $\log(1+x^{-2}) \in \mathcal{R}^{\#}(\mathbb{R}^+)$**.**

17. Define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = \log\Bigl(1 + \frac{1}{x^2}\Bigr).$$

Show that $f \in \mathcal{R}^{\#}(\mathbb{R}^+)$.
Hint. Examine f separately on $(0,1]$ and on $[1,\infty)$.

18. Following up Exercise 6, write

$$\int \log(x^2+1)\,dx = x\log(x^2+1) + G(x),$$

and compute $G'(x) = -2x^2/(x^2+1)$, to get

$$G(x) = -2x + 2\tan^{-1} x.$$

Deduce that

$$\int \log\Bigl(1+\frac{1}{x^2}\Bigr)\,dx = x\log\Bigl(1+\frac{1}{x^2}\Bigr) + 2\tan^{-1} x,$$

and hence that

$$\int_0^\infty \log\Bigl(1+\frac{1}{x^2}\Bigr)\,dx = \pi.$$

Chapter 5. Further topics in Analysis

§5.5. Revised exercises on Newton's method.

Implement Newton's method to get approximate solutions to the following equations.

2. $e^x = 2$.

3. $\tan x = x, \quad \pi < x < \frac{3}{2}\pi$.