Remarks on the Hydrogen Atom Schrödinger Operator

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Contents

1. Introduction

The Schrödinger operator associated to the hydrogen atom has the form

$$H = -\Delta + V, \quad V(x) = -\frac{K}{|x|}, \quad K > 0, \tag{1.1}$$

acting on functions on $\mathbb{R}^3$. More precisely, H is a self-adjoint operator on $L^2(\mathbb{R}^3)$, constructed by the Friedrichs process, with domain

$$\mathcal{D}(H) = \{u \in H^1(\mathbb{R}^3) : -\Delta u - K|x|^{-1}u \in L^2(\mathbb{R}^3)\}. \tag{1.2}$$

We recall some well known facts about this operator, details on which can be found in §7 of [T], and then produce some additional results.

First (cf. Proposition 7.2 of [T]),

$$\mathcal{D}(H) = H^2(\mathbb{R}^3). \tag{1.3}$$

Here and below, $H^s(\mathbb{R}^3)$ will denote L^2-Sobolev spaces of functions or distributions on $\mathbb{R}^3$, and $H^{s,p}(\mathbb{R}^3)$ will denote L^p-Sobolev spaces. Next, (cf. Proposition 7.3 of [T]), the part of $\operatorname{Spec} H$ in $(-\infty, 0)$ is a discrete subset, consisting of eigenvalues of finite multiplicity. In more detail (cf. (7.15)–(7.38) of [T]), these eigenvalues are

$$E_n = -\frac{K^2}{4n^2}, \quad \text{eigenspaces } V_n, \quad \dim V_n = n^2, \tag{1.4}$$

for $n = 1, 2, 3, \cdots$, and V_n has a basis consisting of functions of the form

$$p(x)q(|x|)e^{-2K|x|/n}, \tag{1.5}$$

where p is a harmonic polynomial on $\mathbb{R}^3$, homogeneous of degree k, and q is a polynomial on $\mathbb{R}$ of degree j, with $j + k + 1 = n$.

As is clear from the formula (1.5), these eigenfunctions are Lipschitz, but, in cases where $p(x)$ is constant, they have no higher regularity. In particular, $\mathcal{D}(H^2)$ is not contained in $H^4(\mathbb{R}^3)$. In [T] it is shown that

$$\mathcal{D}(H^2) \subset H^{5/2-\varepsilon}(\mathbb{R}^3), \quad \forall\, \varepsilon > 0. \tag{1.6}$$

This does not quite imply that

$$\mathcal{D}(H^2) \subset \mathrm{Lip}(\mathbb{R}^3). \tag{1.7}$$

In §2 we prove (1.7). Part of our motivation for doing this lies in our investigation of functional calculus for H, discussed in [T2]. It is shown that interesting Littlewood-Paley results follow if one has an estimate

$$\|e^{-tH}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le Ct^{-5/4}, \quad \text{for} \ \ t \in (0,1]. \tag{1.8}$$

We discuss conjectural results related to (1.8) in §3. Work there naturally motivates us to consider fractional powers $\mathcal{H}^{1+a}$, for $a \in (0,1)$, where

$$\mathcal{H} = H + K^2/4 + 1, \tag{1.9}$$

a self-adjoint operator with spectrum in $[1,\infty)$. In §4 we show that, on the one hand,

$$\mathcal{D}(\mathcal{H}^{1+a}) \subset \mathrm{Lip}(\mathbb{R}^3), \quad \text{if} \ \ 1+a > \frac{5}{4}, \tag{1.10}$$

and, on the other hand,

$$\mathcal{D}(\mathcal{H}^{1+a}) = H^{2(1+a)}(\mathbb{R}^3), \quad \text{if} \ \ 1+a < \frac{5}{4}. \tag{1.11}$$

Section 5 contains some results and conjectures on the orthogonal projection P of $L^2(\mathbb{R}^3)$ onto the direct sum of the eigenspaces associated with the negative spectrum of H.

2. Domain of H^2

We next examine the domain of H^2, for H of the form (1.1). As shown in Proposition 7.7 of [T],

$$\mathcal{D}(H^2) \subset H^{5/2-\varepsilon}(\mathbb{R}^3), \quad \forall\, \varepsilon > 0. \tag{2.1}$$

Here we desire to prove the following, which is not quite implied by (2.1).

Proposition 2.1. *We have*

$$\mathcal{D}(H^2) \subset \mathrm{Lip}(\mathbb{R}^3). \tag{2.2}$$

Proof. We start by recalling the proof of Proposition 7.7 of [T]. Pick $\lambda \notin \operatorname{Spec} H$. Take $u \in \mathcal{D}(H^2)$, and set $(H-\lambda)u = f \in \mathcal{D}(H) = H^2(\mathbb{R}^3)$. Also set $R_\lambda = (-\Delta - \lambda)^{-1}$. Then

$$u = -R_\lambda V u + R_\lambda f, \quad R_\lambda f \in H^4(\mathbb{R}^3). \tag{2.3}$$

The next step taken in [T] is to use the fact that

$$M_V : H^2(\mathbb{R}^3) \longrightarrow H^{1/2-\varepsilon}(\mathbb{R}^3), \quad \forall \varepsilon > 0, \tag{2.4}$$

to deduce that $R_\lambda V u \in H^{5/2-\varepsilon}(\mathbb{R}^3)$, yielding (2.1) from (2.3).

We modify the last part of this argument as follows. Pick

$$\begin{gathered} \chi \in C_0^\infty(\mathbb{R}^3), \quad \chi(x) = 1 \text{ for } |x| \le 1, \quad \chi(x) = 0 \text{ for } |x| \ge 2, \\ V = \chi V + (1-\chi)V = V_0 + V_1. \end{gathered} \tag{2.5}$$

Then

$$u \in \mathcal{D}(H) = H^2(\mathbb{R}^3) \Rightarrow V_1 u \in H^2(\mathbb{R}^3) \Rightarrow R_\lambda V_1 u \in H^4(\mathbb{R}^3). \tag{2.6}$$

Meanwhile, since

$$H^2(\mathbb{R}^3) \subset C^\alpha(\mathbb{R}^3), \quad \alpha = \frac{1}{2}, \tag{2.7}$$

where, for $0 < \alpha < 1$,

$$\|v\|_{C^\alpha} = \|v\|_{L^\infty} + \sup\Bigl\{ \frac{|v(x) - v(y)|}{|x-y|^\alpha} : x \ne y \in \mathbb{R}^3,\ |x-y| \le 1 \Bigr\}, \tag{2.8}$$

we can write, for $u \in \mathcal{D}(H)$,

$$u = u(0)\chi + u_1, \quad u_1 \in C^\alpha(\mathbb{R}^3),\ u_1(0) = 0. \tag{2.9}$$

Hence, given $u \in \mathcal{D}(H)$,

$$R_\lambda V_0 u = u(0) R_\lambda V_0 \chi + R_\lambda V_0 u_1. \tag{2.10}$$

Now $V_0 u_1$ is supported on $\{x : |x| \le 2\}$, and

$$|V_0(x) u_1(x)| \le C|x|^{\alpha-1}, \tag{2.11}$$

for $\alpha = 1/2$.

In fact, given $u \in \mathcal{D}(H^2)$, we can use (2.1) to improve the conclusion of (2.9) to $u_1 \in C^\alpha(\mathbb{R}^3)$ for all $\alpha < 1$, and we also have (2.11) for all $\alpha < 1$. Consequently,

$$V_0 u_1 \in L^p(\mathbb{R}^n), \quad \forall\, p \in (1, \infty), \tag{2.12}$$

hence

$$R_\lambda V_0 u_1 \in H^{2,p}(\mathbb{R}^3), \quad \forall\, p \in (1, \infty). \tag{2.13}$$

We deduce that, if $u \in \mathcal{D}(H^2)$, with $(H - \lambda)u = f \in \mathcal{D}(H)$, then

$$\begin{aligned} u &= -R_\lambda\big(u(0)V_0\chi + V_0 u_1\big) \qquad \mod H^4(\mathbb{R}^3) \\ &= -u(0) R_\lambda (V_0\chi) \qquad \mod H^{2,p}(\mathbb{R}^3), \end{aligned} \tag{2.14}$$

for all $p \in [2, \infty)$. Given that

$$\Delta |x| = 2|x|^{-1}, \quad \text{on } \ \mathbb{R}^3, \tag{2.15}$$

we have

$$R_\lambda(V_0\chi) = \frac{K}{2}\,|x|\chi^2 + \frac{\lambda K}{2} R_\lambda(|x|\chi^2) \mod \mathcal{S}(\mathbb{R}^3). \tag{2.16}$$

and

$$R_\lambda(|x|\chi^2) \in \bigcap_{1 \le q < \infty} H^{3,q}(\mathbb{R}^3). \tag{2.16A}$$

We hence have the following result, which is more precise than Proposition 2.1.

Proposition 2.2. *We have*

$$\mathcal{D}(H^2) \subset \operatorname{Span}(|x|\chi(x)) + \bigcap_{2\le p<\infty} H^{2,p}(\mathbb{R}^3). \tag{2.17}$$

In view of the formula (1.5) for elements of V_n, which of course belong to $\mathcal{D}(H^k)$ for all k, this is a pretty good result. However, we take one more pass through the proof of Propositions 2.1 and 2.2, to obtain an improvement of (2.13)–(2.14). To begin, given $u \in \mathcal{D}(H^2)$, we can now say that u_1 in (2.9) satisfies

$$u_1 \in \operatorname{Lip}(\mathbb{R}^3), \quad u_1(0) = 0. \tag{2.18}$$

It follows that

$$|\nabla(V_0 u_1)| \le C|x|^{-1}, \tag{2.19}$$

hence

$$\nabla(V_0 u_1) \in L^{3-\varepsilon}(\mathbb{R}^3), \quad \forall\, \varepsilon \in (0,1], \tag{2.20}$$

so

$$R_\lambda V_0 u_1 \in H^{3,3-\varepsilon}(\mathbb{R}^3), \quad \forall\, \varepsilon \in (0,1]. \tag{2.21}$$

Note that this improves (2.13), since

$$\bigcap_{1<q<3} H^{1,q}(\mathbb{R}^3) \subset \bigcap_{1<p<\infty} L^p(\mathbb{R}^3), \tag{2.22}$$

hence

$$\bigcap_{1<q<3} H^{3,q}(\mathbb{R}^3) \subset \bigcap_{1<p<\infty} H^{2,p}(\mathbb{R}^3). \tag{2.23}$$

Also, since $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, it follows that $H^4(\mathbb{R}^3) \subset H^{3,6}(\mathbb{R}^3) \cap H^{3,2}(\mathbb{R}^3) \subset H^{3,q}(\mathbb{R}^3)$, if $2 \le q \le 6$, so we improve (2.14) to

$$u = -u(0) R_\lambda(V_0\chi), \qquad \operatorname{mod}\ H^{3,q}(\mathbb{R}^3), \tag{2.24}$$

for each $q \in [2,3)$. In light of (2.16)–(2.16A), this gives:

Proposition 2.3. *Improving Proposition 2.2,*

$$\mathcal{D}(H^2) \subset \operatorname{Span}(|x|\chi) + \bigcap_{2\le q<3} H^{3,q}(\mathbb{R}^3). \tag{2.25}$$

3. Conjectural semigroup and resolvent estimates

Estimates on the semigroup e^{-tH} are of great significance. We state a conjecture which, if true, would have nice applications to Littlewood-Paley theory associated to the operator H.

Conjecture 3.1. *The semigroup e^{-tH} satisfies, for $t \in (0,1]$,*

$$\begin{aligned} &\|e^{-tH}\|_{\mathcal{L}(L^2,L^\infty)} \le Ct^{-3/4}, \quad \textit{and} \\ &\|e^{-tH}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le Ct^{-3/4-1/2}. \end{aligned} \tag{3.1}$$

In connection with this, we have the following.

Proposition 3.2. *Set*

$$\mathcal{H} = H + \frac{K^2}{4} + 1, \tag{3.2}$$

which is self-adjoint with spectrum in $[1,\infty)$. Then there exists $C < \infty$ such that

$$\sup_{t\ge 1} \|e^{-t\mathcal{H}}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C. \tag{3.3}$$

Proof. Let $t = 1+s,\ s \ge 0$. Since $e^{-t\mathcal{H}} = e^{-\mathcal{H}}e^{-s\mathcal{H}}$ and $e^{-s\mathcal{H}}$ is a contraction on $L^2(\mathbb{R}^3)$, we have

$$\sup_{t\ge 1} \|e^{-t\mathcal{H}}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le \|e^{-\mathcal{H}}\|_{\mathcal{L}(L^2,\mathrm{Lip})}. \tag{3.4}$$

Since

$$e^{-\mathcal{H}} = \mathcal{H}^{-2}(\mathcal{H}^2 e^{-\mathcal{H}}), \tag{3.5}$$

and $\mathcal{H}^2 e^{-\mathcal{H}}$ is bounded on $L^2(\mathbb{R}^3)$, we have

$$\|e^{-\mathcal{H}}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C_1 \|\mathcal{H}^{-2}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C, \tag{3.6}$$

the last inequality by Proposition 2.1.

Using $\mathcal{H}$, we can rewrite the second conjectural estimate in (3.1) as

$$\|e^{-t\mathcal{H}}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C(t^{-3/4-1/2}+1), \quad t > 0. \tag{3.7}$$

It is useful to connect (3.7) to resolvent estimates.

Proposition 3.3. *If (3.7) holds, then*

$$\|(1+t^2\mathcal{H}^2)^{-1}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C(t^{-3/4-1/2}+1), \quad t>0. \tag{3.8}$$

Proof. We can use

$$(1+t^2\mathcal{H}^2)^{-1} = (1+t\mathcal{H})^{-2}(1+t\mathcal{H})^2(1+t^2\mathcal{H}^2)^{-1} \tag{3.9}$$

to reduce our task to showing that (3.7) implies

$$\|(1+t\mathcal{H})^{-2}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C(t^{-3/4-1/2}+1). \tag{3.10}$$

To prove (3.10), we use the identity

$$(1+t\mathcal{H})^{-\sigma} = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-s}e^{-st\mathcal{H}}s^{\sigma-1}\,ds. \tag{3.11}$$

Then, given (3.7),

$$\begin{aligned}\|(1+t\mathcal{H})^{-\sigma}\|_{\mathcal{L}(L^2,\mathrm{Lip})} &\le C\int_0^{t^{-2}} e^{-s}(st)^{-3/4-1/2}s^{\sigma-1}\,ds \\ &\quad + C\int_{t^{-2}}^\infty e^{-s}s^{\sigma-1}\,ds \\ &\le C_1(t^{-3/4-1/2}+1),\end{aligned} \tag{3.12}$$

with $C_1<\infty$ if $\sigma>3/4+1/2$. In particular, this applies for $\sigma=2$.

We also have a converse, derivable via the following useful result.

Proposition 3.4. *If (3.8) holds, then*

$$\|\Phi(t\mathcal{H})\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C(t^{-3/4-1/2}+1), \quad t>0, \tag{3.13}$$

whenever $\Phi(\lambda)$ *satisfies*

$$|\Phi(\lambda)| \le C(1+|\lambda|)^{-2}. \tag{3.14}$$

Proof. Indeed,

$$\Phi(t\mathcal{H}) = (1+t^2\mathcal{H}^2)^{-1}(1+t^2\mathcal{H}^2)\Phi(t\mathcal{H}), \tag{3.15}$$

and the product of the last two factors is uniformly bounded on $L^2(\mathbb{R}^3)$.

Taking

$$\Phi(\lambda) = e^{-|\lambda|}, \tag{3.16}$$

we see that

$$(3.8) \Longrightarrow (3.7).$$

Remark. Having (3.7), we also see from (3.12) that the estimate (3.13) holds provided

$$|\Phi(\lambda)| \le C(1+|\lambda|)^{-\beta}, \quad \beta>\frac{5}{4}. \tag{3.17}$$

4. Domain of $\mathcal{H}^{1+a}$

By Proposition 3.4 and the subsequent remark, involving (3.17), we see that if Conjecture 3.1 is true, then the following must hold.

Proposition 4.1. *Take $\mathcal{H}$ as in (3.2). Then*

$$\mathcal{D}(\mathcal{H}^{1+a}) \subset \mathrm{Lip}(\mathbb{R}^3), \quad \text{for } 1+a > \frac{5}{4}. \tag{4.1}$$

Our first goal here is to prove (4.1).

To start, take $a \in (0,1)$ and $u \in \mathcal{D}(\mathcal{H}^{1+a})$, and write

$$\mathcal{H}u = f \in \mathcal{D}(\mathcal{H}^a) = H^{2a}(\mathbb{R}^3), \tag{4.2}$$

the last identification by (1.3) (which implies $\mathcal{D}(\mathcal{H}) = H^2(\mathbb{R}^3)$) and interpolation. Now $\mathcal{H} = H - \lambda$, with $\lambda = -1 - K^2/4 \notin \operatorname{Spec} H$, so, parallel to (2.3), we have

$$u = -R_\lambda V u + R_\lambda f, \quad R_\lambda f \in H^{2(1+a)}(\mathbb{R}^3). \tag{4.3}$$

Note that $H^{2(1+a)}(\mathbb{R}^3) \subset \mathrm{Lip}(\mathbb{R}^3)$ precisely for $1 + a > 5/4$. Taking $V = V_0 + V_1$, as in (2.5), we can rewrite (4.3) as

$$\begin{aligned} u &= -R_\lambda V_0 u - R_\lambda V_1 u + R_\lambda f \\ &= -R_\lambda V_0 u \qquad \mod H^{2(1+a)}(\mathbb{R}^3), \end{aligned} \tag{4.4}$$

since $u \in \mathcal{D}(\mathcal{H}) \Rightarrow R_\lambda V_1 u \in H^4(\mathbb{R}^3)$.

Next, parallel to (2.9),

$$u = u(0)\chi + u_1. \tag{4.5}$$

At this point, we can say the following about u_1. Since

$$\mathcal{D}(\mathcal{H}) = H^2(\mathbb{R}^3) \subset C^{1/2}(\mathbb{R}^3), \quad \text{and } \mathcal{D}(\mathcal{H}^2) \subset \mathrm{Lip}(\mathbb{R}^3), \tag{4.6}$$

an interpolation argument gives, for $a \in (0,1)$,

$$u \in \mathcal{D}(\mathcal{H}^{1+a}) \Longrightarrow u \in C^{(1+\alpha)/2}(\mathbb{R}^3), \quad \forall\, \alpha < a, \tag{4.7}$$

so

$$u_1 \in C^{(1+\alpha)/2}(\mathbb{R}^3), \quad \text{and } u_1(0) = 0. \tag{4.8}$$

Now, by (4.4),

$$u = -u(0)R_\lambda(V_0\chi) - R_\lambda V_0 u_1, \quad \text{mod } H^{2(1+a)}(\mathbb{R}^3), \tag{4.10}$$

and, by (4.8),

$$|V_0(x)u_1(x)| \le C\chi(x)|x|^{\beta-1}, \quad \forall\, \beta < \frac{1+a}{2}. \tag{4.11}$$

Next,

$$\begin{aligned} 1+a > \frac{5}{4} &\Rightarrow |V_0(x)u_1(x)| \le C\chi(x)|x|^{-3/8+\varepsilon} \quad (\varepsilon > 0) \\ &\Rightarrow V_0 u_1 \in L^8(\mathbb{R}^3) \\ &\Rightarrow R_\lambda V_0 u_1 \in H^{2,8}(\mathbb{R}^3) \\ &\Rightarrow R_\lambda V_0 u_1 \in C^{1+r}(\mathbb{R}^3), \quad \forall\, r < \frac{5}{8}. \end{aligned} \tag{4.12}$$

Meanwhile, (2.16)–(2.16A) apply to $R_\lambda(V_0\chi)$, so we have the desired result (4.1).

Here is an interesting complement to Proposition 4.1.

Proposition 4.2. *We have*

$$\mathcal{D}(\mathcal{H}^{1+a}) = H^{2(1+a)}(\mathbb{R}^3), \quad \textit{for } \; 1 \le 1+a < \frac{5}{4}. \tag{4.15}$$

Proof. First, take $u \in \mathcal{D}(\mathcal{H}^{1+a})$. The first part of the proof of Proposition 4.1 applies in this setting, to yield (4.10), i.e.,

$$u \in \mathcal{D}(\mathcal{H}^{1+a}) \Rightarrow u = -u(0)R_\lambda(V_0\chi) - R_\lambda V_0 u_1 \;\text{ mod } H^{2(1+a)}(\mathbb{R}^3). \tag{4.16}$$

Also, $1+a \le 5/4 \Rightarrow H^{2(1+a)}(\mathbb{R}^3) \supset H^{5/2}(\mathbb{R}^3)$, so (2.16)–(2.16A) implies $R_\lambda(V_0\chi) = (K/2)|x|\chi^2 \text{ mod } H^{2(1+a)}(\mathbb{R}^3)$. Meanwhile, for the Fourier transform of $|x|\chi^2$ we have

$$\begin{aligned} \mathcal{F}(|x|\chi^2)(\xi) &\sim C\langle\xi\rangle^{-2}\mathcal{F}(|x|^{-1}\chi^2)(\xi) \\ &\sim C\langle\xi\rangle^{-4}, \end{aligned} \tag{4.17}$$

and hence

$$|x|\chi^2 \in H^{5/2-\varepsilon}(\mathbb{R}^3), \quad \forall\, \varepsilon > 0. \tag{4.18}$$

Hence, as long as $1 \le 1+a < 5/4$,

$$u \in \mathcal{D}(\mathcal{H}^{1+a}) \Rightarrow u = -R_\lambda V_0 u_1 \;\text{ mod } H^{2(1+a)}(\mathbb{R}^3). \tag{4.19}$$

At this point, we have $u_1 \in \mathcal{D}(\mathcal{H}) = H^2(\mathbb{R}^3)$ and $V_0 = |x|^{-1}\chi$. Parallel to (4.17)–(4.18), $\widehat{V}_0(\xi) \sim C\langle\xi\rangle^{-2}$, so $V_0 \in H^{1/2-\varepsilon}(\mathbb{R}^3)$, $\forall\,\varepsilon > 0$, hence, for all $\varepsilon > 0$,

$$V_0 \in H^{1/2-\varepsilon}(\mathbb{R}^3), \quad \text{hence} \;\; V_0 u_1 \in H^{1/2-\varepsilon}(\mathbb{R}^3), \tag{4.20}$$

so

$$u \in \mathcal{D}(\mathcal{H}) \Rightarrow R_\lambda V_0 u_1 \in H^{5/2-\varepsilon}(\mathbb{R}^3), \quad \forall\,\varepsilon > 0. \tag{4.21}$$

This proves that $\mathcal{D}(\mathcal{H}^{1+a}) \subset H^{2(1+a)}(\mathbb{R}^3)$ if $1 \le 1 + a < 5/4$. Using

$$\begin{aligned} \mathcal{D}(\mathcal{H}^{1+a}) &= \{u \in \mathcal{D}(\mathcal{H}) : \mathcal{H}u \in \mathcal{H}^a\} \\ &= \{u \in H^2(\mathbb{R}^3) : \mathcal{H}u \in H^{2a}(\mathbb{R}^3)\}, \end{aligned} \tag{4.22}$$

one can produce a similar argument to establish the converse.

Remark. We have

$$\|e^{-t\mathcal{H}}\|_{\mathcal{L}(L^2,\mathcal{D}(\mathcal{H}^{5/4}))} = \|\mathcal{H}^{5/4}e^{-t\mathcal{H}}\|_{\mathcal{L}(L^2)} \le Ct^{-5/4}, \tag{4.23}$$

an estimate that, in light of Propositions 4.1–4.2, is tantalizingly close to the conjectured estimate (3.1), but of course not quite on the mark.

5. Projection onto the negative spectrum of H

Let

$$P = \text{ orthogonal projection of } L^2(\mathbb{R}^3) \text{ onto } \bigoplus_{n\geq 1} V_n, \tag{5.1}$$

where the spaces V_n are the eigenspaces of H described in (1.4). Clearly

$$P : L^2(\mathbb{R}^3) \longrightarrow \bigcap_{k\geq 1} \mathcal{D}(H^k). \tag{5.2}$$

In particular,

$$P : L^2(\mathbb{R}^3) \longrightarrow H^{5/2-\varepsilon}(\mathbb{R}^3), \quad \forall\, \varepsilon > 0, \tag{5.3}$$

and

$$P : L^2(\mathbb{R}^3) \longrightarrow \operatorname{Lip}(\mathbb{R}^3). \tag{5.4}$$

A special case of (1.5) is

$$V_1 = \operatorname{Span} e^{-2K|x|}, \tag{5.5}$$

so $\operatorname{Lip}(\mathbb{R}^3)$ cannot be replaced by $C^1(\mathbb{R}^3)$ in (5.4). Note that (5.3) implies

$$P : L^2(\mathbb{R}^3) \longrightarrow \bigcap_{2\leq p\leq\infty} L^p(\mathbb{R}^3), \tag{5.6}$$

and

$$P : L^2(\mathbb{R}^3) \longrightarrow C_0(\mathbb{R}^3). \tag{5.7}$$

By duality, since $P = P^*$,

$$P : L^q(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3), \quad \forall\, q \in [1, 2], \tag{5.8}$$

and, since also $P^2 = P$, we then have

$$P : L^q(\mathbb{R}^3) \longrightarrow L^p(\mathbb{R}^3), \quad \text{for } \ 1 \leq q \leq 2 \leq p \leq \infty. \tag{5.9}$$

Note that each eigenfunction described in (1.5) belongs to $L^p(\mathbb{R}^3)$ for all $p \in [1, \infty]$. Nevertheless, we hazzard the following:

Conjecture 5.1. *Given* $p \in [1, \infty]$,

$$P : L^2(\mathbb{R}^3) \to L^p(\mathbb{R}^3) \Longrightarrow p \geq 2, \quad \textit{and} \tag{5.10}$$

$$P : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3) \Longrightarrow p = 2. \tag{5.11}$$

Note that the operators $H^k P$ enjoy the mapping properties (5.2)–(5.4) and (5.6)–(5.9). Actually, these operators are milder than P. For example, it follows from (1.4) that

$$HP \text{ is Hilbert-Schmidt, and } H^2 P \text{ is trace class, on } \ L^2(\mathbb{R}^3). \tag{5.12}$$

We therefore hazzard the following:

Conjecture 5.2. *There exist* $p_0, p_1 \in (1, 2)$ *such that*

$$HP : L^2(\mathbb{R}^3) \longrightarrow L^p(\mathbb{R}^3), \ \textit{for} \ \ p \in (p_0, 2], \ \textit{and} \tag{5.13}$$

$$HP : L^p(\mathbb{R}^3) \longrightarrow L^p(\mathbb{R}^3), \ \textit{for} \ \ p \in (p_1, p_1'). \tag{5.14}$$

[T] M. Taylor, Spectral Theory, Chapter 8 of Partial Differential Equations, Vol. 2, Springer-Verlag, New York, 1996 (2nd ed., 2011).

[T2] M. Taylor, Functional calculus and Littlewood-Paley theory for Schrödinger operators, Preprint, 2017.