Potentials in the Kato Class of Measures
And Other Very Singular Potentials

Michael Taylor

Contents

1. Introduction

Let M be a compact Riemannian manifold, of dimension $n \ge 2$, with volume element $d\omega$ and Laplace operator Δ. Set

$$G(x,y) = \gamma_n(d(x,y)), \tag{1.1}$$

where $d(x,y)$ is the Riemannian distance from x to y, and

$$\begin{aligned} \gamma_n(r) &= r^{2-n}, \quad n \ge 3, \\ \log \frac{A}{r}&, \quad n = 2, \end{aligned} \tag{1.2}$$

where $A > \operatorname{diam} M$. An essentially equivalent specification of G is

$$(1-\Delta)^{-1}u(x) = \int_M G(x,y)u(y)\, d\omega(y). \tag{1.3}$$

The Kato class $\mathcal{K}(M)$ of integrable functions, introduced in [K], is defined as follows.

Definition. Given $V \in L^1(M)$, we say $V \in \mathcal{K}(M)$ provided

$$\lim_{r\to 0} \sup_{x\in M} \int_{B_r(x)} G(x,y)|V(y)|\, d\omega(y) = 0. \tag{1.4}$$

Equivalent conditions are

$$\lim_{\lambda\to+\infty} \sup_{x\in M} (\lambda-\Delta)^{-1}|V|(x) = 0, \tag{1.5}$$

hence

$$\lim_{\lambda\to+\infty} \sup_{\|u\|_{L^1}\le 1} |(u, (\lambda-\Delta)^{-1}|V|)| = 0, \tag{1.6}$$

hence

$$\lim_{\lambda\to+\infty} \sup_{\|u\|_{L^1}\le 1} |(|V|(\lambda-\Delta)^{-1}u, 1)| = 0, \tag{1.7}$$

hence (since $(\lambda-\Delta)^{-1}$ is positivity-preserving)

$$\lim_{\lambda\to+\infty} \|V(\lambda-\Delta)^{-1}\|_{\mathcal{L}(L^1(M))} = 0. \tag{1.8}$$

Here is another characterization (a consequence of Theorem 4.15 of [AS]).

Proposition 1.1. *If $V \in L^1(M)$, then*

$$V \in \mathcal{K}(M) \Longleftrightarrow (1-\Delta)^{-1}|V| \in C(M). \tag{1.9}$$

There is also the following property of M_V (defined by $M_V u = Vu$), important for defining $-\Delta + V$ as a self-adjoint operator.

Proposition 1.2. *If $V \in \mathcal{K}(M)$, then*

$$M_V : H^1(M) \longrightarrow H^{-1}(M), \tag{1.10}$$

and for each $\varepsilon > 0$, there exists $C(\varepsilon) < \infty$ such that, for $u \in H^1(M)$,

$$\int_M |V|u^2 \, d\omega \le \varepsilon \|u\|_{H^1}^2 + C(\varepsilon)\|u\|_{L^2}^2. \tag{1.11}$$

For a generalization involving fractional powers of $1-\Delta$, see Theorem 4.2 of [ZY].

The notion of Kato class for integrable functions given above has the following natural extension (introduced in [BM]).

Definition. Let μ be a finite, signed Borel measure on M. We say $\mu \in \mathcal{K}(M)$ provided

$$\lim_{r\to 0} \sup_{x\in M} \int_{B_r(x)} G(x,y)\, d|\mu|(y) = 0, \tag{1.12}$$

where $|\mu|$ denotes the positive "total variation" measure associated with μ, via the Hahn decomposition.

Lots of singular measures satisfy (1.12):

Proposition 1.3. *Assume that there exists $A < \infty$ such that, for all $x \in M,\ r > 0$,*

$$|\mu|(B_r(x)) \le Ar^\alpha, \ \ \alpha > n-2. \tag{1.13}$$

Then (1.12) holds.

Proof. We have

$$\int_{B_r(x)\setminus B_{r/2}(x)} G(x,y)\, d|\mu|(y) \le C\gamma_n(r) r^\alpha. \tag{1.14}$$

For $n \geq 3$, this is dominated by

$$Cr^{\alpha-(n-2)}, \tag{1.15}$$

and for $n = 2$ it is dominated by

$$Cr^{\alpha} \log \frac{A}{r}. \tag{1.16}$$

In either case, we have (1.12).

Remark. In case $n = 2$, the condition (1.13) is

$$|\mu|(B_r(x)) \leq Ar^{\alpha}, \quad \text{for some} \ \ \alpha > 0. \tag{1.17}$$

We see in Appendix A (which arose in the course of writing [CST]) that (1.17) holds for various Cantor sets in a 2D manifold M, endowed with Hausdorff measure. It is also seen in Chapter 7 of [T2] that (1.17) holds when μ is the maximal entropy measure on a Julia set in S^2, associated with a holomorphic map on S^2 of degree ≥ 2. In Appendix A we show directly that

$$M_{\mu} : H^1(M) \longrightarrow H^{-1}(M) \ \text{ is compact}, \tag{1.18}$$

whenever μ is a finite positive measure on an n-dimensional Riemannian manifold M satisfying

$$\mu(B_r(x)) \leq Cr^{\alpha}, \quad \alpha > n - 1 - \frac{1}{n-1}. \tag{1.19}$$

Returning to the description of our analysis of Kato-class measures, in §2 we show that Propositions 1.1 and 1.2 extend from L^1 functions $V \in \mathcal{K}(M)$ to finite, signed measures $\mu \in \mathcal{K}(M)$. We deduce that, for such μ, there exists a constant C such that

$$A = -\Delta + M_{\mu} + C \ \text{ is a positive self-adjoint operator on } \ L^2(M), \tag{1.20}$$

satisfying

$$\mathcal{D}(A^{1/2}) = H^1(M). \tag{1.21}$$

Going further, in §3, we examine self-adjoint extensions of such operators as

$$A = -\Delta + M_{\mu} + FM_{\nu} + C, \tag{1.22}$$

with μ and C as in (1.20), ν a positive measure on M satisfying

$$M_{\nu} : H^1(M) \longrightarrow H^{-1}(M), \tag{1.23}$$

and

$$F \geq 1, \quad F \in L^1(M, \nu). \tag{1.24}$$

In case $\mu = 0$ and $\nu = \omega$, Proposition 3.2 specializes to a classical result of Friedrichs (cf. [CFKS]), Chapter 1.

In §§4–5 we point to further results on Schrödinger operators with singular potentials. We cite various papers that we have not digested.

Appendix A gives a general discussion of multipliers from $H^1(M)$ to $H^{-1}(M)$, taken from [CST].

2. Properties of measures in $\mathcal{K}(M)$

Our first goal here is to extend Proposition 1.1. In preparation for this, let us take $G(x,y)$ as in (1.3) and write (for $t>0$)

$$(1-\Delta)^{-1}e^{t(\Delta-1)}|\mu|(x)=\int\limits_M G_t(x,y)\,d|\mu|(y), \tag{2.1}$$

and

$$R_t(x,y)=G(x,y)-G_t(x,y), \tag{2.2}$$

so

$$\begin{aligned}\int\limits_M R_t(x,y)\,d|\mu|(y)&=(1-\Delta)^{-1}(1-e^{t(\Delta-1)})|\mu|(x)\\ &=\int_0^t e^{s(\Delta-1)}|\mu|(x)\,ds,\end{aligned} \tag{2.3}$$

the last identity yielding $R_t(x,y)\geq 0$. Comparison with (1.12) shows that

$$\begin{aligned}\mu\in\mathcal{K}(M)&\Longleftrightarrow \lim_{t\searrow 0}\ \sup_x \int\limits_M R_t(x,y)\,d|\mu|(y)=0\\ &\Longleftrightarrow \lim_{t\searrow 0}\big\|(1-e^{t(\Delta-1)})(1-\Delta)^{-1}|\mu|\big\|_{\sup}=0.\end{aligned} \tag{2.4}$$

As an aside, we mention that, if μ is a finite signed measure on M, then, for each $\varepsilon>0$, there exists $\delta>0$ such that

$$\psi:=(1-\Delta)^{-1}|\mu|\in H^{2-\varepsilon,1+\delta}(M),\quad \psi\geq 0. \tag{2.5}$$

Here is our extension of Proposition 1.1.

Proposition 2.1. *Given that μ is a finite signed measure on M,*

$$\mu\in\mathcal{K}(M)\Longleftrightarrow(1-\Delta)^{-1}|\mu|\in C(M). \tag{2.6}$$

Proof. Take ψ as in (2.5). Since $\{e^{t(\Delta-1)}:t\geq 0\}$ is a strongly continuous semigroup on $C(M)$ and, for each $\psi\in\mathcal{D}'(M)$, $e^{t(\Delta-1)}\psi\in C^\infty(M)$ for all $t>0$, we have

$$\begin{aligned}\psi\in C(M)&\Longleftrightarrow \lim_{t\searrow 0}\big\|(1-e^{t(\Delta-1)})\psi\big\|_{\sup}=0\\ &\Longleftrightarrow \mu\in\mathcal{K}(M),\end{aligned} \tag{2.7}$$

the last equivalence by (2.4).

Our next goal is to extend Proposition 1.2, and show that if μ is a finite signed measure,

$$\mu \in \mathcal{K}(M) \Longrightarrow M_\mu : H^1(M) \to H^{-1}(M), \tag{2.8}$$

accompanied by estimates of the form (1.11). Our analysis will be adapted from the proof of Theorem 4.2 in [ZY], specialized from $(-\Delta)^{-\alpha/2}$ to the classical case $\alpha = 2$. To start, note that if $\mu = \mu^+ - \mu^-$ is the Hahn decomposition, $\mu \in \mathcal{K}(M)$ implies $\mu^\pm \in \mathcal{K}(M)$, so we can restrict attention to the case

$$\mu \in \mathcal{K}(M) \ \text{ is a positive measure.} \tag{2.9}$$

It will be convenient to pass to

$$\nu = \mu + \omega, \tag{2.10}$$

where ω is the volume measure on M. Clearly also $\nu \in \mathcal{K}(M)$. Parallel to (1.5), we have, for $\lambda > 0$,

$$\sup_{x\in M} (\lambda - \Delta)^{-1}\nu(x) \le \varepsilon(\lambda), \quad \varepsilon(\lambda) \to 0 \ \text{ as } \ \lambda \to +\infty. \tag{2.11}$$

In order to adapt arguments from [ZY], we set

$$V_s = e^{s\Delta}\nu, \quad s > 0, \tag{2.12}$$

so $V_s \in C^\infty(M)$, and

$$V_s = W_s + 1, \quad W_s = e^{s\Delta}\mu \ge 0. \tag{2.13}$$

Here, as is natural, we implement the identification $C^\infty(M) \hookrightarrow \mathcal{D}'(M)$ by $f \mapsto f\omega$. Note that

$$\sup_{x\in M} (\lambda - \Delta)^{-1} V_s(x) \le \varepsilon(\lambda), \tag{2.14}$$

with $\varepsilon(\lambda)$ exactly as in (2.11), since $\{e^{s\Delta} : s \ge 0\}$ is a contraction semigroup on $C(M)$. In particular, the estimate (2.14) is independent of $s \in (0,\infty)$. Hence, parallel to (1.8), we have

$$\|V_s(\lambda - \Delta)^{-1}\|_{\mathcal{L}(L^1(M))} \le \varepsilon(\lambda). \tag{2.15}$$

By duality,

$$\|(\lambda - \Delta)^{-1} V_s\|_{\mathcal{L}(L^\infty(M))} \le \varepsilon(\lambda). \tag{2.16}$$

Now, following [ZY], we bring in weighted L^p-spaces,

$$L^p_s(M) = L^p(M, V_s\, d\omega). \tag{2.17}$$

We see that $M_{V_s^{1/p}} : L^p_s(M) \to L^p(M)$ is an isometric isomorphism. Hence

$$\begin{aligned} \|(\lambda-\Delta)^{-1}V_s\|_{\mathcal{L}(L^1_s(M))} &= \|V_s(\lambda-\Delta)^{-1}\|_{\mathcal{L}(L^1(M))} \le \varepsilon(\lambda), \\ \|(\lambda-\Delta)^{-1}V_s\|_{\mathcal{L}(L^\infty_s(M))} &= \|(\lambda-\Delta)^{-1}V_s\|_{\mathcal{L}(L^\infty(M))} \le \varepsilon(\lambda). \end{aligned} \tag{2.18}$$

Therefore, by the Riesz-Thorin interpolation theorem,

$$\|(\lambda-\Delta)^{-1}V_s\|_{\mathcal{L}(L^2_s(M))} \le \varepsilon(\lambda), \tag{2.19}$$

which translates to

$$\|V_s^{1/2}(\lambda-\Delta)^{-1}V_s^{1/2}\|_{\mathcal{L}(L^2(M))} \le \varepsilon(\lambda). \tag{2.20}$$

Hence, for $f \in L^2(M)$,

$$\begin{aligned} \|(\lambda-\Delta)^{-1/2}V_s^{1/2}f\|^2_{L^2(M)} &= ((\lambda-\Delta)^{-1/2}V_s^{1/2}f, (\lambda-\Delta)^{-1/2}V_s^{1/2}f) \\ &= (V_s^{1/2}(\lambda-\Delta)^{-1}V_s^{1/2}f, f) \\ &\le \varepsilon(\lambda)\|f\|^2_{L^2(M)}, \end{aligned} \tag{2.21}$$

so

$$\|(\lambda-\Delta)^{-1/2}V_s^{1/2}\|_{\mathcal{L}(L^2(M))} \le \varepsilon(\lambda)^{1/2}, \tag{2.22}$$

and, by duality,

$$\|V_s^{1/2}(\lambda-\Delta)^{-1/2}\|_{\mathcal{L}(L^2(M))} \le \varepsilon(\lambda)^{1/2}. \tag{2.23}$$

Hence, for $u \in L^2(M)$,

$$\begin{aligned} &(V_s(\lambda-\Delta)^{-1/2}u, (\lambda-\Delta)^{-1/2}u) \\ &= (V_s^{1/2}(\lambda-\Delta)^{-1/2}u, V_s^{1/2}(\lambda-\Delta)^{-1/2}u) \\ &\le \varepsilon(\lambda)\|u\|^2_{L^2(M)}. \end{aligned} \tag{2.24}$$

Let us set

$$\varphi = (\lambda-\Delta)^{-1/2}u, \quad \|\varphi\|_{H^1(M)} = \|(1-\Delta)^{1/2}\varphi\|_{L^2(M)}. \tag{2.25}$$

Then (2.24) gives (for $\lambda > 1$)

$$
\begin{aligned}
\int_M V_s \varphi^2 \, d\omega &\le \varepsilon(\lambda) \|(\lambda - \Delta)^{1/2} \varphi\|_{L^2}^2 \\
&= \varepsilon(\lambda)((\lambda - \Delta)\varphi, \varphi) \\
&= \varepsilon(\lambda) \Big\{ \|\varphi\|_{H^1(M)}^2 + (\lambda - 1)\|\varphi\|_{L^2(M)}^2 \Big\}.
\end{aligned} \tag{2.26}
$$

Taking $\lambda = 1$ and applying Cauchy's inequality gives, for $\varphi, \psi \in H^1(M)$,

$$
\Big| \int_M V_s \varphi \psi \, d\omega \Big| \le \varepsilon(1) \|\varphi\|_{H^1(M)} \|\psi\|_{H^1(M)}, \tag{2.27}
$$

hence $M_{V_s} : H^1(M) \to H^{-1}(M)$ satisfies

$$
\|M_{V_s} \varphi\|_{H^{-1}(M)} \le \varepsilon(1) \|\varphi\|_{H^1(M)}, \tag{2.28}
$$

an estimate that is independent of $s > 0$. Taking $s \searrow 0$ and applying a little basic functional analysis yields the following.

Proposition 2.2. *Let $\mu \in \mathcal{K}(M)$ be a positive measure, and set $\nu = \mu + \omega$. Then*

$$
M_\nu : H^1(M) \longrightarrow H^{-1}(M), \tag{2.29}
$$

and, for all $\varphi \in H^1(M)$, $\lambda > 1$,

$$
(M_\nu \varphi, \varphi) \le \varepsilon(\lambda) \Big\{ \|\varphi\|_{H^1(M)}^2 + (\lambda - 1)\|\varphi\|_{L^2(M)}^2 \Big\}, \tag{2.30}
$$

with $\varepsilon(\lambda)$ as in (2.11). Consequently,

$$
M_\mu : H^1(M) \longrightarrow H^{-1}(M), \tag{2.31}
$$

with a similar estimate.

If we denote by $Q_\lambda(\varphi)$ the quantity in brackets on the right side of (2.30), then Cauchy's inequality yields

$$
|(M_\nu \varphi, \psi)| \le \varepsilon(\lambda) Q_\lambda(\varphi)^{1/2} Q_\lambda(\psi)^{1/2}, \tag{2.32}
$$

for $\varphi, \psi \in H^1(M)$, $\lambda \ge 1$. Recalling the reduction to (2.9) via the Hahn decomposition $\mu = \mu^+ - \mu^-$, we have the following.

Corollary 2.3. *Let $\mu \in \mathcal{K}(M)$ be a finite signed measure. Then (2.31) holds, and*

$$|(M_\mu \varphi, \psi)| \le \varepsilon(\lambda) Q_\lambda(\varphi)^{1/2} Q_\lambda(\psi)^{1/2}. \tag{2.33}$$

We now discuss how to use these estimates to define $-\Delta + M_\mu$ as a self-adjoint operator, when $\mu \in \mathcal{K}(M)$ is a finite, signed measure. Let us rewrite the case $\psi = \varphi$ of (2.33) as

$$|(M_\mu \varphi, \varphi)| \le \varepsilon \|\varphi\|^2_{H^1(M)} + C(\varepsilon) \|\varphi\|^2_{L^2(M)}. \tag{2.34}$$

Set $C^\#(\varepsilon) = C(\varepsilon) + 1$. It follows that

$$([-\Delta + M_\mu + C^\#(1/2)]\varphi, \varphi) \ge \frac{1}{2} \|\varphi\|^2_{H^1(M)}. \tag{2.35}$$

This together with (2.34) implies (e.g., via the Lax-Milgram theorem) that the self-adjoint map

$$-\Delta + M_\mu + C^\#(1/2) : H^1(M) \longrightarrow H^{-1}(M) \ \text{ is bijective,} \tag{2.36}$$

with inverse

$$[-\Delta + M_\mu + C^\#(1/2)]^{-1} : H^{-1}(M) \longrightarrow H^1(M), \tag{2.37}$$

which is self adjoint. Restriction to $L^2(M)$ yields

$$[-\Delta + M_\mu + C^\#(1/2)]^{-1} : L^2(M) \longrightarrow L^2(M), \tag{2.38}$$

self adjoint and compact, and injective. We can hence apply the classical theory of von Neumann/Friedrichs, to deduce the following.

Proposition 2.4. *Let $\mu \in \mathcal{K}(M)$ be a finite signed measure. Then*

$$A = -\Delta + M_\mu + C^\#(1/2) \tag{2.39}$$

is a positive self-adjoint operator on $L^2(M)$, with domain

$$\mathcal{D}(A) = \{u \in H^1(M) : -\Delta u + M_\mu u \in L^2(M)\}. \tag{2.40}$$

This self-adjoint operator has compact resolvent, hence discrete spectrum. Furthermore,

$$\mathcal{D}(A^{1/2}) = H^1(M). \tag{2.41}$$

3. More singular potentials

Take M as before, and take an operator L, satisfying

$$L : H^1(M) \xrightarrow{\approx} H^{-1}(M), \quad \text{positive and self adjoint.} \tag{3.1}$$

For example, we might have $L = -\Delta + 1$, or more generally

$$L = -\Delta + M_\mu + C, \quad \mu \in \mathcal{K}(M), \text{ finite signed measure,} \tag{3.2}$$

and $C \in (0, \infty)$ sufficiently large. Such L gives rise to a positive, self-adjoint operator on $L^2(M)$, with $\mathcal{D}(L^{1/2}) = H^1(M)$. To go further, take a finite, positive measure ν on M such that

$$M_\nu : H^1(M) \longrightarrow H^{-1}(M). \tag{3.3}$$

Next, take a Borel function F on M, satisfying

$$F \geq 1, \quad F \in L^1(M, \nu). \tag{3.4}$$

Consider the space

$$\mathcal{H}_{F\nu}(M) = \{\varphi \in H^1(M) : \varphi \in L^2(M, F\nu + \omega)\} \tag{3.5}$$

(where, recall, ω is volume measure on M), and set

$$Q(\varphi, \psi) = (L\varphi, \psi) + \int_M \varphi\psi \, (F \, d\nu + d\omega), \tag{3.6}$$

for $\varphi, \psi \in \mathcal{H}_{F\nu}(M)$. We aim to prove the following.

Proposition 3.1. *The quadratic form Q defined by (3.6), with form domain $\mathcal{H}_{F\nu}(M)$, is closed.*

This result has the following implication.

Proposition 3.2. *Under the hypotheses given in (3.1)–(3.6), we have a self-adjoint operator*

$$A = L + M_{F\nu}, \tag{3.7}$$

satisfying

$$\mathcal{D}(A^{1/2}) = \mathcal{H}_{F\nu}(M), \tag{3.8}$$

and

$$\mathcal{D}(A) = \{\varphi \in \mathcal{H}_{F\nu}(M) : L\varphi + FM_\nu\varphi \in L^2(M)\}. \tag{3.9}$$

Proof of Proposition 3.1. Take $\varphi_k \in \mathcal{H}_{F\nu}(M)$, and assume

$$(\varphi_k) \text{ is Cauchy in the } Q\text{-norm.} \tag{3.10}$$

Equivalently,

$$(\varphi_k) \text{ is Cauchy in the } H^1(M)\text{-norm,} \tag{3.11}$$

and

$$(\varphi_k) \text{ is Cauchy in the } L^2(M, F\nu + \omega)\text{-norm.} \tag{3.12}$$

Our task is to show that (φ_k) converges in the Q-norm to an element of $\mathcal{H}_{F\nu}(M)$. Now (3.11)–(3.12) imply that there exist

$$\varphi \in H^1(M) \quad \text{and} \quad \psi \in L^2(M, F\nu + \omega) \tag{3.13}$$

such that

$$\begin{aligned} \varphi_k &\to \varphi, \quad \text{in} \quad H^1(M), \text{ and} \\ \varphi_k &\to \psi, \quad \text{in} \quad L^2(M, F\nu + \omega). \end{aligned} \tag{3.14}$$

Now, thanks to (3.3), the first part of (3.14) implies that

$$\varphi_k \longrightarrow \varphi \quad \text{in} \quad L^2(M, \nu + \omega). \tag{3.15}$$

Meanwhile, the second part of (3.14) implies

$$\varphi_k \longrightarrow \psi \quad \text{in} \quad L^2(M, \nu + \omega). \tag{3.16}$$

Hence

$$\varphi = \psi, \quad (\nu + \omega)\text{-a.e., on} \quad M, \tag{3.17}$$

so we have

$$\varphi_k \longrightarrow \varphi \quad \text{in } Q\text{-norm,} \tag{3.18}$$

and Proposition 3.1 is proved.

4. Schrödinger operators defined via Dirichlet forms

A number of papers, cited in "more references," treat self-adjoint extensions of $-\Delta + M_\mu$ for substantially more singular potentials than discussed in §§2–3. The approach involves more general "Dirichlet forms." To introduce this approach in a simple fashion, we start with a quadratic form

$$Q(\varphi,\psi) = \int_M \langle \nabla\varphi, \nabla\psi\rangle \; A^2 \, d\omega, \tag{4.1}$$

where

$$A \in C^\infty(M), \quad A > 0. \tag{4.2}$$

We have

$$\begin{aligned} Q(\varphi,\psi) &= \int_M \langle \nabla\varphi, A^2\nabla\psi\rangle \, d\omega \\ &= \int_M \langle \varphi, \nabla(A^2\psi) - (\nabla A^2)\psi\rangle \, d\omega \\ &= -\int_M \big[\Delta\varphi + 2A^{-1}\nabla A\cdot\nabla\varphi\big]\psi A^2 \, d\omega. \end{aligned} \tag{4.3}$$

Under the hypotheses in (4.2), we see that Q is a closed quadratic form, with form domain $H^1(M)$, determining the operator

$$L = \Delta + 2A^{-1}\nabla A\cdot\nabla, \tag{4.4}$$

as a self-adjoint operator on $L^2(M, A^2\,\omega)$, with domain $H^2(M)$.

Now we have the unitary operator

$$M_A : L^2(M, A^2\omega) \xrightarrow{\approx} L^2(M,\omega), \tag{4.5}$$

given by $M_A\varphi = A\varphi$, giving rise to a self-adjoint operator

$$M_A L M_A^{-1} \;\text{ on }\; L^2(M,\omega), \tag{4.6}$$

also with domain $H^2(M)$. A calculation gives

$$\Delta(A\varphi) = A\Delta\varphi + 2\langle\nabla A, \nabla\varphi\rangle + (\Delta A)\varphi, \tag{4.7}$$

hence

$$A^{-1}\Delta(A\varphi) = \Delta\varphi + 2A^{-1}\nabla A\cdot\nabla\varphi + A^{-1}(\Delta A)\varphi, \tag{4.8}$$

so

$$M_A^{-1}\Delta M_A = L + M_{A^{-1}(\Delta A)}, \tag{4.9}$$

hence

$$M_A L M_A^{-1} = \Delta - M_{A^{-1}(\Delta A)}. \tag{4.10}$$

If

$$A = (-\Delta+1)^{-1}\gamma, \tag{4.11}$$

with $\gamma \in C^\infty(M),\ \gamma \ge 0\ (\gamma \ne 0)$, we are in the setting of (4.2), and (4.10) becomes

$$\begin{aligned} M_A L M_A^{-1} &= \Delta - M_{A^{-1}(\Delta A - A)} - 1 \\ &= \Delta + M_{A^{-1}\gamma} - 1. \end{aligned} \tag{4.12}$$

Now if (4.11) holds for smooth $\gamma \ge 0$, then the calculations leading to (4.12) are simply a hugely indirect way of defining $\Delta + M_{A^{-1}\gamma}$ as a self-adjoint operator on $L^2(M,\omega)$. To go further, we want to take γ to be singular.

For example, by Proposition 2.1,

$$\gamma \in \mathcal{K}(M),\ \gamma \ge 0,\ \gamma \ne 0 \Longrightarrow A \in C(M),\ A > 0. \tag{4.13}$$

Under these conditions, again Q in (4.1) is a closed quadratic form, with form domain $H^1(M)$, yielding a self-adjoint operator L on $L^2(M, A^2\omega)$ satisfying

$$-(L\varphi,\psi)_{L^2(M,A^2\omega)} = Q(\varphi,\psi), \tag{4.14}$$

with

$$\mathcal{D}((-L)^{1/2}) = H^1(M), \tag{4.15}$$

and

$$\mathcal{D}(L) = \{\varphi \in H^1(M) : |Q(\varphi,\psi)| \le C(\varphi)\|\psi\|_{L^2(M,A^2\omega)}\}. \tag{4.16}$$

Now our principal object of interest is not L, but

$$L_A = M_A L M_A^{-1}, \tag{4.17}$$

as a self-adjoint operator on $L^2(M,\omega)$, with the idea of extending the identity (4.12), and understanding its significance. In connection with this, we have from (4.13) that

$$\gamma \in \mathcal{K}(M),\ \gamma \geq 0,\ \gamma \neq 0 \Longrightarrow A^{-1}\gamma \in \mathcal{K}(M), \tag{4.18}$$

so the results of §2 apply to the operator on the right side of (4.12), in this setting. For a clear connection from this to (4.14)–(4.15), it would be useful to have

$$M_A, M_A^{-1} : H^1(M) \longrightarrow H^1(M). \tag{4.19}$$

To see this, take $\varphi \in H^1(M)$. We have

$$\begin{aligned} \nabla(A\varphi) &= A\nabla\varphi + (\nabla A)\varphi, \\ \nabla(A^{-1}\varphi) &= A^{-1}\nabla\varphi - A^{-2}(\nabla A)\varphi, \end{aligned} \tag{4.20}$$

so (4.19) holds provided

$$M_{\nabla A} : H^1(M) \longrightarrow L^2(M). \tag{4.21}$$

That this holds is a consequence of Theorem 11.1.1 of [MS], which we state.

Proposition 4.1. *Given $\gamma \in \mathcal{D}'(M),\ A = (-\Delta+1)^{-1}\gamma$, we have*

$$M_\gamma : H^1(M) \to H^{-1}(M) \Longleftrightarrow M_{\nabla A} : H^1(M) \to L^2(M). \tag{4.22}$$

Actually, the result of [MS] is done in the Euclidean space setting. We should look into checking the manifold case. We note the following related result, Theorem 11.2.2 of [MS] (again, formulated here in the manifold setting).

Proposition 4.2. *Given $\gamma \in \mathcal{D}'(M),\ B = (-\Delta+1)^{-1/2}\gamma$, we have*

$$M_\gamma : H^1(M) \to H^{-1}(M) \Longleftrightarrow M_B : H^1(M) \to L^2(M). \tag{4.23}$$

The proof in [MS] makes use of a result of [MV] that suggests the following.

Claim 4.3. *Let $B \in L^2(M),\ P \in OPS^0(M)$. Then*

$$M_B : H^1(M) \to L^2(M) \Longrightarrow M_{PB} : H^1(M) \to L^2(M). \tag{4.24}$$

In fact, the following result is suggested by Lemmas 12.1.3–12.1.4 of [MS] (which cites [MV] for proofs):

Claim 4.4. *Assume*

$$p \in (1,\infty), \quad 0 < s < \frac{n}{p}. \tag{4.25}$$

Then, for $B \in L^p(M),\ P \in OPS^0(M)$,

$$M_B : H^{s,p}(M) \to L^p(M) \Longrightarrow M_{PB} : H^{s,p}(M) \to L^p(M). \tag{4.26}$$

According to Theorem 2.3.2 of [MS], given $m \in \mathbb{N},\ p \in (1,\infty)$, and $B \in L^p(M)$, we have $M_B : H^{m,p}(M) \to L^p(M)$ if and only if there exists $a < \infty$ such that, for each compact set K,

$$\int_K |B|^p \, dV(x) \le aC_{p,m}(K), \tag{4.27}$$

where $C_{p,m}(K)$ is a certain capacity. One might also want to check whether this characterization (in case $m = 1, p = 2$) is directly applicable to (4.21).

So we see that the approach via Dirichlet forms applies to potentials given by measures in the Kato class. A number of works have gone beyond this, and treated more singular potentials. Papers that do this include [AM], [ABR], [BM], [Br], [BrT], [Her], [KT]. Unfortunately, lots of these papers are tucked away in conference proceedings and are hard to access.

5. Point scatterers in 2D and 3D

There seem to be treatments of this topic using Dirichlet forms in papers cited at the end of §4, but I have not been able to verify their claims.

Another approach to point scattering in 2D and 3D can be found in [RU] and [U]. I have not figured out how these papers tie in with other works.

A. Measures that multiply $H^1(M)$ to $H^{-1}(M)$

Let M be a compact, n-dimensional Riemannian manifold, and $\Omega \subset M$ a connected open subset. Let μ be a finite, positive measure on $\overline{\Omega}$. We want to give conditions that imply

$$M_\mu : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega), \quad \text{compactly, where} \quad M_\mu f = f\mu. \tag{A.1}$$

It will actually be convenient to let μ be a positive finite measure on M, and ask when

$$M_\mu : H^1(M) \longrightarrow H^{-1}(M), \quad \text{compactly.} \tag{A.2}$$

Since $H_0^1(\Omega)$ is a closed linear subspace of $H^1(M)$, we always get (A.1) from (A.2), by restriction (also restricting μ to $\overline{\Omega}$). We emphasize that the support of μ can have nonempty intersection with $\partial\Omega$.

Note that, if $\dim M = n \geq 2$,

$$f, g \in H^1(M) \Rightarrow fg \in H^{1,p}(M), \quad \begin{array}{l} \text{for } p = \dfrac{n}{n-1} \ \text{ if } \ n \geq 3, \\ \text{for all } p < 2 \ \text{ if } \ n = 2. \end{array} \tag{A.3}$$

Since $C(M)$ is dense in the duals of such spaces $H^{1,p}(M)$, we deduce that (A.2) holds under the following conditions.

$$\begin{array}{l} \mu \in H^{1,n/(n-1)}(M)^* = H^{-1,n}(M), \quad \text{if } \ n \geq 3, \\ \mu \in H^{-1,r}(M), \quad \text{for some } \ r > 2, \quad \text{if } \ n = 2. \end{array} \tag{A.4}$$

We can use this to obtain the following class of singular measures satisfying (A.2).

Proposition A.1. *Let a compact $S \subset M$ be locally a Lipschitz graph, of dimension $n-1$, equipped with surface measure, i.e., $(n-1)$-dimensional Haudorff measure, σ_S. Let*

$$\mu = h\,\sigma_S, \tag{A.5}$$

with

$$\begin{array}{ll} h \in L^{n-1}(S, d\sigma_S), & \text{if } \ n \geq 3, \\ h \in L^{1+\delta}(S, d\sigma_S), & \text{for some } \delta > 0, \text{ if } n = 2. \end{array} \tag{A.6}$$

Then (A.4) holds, hence (A.2) holds.

Proof. We can apply the trace theorem, followed by the embedding theorem,

$$\mathrm{Tr} : H^{1,p}(M) \longrightarrow B^s_{p,p}(S) \subset L^q(S), \tag{A.7}$$

with

$$\begin{aligned} p &= \frac{n}{n-1}, \quad s = 1 - \frac{1}{p}, \\ q &= \frac{(n-1)p}{n-1-sp} = \frac{n-1}{n-2}, \quad q' = n-1, \end{aligned} \tag{A.8}$$

if $n \geq 3$. For $n = 2$, (A.7) applies for all $p \in (1,2)$, again with $s = 1 - 1/p$, and taking $p \nearrow 2$ yields $q \nearrow \infty$, hence $q' \searrow 1$. Thus (A.5)–(A.6) imply (A.4).

Note. The trace result is perhaps better known when S is smooth. (Cf. [BL], Theorem 6.6.1.) However, all the function spaces involved are invariant under bi-Lipschitz maps.

Positive measures satisfying (A.2) can have much wilder support than a Lipschitz surface. For example, one can take an infinite sequence of measures μ_k satisfying the hypotheses of Proposition A.1, supported on surfaces S_k, and set $\mu = \sum_{k=1}^{\infty} a_k \mu_k$, with positive a_k decreasing sufficiently fast.

Here is another class of examples. Let $\mathcal{O} \subset M$ be an open set whose boundary $\partial\mathcal{O}$ is locally the graph of a continuous function. Then one can take a smooth vector field X on M, vanishing nowhere on $\partial\mathcal{O}$, whose flow $\mathcal{F}^t_X$ has the property that, for each $y \in \partial\mathcal{O}$, $\mathcal{F}^t_X y$ belongs to $\mathcal{O}$ for small $t > 0$. Then

$$\mu = X\chi_{\mathcal{O}} \tag{A.9}$$

is a positive measure, supported by $\partial\mathcal{O}$, and it belongs to $H^{-1,\infty}(M)$. The positivity of μ is a consequence of the fact that

$$\chi_{\mathcal{O}} \circ \mathcal{F}^t_X \geq \chi_{\mathcal{O}}, \tag{A.10}$$

for all small $t > 0$, since

$$t^{-1}(\chi_{\mathcal{O}} \circ \mathcal{F}^t_X - \chi_{\mathcal{O}}) \longrightarrow X\chi_{\mathcal{O}} \ \text{ in } \ \mathcal{D}'(M), \quad \text{as} \ \ t \to 0. \tag{A.11}$$

In the last class of examples, (A.9), the support of μ has topological dimension $n-1$, but its Hausdorff dimension can be $> n-1$. We next produce measures satisfying (A.4) and supported on "fractal" sets of Hausdorff dimension $< n-1$. We make use of the following result, contained in Theorem 4.7.4 of [Zie]. Here, $B_r(x)$ denotes the ball of radius r centered at x.

Lemma. *Let* μ *be a positive measure on* M *with the property that there exist* $A < \infty$ *and* $\varepsilon > 0$ *such that*

$$\mu(B_r(x)) \le A r^{n-q+\varepsilon}, \quad \forall\, r \in (0,1],\ x \in M. \tag{A.12}$$

Assume $q \in (1,n)$. *Then*

$$\mu \in H^{-1,p}(M), \quad p = q'. \tag{A.13}$$

We then see that (A.4) holds whenever

$$\mu(B_r(x)) \le A r^{\alpha}, \quad \alpha > n - 1 - \frac{1}{n-1}. \tag{A.14}$$

In particular, for $n = 2$, it suffices to have (A.14) for some $\alpha > 0$.

We will give some explicit examples of a compactly supported measure on $\mathbb{R}^2$ satisfying (A.14). It will be clear that many other examples can be constructed. We start with the Cantor middle third set $\mathcal{K} \subset [0,1]$. Now put $[0,1] \subset \mathbb{R} \subset \mathbb{R}^2$, say as part of the x-axis, so now $\mathcal{K} \subset \mathbb{R}^2$. As is well known (cf. [T], p. 170), there is the α-dimensional Hausdorff measure computation

$$\mathcal{H}^{\alpha}(\mathcal{K}) = \gamma_\alpha, \quad \text{for} \ \ \alpha = \frac{\log 2}{\log 3} \approx 0.6309, \tag{A.15}$$

with $0 < \gamma_\alpha < \infty$ (in fact, $\gamma_\alpha = \pi^{\alpha/2} 2^{-\alpha} / \Gamma(\alpha/2 + 1)$). Set

$$\mu = \mathcal{H}^{\alpha} \lfloor \mathcal{K}, \tag{A.16}$$

i.e., $\mu(S) = \mathcal{H}^{\alpha}(\mathcal{K} \cap S)$, for Borel sets $S \subset \mathbb{R}^2$. The self similarity of $\mathcal{K}$ enables one to show that

$$\mu(B_{3^{-k}}(x)) \le C\, 2^{-k}, \tag{A.17}$$

which readily leads to (A.14), with α as in (A.15).

The Cantor middle third set described above is one of a family of Cantor sets $\mathcal{K}(\vartheta) \subset [0,1]$, defined for $\vartheta \in (0,1)$ as follows. Remove from $[0,1] = I$ the open interval of length $\vartheta \ell(I)$, with the same center as I, and repeat this process with the other closed subintervals that remain. (Thus $\mathcal{K} = \mathcal{K}(1/3)$.) This time (cf. [T], p. 171), one has

$$\mathcal{H}^{\alpha}(\mathcal{K}(\vartheta)) = \gamma_\alpha, \quad \alpha = \frac{\log 2}{\log b}, \quad b = \frac{2}{1-\vartheta}, \tag{A.18}$$

and again self-similarity yields

$$\mu(B_r(x)) \le A r^{\alpha}, \tag{A.19}$$

with α as in (A.18), when

$$\mu = \mathcal{H}^{\alpha}\lfloor\mathcal{K}(\vartheta). \tag{A.20}$$

Note that

$$\vartheta \searrow 0 \Rightarrow \alpha \nearrow 1, \quad \vartheta \nearrow 1 \Rightarrow \alpha \searrow 0. \tag{A.21}$$

As before, we put $\mathcal{K}(\vartheta) \subset [0,1] \subset \mathbb{R} \subset \mathbb{R}^2$, and regard μ in (A.20) as a compactly supported measure on $\mathbb{R}^2$. Thus the push-forward of μ to a measure on a compact two-dimensional manifold M, via a locally bi-Lipschitz map, yields a measure on M satisfying (A.4), hence (A.2), whenever $0 < \vartheta < 1$.

One way to get measures on higher dimensional spaces satisfying (A.14) is to take products. Say $n = n_1 + n_2$ and μ_j are compactly supported measures on $\mathbb{R}^{n_j}$ satisfying

$$\mu_j(B_r(x_j)) \le C_j r^{\alpha_j}, \quad j = 1, 2, \quad x_j \in \mathbb{R}^{n_j}. \tag{A.22}$$

If $x = (x_1, x_2) \in \mathbb{R}^n$, note that $B_r(x) \subset B_r(x_1) \times B_r(x_2)$, so if

$$\mu = \mu_1 \times \mu_2 \tag{A.23}$$

is the product measure on $\mathbb{R}^n$, we have

$$\mu(B_r(x)) \le \mu_1(B_r(x_1))\mu_2(B_r(x_2)) \le C_1 C_2 r^{\alpha_1+\alpha_2}, \quad x \in \mathbb{R}^n. \tag{A.24}$$

If $\alpha = \alpha_1 + \alpha_2$ satisfies the condition on α in (A.14), we get a compactly supported measure on $\mathbb{R}^n$ whose push-forward to a measure on an n-dimensional compact manifold M, via a locally bi-Lipschitz map, satisfies (A.14), hence (A.4), hence (A.2).

For example, take $\vartheta_j \in (0,1)$, and set

$$\mu_j = \mathcal{H}^{\alpha_j}\lfloor\mathcal{K}(\vartheta_j), \tag{A.25}$$

with α_j as in (A.18), i.e.,

$$\alpha_j = \frac{\log 2}{\log b_j}, \quad b_j = \frac{2}{1-\vartheta_j}. \tag{A.26}$$

We regard μ_1 as a measure on $\mathbb{R}^2$ and μ_2 as a measure on $\mathbb{R}$, via $\mathcal{K}(\vartheta_1) \subset [0,1] \subset \mathbb{R} \subset \mathbb{R}^2$ and $\mathcal{K}(\vartheta_2) \subset [0,1] \subset \mathbb{R}$. Thus $\mu = \mu_1 \times \mu_2$ is a compactly supported measure on $\mathbb{R}^3$ (actually supported on a 2D linear subspace of $\mathbb{R}^3$). In this case, the condition for (A.18) to hold is

$$\alpha_1 + \alpha_2 > \frac{3}{2}. \tag{A.27}$$

Looking at (A.15), we see that (A.27) fails when $\vartheta_1 = \vartheta_2 = 1/3$. In case $\vartheta_1 = \vartheta_2 = \vartheta$, the condition that (A.27) hold is that

$$2\,\frac{\log 2}{\log 2 - \log(1-\vartheta)} > \frac{3}{2}, \tag{A.28}$$

or equivalently,

$$\vartheta < 1 - 2^{-1/3} \approx 0.2063. \tag{A.29}$$

Note. When the measures μ_j are given by (A.25), $\mu = \mu_1 \times \mu_2$ is supported on $\mathcal{K}(\vartheta_1) \times \mathcal{K}(\vartheta_2)$, but it is generally *not* $(\alpha_1 + \alpha_2)$-dimensional Hausdorff measure on this set. See pp. 70–74 of [Fal] for a discussion of this matter.

Returning to generalities, we mention that while (A.12) is a sufficient condition for (A.13), it is not quite necessary. There is a (somewhat more elaborate) necessary and sufficient condition for (A.13) to hold, provided $1 < q < n$, given in terms of an estimate on $\mu(B_r(x))$. See Theorem 4.7.5 of [Zie]. The condition is subtly weaker than (A.12).

We also mention conditions for

$$M_\mu : H^1(M) \longrightarrow H^{-1}(M), \tag{A.30}$$

i.e., (A.2) but disregarding compactness. If we take $f \in H^1(M)$ and pair $M_\mu f$ with f, since μ is a positive measure, a necessary condition for (A.30) is that

$$\int\limits_M |f|^2\, d\mu \le C\|f\|^2_{H^1(M)}, \quad \forall\, f \in H^1(M). \tag{A.31}$$

Applying Cauchy's inequality to $\int_M f\overline{g}\, d\mu$, we see that (A.31) is also sufficient for (A.30). Furthermore, it follows from Theorem 1.2.2 of [MS] that the existence of $A < \infty$ such that

$$\mu(S) \le A\,\mathrm{Cap}(S), \quad \forall\,\text{Borel } S \subset M \tag{A.32}$$

is necessary and sufficient for (A.31) to hold, hence for (A.30) to hold, where $\mathrm{Cap}(S)$ is a varant of electrostatic capacity, appropriate for this situation.

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