Euler equation on a rotating surface

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Notes available on my website, under Downloadable Lecture Notes

8. Seminar talks and AMS talks

See also
21. Euler and Navier-Stokes equations

Euler equation for ideal, incompressible flow

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p, \quad \operatorname{div} u = 0.$$

$u =$ fluid velocity, $p =$ pressure.

Valid on $\mathbb{R}^n$, also on n-dimensional Riemannian manifold M.
Then ∇_u denotes the covariant derivative.

If $\partial M \neq \emptyset$, u is tangent to ∂M.

Euler equation on a rotating 2D surface $M \subset \mathbb{R}^3$.

$$\frac{\partial u}{\partial t} + \nabla_u u = \Omega \chi(x) J u - \nabla p, \quad \text{div}\, u = 0.$$

Assume constant angular velocity $\omega = -\Omega/2$, rotation about x_3 axis.

Term $\Omega \chi(x) J u$ reflects Coriolis force.

$$\chi(x) = e_3 \cdot \nu(x).$$

Outward pointing normal $\nu(x)$ to $x \in M$.

$$M = S^2 \Longrightarrow \chi(x) = x_3.$$

$$J : T_x M \to T_x M, \text{ counterclockwise rotation by } 90^\circ.$$

Standing hypotheses on $M \subset \mathbb{R}^3$.

M is diffeomorphic to S^2 and has positive Gauss curvature.

M is rotationally symmetric about x_3-axis.

Conservation of energy.

Assume u solves the 2D rotational Euler equation. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 &= (\partial_t u, u) \\ &= -(\nabla_u u, u) - (u, \nabla p) + \Omega(J\chi u, u).\end{aligned}$$

We claim the three terms on the right are all zero. First,

$$\nabla_u^* v = -\nabla_u v - (\operatorname{div} u)v,$$

for vector fields u and v. Next,

$$(u, \nabla p) = -(\operatorname{div} u, p).$$

Finally,

$$Ju(x) \perp u(x), \quad \forall x \in M.$$

Consequently we have conservation of energy:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0.$$

Differential form presentation, $u \mapsto \tilde{u}$ (1-form).

$$\frac{\partial \tilde{u}}{\partial t} + \nabla_u \tilde{u} = \Omega\chi * \tilde{u} - dp, \quad \delta \tilde{u} = 0.$$

Alternative

$$\frac{\partial \tilde{u}}{\partial t} + \mathcal{L}_u \tilde{u} = \Omega\chi * \tilde{u} + d\Big(\frac{1}{2}|u|^2 - p\Big).$$

$\mathcal{L}_u$ is the Lie derivative. It commutes with the exterior derivative, d.

Vorticity

$$d\tilde{u} = \tilde{w} = w\alpha, \quad w = \operatorname{rot} u,$$

α is the area form on M.

Vorticity Equation.

Apply d to the Euler equation. Get

$$\frac{\partial \tilde{w}}{\partial t} + \mathcal{L}_u \tilde{w} = \Omega d(\chi * \tilde{u}).$$

Right side equals $\Omega(d\chi \wedge *\tilde{u}) = \Omega\langle d\chi, u\rangle\alpha = \Omega\nabla_u\chi$. Hence

$$\frac{\partial w}{\partial t} + \nabla_u w = \Omega\nabla_u\chi.$$

Alternative

$$\frac{\partial}{\partial t}(w - \Omega\chi) + \nabla_u(w - \Omega\chi) = 0.$$

Conservation law.

Stream function.

Given that $\mathcal{H}^1(M) = 0$,

$$\begin{aligned} \operatorname{div} u = 0 &\Longrightarrow d * \tilde{u} = 0 \\ &\Longrightarrow \tilde{u} = *df \\ &\Longrightarrow u = J\nabla f, \end{aligned}$$

with f (the stream function) determined up to an additive constant.

We have

$$w = \Delta f,$$

and the vorticity equation can be written as

$$\frac{\partial w}{\partial t} + \langle J\nabla f, \nabla(w - \Omega\chi)\rangle = 0.$$

Another conservation law (from rotational symmetry of M).

X_3 generates 2π-periodic rotation about x_s-axis.

Generates flow by isometries on M. Hence $\operatorname{div} X_3 = 0$.

Hence there exists $\xi \in C^\infty(M)$ such that

$$J\nabla\xi = -X_3.$$

Note that

$$M = S^2 \Longrightarrow \xi = \chi = x_3.$$

More generally, under our hypotheses, ξ and χ are smooth functions of x_3, with positive x_3-derivatives.

Theorem. If u solves the rotational Euler equation and $w = \operatorname{rot} u$, then

$$\int_M \xi(x) w(t,x)\, dS(x) \text{ is independent of } t.$$

Proof uses the vorticity equation, and the fact that ∇_u and X_3 are skew-adjoint on scalar functions. Also it brings in the stream function f, and shows that

$$\frac{d}{dt} \int_M \xi w(t)\, dS = (X_3 f, \Delta f) = 0.$$

Another presentation of Euler's equation. Eliminate the pressure.

$$\frac{\partial \tilde{u}}{\partial t} + P\nabla_u \tilde{u} = \Omega B \tilde{u},$$

where

$$B\tilde{u} = P(\chi * P\tilde{u}).$$

The Leray projection

$$P : L^2(M, \Lambda^1) \longrightarrow \{\tilde{u} \in L^2(M, \Lambda^1) : \delta \tilde{u} = 0\},$$

orthogonal projection that annihilates dp. Via Hodge decomposition,

$$P = -\delta \Delta_2^{-1} d.$$

We have

$$B^* = -B, \quad B \in OPS^{-1}(M),$$

via

$$B\tilde{u} = -\delta\Delta_2^{-1} d(\chi * P\tilde{u}) = -\delta \Delta_2^{-1}(d\chi \wedge *P\tilde{u}).$$

Existence for initial value problem,

$$\tilde{u}(0) = \tilde{u}_0 \in H^s(M), \quad \delta \tilde{u}_0 = 0.$$

Construct mollified evolution equation, using

$$J_\varepsilon = \varphi(\varepsilon \Delta_1), \quad \varphi \in C_0^\infty(\mathbb{R}),\ \varphi(0) = 1.$$

Solve

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t} + PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon \tilde{u}_\varepsilon = \Omega J_\varepsilon B J_\varepsilon \tilde{u}_\varepsilon,$$

$$P\tilde{u}_\varepsilon = \tilde{u}_\varepsilon, \quad \tilde{u}_\varepsilon = J_\varepsilon \tilde{u}_0.$$

Need to estimate $\tilde{u}_\varepsilon$ in $H^s(M)$, for $t \in I$, independent of ε, and take $\varepsilon \to 0$, for short time existence.

First step,

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_\varepsilon(t)\|_{L^2}^2 = -(PJ_\varepsilon \nabla_{u_\varepsilon} J\tilde{u}_\varepsilon, \tilde{u}_\varepsilon) + \Omega(J_\varepsilon B J_\varepsilon \tilde{u}_\varepsilon, \tilde{u}_\varepsilon) = 0.$$

Hence

$$\|\tilde{u}_\varepsilon(t)\|_{L^2} \equiv \|J_\varepsilon \tilde{u}_0\|_{L^2}.$$

Higher order Sobolev estimates. Take $A = (-\Delta_1)^{1/2}$.

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}_\varepsilon(t)\|^2_{H^s} = -(A^s P J_\varepsilon \nabla_{u_\varepsilon} J \tilde{u}_\varepsilon, A^s \tilde{u}_\varepsilon) + \Omega(A^s J_\varepsilon B J_\varepsilon \tilde{u}_\varepsilon, A^s \tilde{u}_\varepsilon).$$

Need commutator estimates:

$$\|[A^s, \nabla_{u_\varepsilon}] J_\varepsilon \tilde{u}_\varepsilon\|_{L^2} \le C\|\tilde{u}_\varepsilon(t)\|_{C^1}\|\tilde{u}_\varepsilon(t)\|_{H^s}.$$

Moser estimates, for $s = 2k \ge 4$. Kato-Ponce estimates, for real $s > 2$.

Proposition. Given $s > 2$, $\tilde{u}_0 \in H^s(M)$, $\delta \tilde{u}_0 = 0$, there is a unique solution to the Euler equation on an interval I about 0, satisfying

$$\tilde{u} \in C(I, H^s(M)) \cap C^1(I, H^{s-1}(M)), \quad \tilde{u}(0) = \tilde{u}_0.$$

The solution depends continuously on the initial data $\tilde{u}_0$. Furthermore, if $\tilde{u}$ is such a solution on $I = (-a, b)$, then $\tilde{u}$ continues beyond the endpoints unless $\|\tilde{u}(t)\|_{C^1(M)}$ blows up at an endpoint.

Global existence.

Theorem. Given $s > 2$, $\tilde{u}_0 \in H^s(M)$, $\delta \tilde{u}_0 = 0$, the interval for existence for the Euler equation is $I = \mathbb{R}$. Furthermore, one has the global estimate

$$\|\tilde{u}(t)\|_{H^s}^2 \le C \|\tilde{u}_0\|_{H^s}^2 \; exp\, exp\big(C_s A|t|\big),$$

with

$$A = \|\operatorname{rot} u_0\|_{L^\infty} + C|\Omega| + 1.$$

Proof makes use of the estimate

$$\frac{d}{dt}\|\tilde{u}(t)\|_{H^s}^2 \le C\|\tilde{u}(t)\|_{C^1}\|\tilde{u}(t)\|_{H^s}^2 + C|\Omega| \cdot \|\tilde{u}(t)\|_{H^{s-1}}^2,$$

hence involves bounding $\|u(t)\|_{C^1}$.

To start, vorticity equation yields

$$\|w(t)\|_{L^\infty} \le \|w(0)\|_{L^\infty} + 2|\Omega|, \quad w(t) = \operatorname{rot} u(t).$$

Now elliptic regularity gives

$$\|u(t)\|_{bmo^1} \le C\|\operatorname{rot} u(t)\|_{L^\infty},$$

given $\operatorname{div} u(t) = 0$, but it fails to produce such a bound for $\|u(t)\|_{C^1}$.

Beale-Kato-Majda approach, [BKM]. Bring in

$$\|u\|_{C^1} \le C\Bigl(\log \frac{A\|u\|_{H^s}}{\|\operatorname{rot} u\|_{L^\infty}}\Bigr)\|\operatorname{rot} u\|_{L^\infty},$$

with $s > 2$ ($\dim\ M = 2$).

Then for $y(t) = \|\tilde{u}(t)\|^2_{H^s}$, we have

$$\frac{dy}{dt} \le C_s A(1 + \log^+ y(t))y(t),$$

and via Gronwall's inequality we obtain the conclusion of the theorem.

Stationary solutions

$$\nabla_u u = \Omega\chi J u - \nabla p, \quad \operatorname{div} u = 0.$$

Vorticity equation for $w = \operatorname{rot} u$,

$$\nabla_u (w - \Omega\chi) = 0,$$

or

$$\langle J\nabla f, \nabla(\omega - \Omega\chi)\rangle = 0,$$

$f =$ stream function,

$$w = \Delta f, \quad u = J\nabla f \quad (\tilde{u} = *df).$$

Equivalently

$$\nabla(\Delta f - \Omega\chi) \parallel \nabla f.$$

Zonal solutions.

M rotationally symmetric, Killing field X_3, positive Gauss curvature.

Definition. If $X_3 f = 0$ (so $f = f(x_3)$), we say f is a zonal function. Then $u = J\nabla f$ is a zonal field.

Proposition. If f is a zonal function, then $u = J\nabla f$ is a stationary solution to the Euler equation.
Proof. We have

$$\chi = \chi(x_3), \ f = f(x_3), \ w = w(x_3),$$

hence ∇f, and $\nabla(\Delta f - \Omega\chi)$ are both parallel to the x_3-axis.

Non-zonal solutions. Rossby-Haurwitz waves.

Specialize to $M = S^2$. Look for solutions f to

$$\Delta f = \psi(f) + \Omega x_3.$$

Take $\psi(f) = -\lambda_k f,\ k \geq 2$, where

$$Spec(-\Delta) = \{\lambda_k = k^2 + k : k = 0, 1, 2, \dots\}.$$

Note that $\Delta x_3 = -\lambda_1 x_3$ on S^2. Solve

$$(\Delta + \lambda_k) f = \Omega x_3,$$

Solution

$$f = \frac{\Omega}{\lambda_k - 2} x_3 + g_k, \quad g_k \in Ker(\Delta + \lambda_k).$$

$g_k =$ restriction to S^2 of harmonic polynomial, homogeneous of degree k.

Examples:

$$k = 2, \quad \lambda_k = 6, \quad g_k(x) = x_1^2 - x_2^2,$$
$$k = 3, \quad \lambda_k = 12, \quad g_k(x) = Re(x_1 + ix_2)^3,$$

and so on.
Given such a stream function f, we have stationary Euler flow

$$u = -\frac{\Omega}{\lambda_k - 2} X_3 + J\nabla g_k.$$

Known as Rossby-Hauritz waves of degree k. (Particularly $k = 2$.) Instability of such solutions impacts the difficulty of long-term weather forecasting.

Stability criterion for zonal flows.

Variant of Arn'old stability method. Seek stable critical points of

$$\mathcal{H}(u) = \int\limits_M \Big\{ \frac{1}{2}|u|^2 + \varphi(w - \Omega\chi) + \gamma\xi w \Big\}\, dS,$$

with $w = \operatorname{rot} u$, φ and γ tuned to the stationary solution u.

Note $\mathcal{H}(u(t))$ independent of t when $u(t)$ solves Euler equation.

Take $u = J\nabla f, w = \Delta f$, and rewrite functional as

$$H(f) = \int\limits_M \Big\{ \frac{1}{2}|\nabla f|^2 + \varphi(\Delta f - \Omega\chi) + \gamma\xi\Delta f \Big\}\, dS.$$

Get

$$\partial_s H(f + sg)\big|_{s=0} = \int\limits_M \Big\{ -f + \varphi'(\Delta f - \Omega\chi) + \gamma\xi \Big\} \Delta g \, dS.$$

Vanishes for all g if and only if

$$f = \varphi(\Delta f - \Omega\chi) + \gamma\xi,$$

up to an additive constant, which we can take to be 0.

Critical point condition

$$f = \varphi(\Delta f - \Omega\chi) + \gamma\xi.$$

If $\nabla f \parallel \nabla\xi$, the critical point condition implies

$$\nabla(\Delta f - \Omega\chi) \parallel \nabla f,$$

hence

$$\langle J\nabla f, \nabla(w - \Omega\chi)\rangle = 0,$$

which is the stationary vorticity equation, implying that $u = J\nabla f$ is a stationary Euler flow. Works when f is zonal.

For f not zonal, this method works only if $\gamma = 0$, and yields weaker results.

For zonal f, write $f = f(\xi)$, $w = \Delta f = w(\xi)$, and rewrite critical point equation as

$$\varphi'(w(\xi) - \Omega\chi(\xi)) = f(\xi) - \gamma\xi.$$

Given $\gamma \in \mathbb{R}$, this uniquely specifies φ' provided

$$w'(\xi) - \Omega\chi'(\xi) \text{ is bounded away from 0,}$$

so

$$w(\xi) - \Omega\chi(\xi) \text{ is strictly monotone in } \xi.$$

Call this Condition A. If it holds, φ is determined, up to an additive constant, and

$$\varphi''(w - \Omega\chi) = \frac{\gamma - f'(\xi)}{\Omega\chi'(\xi) - w'(\xi)}.$$

Stability condition.

Assume f is zonal, and assume Condition A holds. Choose φ so that critical point equation holds. Then

$$\partial_s^2 H(f+sg)\big|_{s=0} = \int_M \Big\{ |\nabla g|^2 + \varphi''(\Delta f - \Omega\chi)(\Delta g)^2 \Big\}\, dS$$

$$= \int_M \Big\{ |\nabla g|^2 + \frac{\gamma - f'(\xi)}{\Omega\chi'(\xi) - w'(\xi)}(\Delta g)^2 \Big\}\, dS.$$

Give Condition A, either

$$\Omega\chi'(\xi) - w'(\xi) \ \text{ is} \geq b > 0 \text{ or } \leq -b < 0$$

on M. In either case, can make

$$K(\xi) = \frac{\gamma - f'(\xi)}{\Omega\chi'(\xi) - w'(\xi)} \geq c > 0$$

on M, by taking γ sufficiently positive in the first case and sufficiently negative in the second case.

Conclusion. Let f be zonal and satisfy Condition A. Then one can choose γ (and then φ) so that f is a critical point of H, and

$$\partial_s^2 H(f+sg)\big|_{s=0} \geq \|\nabla g\|_{L^2}^2 + C\|\Delta g\|_{L^2}^2,$$

with $C > 0$, for all $g \in H^2(M)$.

This implies stability of f (defined only up to an additive constant) in $H^2(M)$.

Theorem. Given a smooth $f(\xi)$, $u = J\nabla f$ is a stable stationary solution to the Euler equation in $H^1(M)$, as long as Ω is such that Condition A holds, with $w = \Delta f$.

Note. Condition A always holds for $|\Omega|$ sufficiently large.

Perturbations of stable zonal flows - other estimates.

If u_s is a zonal flow, $u^\varepsilon(t)$ an Euler flow with $u^\varepsilon(0)$ close to u_s, since $\mathcal{H}(u^\varepsilon(t))$ is independent of t, the estimate

$$|\mathcal{H}(u^\varepsilon(t)) - \mathcal{H}(u_s)| \approx \|u^\varepsilon(t) - u_s\|_{H^1}^2,$$

which holds if the stability criterion holds, implies

$$\|u^\varepsilon(t) - u_s\|_{H^1} \le C\|u^\varepsilon(0) - u_s\|_{H^1},$$

for all t.

Next goal: look for bounds on

$$w^\varepsilon(t) = u^\varepsilon(t) - u_s$$

in other norms.

Tools for the analysis. [BKM] estimate

$$\|w\|_{C^1} \le C\Bigl(\log \frac{A\|w\|_{H^s}}{\|\operatorname{rot} w\|_{L^\infty}}\Bigr)\|\operatorname{rot} w\|_{L^\infty},$$

and [BW] type estimate

$$\|w\|_{L^\infty} \le C\Bigl(\log \frac{A\|\operatorname{rot} w\|_{L^\infty}}{\|\operatorname{rot} w\|_{L^2}}\Bigr)^{1/2}\|\operatorname{rot} w\|_{L^2},$$

with $s > 2$, in both cases assuming $\operatorname{div} w = 0$. Then

$$\|\operatorname{rot} w\|_{L^2} \approx \|w\|_{H^1}, \quad \|\operatorname{rot} w\|_{L^\infty} \ge \|\operatorname{rot} w\|_{bmo} \approx \|u\|_{bmo^1}.$$

Another tool, vorticity equation (conservation law):

$$\|\operatorname{rot} u^\varepsilon(t) - \Omega\chi\|_{L^\infty} \equiv \|\operatorname{rot} u^\varepsilon(0) - \Omega\chi\|_{L^\infty}.$$

Follow strategy of [T2].

First estimate on $w^\varepsilon(t) = u^\varepsilon(t) - u_s$.

$$\|w^\varepsilon(t)\|_{L^\infty} \le C\Bigl(\log \frac{AK(u^\varepsilon(0), u_s)}{\|w^\varepsilon(0)\|_{H^1}}\Bigr)^{1/2} \|w^\varepsilon(0)\|_{H^1},$$

with

$$K(u^\varepsilon(0), u_s) = \|\operatorname{rot} u^\varepsilon(0)\|_{L^\infty} + \|\operatorname{rot} u_s\|_{L^\infty} + 2|\Omega|.$$

Second estimate.

$$\|\operatorname{rot} w^\varepsilon(t)\|_{L^\infty} \le \|\operatorname{rot} w^\varepsilon(0)\|_{L^\infty} + K_2(u^\varepsilon(0), u_s)\cdot|t|,$$

with

$$K_2(u^\varepsilon(0), u_s) = \|\operatorname{rot} u_s - \Omega\chi\|_{C^1}\cdot \sup_s \|w^\varepsilon(s)\|_{L^\infty},$$

last factor amenable to first estimate. Effective for

$$|t| \lesssim K_2(u^\varepsilon(0), u_s)^{-1}.$$

Otherwise, just use vorticity conservation.

Third estimate.

$$\|w^\varepsilon(t)\|_{C^1} \le C\big(\log^+ A\|w^\varepsilon(t)\|_{H^s}\big)\|\operatorname{rot} w^\varepsilon(t)\|_{L^\infty} + C\Big(\log^+ \frac{1}{\|\operatorname{rot} w^\varepsilon(t)\|_{L^\infty}}\Big)\|\operatorname{rot} w^\varepsilon(t)\|_{L^\infty}.$$

Fix $s > 2$.

Double exponential estimate on $\|u^\varepsilon(t)\|_{H^s}$ yields exponential bound on $\|w^\varepsilon(t)\|_{C^1}$.

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Appendix

To compute ∇_u^*, write

$$\int_M \langle \nabla_u v, w \rangle \, dV = \int_M \Big[\nabla_u \langle v, w \rangle - \langle v, \nabla_u w \rangle \Big] \, dv.$$

To finish, use

$$\int_M (\nabla_u f) \, dV = \int_M \langle u, \nabla f \rangle \, dV = -\int_M (\operatorname{div} u) f \, dV.$$

Hence

$$\nabla_u^* v = -\nabla_u v - (\operatorname{div} u) v.$$

To relate $\mathcal{L}_u \tilde{u}$ to $\nabla_u \tilde{u}$, write

$$\begin{aligned}
\langle \mathcal{L}_u \tilde{u}, v \rangle &= \mathcal{L}_u \langle \tilde{u}, v \rangle - \langle \tilde{u}, [u, v] \rangle \\
&= \nabla_u \langle \tilde{u}, v \rangle - \langle \tilde{u}, \nabla_u v - \nabla_v u \rangle \\
&= \langle \nabla_u \tilde{u}, v \rangle + \langle \tilde{u}, \nabla_v u \rangle \\
&= \langle \nabla_u \tilde{u}, v \rangle + \frac{1}{2} \nabla_v \langle u, u \rangle \\
&= \langle \nabla_u \tilde{u}, v \rangle + \frac{1}{2} \langle d|u|^2, v \rangle,
\end{aligned}$$

obtaining

$$\mathcal{L}_u \tilde{u} = \nabla_u \tilde{u} + \frac{1}{2} d|u|^2.$$

Variant of [BKM] estimate.

$$\|u\|_{C^k} \leq C\left(\log \frac{A\|u\|_{H^s}}{\|u\|_{C^k_*}}\right)\|u\|_{C^k_*},$$

for $s > k + n/2$, on n-dimensional M.

Take $\Psi \in C_0^\infty(\mathbb{R}),\ \Psi(\lambda) = 1$ for $|\lambda| < 1$, 0 for $|\lambda| > 2$.

$$u = \Psi(\varepsilon D)u + (I - \Psi(\varepsilon D))u.$$

Establish 2 estimates:

$$\begin{aligned}\|(I - \Psi(\varepsilon D))u\|_{C^k} &\leq C\|(I - \Psi(\varepsilon D))u\|_{H^{s-\delta}} \\ &\leq C\varepsilon^\delta\|(I - \Psi(\varepsilon D))u\|_{H^s},\end{aligned}$$

$$\Psi(\varepsilon D)u\|_{C^k} \leq C\left(\log \frac{1}{\varepsilon}\right)\|u\|_{C^k_*}.$$

Pick

$$\varepsilon^\delta = \frac{\|u\|_{C^k_*}}{\|u\|_{H^s}}.$$

First estimate is elementary, second follows from Littlewood-Paley characterization of C^k_*.

Variant of [BW] estimate.

$$\|u\|_{L^\infty} \le C\Big(\log \frac{A\|u\|_{H^{n/p,q}}}{\|u\|_{H^{n/p,p}}}\Big)^{1-1/p} \|u\|_{H^{n/p,p}}$$

on n-dimensional M, given $1 < p < \infty,\ p < q < \infty$. Present case: $n = p = 2$.

Take $\Psi \in C_0^\infty(\mathbb{R}),\ \Psi(\lambda) = 1$ for $|\lambda| \le 1$, 0 for $|\lambda| > 2$.

$$u = \Psi(\varepsilon D)u + (I - \Psi(\varepsilon D))u.$$

Establish 2 estimates:

$$\|(I - \Psi(\varepsilon D))u\|_{L^\infty} \le C\varepsilon^r \|u\|_{C^r} \le C\varepsilon^r \|u\|_{H^{n/p,q}},$$

with $r = n/p - n/q > 0$, and

$$\|\Psi(\varepsilon D)u\|_{L^\infty} \le C\Big(\log \frac{1}{\varepsilon}\Big)^{1-1/p} \|u\|_{H^{n/p,p}}.$$

Pick

$$\varepsilon^r = \frac{\|u\|_{H^{n/p,p}}}{\|u\|_{H^{n/p,q}}}.$$

For first estimate, use Littlewood-Paley characterization of C^r.

For second, use

$$\Lambda^{-s} v = J_s * v, \quad J_s(x) \sim c|x|^{s-n} \text{ for } |x| \le 1,$$

for $0 < s < n$, especially $s = n/p$. This yields

$$\Psi(\varepsilon D)\Lambda^{-s} v = K_{s,\varepsilon} * v,$$

with

$$\begin{aligned} |K_{s,\varepsilon}(x)| \le C\,\varepsilon^{s-n}, &\quad |x| \le \varepsilon, \\ C|x|^{s-n}, &\quad |x| \in [\varepsilon, 1], \end{aligned}$$

in turn giving the second estimate.