Regularity of Solutions to Nonlinear Overdetermined Elliptic PDE

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Assume $r > 0, m \in \mathbb{N}$, and $u \in C^{m+r}(\Omega)$ solves the nonlinear system of PDE

$$P(x, D^m u) = g. \tag{1}$$

Also assume P is a smooth function of its arguments and

$$P(x, D^m u) \text{ is overdetermined elliptic at } u. \tag{2}$$

We aim to prove the following.

Proposition. *Under the hypotheses stated above,*

$$g \in C^\infty(\Omega) \Longrightarrow u \in C^\infty(\Omega). \tag{3}$$

Proof. Assume also $P(x, 0) = 0$ (otherwise, subtract this off). We apply the paralinearization method of J.-M. Bony (as tweaked by Y. Meyer) to write

$$F(x, D^m u) = M(u; x, D)u. \tag{4}$$

See Proposition 3.3.A of [T]. Furthermore, as in (3.3.9)–(3.3.10) of [T], pick $\delta \in (0, 1)$ and apply symbol smoothing to obtain

$$M(u; x, D) = M^\#(x, D) + M^b(x, D), \tag{5A}$$

where, if $u \in C^{m+r}$,

$$M^\#(x, \xi) \in S^m_{1,\delta}, \quad M^b(x, \xi) \in S^{m-r\delta}_{1,1}. \tag{5B}$$

See also Chapter II, §§4–5 of [T2]. The hypothesis (2) implies

$$M^\#(x, D) \text{ is overdetermined elliptic.} \tag{6}$$

Now (1) gives

$$M^\#(x, D)u = g - M^b(x, D)u = h, \tag{7}$$

and Theorem 2.1.A of [T] gives

$$u \in C^{m+r} \Rightarrow M^b(x, D)u \in C^{r+r\delta}_* \Rightarrow h \in C^{r+r\delta}_*. \tag{8}$$

Here C^s_* denotes a Zygmund space, equal to C^s if $s \in (0,\infty) \setminus \mathbb{N}$, and slightly different if $s \in \mathbb{N}$.

Now we exploit (6). We have

$$M^\#(x,D)^* M^\#(x,D) \in OPS^{2m}_{1,\delta}, \quad \text{determined elliptic,} \tag{9}$$

hence this operator has a parametrix $E \in OPS^{-2m}_{1,\delta}$. We set

$$F = EM^\#(x,D)^* \in OPS^{-m}_{1,\delta}, \tag{10}$$

giving

$$FM^\#(x,D) = I + R, \quad R \in OPS^{-\infty}. \tag{11}$$

Now, by Corollary 2.1.B of [T],

$$F : C^s_* \longrightarrow C^{s+m}_*, \quad \forall\, s \in \mathbb{R}, \tag{12}$$

so we apply F to (7), obtaining (mod C^∞)

$$u = Fh \in C^{m+r+r\delta}_*. \tag{13}$$

Having this improvement over the hypothesis $u \in C^{m+r}$, we iterate this argument, obtaining $u \in C^{m+s}_*$ for all $s < \infty$, hence (3).

References

[T] M. Taylor, Pseudodifferential Operators and Nonlinear PDE, Birkhauser, Boston, 1991.

[T2] M. Taylor, Short Course on Pseudodifferential Operators, Notes, available at https://mtaylor.web.unc.edu/notes, item #1.