Surface Quasigeostrophic Equation on a Rotating Body

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1. Introduction

The surface quasigeostrophic (SQG) equation, for an active scalar $\theta(t,x)$ and velocity $u(t,x)$, $x \in \mathbb{R}^2$, takes the following form:

$$
\begin{aligned}
&\partial_t \theta + \nabla_u \theta = 0, \\
&u = -J\nabla(\Delta)^{-1/2}\theta.
\end{aligned}
\tag{1.1}
$$

Here, J acts on vectors by counterclockwise rotation by 90°. The vector field $J\nabla\theta$ behaves somewhat like vorticity in 3D:

$$
(\partial_t + \nabla_u)(J\nabla\theta) = (\nabla u)(J\nabla\theta), \tag{1.2}
$$

and features of the solution to (1.1) are seen to describe behavior more closely analogous to 3D fluid motion than the 2D Euler equations, while remaining in the setting of 2D. That said, it is also folklore that (1.1) produces a better reflection of fluid behavior in the tropics than at other latitudes. This suggests several extensions of (1.1).

First, (1.1) as stated works fine when the Euclidean plane $\mathbb{R}^2$ is replaced by a compact, oriented, 2D Riemannian manifold, such as the sphere S^2. Going further, it is natural to set this surface spinning and modify (1.1) to take account of the Coriolis force. Here we investigate such a modification of SQG.

To set this up, we recall that SQG can be placed within a continuum of evolution equations:

$$
\partial_t \theta + \nabla_u \theta = 0, \quad u = -J\nabla(-\Delta)^{-\beta}\theta, \tag{1.3}
$$

with $\beta \in [1/2, 1]$. The case $\beta = 1/2$ is SQG. The case $\beta = 1$ yields the vorticity equation for 2D Euler, with $\theta = w = \operatorname{rot} u$. Now, 2D Euler with the Coriolis force on an axially symmetric surface $M \subset \mathbb{R}^3$, rotating about the x_3-axis, has the form

$$
\partial_t u + \nabla_u u = \Omega\chi(x)Ju - \nabla p, \quad \operatorname{div} u = 0, \tag{1.4}
$$

with $\Omega \in \mathbb{R}$ proportional to the angular velocity of M and

$$
\chi(x) = e_3 \cdot \nu(x), \tag{1.5}
$$

$\nu(x)$ denoting the outward unit normal to M at x, and e_3 the unit vector along the positive x_3-axis. This equation yields the following vorticity equation for $w = \operatorname{rot} u$ (cf. [T]):

$$
(\partial_t + \nabla_u)(w - \Omega\chi) = 0. \tag{1.6}
$$

In light of this, we propose the following surface quasigeostrophic equation with Coriolis force (SQGC), on the rotating, axially symmetric surface M:

$$\begin{aligned}(\partial_t + \nabla_u)(\theta - \Omega\chi) &= 0,\\ u &= -J\nabla(-\Delta)^{-1/2}\theta,\end{aligned} \tag{1.7}$$

with $\Omega \in \mathbb{R}$ and $\chi \in C^\infty(M)$ given by (1.5).

It is of interest to reformulate the evolution equation as follows. Note that the formula for u yields

$$\operatorname{div} u = 0. \tag{1.8}$$

Hence

$$\nabla_u \varphi = \operatorname{div}(\varphi u), \tag{1.9}$$

for real-valued functions φ. Thus the evolution equation takes the form

$$\partial_t \varphi + \operatorname{div}(\varphi u) = 0, \quad \varphi = \theta - \Omega\chi. \tag{1.10}$$

We can rewrite the latter half of (1.7) as

$$\begin{aligned}u &= -J\nabla(-\Delta)^{-1/2}(\varphi + \Omega\chi)\\ &= R^\perp(\varphi + \Omega\chi).\end{aligned} \tag{1.11}$$

Then another formulation of (1.10) is

$$\partial_t \varphi + \operatorname{div}(\varphi R^\perp \varphi) = -\Omega \operatorname{div}(\varphi R^\perp \chi), \tag{1.12}$$

with

$$R^\perp = -J\nabla\Lambda^{-1}, \quad \Lambda = (-\Delta)^{1/2}. \tag{1.13}$$

Note that

$$R^\perp \in OPS^0(M), \tag{1.14}$$

and

$$(R^\perp)^* R^\perp = I - P_0, \tag{1.15}$$

where $P_0 : L^2(M) \to L^2(M)$ is the orthogonal projection onto the space of constant functions on M.

A. Some conservation laws

The equation (1.7), with smooth initial data, has a short time smooth solution. This will be established in §B. Here we take note of some conserved quantities that such a solution possesses. First, directly from (1.7), we have

$$\|\theta(t,\cdot)-\Omega\chi\|_{L^\infty(M)} \ \text{ is independent of } t. \tag{A.1}$$

Furthermore, thanks to (1.8),

$$\int_M G(\theta(t,x)-\Omega\chi(x))\, dS(x) \ \text{ is independent of } t, \tag{A.2}$$

for each continuous function $G:\mathbb{R}\to\mathbb{R}$. Taking $G(s)=s$ yields

$$\int_M \theta(t,x)\, dS(x) \ \text{ independent of } t. \tag{A.3}$$

Note that if θ satisfies (1.7) and $\tilde\theta=\theta+a$, $a\in\mathbb{R}$, then $\tilde\theta$ also satisfies (1.7), with u unchanged. We will normalize so that

$$\int_M \theta(t,x)\, dS(x)\equiv 0. \tag{A.4}$$

Note incidentally that, by the divergence theorem, the formula (1.5) for χ yields

$$\int_M \chi(x)\, dS(x)=0. \tag{A.5}$$

Let us examine the not-quite-conservation law we get for $\|u(t)\|_{L^2}$. Since $u=R^\perp\theta$,

$$\|u(t)\|^2_{L^2}=(R^\perp\theta(t),R^\perp\theta(t))=\|\theta(t)\|^2_{L^2}, \tag{A.6}$$

thanks to (1.15) and (A.4). Meanwhile, (A.2) implies the following is independent of t:

$$\|\theta(t)-\Omega\chi\|^2_{L^2}=\|\theta(t)\|^2-2\Omega(\theta(t),\chi)+\|\Omega\chi\|^2_{L^2}, \tag{A.7}$$

so

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2_{L^2}&=\Omega\,\frac{d}{dt}(\theta(t),\chi)\\&=\Omega(\partial_t\theta(t),\chi)\\&=-\Omega(\nabla_u\theta(t),\chi)\\&=\Omega(\theta(t),\nabla_u\chi).\end{aligned} \tag{A.8}$$

B. Short time existence

Here we establish existence of solutions to the equation

$$\partial_t \varphi + \nabla_u \varphi = 0, \quad \varphi(0) = \varphi_0, \tag{B.1}$$

with

$$\varphi = \theta - \Omega\chi, \quad u = R^\perp \theta \tag{B.2}$$

(as in (1.7)-(1.13)), on some interval $|t| < A$, given $\varphi_0 \in H^s(M)$, with conditions on s described below. Such a result works basically like the case $\Omega = 0$, which is well known, but we include a treatment here for the sake of completeness. The method we use is close to the treatment short-time existence for quasilinear symmetric hyperbolic equations given in Chapter 16 of [T2] (and in Chapter 5 of [T3]).

Our approach is to take a mollifier $J_\varepsilon = \psi(\varepsilon\Delta)$, ψ real valued and in $C_0^\infty(\mathbb{R})$, with $\psi(0) = 1$, and consider

$$\frac{\partial \varphi_\varepsilon}{\partial t} + J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon \varphi_\varepsilon = 0, \quad \varphi_\varepsilon(0) = J_\varepsilon \varphi_0, \tag{B.3}$$

where

$$u_\varepsilon = R^\perp (J_\varepsilon \varphi_\varepsilon + \Omega\chi), \quad \text{so} \quad \operatorname{div} u_\varepsilon = 0. \tag{B.4}$$

Given $\varepsilon > 0$, the short-time solvability of (B.3) is elementary, since this is essentially a finite system of ODEs. We aim to obtain estimates for $\varphi_\varepsilon(t)$ in $H^s(M)$ for t in some interval that is independent of ε, and pass to the limit, assuming $\varphi_0 \in H^s(M)$ and s is large enough.

To start, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\varphi_\varepsilon(t)\|_{L^2}^2 &= (\partial_t \varphi_\varepsilon, \varphi_\varepsilon) \\ &= -(J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon \varphi_\varepsilon, \varphi_\varepsilon) \\ &= -(\nabla_{u_\varepsilon} J_\varepsilon \varphi_\varepsilon, J_\varepsilon \varphi_\varepsilon) \\ &= 0, \end{aligned} \tag{B.5}$$

the last identity holding because $\operatorname{div} u_\varepsilon = 0$. This guarantees global existence of solutions to (B.3), for each $\varepsilon > 0$.

To estimate higher-order Sobolev norms, we bring in

$$A^s = (1 - \Delta)^{s/2} \in OPS^s(M), \quad \|\varphi\|_{H^s} = \|A^s \varphi\|_{L^2}. \tag{B.6}$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_\varepsilon(t)\|_{H^s}^2 &= (A^s\partial_t\varphi_\varepsilon, A^s\varphi_\varepsilon)\\ &= -(A^sJ_\varepsilon\nabla_{u_\varepsilon}J_\varepsilon\varphi_\varepsilon, A^s\varphi_\varepsilon)\\ &= -(A^s\nabla_{u_\varepsilon}J_\varepsilon\varphi_\varepsilon, A^sJ_\varepsilon\varphi_\varepsilon),\end{aligned} \tag{B.7}$$

the latter identity holding since A^s and J_ε commute. We can write the last line of (B.7) as

$$-(\nabla_{u_\varepsilon}A^sJ_\varepsilon\varphi_\varepsilon, A^sJ_\varepsilon\varphi_\varepsilon) - ([A^s,\nabla_{u_\varepsilon}]J_\varepsilon\varphi_\varepsilon, A^sJ_\varepsilon\varphi_\varepsilon). \tag{B.8}$$

The first term in (B.8) vanishes since $\operatorname{div} u_\varepsilon = 0$. Our next task is to estimate

$$\|[A^s,\nabla_{u_\varepsilon}]J_\varepsilon\varphi_\varepsilon\|_{L^2}. \tag{B.9}$$

For this, we use the Kato-Ponce estimate, established in [KP] in the Euclidean space setting, and in the manifold setting in §3.6 of [T3].

In more detail, the KP-estimate gives, for $s > 0$,

$$\|A^s(fv) - fA^sv\|_{L^2} \le C\|f\|_{C^1}\|v\|_{H^{s-1}} + C\|f\|_{H^s}\|v\|_{L^\infty}. \tag{B.10}$$

We take $v = X\varphi$, where X is a first-order differential operator, and write

$$A^s(fX\varphi) - fX(A^s\varphi) = A^s(fX\varphi) - fA^s(X\varphi) + f[A^s,X]\varphi. \tag{B.11}$$

Then (B.10) applies to estimate the first two terms on the right side of (B.11), and the L^2-norm of the last term is bounded by $C\|f\|_{L^\infty}\|\varphi\|_{H^s}$, since

$$[A^s,X] \in OPS^s(M). \tag{B.12}$$

Thus

$$\|[A^s,fX]\varphi\|_{L^2} \le C\|f\|_{C^1}\|\varphi\|_{H^s} + C\|f\|_{H^s}\|\varphi\|_{C^1}. \tag{B.13}$$

Hence our estimate on (B.9) is

$$\|[A^s,\nabla_{u_\varepsilon}]J_\varepsilon\varphi_\varepsilon\|_{L^2} \le C\|u_\varepsilon\|_{C^1}\|\varphi_\varepsilon\|_{H^s} + C\|u_\varepsilon\|_{H^s}\|\varphi_\varepsilon\|_{C^1}. \tag{B.14}$$

Recalling that u_ε is related to φ_ε by (B.4), and $R^\perp \in OPS^0(M)$, we have

$$\begin{aligned}\|u_\varepsilon\|_{H^s} &\le C\|\varphi_\varepsilon\|_{H^s} + C|\Omega|,\\ \|u_\varepsilon\|_{C^1} &\le C\|\varphi_\varepsilon\|_{C^1} + C\|R^\perp\varphi_\varepsilon\|_{C^1} + C|\Omega|\\ &=: C\|\varphi_\varepsilon\|_{C^1_\#} + C|\Omega|.\end{aligned} \tag{B.15}$$

Returning to (B.7)–(B.9), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_\varepsilon(t)\|_{H^s}^2 \le C\Bigl(\|\varphi_\varepsilon(t)\|_{C^1_\#}+|\Omega|\Bigr)\|\varphi_\varepsilon(t)\|_{H^s}^2 \\ + C\|\varphi_\varepsilon(t)\|_{C^1}\Bigl(\|\varphi_\varepsilon(t)\|_{H^s}+|\Omega|\Bigr)\|\varphi_\varepsilon(t)\|_{H^s}.\end{aligned} \tag{B.16}$$

To proceed, fix a number

$$\sigma > 2, \quad \text{so} \quad \|\varphi\|_{C^1_\#} \le C\|\varphi\|_{H^\sigma}, \tag{B.17}$$

and then (B.16) yields, for the solution φ_ε to (B.3), the estimate

$$\frac{d}{dt}\|\varphi_\varepsilon(t)\|_{H^\sigma}^2 \le C_\sigma\Bigl(\|\varphi_\varepsilon(t)\|_{H^\sigma}^3 + |\Omega|\|\varphi_\varepsilon(t)\|_{H^\sigma}^2\Bigr), \tag{B.18}$$

with C_σ independent of $\varepsilon \in (0,1]$. By Gronwall's inequality, we have, for $t \ge 0$,

$$\|\varphi_\varepsilon(t)\|^2 \le y(t), \tag{B.19}$$

where $y(t)$ solves

$$\frac{dy}{dt} = C_\sigma(y^{3/2} + |\Omega|y), \quad y(0) = \|\varphi_0\|_{H^\sigma}^2. \tag{B.20}$$

In particular,

$$\{\varphi_\varepsilon(t) : 0 \le t \le T\} \ \text{ is uniformly bounded in } \ H^\sigma(M), \tag{B.21}$$

independent of $\varepsilon \in (0,1]$, for

$$T < T(y(0)) = \frac{1}{C_\sigma}\int_{y(0)}^\infty \frac{dy}{y^{3/2}+|\Omega|y}. \tag{B.22}$$

For t in this interval, we have

$$\int_{y(0)}^{y(t)} \frac{dy}{y^{3/2}+|\Omega|y} = C_\sigma t. \tag{B.23}$$

In connection with this, note that

$$\begin{aligned}\Xi(y) = \int_y^\infty \frac{d\eta}{\eta^{3/2}+|\Omega|\eta} &= 2\int_{y^{1/2}}^\infty \frac{dt}{(t+|\Omega|)t} \\ &= \frac{2}{|\Omega|}\int_{y^{1/2}}^\infty \Bigl(\frac{1}{t} - \frac{1}{t+|\Omega|}\Bigr)\,dt \\ &= \frac{2}{|\Omega|}\log(1+|\Omega|y^{-1/2}),\end{aligned} \tag{B.24}$$

so

$$T(y(0)) = \frac{2}{C_\sigma |\Omega|} \log\bigl(1 + |\Omega| y(0)^{-1/2}\bigr). \tag{B.25}$$

Taking $|\Omega| \to 0$ yields

$$T(y(0)) = \frac{2}{C_\sigma} y(0)^{-1/2}, \tag{B.26}$$

in this case, which of course also follows directly from (B.22).

Similar estimates of $\varphi_\varepsilon(-t)$ allow us to elevate (B.21) to the result that

$$\{\varphi_\varepsilon(t) : |t| \le T\} \ \text{ is uniformly bounded in } \ H^\sigma(M), \tag{B.27}$$

independently of $\varepsilon \in (0,1]$, provided (B.22) holds. In such a case, u_ε enjoys the same sort of bound, and we have from (B.3) that

$$\{\partial_t \varphi_\varepsilon(t) : |t| \le T\} \ \text{ is uniformly bounded in } \ H^{\sigma-1}(M). \tag{B.28}$$

Note that

$$\nabla_{u_\varepsilon} J_\varepsilon \varphi_\varepsilon = \operatorname{div}(u_\varepsilon J_\varepsilon \varphi_\varepsilon), \tag{B.29}$$

in such a case. Also, for $\sigma > 2$, $H^{\sigma-1}(M)$ is an algebra under pointwise multiplication.

Having these estimates, one can apply standard techniques, discussed in Chapter 16 of [T2], and Chapter 5 of [T3], to obtain a solution φ to (B.1)–(B.2) in $C([-T,T], H^s(M))$, given initial data in $H^s(M)$, as long as $s \ge \sigma$ and T satisfies (B.22). We do not need to shrink the interval further for $s > \sigma$. Also estimates parallel to those produced above establish uniqueness of solutions $\varphi(t)$ and continuous dependence on the initial data φ_0.

2. Mock vorticity equation

Let us write the evolution equation (1.7) as

$$(\partial_t + \mathcal{L}_u)\varphi = 0, \quad \varphi = \theta - \Omega\chi. \tag{2.1}$$

We apply the exterior derivative d and use the fact that $d\mathcal{L}_u = \mathcal{L}_u d$ to obtain

$$(\partial_t + \mathcal{L}_u)(d\varphi) = 0. \tag{2.2}$$

Now, if X is a vector field on M,

$$\begin{aligned} \langle \mathcal{L}_u(d\varphi), X\rangle &= \mathcal{L}_u\langle d\varphi, X\rangle - \langle d\varphi, \mathcal{L}_u X\rangle \\ &= \mathcal{L}_u\langle \nabla\varphi, X\rangle - \langle \nabla\varphi, [u, X]\rangle \\ &= \langle \nabla_u(\nabla\varphi), X\rangle + \langle \nabla\varphi, \nabla_u X\rangle - \langle \nabla\varphi, [u, X]\rangle. \end{aligned} \tag{2.3}$$

In view of the identity $\nabla_u X - \nabla_X u = [u, X]$, we get

$$\langle \mathcal{L}_u(d\varphi), X\rangle = \langle \nabla_u(\nabla\varphi), X\rangle + \langle \nabla\varphi, \nabla_X u\rangle. \tag{2.4}$$

Thus (2.2) yields, for all vector fields X,

$$\begin{aligned} 0 &= \langle (\partial_t + \mathcal{L}_u)(d\varphi), X\rangle \\ &= \langle \partial_t(\nabla\varphi), X\rangle + \langle \nabla_u(\nabla\varphi), X\rangle + \langle \nabla\varphi, (\nabla u)X\rangle, \end{aligned} \tag{2.5}$$

hence

$$(\partial_t + \nabla_u)(\nabla\varphi) = -(\nabla u)^t\nabla\varphi. \tag{2.6}$$

Note that the 2D surface M has the Kahler property,

$$J\nabla_u Y = \nabla_u(JY), \tag{2.7}$$

for each smooth vector fields u and Y. Hence (2.6) gives

$$(\partial_t + \nabla_u)(J\nabla\varphi) = -J(\nabla u)^t J^{-1}(J\nabla\varphi). \tag{2.8}$$

This is the mock vorticity equation.

3. Stationary solutions to SQGC

As before, $M \subset \mathbb{R}^3$ is a compact, smooth surface, axially symmetric about the x_3-axis, so the group of rotations about this axis acts on M. Then the function $\chi \in C^\infty(M)$ defined in (1.5) is a zonal function, i.e., invariant under rotations about the x_3-axis. Hence $\nabla\chi$ is orthogonal to the orbits of this rotation action (curves of latitude), hence $J\nabla\chi$ is parallel to such orbits. Also, for $\theta \in C^\infty(M)$,

$$\begin{aligned}\theta \text{ zonal} &\Longrightarrow u = -J\nabla(-\Delta)^{-1/2}\theta \text{ parallel to curves of latitude} \\ &\Longrightarrow \nabla_u\psi = 0, \quad \forall \text{ zonal } \psi \\ &\Longrightarrow \nabla_u(\theta - \Omega\chi) = 0 \\ &\Longrightarrow \theta \text{ is a stationary solution to SQGC.}\end{aligned} \tag{3.1}$$

We next seek examples of stationary solutions to SQGC that are not zonal functions. For this, let us write

$$\theta = \Lambda f = (-\Delta)^{1/2} f, \quad \int\limits_M f\, dS = 0, \tag{3.2}$$

and rewrite the equation $\nabla_u(\theta - \Omega\chi) = 0$ as

$$\langle J\nabla f, \nabla(\Lambda f - \Omega\chi\rangle = 0, \tag{3.3}$$

equivalently

$$\nabla f(x) \parallel \nabla(\Lambda f(x) - \Omega\chi(x)), \quad \forall\, x \in M, \tag{3.4}$$

a result that holds whenever

$$\Lambda f - \Omega\chi = \psi(f), \tag{3.5}$$

for some $\psi \in C^\infty(\mathbb{R})$. For an example, we take

$$\psi(f) = \lambda_k f, \quad \lambda_k \in \operatorname{Spec}\Lambda, \quad \lambda_k > 0, \tag{3.6}$$

and seek a solution f to

$$(\Lambda - \lambda_k) f = \Omega\chi. \tag{3.7}$$

If $\Omega \neq 0$, such a solution exists if and only if χ is orthogonal to $\operatorname{Ker}(\Lambda - \lambda_k)$.

To proceed, we specialize to $M = S^2$, so

$$\chi(x) = x_3, \tag{3.8}$$

which is an eigenfunction of Λ:

$$-\Delta\chi = 2\chi, \quad \text{so} \ \ \Lambda\chi = \lambda_1\chi, \quad \lambda_1 = \sqrt{2}. \tag{3.9}$$

As long as $k \geq 2$, (3.7) has solutions

$$f = \frac{\Omega}{\lambda_1 - \lambda_k}\chi + g_k, \quad g_k \in \operatorname{Ker}(\Lambda - \lambda_k). \tag{3.10}$$

We can pick g_k not to be zonal, and then $\theta = \Lambda f$ is a stationary solution to SQGC that is not zonal.

4. Another conservation law

As usual, $M \subset \mathbb{R}^3$ is a smooth compact surface that is invariant under the group of rotations about the x_3-axis, generated by X_3. We also assume M is diffeomorphic to S^2 and has positive Gauss curvature everywhere. As a consequence, χ is a smooth function of x_3 and

$$\frac{d\chi}{dx_3} \geq b > 0, \quad \text{for} \ \ x_3 \in [-a, a], \tag{4.1}$$

where we arrange

$$a = \max_M x_3, \quad -a = \min_M x_3.$$

Since X_3 generates a flow by isometries on M, we have $\operatorname{div} X_3 = 0$ on M, so there exists $\xi \in C^\infty(M)$ such that

$$J\nabla\xi = -X_3. \tag{4.2}$$

Clearly $X_3\xi = 0$. As a further consequence of our geometric hypotheses,

$$\begin{aligned} &\xi \text{ is a smooth function of } x_3, \text{ and} \\ &\frac{d\xi}{dx_3} \geq b > 0 \text{ for } \ x_3 \in [-a, a]. \end{aligned} \tag{4.3}$$

We note that

$$M = S^2 \Longrightarrow \chi = \xi = x_3. \tag{4.4}$$

We aim to establish the following.

Proposition 4.1. *Under the hypotheses on M made above, if θ, u satisfy (1.7), then*

$$\int\limits_M \xi(x)\theta(t,x)\,dS(x) \quad \textit{is independent of } t. \tag{4.5}$$

Proof. Applying ∂_t to (4.5) yields

$$\begin{aligned} \int\limits_M \xi\,\partial_t\theta\,dS &= \int\limits_M \xi\nabla_u(\Omega\chi - \theta)\,dS \\ &= -\int\limits_M \xi(\nabla_u\theta)\,dS + \Omega\int\limits_M \xi\nabla_u\chi\,dS. \end{aligned} \tag{4.6}$$

Note that (4.1)–(4.3) imply that ξ is a smooth function of χ, with $\xi = \xi(\chi)$. Thus $\xi \nabla_u \chi = \nabla_u G(\chi)$, where $G'(\chi) = \xi(\chi)$. Hence

$$\int\limits_M \xi \nabla_u \chi \, dS = \int\limits_M \nabla_u G(\chi) \, ds = 0, \tag{4.7}$$

sice $\operatorname{div} u = 0$ implies ∇_u is skew-adjoint and $\nabla_u 1 = 0$, Next,

$$\int\limits_M \xi(\nabla_u \theta) \, dS = -\int\limits_M (\nabla_u \xi)\theta \, dS, \tag{4.8}$$

which we can analyze using

$$\theta = \Lambda f, \quad u = -J\nabla f \tag{4.9}$$

(defining f as the "stream function"). We have (4.8) equal to

$$\begin{aligned} &\int\limits_M \langle J\nabla f, \nabla \xi \rangle (\Lambda f) \, dS \\ &= \int\limits_M (X_3 f)(\Lambda f) \, dS \\ &= (X_3 f, \Lambda f). \end{aligned} \tag{4.10}$$

Now, since X_3 commutes with all powers of Λ, and is skew-adjoint,

$$(X_3 f, \Lambda f) = (X_3 \Lambda^{1/2} f, \Lambda^{1/2} f) = 0. \tag{4.11}$$

It follows that

$$\frac{d}{dt} \int\limits_M \xi(x) \theta(t,x) \, dS(x) = 0, \tag{4.12}$$

proving Proposition 4.1.

As shown in §3.5 of [T], if M has the geometrical properties hypothesized above, then

$$\Psi \in C^\infty(M), \text{ zonal} \Longrightarrow \Psi(x) = \psi(x_3), \text{ with } \psi \in C^\infty([-a,a]). \tag{4.13}$$

5. Linearization about a stationary solution

Let $M \subset \mathbb{R}^3$ be a compact surface, rotationally symmetric about the x_3-axis, with positive Gauss curvature, and let θ be a stationary solution to SQGC, i.e.,

$$\nabla_u(\theta - \Omega\chi) = 0, \quad u = -J\nabla f, \quad \theta = \Lambda f, \quad f = f(x). \tag{5.1}$$

Let us set

$$f_\varepsilon(t) = f + \varepsilon\eta(t) + \cdots, \quad \theta_\varepsilon(t) = \theta + \varepsilon\zeta(t) + \cdots, \quad \zeta = \Lambda\eta. \tag{5.2}$$

Inserting this into

$$\partial_t\theta_\varepsilon - \langle J\nabla f_\varepsilon, \nabla(\theta_\varepsilon - \Omega\chi)\rangle = 0, \tag{5.3}$$

using (5.1), and discarding higher powers of ε produces the linearized equation

$$\partial_t\zeta - \langle J\nabla f, \nabla\zeta\rangle - \langle J\nabla\eta, \nabla(\theta - \Omega\chi)\rangle = 0. \tag{5.4}$$

Now

$$\begin{aligned}\langle J\nabla\eta, \nabla(\theta - \Omega\chi)\rangle &= -\langle\nabla\eta, J\nabla(\theta - \Omega\chi)\rangle \\ &= -\nabla_{J\nabla(\theta-\Omega\chi)}\eta.\end{aligned} \tag{5.5}$$

Since $\eta = \Lambda^{-1}\zeta$, where we define Λ^{-1} to annihilate constants and to have range orthogonal to constants, (5.4) becomes the linear equation

$$\frac{\partial\zeta}{\partial t} = \Gamma\zeta, \tag{5.6}$$

where

$$\Gamma\zeta = \nabla_{J\nabla f}\zeta - \nabla_{J\nabla(\theta-\Omega\chi)}\Lambda^{-1}\zeta. \tag{5.7}$$

6. Linearized stability – Rayleigh type criterion

The question of linear stability of a solution to (5.1) is the question of whether the operator Γ, given by (5.7), generates a uniformly bounded group of operators on

$$L^2_b(M) = \Big\{\zeta \in L^2(M) : \int\limits_M \zeta \, dS = 0\Big\}. \tag{6.1}$$

Let us retain the hypotheses on M from §5. Then $\chi = \chi(\xi)$, with ξ as in (4.2), i.e., $J\nabla\xi = -X_3$. Let us also assume f is a zonal function, i.e., $X_3 f = 0$, so $f = f(\xi)$. This also implies $X_3\theta = 0$, hence $\theta = \theta(\xi)$. Then

$$J\nabla f = -f'(\xi)X_3, \quad J\nabla(\theta - \Omega\chi) = \big[\Omega\chi'(\xi) - \theta'(\xi)\big]X_3, \tag{6.2}$$

and (5.7) becomes

$$\Gamma\zeta = f'(\xi)X_3\zeta - \big[\Omega\chi'(\xi) - \theta'(\xi)\big]X_3\Lambda^{-1}\zeta. \tag{6.3}$$

In such a case, Γ commutes with X_3. Hence we can decompose

$$L^2_b(M) = \bigoplus_k V_k, \tag{6.4}$$

where, for $k \in \mathbb{Z}$,

$$V_k = \{\zeta \in L^2_b(M) : X_3\zeta = ik\zeta\}, \tag{6.5}$$

and we have

$$\Gamma = \bigoplus_k \Gamma_k, \quad \Gamma_k : V_k \to V_k, \tag{6.6}$$

where

$$\Gamma_k\zeta = ik\Big\{f'(\xi)\zeta - \big[\Omega\chi'(\xi) - \theta'(\xi)\big]\Lambda^{-1}\zeta\Big\}. \tag{6.7}$$

Note also that

$$\Lambda^{-1} : V_k \longrightarrow V_k \ \text{ is compact}, \tag{6.8}$$

for each k, so each Γ_k is a compact perturbation of a bounded, skew-adjoint operator on V_k. In light of this, basic analytic Fredholm theory yields the following.

Proposition 6.1. *For each k,*

$$\operatorname{Spec}\Gamma_k \subset ik\Sigma \cup S_k, \tag{6.9}$$

where

$$\Sigma = \{f'(\lambda) : \alpha_0 \le \lambda \le \alpha_1\}, \quad \alpha_0 = \min_M \xi, \quad \alpha_1 = \max_M \xi, \tag{6.10}$$

and S_k is a countable set of points in $\mathbb{C}$ whose accumulation points all must lie in $ik\Sigma$. Each $\mu \in S_k$ is an eigenvalue of Γ_k, and the associated generalized eigenspace is finite dimensional.

In fact, for each $\mu \in \mathbb{C} \setminus ik\Sigma$, $\Gamma_k - \mu I$ is a bounded operator on V_k that is Fredholm of index 0, and it is clearly invertible for $|\mu| > \|\Gamma_k\|$.

Corollary 6.2. *Assume Γ has the form (6.3). If $\operatorname{Spec}\Gamma$ is not contained in the imaginary axis, then some Γ_k has an eigenvalue with nonzero real part.*

Now, having $\operatorname{Spec}\Gamma \subset i\mathbb{R}$ would not guarantee that Γ generates a bounded group of operators on $L^2_b(M)$, but not having this inclusion definitely guarantees (thanks to Corollary 6.2) that the associated group of operators is not uniformly bounded. Thus Corollary 6.2 points to an approach to finding cases that are linearly unstable.

Actually establishing such cases of linear instability is not so straightforward. We proceed to derive some necessary conditions for such linear instability to hold, i.e., for some Γ_k to have an eigenvalue with nonzero real part.

Of course $\Gamma_0 = 0$. Suppose $k \neq 0$ and Γ_k has an eigenvalue $\mu = ik\beta$, $\beta \notin \mathbb{R}$. Then there exists a nonzero $\zeta \in V_k$ such that

$$(f'(\xi) - \beta)\zeta = \big[\Omega\chi'(\xi) - \theta'(\xi)\big]\Lambda^{-1}\zeta, \tag{6.11}$$

hence (with $\eta = \Lambda^{-1}\zeta$)

$$\Lambda\eta = -\frac{\theta'(\xi) - \Omega\chi'(\xi)}{f'(\xi) - \beta}\,\eta. \tag{6.12}$$

If $\beta \notin \mathbb{R}$, the denominator on the right side of (6.12) is nowhere vanishing. In (6.11)–(6.12), ζ and η would not be real valued. Taking the inner product of both sides of (6.12) with η yields

$$\begin{aligned}(\Lambda\eta, \eta) &= -\int_M \frac{\theta'(\xi) - \Omega\chi'(\xi)}{f'(\xi) - \beta}|\eta|^2\,dS \\ &= -\int_M \frac{\theta'(\xi) - \Omega\chi'(\xi)}{|f'(\xi) - \beta|^2}\big[f'(\xi) - \overline{\beta}\big]|\eta|^2\,dS.\end{aligned} \tag{6.13}$$

Now $(\Lambda\eta, \eta)$ is real and positive, but $\operatorname{Im} \overline{\beta} \neq 0$. Hence taking the imaginary part of (6.13) yields

$$\int\limits_M \frac{\theta'(\xi) - \Omega\chi'(\xi)}{|f'(\xi) - \beta|^2} |\eta|^2 \, dS = 0. \tag{6.14}$$

Using this in (6.13) gives

$$-(\Lambda\eta, \eta) = \int\limits_M \frac{\theta'(\xi) - \Omega\chi'(\xi)}{|f'(\xi) - \beta|^2} \big[f'(\xi) - K\big] |\eta|^2 \, dS < 0, \quad \forall\, K \in \mathbb{R}. \tag{6.15}$$

We have (6.14) and (6.15) as necessary conditions for Γ_k to have an eigenvalue with nonzero real part, with associated eigenfunction $\zeta = \Lambda\eta,\ \eta \in V_k$. These results in turn imply the following.

Proposition 6.3. *If* Γ *has an eigenvalue with nonzero real part, then*

$$\theta'(s) - \Omega\chi'(s) \ \textit{must change sign in } s \in (\alpha_0, \alpha_1), \tag{6.16}$$

with α_j *as in (6.10), and*

$$\begin{gathered} \forall\, K \in \mathbb{R}, \quad \exists\, s \in (\alpha_0, \alpha_1) \ \textit{such that} \\ (\theta'(s) - \Omega\chi'(s))\,(f'(s) - K) < 0. \end{gathered} \tag{6.17}$$

These results are parallel to classical results that apply to planar 2D Euler flows, with $\Omega = 0$. Indeed, (6.16) is the counterpart (in the SQGC setting) of the "Rayleigh criterion" for linear instability, and (6.17) is the counterpart of the "Fjortoft criterion." See [R] and [MP], pp. 122–123.

7. Arnold-type stability estimate

Let $M \subset \mathbb{R}^3$ be a compact, axially symmetric surface, with positive Gauss curvature, and let $\theta \in C^\infty(M)$ be a zonal function (with mean value 0), yielding a stationary solution of SQGC. We seek a condition yielding L^2-stability of such a solution. To this end, we bring in functionals of the form

$$\mathcal{H}(\theta) = \int_M \Big\{ G(\theta - \Omega\chi) + \xi\theta \Big\}\, dS, \tag{7.1}$$

with $G \in C^\infty(\mathbb{R})$ to be determined, and $\xi \in C^\infty(M)$ given by (4.2). As a consequence of previous results on conservation laws, if $\tilde{\theta}(t)$ is a sufficiently smooth solution to SQGC, then $\mathcal{H}(\tilde{\theta}(t))$ is independent of t. We want to produce $\mathcal{H}(\theta)$, of the form (7.1), such that θ is a critical point of $\mathcal{H}$, and examine when it can be arranged to have either positive definite or negative definite second derivative.

To begin, if $\psi \in C^\infty(M)$ (also with mean value 0), we have

$$\partial_s \mathcal{H}(\theta + s\psi) = \int_M \Big\{ G'(\theta + s\psi - \Omega\chi) + \xi\psi \Big\}\, dS, \tag{7.2}$$

hence

$$\partial_s \mathcal{H}(\theta + s\psi)\big|_{s=0} = \int_M \Big\{ G'(\theta - \Omega\chi) + \xi\psi \Big\}\, dS, \tag{7.3}$$

and

$$\partial_s^2 \mathcal{H}(\theta + s\psi)\big|_{s=0} = \int_M G''(\theta - \Omega\chi)\psi^2\, dS. \tag{7.4}$$

From (7.3), we see that θ is a critical point of $\mathcal{H}$ provided

$$G'(\theta - \Omega\chi) = -\xi. \tag{7.5}$$

From our hypotheses on M and θ, we have $\theta = \theta(\xi),\ \chi = \chi(\xi)$, and (7.5) becomes

$$G'(\theta(\xi) - \Omega\chi(\xi)) = -\xi. \tag{7.6}$$

We can find such a smooth function G provided the following holds:

$$\theta(\xi) - \Omega\chi(\xi) \text{ is strictly monotone in } \xi, \tag{7.7}$$

more precisely

$$\theta'(\xi) - \Omega\chi'(\xi) \text{ is bounded away from } 0. \tag{7.8}$$

If (7.8) holds, we pick smooth G to satisfy (7.6), and then apply $d/d\xi$ to (7.6), obtaining

$$G''(\theta - \Omega\chi) = -\frac{1}{\theta'(\xi) - \Omega\chi'(\xi)}, \tag{7.9}$$

which is smooth and bounded away from 0. In such a case, either

$$\begin{aligned} &\partial_s^2 \mathcal{H}(\theta + x\psi)\big|_{s=0} \geq A\|\psi\|_{L^2}^2, \quad \text{or} \\ &\partial_s^2 \mathcal{H}(\theta + s\psi)\big|_{s=0} \leq -A\|\psi\|_{L^2}^2, \end{aligned} \tag{7.10}$$

for some $A > 0$. This implies stability of θ in $L^2(M)$ as a critical point of (7.1). We summarize.

Proposition 7.1. *Given a smooth $\theta(\xi)$ (with mean value 0), this is a stable stationary solution to SQGC, in $L^2(M)$, as long as Ω is such that (7.8) holds.*

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