The Zak Transform

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The Zak transform arises in the study of a pair of unitary operators on $L^2(\mathbb{R})$,

$$Tu(x) = u(x+2\pi), \quad Mu(x) = e^{ix}u(x). \tag{1}$$

These two unitary operators commute, and one seeks a joint spectral representation. As an intermediate step, take

$$\mathcal{S}u(x,k) = u(x+2\pi k), \quad x \in I = [0,2\pi],\ k \in \mathbb{Z}. \tag{2}$$

Then $\mathcal{S} : L^2(\mathbb{R}) \to L^2(I \times \mathbb{Z})$ is unitary and

$$\mathcal{S}Tu(x,k) = u(x,k+1), \quad \mathcal{S}Mu(x,k) = e^{ix}u(x,k). \tag{3}$$

The Zak transform is a unitary map

$$\mathcal{Z}f : L^2(\mathbb{R}) \longrightarrow L^2(I \times \mathbb{T}, dx\,d\varphi/2\pi), \quad \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \tag{4}$$

given by

$$\begin{aligned} \mathcal{Z}u(x,\varphi) &= \sum_{k\in\mathbb{Z}} \mathcal{S}u(x,k)e^{ik\varphi} \\ &= \sum_{k\in\mathbb{Z}} u(x+2\pi k)e^{ik\varphi}, \quad x \in I,\ \varphi \in \mathbb{T}. \end{aligned} \tag{5}$$

We have

$$\mathcal{Z}Tu(x,\varphi) = e^{-i\varphi}\mathcal{Z}u(x,\varphi), \quad \mathcal{Z}Mu(x,\varphi) = e^{ix}\mathcal{Z}u(x,\varphi). \tag{6}$$

The Fourier inversion formula gives

$$\frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{Z}u(x,\varphi)e^{-ik\varphi}\,d\varphi = \mathcal{S}u(x,k) = u(x+2\pi k), \tag{7}$$

for $x \in I,\ k \in \mathbb{Z}$.

This setting has natural extensions. For example, let X be a Riemannian manifold, having a discrete group G of isometries. Assume there exists a fundamental domain $D \subset X$, having the property

$$X = \bigcup_{g\in G} Dg, \quad m(Dg_1 \cap Dg_2) = 0 \text{ for } g_1 \neq g_2. \tag{8}$$

(We write the g-action as a right action.) Then we have a unitary map

$$\mathcal{S} : L^2(X) \longrightarrow L^2(D \times G), \quad \mathcal{S}u(x,g) = u(xg), \quad x \in D,\ g \in G, \tag{9}$$

satisfying

$$\mathcal{S}T_h u(x,g) = u(x,gh), \quad \mathcal{S}M_a u(x,g) = a(x)u(x,g), \tag{10}$$

where

$$T_h u(x) = u(xh),\ x \in X, h \in G, \quad M_a u(x) = a(x)u(x), \tag{11}$$

and we require that the function a be G-invariant:

$$a(xg) = a(x), \quad \forall\, x \in X, g \in G. \tag{12}$$

Then we define the Zak transform of u as a function on $D \times \widehat{G}$, where $\widehat{G}$ consists of a complete set of irreducible unitary representations of G, by

$$\begin{aligned} \mathcal{Z}u(x,\pi) &= \sum_{g \in G} \mathcal{S}u(x,g)\pi(g) \\ &= \sum_{g \in G} u(xg)\pi(g), \quad x \in D,\ \pi \in \widehat{G}. \end{aligned} \tag{13}$$

We have

$$\mathcal{Z}T_h u(x,\pi) = \mathcal{Z}u(x,\pi)\pi(h)^{-1}, \quad \mathcal{Z}M_a u(x,\pi) = a(x)\mathcal{Z}u(x,\pi). \tag{14}$$