WORKSHEETS for MATH 522
Advanced Calculus II

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Contents

Introduction

These worksheets serve to guide the student through the text for Math 522 *Introduction to Analysis in Several Variables - Advanced Calculus*, by M. Taylor. Each worksheet deals with material in a designated section of the text, and the idea is that a student can do the exercises in a worksheet in consultation with the text, and in that manner master the material in the text.

Worksheet 1

§1.1, One variable calculus

1. Let $I = [a, b]$ be a closed, bounded interval in $\mathbb{R}$. Read the definition of a *partition* $\mathcal{P}$ of I into intervals $J_k = [x_k, x_{k+1}]$, associated to a collection of points $\{x_j\}$, satisfying

$$a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b.$$

Write down the definitions of maxsize($\mathcal{P}$), $\ell(J_k)$, and $\mathcal{Q} \succ \mathcal{P}$.

2. Let $f : I \to \mathbb{R}$ be a bounded function. Read (1.1.1)–(1.1.7), and write down formulas for

$$\overline{I}_{\mathcal{P}}(f), \quad \underline{I}_{\mathcal{P}}(f), \quad \overline{I}(f), \quad \underline{I}(f).$$

3. Take f as in #2. Note that

$$\underline{I}(f) \le \overline{I}(f).$$

We say that f is Riemann integrable, and write $f \in \mathcal{R}(I)$, provided

$$\underline{I}(f) = \overline{I}(f).$$

Then we write

$$\int_a^b f(x)\,dx = \int_I f\,dx = \overline{I}(f) = \underline{I}(f).$$

4. Proposition 1.1.1 says that $f, g \in \mathcal{R}(I) \Rightarrow f + g \in \mathcal{R}(I)$ and

$$\int_I (f+g)\,dx = \int_I f\,dx + \int_I g\,dx.$$

Read its proof.

5. Proposition 1.2.2 says that if $f : I \to \mathbb{R}$ is continuous (we write $f \in C(I)$), then $f \in \mathcal{R}(I)$. Read its proof.

6. Write down Darboux's theorem, Theorem 1.1.4. One implication is that, if $f \in \mathcal{R}(I)$, and if we have a sequence

$$\mathcal{P}_\nu = \{J_{\nu k} : 1 \le k \le \nu\}$$

of partitions of I, satisfying maxsize($\mathcal{P}_\nu$) $\to 0$, then

$$\int_a^b f(x)\,dx = \lim_{\nu \to \infty} \sum_{k=1}^{\nu} f(\xi_{\nu k})\,\ell(J_{\nu k}),$$

where we take arbitrary $\xi_{\nu k} \in J_{\nu k}$. These sums are called Riemann sums.

7. State the Fundamental Theorem of Calculus, given in Theorems 1.1.6 and 1.1.7, and follow the proof. Note the use of the Mean Value Theorem in the proof of Theorem 1.1.7.

8. Take a set $S \subset I$. Write down the definitions of upper content and outer measure,

$$\mathrm{cont}^+(S), \quad \text{and} \quad m^*(S),$$

given in (1.1.21) and (1.1.22).

9. Write down the sufficient conditions for a bounded function $f : I \to \mathbb{R}$ to be Riemann integrable, given in Proposition 1.1.11 and in Proposition 1.1.12. These involve two different evaluations of the "size" of the set S of points in I at which f is not continuous, namely $\mathrm{cont}^+(S)$ and $m^*(S)$.

10. Note the example of a bounded function that is not Riemann integrable, given in (1.1.16). Note the examples of bounded, discontinuous functions that are Riemann integrable, given in Exmples 1 and 2, after Proposition 1.1.11.

11. Do exercise 4, at the end of §1.1.

Worksheet 2

§1.2, Euclidean spaces

1. Define the vector operations on $\mathbb{R}^n$.

2. Given $x, y \in \mathbb{R}^n$, define the dot product $x \cdot y$.

3. Given $x \in \mathbb{R}^n$, we define the norm $|x| \in [0, \infty)$ by

$$|x| = \sqrt{x \cdot x}.$$

Consult Proposition 1.2.1 and show that the triangle inequality

$$|x + y| \le |x| + |y|$$

follows from Cauchy's inequality

$$|x \cdot y| \le |x|\,|y|.$$

4. Consult Proposition 1.2.2 for the proof of Cauchy's inequality.

5. Given $p_j \in \mathbb{R}^n$, define what it means to say

$$\begin{aligned} &p_j \text{ converges to } p \text{ as } j \to \infty. \\ &(p_j) \text{ is Cauchy.} \end{aligned}$$

6. Given $S \subset \mathbb{R}^n$, define what it means to say

$$\begin{aligned} &S \text{ is closed,} \\ &S \text{ is open.} \end{aligned}$$

7. Given $x, y \in \mathbb{R}^n$, we say

$$x \perp y \Longleftrightarrow x \cdot y = 0.$$

Show that

$$|x + y|^2 = |x|^2 + |y|^2 \Longleftrightarrow x \perp y.$$

8. Check out the notion of compactness, the Bolzano-Weierstrass theorem, and the Heine-Borel theorem.

Worksheet 3

§1.3, Vector spaces and linear transformations

1. Define the concept of a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Note that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. ($\mathbb{F}^n$ is a vector space over $\mathbb{F}$.)

2. Let $S = \{v_1, \dots, v_k\} \subset V$, a vector space. Define what it means to say

$$
\begin{aligned}
&S \text{ spans } V,\\
&S \text{ is linearly independent},\\
&S \text{ is a basis of } V.
\end{aligned}
$$

3. Study Lemma 1.3.1 and Proposition 1.3.2, whose content is that
If V has a basis $\{v_1, \dots, v_k\}$ and if $\{w_1, \dots, w_\ell\} \subset V$ is linearly independent, then $\ell \le k$.
Show that this leads to Corollary 1.3.3:
If V is finite dimensional, then any two bases of V have the same number of elements.
In such a case, $\dim V$ denotes the number of elements in a basis of V.

4. State Propositions 1.3.4 and 1.3.5.

5. State Proposition 1.3.6, the Fundamental Theorem of Linear Algebra, and show how this follows from Propositions 1.3.4 and 1.3.5.

6. Deduce from the Fundamental Theorem of Linear Algebra that if V is finite dimensional and $A : V \to V$ is linear, then

$$A \text{ injective } \Leftrightarrow A \text{ surjective } \Leftrightarrow A \text{ isomorphism}.$$

7. State Proposition 1.3.9, characterizing when a matrix $A \in M(n, \mathbb{F})$ is invertible, in terms of the behavior of its columns.

Worksheet 4

§1.4, Determinants

1. Given

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{we set} \quad \det A = ad - bc.$$

Show that $A : \mathbb{F}^2 \to \mathbb{F}^2$ is invertible if and only if $\det A \neq 0$. If this holds,

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

2. Consult Proposition 1.4.1 and define $\det A$ for $A \in M(n, \mathbb{F})$.

3. Show that the formula (1.4.30) for $\det A$ implies

$$\det A = \det A^t.$$

4. Read the proof of Proposition 1.4.3, that if $A, B \in M(n, \mathbb{F})$,

$$\det(AB) = (\det A)(\det B).$$

Show that this implies Corollary 1.4.4, i.e.,

$$A \text{ invertible} \implies \det A \neq 0.$$

5. Read the proof of Proposition 1.4.6, that if $A \in M(n, \mathbb{F})$,

$$A \text{ invertible} \iff \det A \neq 0.$$

See how this completes the result of Exercise 4.

6. Study Exercises 1–3 at the end of §1.4, treating the expansion of $\det A$ by minors down the kth column, given $A \in M(n, \mathbb{F})$.

7. Use an expansion by minors to evaluate $\det A$ for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

Supplementary worksheet
§1.4, More on determinants

1. Verify the following method of computing 3×3 determinants. Given

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

form a 3×5 rectangular matrix by copying the first two columns of A to the right. The products in (1.4.16) with plus signs are the products of each of the three downward sloping diagonals marked in bold below.

$$\begin{pmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & a_{11} & a_{12} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{21}} & a_{22} \\ a_{31} & a_{32} & \mathbf{a_{33}} & \mathbf{a_{31}} & \mathbf{a_{32}} \end{pmatrix}.$$

The products in (1.4.16) with minus signs are the products of each of the three upward sloping diagonals marked in bold below.

$$\begin{pmatrix} a_{11} & a_{12} & \mathbf{a_{13}} & \mathbf{a_{11}} & \mathbf{a_{12}} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{21}} & a_{22} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} & a_{31} & a_{32} \end{pmatrix}.$$

2. Use the method described above to compute the determinants of

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

3. Given $A = (0\ 1\ 2)$, compute $\det A^t A$ and $\det AA^t$.

4. Compute the determinant of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Worksheet 5

§2.1, The derivative in several variables

As defined in §2.1, if $\mathcal{O} \subset \mathbb{R}^n$ is open, a function $F : \mathcal{O} \to \mathbb{R}^m$ is differentiable at $x \in \mathcal{O}$, with derivative $DF(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, if and only if

$$F(x+y) = F(x) + DF(x)y + R(x,y), \quad R(x,y) = o(\|y\|). \tag{1}$$

One compares this with the partial derivative,

$$\frac{\partial F}{\partial x_j}(x) = \lim_{h\to 0} \frac{1}{h}\big[F(x+he_j) - F(x)\big], \tag{2}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

1. Verify the identity (2.1.4) connecting (1) and (2), when F is differentiable.

2. Study the argument in (2.1.12)–(2.1.14) that, if

$$S : M(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad S(X) = X^2,$$

then

$$DS(X)Y = XY + YX.$$

Then look at Exercise 2, at the end of §2.1. Show that

$$P_3(X) = X^3 \Longrightarrow DP_3(X)Y = YX^2 + XYX + X^2Y.$$

3. Study the argument in (2.1.16)–(2.1.21) that, if

$$\Phi : Gl(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad \Phi(X) = X^{-1},$$

then

$$D\Phi(I)Y = -Y.$$

Note the use of the evaluation of the infinite series

$$(I+Y)^{-1} = \sum_{k=0}^{\infty} (-1)^k Y^k, \quad \text{for} \quad \|Y\| < 1.$$

Going further, as indicated in Exercise 3 at the end of §2.1, show that

$$D\Phi(X)Y = -X^{-1}YX^{-1}.$$

Worksheet 6

§2.1, The derivative in several variables II

1. Proposition 2.1.1 says that if F is of class C^1 (i.e., $\partial F/\partial x_j$ is continuous on $\mathcal{O}$ for each j) then F is differentiable at each $x \in \mathcal{O}$. Study its proof.

2. The chain rule is given in (2.1.25). If $F : \mathcal{O} \to U$ and $G : U \to \mathbb{R}^k$ are differentiable, then $G \circ F$ is differentiable, and

$$D(G \circ F)(x) = DG(F(x))DF(x).$$

Study its proof.

3. Look at Exercise 0 at the end of §2.1, dealing with a strengthening of Proposition 4.1.1.

4. Look at Exercise 14 at the end of §2.1, describing a function that is not differentiable at $(0, 0) \in \mathbb{R}^2$, despite the fact that both its partial derivatives exist (but they are not continuous).

5. Proposition 2.1.2 says that if $\mathcal{O} \subset \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}^m$ is of class C^2, then

$$\partial_j \partial_k F(x) = \partial_k \partial_j F(x), \quad x \in \mathcal{O}, \ j, k \in \{1, \dots, n\}. \tag{1}$$

Study the proof. It involves difference quotients and the mean value theorem.

6. Check out Exercise 15, at the end of §2.1, regarding a function $g \in C^1(\mathbb{R}^2)$ for which $\partial_x \partial_y g$ and $\partial_y \partial_x g$ exist at each point of $\mathbb{R}^2$, but

$$\partial_x \partial_y g(0, 0) \neq \partial_y \partial_x g(0, 0),$$

in contrast to (1) above. (In this case, $g \notin C^2$.)

Worksheet 7

§2.1, Higher derivatives and power series

1. Formulas (2.1.35)–(2.1.36) present two different multi-index notations for derivatives of order k:

$$
\begin{aligned}
&f^{(J)}(x) = \partial_{j_k} \cdots \partial_{j_1} f(x), \quad J = (j_1, \dots, j_k),\ |J| = k, \\
&f^{(\alpha)}(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f(x), \quad \alpha = (\alpha_1, \dots, \alpha_n),\ |\alpha| = \alpha_1 + \cdots + \alpha_n = k.
\end{aligned} \tag{1}
$$

Become familiar with these notations.

2. Review the treatment of power series for functions of one variable in Chapter 1, §1.1, Exercises 9–10. The formula (1.1.50) can be written

$$
f(t) = \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} t^j + R_k(t), \quad t \in I = (-R, R), \tag{2}
$$

and two integral formulas are derived for the remainder $R_k(t)$, depending on whether $f \in C^{k+1}(I)$ or $f \in C^k(I)$. See (1.1.51) and (1.1.54).

Note also the Cauchy and Lagrange formulas for the remainder, given in Appendix A.4.

3. Study formulas (2.1.39)–(2.1.45), leading to the power series formula

$$
F(x) = \sum_{|J| \le k} \frac{1}{|J|!} F^{(J)}(0) x^J + R_k(x), \tag{3}
$$

with the remainder $R_k(x)$ given by (2.1.46).

4. Study formulas (2.1.47)–(2.1.52), leading to the power series formula

$$
F(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_k(x), \tag{4}
$$

with $R_k(x)$ given by (2.1.53).

5. The results (2.1.46) and (2.1.53) require F to be of class C^{k+1}. Study Proposition 2.1.5, which establishes (4), with $R_k(x)$ given by (2.1.56), for F of class C^k.

Worksheet 8

§2.1, Higher derivatives II, critical points

1. Study formulas (2.1.58)–(2.1.62), in which it is established that, if $\mathcal{O} \subset \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}$ is C^2, then, for $y \in \mathcal{O}$,

$$(1) \qquad F(x) = F(y) + DF(y)(x-y) + \frac{1}{2}(x-y) \cdot D^2F(y)(x-y) + R_2(x,y),$$

where $D^2F(y)$ is the $n \times n$ Hessian matrix, given by (2.1.59), and

$$(2) \qquad R_2(x,y) = o(|x-y|^2).$$

2. In the setting of Exercise 1, we say $x_0 \in \mathcal{O}$ is a critical point of F if $DF(x_0) = 0$. Proposition 2.1.6 says that if $F : \mathcal{O} \to \mathbb{R}$ is C^2 and $x_0 \in \mathcal{O}$ is a critical point, then
(i) $D^2F(x_0)$ positive definite $\Rightarrow F$ has a local min at x_0,
(ii) $D^2F(x_0)$ negative definite $\Rightarrow F$ has a local max at x_0,
(iii) $D^2F(x_0)$ strongly indefinite $\Rightarrow x_0$ is a saddle point for F.
Show how this result follows from (1)–(2) above.

3. Study Proposition 2.1.7, characterizing when a matrix $A = A^t \in M(n, \mathbb{R})$ is positive definite, in terms of the behavior of the determinants of all the $\ell \times \ell$ upper left submatrices, $1 \le \ell \le n$.

4. Returning to the setting of Exercise 1 above, check out the remainder formula in (2.1.72).

5. Check out Proposition 2.1.10, regarding smoothness and derivatives of functions given by convergent power series on a domain $\widetilde{R} \subset \mathbb{R}^n$.

Worksheet 9

§2.2, Inverse function theorem

1. Theorem 2.2.1 says that if $\Omega \subset \mathbb{R}^n$ is open,

$$F : \Omega \to \mathbb{R}^n \ \text{ is } \ C^1,$$

$p_0 \in \Omega$, and $DF(p_0)$ is invertible, then F maps some open neighborhood U of p_0 one-one and onto a neighborhood V of $q_0 = F(p_0)$, and the inverse map

$$F^{-1} : V \longrightarrow U \ \text{ is } \ C^1.$$

Study its proof.

2. Proposition 2.2.2 gives a condition that guarantees that a C^1 map $f : \Omega \to \mathbb{R}^n$ be one-one. State it and write down its proof. Describe the role it plays in the proof of Theorem 2.2.1.

3. Theorem 2.2.3 is called the contraction mapping theorem. State it and write down its proof. Describe its role in the proof of Theorem 2.2.1.

4. The map $F : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ in (2.2.21) defines polar coordinates. Study how it illustrates the inverse function theorem.

5. Compare the iterative method (2.2.19) for solving $F(x) = y$ for x with that given in Exercise 1 at the end of §2.2 (Newton's method).

6. Do Exercise 2 at the end of §2.2.

Worksheet 10

§2.2, Implicit function theorem

1. Explain how

$$x^2 + y^2 = 1$$

defines y implicitly as a smooth function of x, in two ways, for $x \in (-1, 1)$.

2. Theorem 2.2.5 is the implicit function theorem. It says that if $x_0 \in U$, open in $\mathbb{R}^m$, $y_0 \in V$, open in $\mathbb{R}^\ell$, and

$$F : U \times V \longrightarrow \mathbb{R}^\ell \ \text{ is } \ C^k, \quad F(x_0, y_0) = u_0, \tag{1}$$

and if

$$D_y F(x_0, y_0) \ \text{ is invertible} \tag{2}$$

(as an $\ell \times \ell$ matrix), then the equation

$$F(x, y) = u_0 \tag{3}$$

defines

$$y = g(x, u_0), \tag{4}$$

for x near x_0 (satisfying $g(x_0, u_0) = y_0$), and g is a C^k map.

Study the proof given in (2.2.39)–(2.2.45), which brings in

$$H : U \times V \to \mathbb{R}^m \times \mathbb{R}^\ell, \quad H(x, y) = \begin{pmatrix} x \\ F(x, y) \end{pmatrix}, \quad H(x_0, y_0) = \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}. \tag{5}$$

See from

$$DH = \begin{pmatrix} I & 0 \\ D_x F & D_y F \end{pmatrix} \tag{6}$$

that the hypothesis (2) implies $DH(x_0, y_0)$ is invertible. See how the inverse function theorem (Theorem 2.2.1) yields a smooth inverse

$$G : \mathcal{O} \longrightarrow U \times V \tag{7}$$

to H, where $\mathcal{O}$ is a neighborhood of (x_0, u_0) in $\mathbb{R}^m \times \mathbb{R}^\ell$, and that $G(x, u)$ has the form

$$G(x, u) = \begin{pmatrix} x \\ g(x, u) \end{pmatrix}, \tag{8}$$

yielding the identity

$$F(x, g(x, u)) = u, \tag{9}$$

and hence satisfying (3)–(4).

3. Check out Proposition 2.2.6, which treats a C^k map

$$F : \Omega \longrightarrow \mathbb{R}, \quad F(x_0) = u_0, \quad \Omega \subset \mathbb{R}^n \text{ open},$$

under the hypothesis that

$$\nabla F(x_0) \neq 0.$$

It shows that if in particular

$$\partial_n F(x_0) \neq 0,$$

then you can solve $F(x) = u_0$ for

$$x_n = g(x_1, \dots, x_{n-1}),$$

with $(x_{10}, \dots, x_{n-1,0}, x_{n0}) = x_0$, for a C^k function g.

4. Check out the relevance of the material of #3 to the example introduced in #1.

5. Do Exercise 7, at the end of §2.2.

Worksheet 11

§2.3, Systems of differential equations and vector fields

Material to be provided later.

Worksheet 12

§3.1, The Riemann integral in n variables

1. Let $R = I_1 \times \cdots \times I_n$ be a cell in $\mathbb{R}^n$, where each $I_\nu = [a_\nu, b_\nu]$ is a closed, bounded interval in $\mathbb{R}$. Study the definition of a partition $\mathcal{P}$ of R, into cells R_α, $\alpha = (\alpha_1, \dots, \alpha_n)$, given in the beginning of §3.1. Write down the definitions of

$$\text{maxsize}(\mathcal{P}), \quad V(R_\alpha), \quad \text{and} \quad \mathcal{Q} \succ \mathcal{P}.$$

2. Let $f : R \to \mathbb{R}$ be a bounded function. Read (3.1.1)–(3.1.8), and write down formulas for

$$\overline{I}_{\mathcal{P}}(f), \quad \underline{I}_{\mathcal{P}}(f), \quad \overline{I}(f), \quad \underline{I}(f).$$

3. Take f as in #2. Note that

$$\underline{I}(f) \le \overline{I}(f).$$

We say f is Riemann integrable, and write $f \in \mathcal{R}(R)$, provided

$$\underline{I}(f) = \overline{I}(f).$$

Then we write

$$\int\limits_R f(x)\, dV(x) = \overline{I}(f) = \underline{I}(f).$$

4. Proposition 3.1.1 is the multi-D Darboux theorem. One implication is that if $f \in \mathcal{R}(R)$, and if we have a sequence

$$\mathcal{P}_\nu = \{R_{\nu\alpha} : \alpha \in S_\nu\}$$

of partitions of R, satisfying maxsize$(\mathcal{P}_\nu) \to 0$, then

$$\int\limits_R f(x)\, dV(x) = \lim_{\nu \to 0} \sum_{\alpha \in S_\nu} f(\xi_{\nu\alpha})\, V(R_{\nu\alpha}),$$

where we take arbitrary $\xi_{\nu\alpha} \in R_{\nu\alpha}$. These sums are called Riemann sums. Compare this result with Theorem 1.1.4. A key ingredient behind this result is that, whenever $g : R \to \mathbb{R}$ is bounded,

$$\overline{I}_{\mathcal{P}_\nu}(g) \to \overline{I}(g), \quad \underline{I}_{\mathcal{P}_\nu}(g) \to \underline{I}(g).$$

5. Proposition 5.1.2 says that $f_j \in \mathcal{R}(R),\ c_j \in \mathbb{R} \Rightarrow c_1 f_1 + c_2 f_2 \in \mathcal{R}(R)$ and

$$\int\limits_R (c_1 f_1 + c_2 f_2)\, dV = c_1 \int\limits_R f_1\, dV + c_2 \int\limits_R f_2\, dV.$$

Compare this result with Proposition 1.1.1.

6. Proposition 3.1.3 says that if $f : R \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}(R)$. Read its proof. Compare this with Proposition 1.1.2.

7. Take a set $S \subset R$, where R is a cell. Write down the definitions of upper content, lower content, and volume,

$$\mathrm{cont}^+(S), \quad \mathrm{cont}^-(S), \quad \text{and} \quad V(S),$$

given in (3.1.13)–(3.1.14).

8. A set $S \subset R$ is called *contented* provided $\mathrm{cont}^+(S) = \mathrm{cont}^-(S)$, in which case the common value is denoted $V(S)$. Proposition 3.1.4 says that a set $S \subset R$ is contented if and only if its boundary bS satisfies

$$\mathrm{cont}^+(bS) = 0.$$

Read its proof.

9. Proposition 3.1.6 says that if $f : R \to \mathbb{R}$ is bounded and if S is the set of its points of discontinuity, then

$$\mathrm{cont}^+(S) = 0 \Longrightarrow f \in \mathcal{R}(R).$$

Read its proof.
NOTE. A stronger result is established in Proposition 3.1.31, namely $m^*(S) = 0 \Rightarrow f \in \mathcal{R}(R)$.

10. Propositions 3.1.7–3.1.8 give sufficient conditions that a set $S \subset R$ have upper content 0 (we then say S is a nil set). Write down these conditions.

Worksheet 13

§3.1, The Riemann integral in n variables II
Iterated integrals

1. Theorem 3.1.9 takes a closed, bounded, contented set $\Sigma \subset \mathbb{R}^{n-1}$

$$\Omega = \{(x, y) \in \mathbb{R}^n : x \in \Sigma,\ g_0(x) \le y \le g_1(x)\},$$

where g_j are continuous on Σ, $g_0 < g_1$. One is given $f : \Omega \to \mathbb{R}$, continuous, and

$$\varphi(x) = \int_{g_0(x)}^{g_1(x)} f(x, y)\, dy,$$

which is continuous on Σ. The conclusion is that

$$\int\limits_{\Omega} f\, dV_n = \int\limits_{\Sigma} \varphi\, dV_{n-1}.$$

Study its proof. Note the role of Proposition 3.1.8. A corollary of Theorem 3.1.9 is

$$V(\Omega) = \int\limits_{\Sigma} [g_1(x) - g_0(x)]\, dx.$$

2. Study the application of these results to

$$A(D) = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \pi.$$

Here, $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

3. Study the application of results of # 1 to

$$V(B^n) = 2 \int\limits_{B^{n-1}} \sqrt{1 - |x|^2}\, dx,$$

where $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$, hence to

$$\begin{aligned} V(B^3) &= 2 \int\limits_{D} \sqrt{1 - |x|^2}\, dx \\ &= 2 \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dy\, dx \\ &= \frac{4}{3}\pi. \end{aligned}$$

4. Here is an extension of Theorem 3.1.9. We take $\Sigma \subset \mathbb{R}^k$, closed, bounded, and contented, $g_j : \Sigma \to [0, \infty)$ continuous, $g_0 < g_1$, and

$$\Omega = \{(x, y) \in \mathbb{R}^n : x \in \Sigma,\ y \in \mathbb{R}^{n-k},\ g_0(x) \le |y| \le g_1(x)\}.$$

We take $f : \Omega \to \mathbb{R}$ continuous. The conclusion is that

$$\varphi(x) = \int\limits_{g_0(x) \le |y| \le g_1(x)} f(x, y)\, dy$$

is continuous on Σ and

$$\int\limits_{\Omega} f\, dV_n = \int\limits_{\Sigma} \varphi\, dV_k.$$

Consider how the proof of Theorem 3.1.9 adapts to this situation.

5. Consider the application of #4 to solids of revolution

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : a \le x \le b,\ \sqrt{y^2 + z^2} \le g(x)\},$$

including

$$V(\Omega) = \pi \int_a^b g(x)^2\, dx,$$

and a derivation of the formula

$$V(B^3) = \pi \int_{-1}^{1} (1 - x^2)\, dx = \frac{4}{3}\pi.$$

6. Study the application of #4 to the recursive formula

$$V(B^n) = \beta_n V(B^{n-1}), \quad \beta_n = \int_{-1}^{1} (1 - x^2)^{(n-1)/2}\, dx,$$

including

$$V(B^4) = \beta_4 V(B^3), \quad \beta_4 = 2 \int_0^{\pi/2} \cos^4 t\, dt.$$

7. State the general Fubini theorem, Proposition 3.1.32. See how it implies Theorem 5.1.10 and the result of #4.

Worksheet 14

§3.1, The Riemann integral in n variables III
Change of variable formulas

The central result of this part of §3.1, encompassing Propositions 3.1.10–3.1.14, is Theorem 3.1.15, which says the following. Take

$$\mathcal{O}, \Omega \subset \mathbb{R}^n \text{ open}, \quad G : \mathcal{O} \to \Omega \ \text{ a } C^1 \text{ diffeomorphism.}$$

Assume $f : \Omega \to \mathbb{R}$ is supported on a compact subset of Ω and is Riemann integrable (we say $f \in \mathcal{R}_c(\Omega)$). Then $f \circ G \in \mathcal{R}_c(\mathcal{O})$, and

$$\int_{\Omega} f(y)\, dV(y) = \int_{\mathcal{O}} f(G(x))\, |\det DG(x)|\, dV(x). \tag{1}$$

This is established in stages.

1. Proposition 3.1.10 says that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and has compact support (we say $f \in C_c(\mathbb{R}^n)$), and if $A \in G\ell(n, \mathbb{R})$ (invertible matrix), then

$$\int f(x)\, dx = |\det A| \int f(Ax)\, dx. \tag{2}$$

Study its proof. Note the role of Proposition 3.1.9.

2. Proposition 3.1.11 establishes other characterizations of $\overline{I}(f)$ and $\underline{I}(f)$, given bounded $f : R \to \mathbb{R}$. In particular,

$$\begin{aligned} \overline{I}(f) &= \inf \Big\{ \int_R g\, dV : g \in C(R), g \geq f \Big\}, \\ \underline{I}(f) &= \sup \Big\{ \int_R g\, dV : g \in C(R), g \leq f \Big\}. \end{aligned} \tag{3}$$

Study its proof.

3. See how (3) leads to the extension of (2) to all compactly supported Riemann integrable functions f on $\mathbb{R}^n$ (we say $f \in \mathcal{R}_c(\mathbb{R}^n)$), in Proposition 3.1.12.

Note also Corollary 3.1.13:

$$V(A(\Sigma)) = |\det A|\, V(\Sigma), \tag{4}$$

when $\Sigma \subset \mathbb{R}^n$ is compact and contented, and $A \in G\ell(n, \mathbb{R})$.

4. Proposition 3.1.14 establishes Theorem 3.1.15, under the additional hypothesis that f is continuous. Study its proof, following (3.1.48)–(3.1.56).

5. See how Theorem 3.1.15 is derived from Proposition 3.1.14, via (3).

6. Check out the use of polar coordinates to do double integrals, in (3.1.62)–(3.1.65), including the formula

$$\int_{D_\rho} f(x,y)\,dx\,dy = \int_0^\rho \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\,d\theta\,dr, \tag{5}$$

where $D_\rho = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \rho^2\}$. Note that one can use polar coordinates to establish

$$A(D_\rho) = \pi\rho^2,$$

and also to compute $V(B^3)$,

$$\begin{aligned} V(B^3) &= 2\int_D \sqrt{1 - x^2 - y^2}\,dx\,dy \\ &= \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2}\,r\,dr\,d\theta \\ &= \frac{4}{3}\pi. \end{aligned} \tag{6}$$

Compare approaches described in #3 and #5 of Worksheet 13.

7. Study the use of spherical polar coordinates on $\mathbb{R}^3$ in Exercise 5 of §3.1, to obtain

$$\int_{B^3} f(x)\,dV(x) = \int_0^{2\pi} \int_0^\pi \int_0^1 f(G(\rho,\theta,\psi))\rho^2 \sin\theta\,d\rho\,d\theta\,d\psi. \tag{7}$$

See from this yet a fourth derivation of the formula (6).

8. Check out the extension of various results to integrals over all of $\mathbb{R}^n$. In particular, see (3.1.71)–(3.1.74), deriving the identity

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = I = \sqrt{\pi} \tag{8}$$

by representing I^2 as an integral over $\mathbb{R}^2$ and switching to polar coordinates. More generally, show that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = I^n = \pi^{n/2}. \tag{9}$$

Worksheet 15

§3.2, Surfaces and surface integrals

1. A C^k smooth m-dimensional surface M in $\mathbb{R}^n$ is covered by coordinate charts. Each $p \in M$ has a neighborhood $U \subset M$ for which there is a C^k map

$$\varphi : \mathcal{O} \longrightarrow U, \quad \text{one-to-one and onto,} \tag{1}$$

with $\mathcal{O} \subset \mathbb{R}^m$ open, such that

$$D\varphi(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \ \text{ is injective, for all } \ x \in \mathcal{O}. \tag{2}$$

Given $p \in U$, we set

$$\begin{aligned} T_pM &= \text{ Range } D\varphi(x_0), \quad \varphi(x_0) = p, \\ N_pM &= \perp \ \text{ complement of } T_pM \text{ in } \ \mathbb{R}^n. \end{aligned} \tag{3}$$

If

$$\psi : \Omega \longrightarrow U, \quad \text{one-to-one and onto,} \tag{4}$$

is another C^k coordinate chart, we set

$$F = \psi^{-1} \circ \varphi : \mathcal{O} \longrightarrow \Omega. \tag{5}$$

Check out Lemma 3.2.1, saying F is a C^k diffeomorphism, and study its proof. Note the role of the Inverse Function Theorem. See that also

$$T_pM = \text{ Range } D\psi(x_1), \quad \text{if } \ \psi(x_1) = p. \tag{6}$$

2. If $M \subset \mathbb{R}^n$ is a smooth m-dimensional surface, as in #1, we associate to each coordinate chart φ, as in (1), an $m \times m$ matrix function of the form

$$G(x) = D\varphi(x)^t D\varphi(x), \tag{7}$$

called a metric tensor. See in (3.2.8) the connection with the inner product on T_pM ($p = \varphi(x)$), inherited from the dot product on $\mathbb{R}^n$. Note that

$$v \cdot G(x)w = D\varphi(x)w \cdot D\varphi(x)v, \tag{7A}$$

for $v, w \in \mathbb{R}^m$, and that

$$G(x) = \big(g_{jk}(x)\big), \quad g_{jk}(x) = \partial_j\varphi(x) \cdot \partial_k\varphi(x). \tag{7B}$$

Show via the chain rule that if we have another coordinate chart ψ (connected with φ via (5)), with metric tensor

$$H(y) = D\psi(y)^t D\psi(y), \tag{8}$$

then

$$G(x) = DF(x)^t \, H(y) \, DF(x), \quad \text{for} \;\; y = F(x). \tag{9}$$

3. If $f : M \to \mathbb{R}$ is a continuous function supported in a coordinate patch U (as in (1)), we define the surface integral of f as

$$\int_M f \, dS = \int_{\mathcal{O}} f(\varphi(x)) \sqrt{g(x)} \, dx, \quad g(x) = \det G(x), \tag{10}$$

with $G(x)$ given by (7). Check out (3.2.14)–(3.2.15) to see that if $\psi : \Omega \to U$ in (4) is another coordinate chart, then (10) is equal to

$$\int_\Omega f(\psi(y)) \sqrt{h(y)} \, dy, \quad h(y) = \det H(y), \tag{11}$$

with $H(y)$ given by (8). Note the role of the change of variable formula, addressed in Worksheet 14.

4. Check out (3.2.18)–(3.2.20), which says that if $M \subset \mathbb{R}^3$ is a 2D surface with coordinate chart $\varphi : \mathcal{O} \to U \subset M$, and f is supported on U, then

$$\int_M f \, dS = \int_{\mathcal{O}} f(\varphi(x)) \, |\partial_1 \varphi \times \partial_2 \varphi| \, dx_1 \, dx_2. \tag{12}$$

5. Check out (3.2.21)–(3.2.22), which says that if $\Omega \subset \mathbb{R}^{n-1}$ is open and $M \subset \mathbb{R}^n$ is the graph of $z = u(x)$, then

$$\varphi(x) = (x, u(x)) \tag{13}$$

provides a coordinate chart, in which the metric tensor formula (7B) becomes

$$g_{jk}(x) = \delta_{jk} + \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k}, \tag{14}$$

and in such a case

$$\sqrt{g} = (1 + |\nabla u|^2)^{1/2}. \tag{15}$$

Worksheet 16

§3.2, Surfaces and surface integrals II

1. Show that the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is a smooth surface. Show that solving

$$x_1^2 + \cdots + x_k^2 + \cdots + x_n^2 = 1$$

for x_k yields coordinate charts

$$\varphi_k^\pm : B^{n-1} \longrightarrow U_k^\pm \subset S^{n-1},$$

where

$$U_k^\pm = \{x \in S^{n-1} : \pm x \cdot e_k > 0\},$$

and $\{e_1, \dots, e_n\}$ denotes the standard orthonormal basis of $\mathbb{R}^n$.

2. Writing φ_n^+ in #1 as

$$\varphi_n^+(x) = (x, u(x)) = (x, \sqrt{1 - |x|^2}), \quad x \in B^{n-1},$$

show that

$$|\nabla u(x)|^2 = \frac{|x|^2}{1 - |x|^2},$$

and hence, from #5 of Worksheet 15,

$$\sqrt{g(x)} = (1 - |x|^2)^{-1/2},$$

in this coordinate system. Deduce that the area A_{n-1} of S^{n-1} is given by

$$A_{n-1} = 2 \int_{B^{n-1}} (1 - |x|^2)^{-1/2} \, dx.$$

3. Deduce from #2 that

$$\begin{aligned} A_2 &= 2 \int_D (1 - |x|^2)^{-1/2} \, dx \\ &= 2 \int_0^{2\pi} \int_0^1 (1 - r^2)^{-1/2} \, r \, dr \, d\theta \\ &= 4\pi. \end{aligned}$$

4. Check out the argument in (3.2.24)–(3.2.28), yielding

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_{S^{n-1}} \Big[\int_0^\infty f(r\omega) r^{n-1} \, dr \Big] \, dS(\omega), \tag{4}$$

for f integrable on $\mathbb{R}^n$. If f is radial, i.e., $f(x) = \varphi(|x|)$, deduce that

$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\,dx = A_{n-1} \int_0^\infty \varphi(r) r^{n-1}\,dr, \tag{5}$$

where A_{n-1} is the area of S^{n-1}.

5. Deduce from (5) that

$$V(B^n) = \frac{1}{n} A_{n-1}. \tag{6}$$

In particular, $V(B^3) = A_2/3$. Compare the computations of $V(B^3)$ in worksheets 20–21 and the computation of A_2 in #3 above.

6. Combine (2), with $n-1$ replaced by n, with (5) to show that

$$A_n = 2A_{n-1} \int_0^1 (1-r^2)^{-1/2} r^{n-1}\,dr. \tag{7}$$

Use this with $n = 2$ to relate A_2 to $A_1 = 2\pi$. Compare the calculation in #3 above.

7. See #8 of Worksheet 14 for a derivation of the identity

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx = I^n = \pi^{n/2}. \tag{8}$$

Take $\varphi(r) = e^{-r^2}$ in (5), and follow the arguments in (3.2.30)–(3.2.32) to see the formula

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{9}$$

where $\Gamma(z)$ is Euler's gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\,ds. \tag{10}$$

8. Study the treatment of $\Gamma(z)$ in (3.2.33)–(3.2.37), including Lemma 3.2.2, which uses integration by parts to establish

$$\Gamma(z+1) = z\Gamma(z), \quad z > 0, \tag{11}$$

and also the particular identities

$$\Gamma(1) = 1, \quad \Gamma\Big(\frac{1}{2}\Big) = \pi^{1/2}, \quad \Gamma(k) = (k-1)! \tag{12}$$

See also how these identities together with (9) yield the formulas

$$A_{2k-1} = \frac{2\pi^k}{(k-1)!}, \quad A_{2k} = \frac{2\pi^k}{(k-\frac{1}{2})\cdots(\frac{1}{2})}. \tag{13}$$

9. Study Proposition 3.2.5, and its special case, arising in Exercise 10 at the end of §3.2, which says that if $\Omega \subset \mathbb{R}^n$ is open, $c \in \mathbb{R}$, and

$$\begin{aligned} &u : \Omega \longrightarrow \mathbb{R} \;\text{ is }\; C^k, \\ &S = \{x \in \Omega : u(x) = c\}, \\ &S \neq \emptyset, \;\text{ and }\; x \in S \Rightarrow \nabla u(x) \neq 0, \end{aligned} \tag{14}$$

then S is a C^k smooth, $(n-1)$-dimensional surface. Also, for $x \in S$,

$$\begin{aligned} N_x S &= \text{ span of } \nabla u(x), \\ T_x S &= \perp \text{ complement of } N_x S. \end{aligned} \tag{15}$$

Apply this to

$$S^{n-1} = \{x \in \mathbb{R}^n : u(x) = 1\}, \quad u(x) = |x|^2. \tag{16}$$

10. Apply #9 to

$$S = \{(x', f(x')) : x' \in \mathbb{R}^{n-1}\},$$

which takes the form (14) with

$$u(x) = x_n - f(x'), \quad x' = (x_1, \dots, x_{n-1}), \quad c = 0.$$