Remarks on Nonsmooth Riemannian Manifolds

Michael Taylor

1. Introduction

We make a few observations about nonsmooth Riemannian manifolds. In §2 we show that a k-dimensional surface of class C^2 in $\mathbb{R}^n$ has additional intrinsic smoothness, and that this implies the geodesic flow is locally uniquely defined on such a surface. This comes about because such a surface has bounded curvature. Hence, in harmonic coordinates, the metric tensor is seen to be almost (but not quite) of class $C^{1,1}$. In §3 we produce a metric tensor on a region $\Omega \subset \mathbb{R}^2$ such that the curvature is bounded but the metric tensor is not of class $C^{1,1}$ in any harmonic coordinate system. In the standard coordinates on $\mathbb{R}^2$ we specify the metric tensor, mod $C^{3-\varepsilon}$, in §3, and in §4 we study it further, and specify it mod $C^{5-\varepsilon}$. Section 5 is devoted to some idle speculations.

2. Intrinsic smoothness of C^2 surfaces

Let M be a C^2 surface of dimension k in $\mathbb{R}^n$. That is (after a re-ordering of the standard coordinates on $\mathbb{R}^n$) M is locally the graph of a C^2 function $\varphi : U \to \mathbb{R}^{n-k}$, with U open in $\mathbb{R}^k$. Note that the Riemannian metric tensor induced on U via φ is only C^1. Our first goal in this section is to establish the following.

Proposition 2.1. *The manifold M is covered by local harmonic coordinate systems, in each of which the metric tensor satisfies*

$$\nabla^2 g_{ij} \in bmo. \tag{2.1}$$

To begin the proof, we start with the coordinate system described above, in which the metric tensor is C^1. This is more than enough regularity for classical results yielding local harmonic coordinate systems. In fact (cf. Proposition 9.5 in Chapter 3 of [T2]) we can say that these harmonic coordinate functions (as functions on $U \subset \mathbb{R}^k$) have 2 derivatives in L^p for each $p < \infty$, and, in any harmonic coordinate system, the metric tensor satisfies

$$\nabla g_{ij} \in L^p, \quad \forall\, p < \infty. \tag{2.2}$$

So far it seems we have lost ground, but now we bring in considerations of the Riemann curvature tensor of M. Since $M \subset \mathbb{R}^n$ is of class C^2, its Gauss map is of class C^1. Thus its Weingarten map is a well defined continuous section of $\mathrm{Hom}(\nu(M) \otimes TM, TM)$. Thus, by the Gauss Theorema Egregium, the Riemann curvature tensor of M is continuous. In particular its components are continuous in any harmonic coordinate system, and so are the components of the Ricci tensor.

Now, in harmonic coordinates, the metric tensor and the Ricci tensor have a relation of the form

$$-\frac{1}{2}\sum_{i,j} \partial_i g^{ij}(x)\partial_j g_{\ell m} + Q_{\ell m}(g, \nabla g) = \mathrm{Ric}_{\ell m}, \tag{2.3}$$

which gives an elliptic system for $(g_{\ell m})$. Results going back to [DeTK] yield regularity properties for $g_{\ell m}$. In the current setting, the result (2.2) allows us to apply Propositions 10.1–10.2 in Chapter 3 of [T2], and we have the conclusion (2.1).

Remark 2.2. As mentioned, any local harmonic coordinate system on our C^2 surface M is smooth of class $H^{2,p}$ ($\forall\, p < \infty$) with respect to the coordinate system $\varphi : U \to M$. It follows from the chain rule that any two such local harmonic coordinate systems are regular of class $H^{2,p}$, $\forall\, p < \infty$, with respect to each other. Now we can say more about how two local harmonic coordinate systems are related when the metric tensor satisfies (2.1). Namely each harmonic coordinate system is regular of class $H^{3,p}$ ($\forall\, p < \infty$) with respect to any other one.

In fact it is worthwhile to keep in mind the following more general result. Suppose $(u_1, \dots, u_k)$ is a local harmonic coordinate system on $\mathcal{O} \subset M$ and suppose $(x_1, \dots, x_k)$ is another local coordinate system (harmonic or not) on which the metric tensor h_{ij} satisfies $\nabla^2 h_{ij} \in bmo$. Since the functions u_ν satisfy

$$\partial_i(h^{1/2}h^{ij}\partial_j u_\nu) = 0, \tag{2.4}$$

we have the map $(x_1, \dots, x_k) \mapsto (u_1, \dots, u_k)$ smooth of class $H^{3,p}$, $\forall\, p < \infty$.

If $k = 2$, there is a special class of harmonic coordinate systems, namely isothermal coordinate systems, and Proposition 2.1 applies to these. Of course any two isothermal coordinate systems on $\mathcal{O} \subset M$ are conformally related to each other, and so they put on M the structure of a real-analytic manifold, with metric tensor satisfying (2.1), if we start with a C^2 two-dimensional surface $M \subset \mathbb{R}^n$.

One advantage of the result (2.1) over the weaker result that $\nabla^2 g_{ij} \in L^p$, $\forall\, p < \infty$ is the following, of obvious geometrical significance.

Proposition 2.3. *In the setting of Proposition 2.1, M has a uniquely defined local geodesic flow.*

Proof. We use a local harmonic coordinate system $(u_1, \dots, u_k)$. The geodesic equations can be written in Hamiltonian form:

$$\dot{u}_i = g^{ij}(u)\,\xi_j, \quad \dot{\xi}_i = -\frac{1}{2}\frac{\partial g^{j\ell}}{\partial u_i}\,\xi_j\xi_\ell. \tag{2.5}$$

Hence we are considering the flow generated by a vector field whose components have one derivative in bmo, if (2.1) holds. Such components might not quite be Lipschitz, but they do have a log-Lipschitz modulus of continuity, which is enough for Osgood's theorem (cf. Chapter 1, Appendix A of [T1]) to apply, giving a uniquely defined local flow.

Remark 2.4. We can easily extend the scope of this discussion from C^2 surfaces to surfaces of type $C^{1,1}$ in $\mathbb{R}^n$. In this more general setting, the Gauss map is Lipschitz and the Riemann curvature tensor is bounded, and Propositions 2.1–2.3 continue to hold.

We can extend the scope of Proposition 2.1 as follows. Assume that $M^k \subset \mathbb{R}^n$ is locally represented as the graph of a map $\varphi : U \to \mathbb{R}^{n-k}$ satisfying

$$\varphi \in C^{1,r}(U) \cap H^{2,p}(U), \quad 2 < p < \infty, \ 0 < r < 1. \tag{2.6}$$

Then the metric tensor on M pulled back to U via φ satisfies

$$g_{ij} \in C^r \cap H^{1,p}. \tag{2.7}$$

Generally, whenever (2.7) holds, the connection coefficients satisfy $\Gamma^j_{k\ell} \in L^p$, and the curvature satisfies $R_{jk\ell m} \in H^{-1,p} + L^{p/2}$. We can also say there are local harmonic coordinates in which the metric tensor satisfies (2.7). Moreover, under our hypotheses on $M \subset \mathbb{R}^n$, the Gauss map is of class $C^r \cap H^{1,p}$ and the Weingarten map is of class L^p, so $R_{jk\ell m} \in L^{p/2}$, and also $\mathrm{Ric}_{ij} \in L^{p/2}$. Hence, in harmonic coordinates, we have

$$g_{ij} \in H^{2,p/2}. \tag{2.8}$$

When $p > k$, (2.8) is strictly stronger than (2.7).

3. Surfaces with bounded curvature: some examples

Here we will construct metric tensors of the form

$$g_{jk} = e^{2u}\, \delta_{jk} \tag{3.1}$$

on an open set $\Omega \subset \mathbb{R}^2$, such that the Gauss curvature is bounded, but the metric tensor g_{jk} will not be of class $C^{1,1}$ in the standard coordinate system on $\mathbb{R}^2$. In fact the metric tensor will not be of class $C^{1,1}$ in any harmonic coordinate system covering Ω.

We recall that the Gauss curvature K of Ω, with metric (3.1), is given by

$$\Delta u = -K\, e^{2u}, \tag{3.2}$$

where Δ is the flat Laplacian, i.e., $\Delta u = \partial_1^2 u + \partial_2^2 u$. Let us take $\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$. To construct g_{jk} in (3.1), we will pick $K \in L^\infty(\Omega)$ and solve (3.2). We pick $K(x)$ as follows. Pick

$$H \in C^\infty(\mathbb{R}^2 \setminus 0), \quad \text{homogeneous of degree } 0, \quad \|H\|_{L^\infty} \le 1, \tag{3.3}$$

such that

$$\Delta^{-1} H \notin C^{1,1}. \tag{3.4}$$

We will say more about the construction of such H below. Then set

$$K(x) = -1 - \tfrac{1}{2} H(x). \tag{3.5}$$

Then $K < 0$ on Ω, and as is well known, if we pick $\varphi \in C^\infty(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ to

$$\Delta u = -K\, e^{2u}, \quad u\big|_{\partial\Omega} = \varphi. \tag{3.6}$$

Cf., e.g., Proposition 1.7 in Chapter 14 of [T1]. Other regularity results on u are readily established:

$$u \in C^\infty(\overline{\Omega} \setminus 0), \quad u \in H^{2,p}(\Omega), \ \forall\, p < \infty. \tag{3.7}$$

To establish the latter result, note that $u \in H^1(\Omega) \Rightarrow e^{2u} \in L^p(\Omega), \ \forall\, p < \infty$. At this point we have $u \in C^{2-\varepsilon}(\overline{\Omega})$ for all $\varepsilon > 0$; hence the right side of the PDE in (3.6) is bounded, so we have

$$\partial_j \partial_k u \in bmo, \tag{3.8}$$

reproducing the conclusion of Proposition 2.1, in this more restricted setting.

To look more closely at u, we compare it with a solution v to

$$\Delta v = K, \quad v\big|_{\partial\Omega} = \psi, \tag{3.9}$$

where we take $\psi \in C^\infty(\partial\Omega)$. As with (3.4), we will have $v \notin C^{1,1}$. Now set $A = e^{2u(0)}$ and write (3.6) as

$$\Delta u = -AK + K(A - e^{2u}). \tag{3.10}$$

Thus

$$u = -Av + w, \tag{3.11}$$

where w solves

$$\Delta w = K(A - e^{2u}), \quad w\big|_{\partial\Omega} = \varphi + A\psi. \tag{3.12}$$

Now we know that $A - e^{2u}$ is Lipschitz and vanishes at $x = 0$, so $K(A - e^{2u})$ is Lipschitz on $\overline{\Omega}$. Hence

$$w \in H^{3,p}(\Omega), \quad \forall\, p < \infty. \tag{3.13}$$

In particular $w \in C^{3-\varepsilon}(\overline{\Omega})$ for all $\varepsilon > 0$. We are now in a position to establish:

Proposition 3.1. *The metric tensor on $\Omega \subset \mathbb{R}^2$ constructed above has bounded Gauss curvature but is not of class $C^{1,1}$ in any harmonic coordinate system covering Ω.*

Proof. Granted that $v \notin C^{1,1}$, the result (3.13) implies that u is not of class $C^{1,1}$ with respect to the standard coordinate functions (x_1, x_2) on $\mathbb{R}^2$ (which are harmonic for metric tensors of the form (3.1)). Suppose now that (y_1, y_2) is another harmonic coordinate system covering $\{x_1 = x_2 = 0\}$. Since y_ν are harmonic for the metric tensor $g_{jk} = e^{2u}\delta_{jk}$, they are also harmonic for the flat metric δ_{jk}, so the map $(x_1, x_2) \mapsto (y_1, y_2)$ is a real-analytic diffeomorphism. If the metric tensor were $C^{1,1}$ in the (y_1, y_2)-coordinate system, it would have to be $C^{1,1}$ in the (x_1, x_2)-coordinate system, so Proposition 3.1 is proven.

Let us now describe a construction of a function H, satisfying (3.3)–(3.4). In fact, setting

$$v_0(x) = \frac{1}{4} x_1 x_2 \log(x_1^2 + x_2^2), \tag{3.14}$$

we see that

$$\Delta v_0(x) = 2 \frac{x_1 x_2}{x_1^2 + x_2^2}, \tag{3.15}$$

while

$$\partial_1 \partial_2 v_0(x) = \frac{1}{4} \log(x^2 + y^2) + G_0(x), \tag{3.16}$$

with G_0 smooth and homogeneous of degree zero on $\mathbb{R}^2 \setminus 0$. Thus we can take $H(x)$ in (3.3) to be given by the right side of (3.15), i.e.,

$$K(x) = -1 - \frac{x_1 x_2}{x_1^2 + x_2^2}. \tag{3.17}$$

Then we can take $v(x)$ in (3.9) to be given as

$$\begin{aligned} v(x) &= -\frac{1}{2} v_0(x) - \frac{1}{2}(x_1^2 + x_2^2) \\ &= -\frac{1}{8} x_1 x_2 \log(x_1^2 + x_2^2) - \frac{1}{2}(x_1^2 + x_2^2), \end{aligned} \tag{3.18}$$

and we have the representation (3.11) for u in (3.1), with the remainder term satisfying (3.13).

4. Finer study of the metrics of §3

We want to look more closely at the behavior of the singularities of u and e^{2u}, with u solving (3.2), with K given by (3.17). So far we have

$$u(x)=-\frac{1}{8}x_1x_2\,\log(x_1^2+x_2^2),\quad \text{mod}\ \ C^{3-\varepsilon}, \tag{4.1}$$

for all $\varepsilon>0$. In other words, we can write

$$u(x)=u(0)+\beta\cdot x-\frac{1}{8}x_1x_2\,\log(x_1^2+x_2^2)+x\cdot Qx+R(x), \tag{4.2}$$

with $\beta\in\mathbb{R}^2$, Q a symmetric 2×2 matrix, and $R\in C^{3-\varepsilon}$ satisfying

$$|R(x)|\le C|x|^{3-\varepsilon},\quad |\nabla R(x)|\le C|x|^{2-\varepsilon},\quad |\nabla^2R(x)|\le C|x|^{1-\varepsilon}. \tag{4.3}$$

A calculation gives

$$e^{2u}=A\Big(1+2\beta\cdot x-\frac{1}{4}x_2x_2\,\log(x_1^2+x_2^2)+x\cdot\widetilde{Q}x\Big)+R_1(x), \tag{4.4}$$

where $A=e^{2u(0)}$, $x\cdot\widetilde{Q}x=2(\beta\cdot x)^2+2x\cdot Qx$, and $R_1\in C^{3-\varepsilon}$ satisfies estimates like (4.3). Thus the equation (3.2) has the form

$$\Delta u=A\Big(1+\frac{x_1x_2}{x_1^2+x_2^2}\Big)\Big(1+2\beta\cdot x-\frac{1}{4}x_2x_2\log(x_1^2+x_2^2)+x\cdot\widetilde{Q}x\Big)+R_2(x), \tag{4.5}$$

with similarly behaved $R_2(x)$. This yields a formula for u, mod $C^{5-\varepsilon}$.

To analyze solutions to (4.5), we look more generally at equations of the form

$$\Delta u_0=h(x),\quad \Delta u_1=h(x)\,\log|x|, \tag{4.6}$$

with $h\in C^\infty(\mathbb{R}^n\setminus 0)$ homogeneous, say of degree $k-2$. Writing the Laplace operator on $\mathbb{R}^n$ in spherical polar coordinates:

$$\Delta u=\frac{\partial^2u}{\partial r^2}+\frac{n-1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Delta_Su, \tag{4.7}$$

where Δ_S is the Laplace operator on S^{n-1}, we obtain, for $\varphi\in C^\infty(S^{n-1})$,

$$\begin{aligned}\Delta(r^k\varphi(\omega))&=r^{k-2}\big[\Delta_S\varphi(\omega)+k(k+n-2)\varphi(\omega)\big],\\ \Delta(r^k\,\log r\,\varphi(\omega))&=r^{k-2}\log r\big[\Delta_S\varphi(\omega)+k(k+n-2)\varphi(\omega)\big]\\ &\quad+(2k+n-2)r^{k-2}\varphi(\omega).\end{aligned} \tag{4.8}$$

Recall that

$$\text{Spec}\,(-\Delta_S) = \{\nu_k = k(k+n-2) : k = 0,1,2,\dots\}. \tag{4.9}$$

Let P_k denote the orthogonal projection of $L^2(S^{n-1})$ onto the ν_k-eigenspace of $-\Delta_S$. Assume $k-2 \in \mathbb{Z}^+$, and say $h(x) = r^{k-2}\psi(\omega)$. Then a solution u_0 to the first equation in (4.6) is given by

$$\begin{aligned} u_0(x) = r^k &\big(\Delta_S + k(k+n-2)\big)^{-1}(I-P_k)\psi(\omega) \\ &+ (2k+n-2)^{-1} r^k \log r\, P_k\psi(\omega), \end{aligned} \tag{4.10}$$

and a solution u_1 to the second equation in (4.6) is given by

$$\begin{aligned} u_1(x) = r^k &\log r \big(\Delta_S + k(k+n-2)\big)^{-1}\psi(\omega) \\ &- (2k+n-2) r^k \big(\Delta_S + k(k+n-2)\big)^{-1}\psi(\omega), \end{aligned} \tag{4.11}$$

in this case provided $P_k\psi = 0$. However, this condition fails for

$$\Delta u_1 = -\frac{1}{4}\frac{x_1^2x_2^2}{x_1^2+x_2^2}\,\log(x_1^2+x_2^2), \tag{4.12}$$

where $k=4$ and $\psi(\omega) = \cos^2\omega\,\sin^2\omega = (1-\cos 4\omega)/4,\ \omega \in S^1 \approx \mathbb{R}/(2\pi\mathbb{Z})$. Thus we must supplement (4.8) with

$$\begin{aligned} \Delta(r^k(\log r)^2\varphi(\omega)) = r^{k-2}&(\log r)^2\big[\Delta_S\varphi(\omega) + k(k+n-2)\varphi(\omega)\big] \\ &+ 2(2k+n-2)r^{k-2}\,\log r\,\varphi(\omega) \\ &+ 2r^{k-2}\varphi(\omega). \end{aligned} \tag{4.13}$$

Thus if $\psi = \psi_k = P_k\psi_k$, a solution to the equation for u_1 in (4.6) is given by

$$\begin{aligned} u_1(x) = (4k+2n-4)^{-1}&r^k(\log r)^2\psi_k(\omega) \\ &+ (2k+n-2)^{-2}r^k(\log r)\psi_k(\omega). \end{aligned} \tag{4.14}$$

Returning to the equation (4.8), we see that, mod $C^{5-\varepsilon}$,

$$\begin{aligned} u(x) = -&\frac{1}{8}x_1x_2\log(x_1^2+x_2^2) + h_3(x) + p_3(x)\,\log(x_1^2+x_2^2) \\ &+ h_4(x) + h_4^{\#}(x)\log(x_1^2+x_2^2) + p_4(x)(\log(x_1^2+x_2^2))^2, \end{aligned} \tag{4.15}$$

where $h_\ell(x)$ and $h_\ell^{\#}(x)$ are in $C^\infty(\mathbb{R}^2\setminus 0)$ and homogeneous of degree ℓ and $p_\ell(x)$ is a polynomial, homogeneous of degree ℓ.

5. Speculations

In §3 we constructed a metric tensor of the form $e^{2u}\delta_{jk}$ on $\Omega \subset \mathbb{R}^2$ whose curvature is bounded but which is not of class $C^{1,1}$ in any harmonic coordinate system. There remains a question: could there be a non-harmonic coordinate syatem (say (y_1, y_2)) in which the metric tensor is of class $C^{1,1}$?

By results of §2, any coordinate system in which the metric tensor satisfies (2.1) must be $H^{3,p}$-equivalent to the standard coordinate system (x_1, x_2), for all $p < \infty$. Thus the induced coordinates on $T^*\Omega$ are $H^{2,p}$-equivalent, for all $p < \infty$. Now if the metric tensor were of class $C^{1,1}$ in the (y_1, y_2) coordinates, the geodesic flow must be a flow of Lipschitz maps on $T^*\Omega$, both in the (y_1, y_2) coordinates and in the (x_1, x_2) coordinates.

So we should test whether the geodesic flow arising from the metric produced in §3 is Lipschitz. In this case the geodesic equations (2.5) take the form

$$\dot{x}_j = e^{-2u}\xi_j, \quad \dot{\xi}_j = e^{-2u}\frac{\partial u}{\partial x_j}(\xi_1^2 + \xi_2^2). \tag{5.1}$$

If we set

$$z = x_1 + ix_2, \quad \zeta = \xi_1 + i\xi_2, \tag{5.2}$$

we can write this system as

$$\dot{z} = e^{-2u}\zeta, \quad \dot{\zeta} = 2e^{-2u}|\zeta|^2\frac{\partial u}{\partial \overline{z}}. \tag{5.3}$$

Here $\partial u/\partial \overline{z} = (1/2)(\partial u/\partial x_1 + i\partial u/\partial x_2)$. Note that (5.3) yields

$$\frac{d\zeta}{dz} = 2\overline{\zeta}\frac{\partial u}{\partial \overline{z}}, \tag{5.4}$$

which can be rewritten as

$$\frac{d\zeta}{\overline{\zeta}} = 2\frac{\partial u}{\partial \overline{z}}\,dz. \tag{5.5}$$

This looks like variables are "separated," but its significance is not clear.

NOTE. Actually the geodesic flow will be Lipschitz. See [Sm].

References

[DeTK] D. DeTurk and J. Kazdan, Some regularity theorems in Riemannian geometry, Ann. Sci. Ecole Norm. Sup. 14 (1981), 249–260.

[Sm] H. Smith, Strichartz and nullform estimates for metrics of bounded curvature, Preprint, 1999.

[T1] M. Taylor, Partial Differential Equations, Vols. 1–3, Springer-Verlag, New York, 1996.

[T2] M. Taylor, Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials, AMS, Providence, R.I., 2000.