How Smooth is a C^2 Surface?

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1. Introduction

Let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold, smooth of class C^2, i.e., locally the graph of C^2 maps $V \to V^\perp$, $V \subset \mathbb{R}^n$ a linear subspace of dimension k. We ask the question

$$\text{how smooth is } M? \tag{1.1}$$

This question might appear to be trivial, but actually it's not. As we proceed, we intend to show that M is smoother than you might think.

To start, we call local coordinates on M induced by such maps $V \to V^\perp$ graph coordinates. Each such graph coordinate yields a C^2 coordinate chart

$$\varphi : U \longrightarrow M \subset \mathbb{R}^n, \tag{1.2}$$

with $U \subset \mathbb{R}^k$ open, $D\varphi(x) : \mathbb{R}^k \to \mathbb{R}^n$ injective. The metric tensor on M is given in such local coordinates by

$$(g_{jk}(x)) = G(x) = D\varphi(x)^t D\varphi(x) \in M(k, \mathbb{R}). \tag{1.3}$$

In particular, the graph coordinates endow M with a C^1 metric tensor. Thus one answer to (1.1) is that M gets a C^1 metric tensor. We'll see that, in a certain more optimal coordinate system, the metric tensor is substantially smoother. One key to perceiving this is to show that the curvature tensor of M is continuous.

Now the curvature tensor of a Riemannian manifold involves second-order derivatives of the metric tensor. Let's see how the curvature tensor of a manifold with a C^1 metric tensor is defined. For starters (cf. (6.2.57) of [T8]),

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.4}$$

In local coordinates, featuring the vector fields $D_j = \partial/\partial x_j$, one has, using the summation convention (cf. (6.1.69)–(6.1.70) of [T8]),

$$\nabla_{D_k} D_b = \Gamma^a{}_{bk} D_a, \tag{1.5}$$

where $\Gamma^a{}_{bk}$ are the Christoffel symbols, given by

$$g_{k\ell}\Gamma^\ell{}_{ij} = \frac{1}{2}\bigl(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\bigr). \tag{1.6}$$

With $R(D_j, D_k)D_b = R^a{}_{bjk} D_a$, we obtain (cf. (6.2.65) of [T8])

$$R^a{}_{bjk} = \partial_j \Gamma^a{}_{bk} - \partial_k \Gamma^a{}_{bj} + \Gamma^a{}_{cj}\Gamma^c{}_{bk} - \Gamma^a{}_{ck}\Gamma^c{}_{bj}. \tag{1.7}$$

Shorthand formulas arise as follows. We define $k \times k$ matrices

$$\Gamma_j = (\Gamma^a{}_{bj}), \quad \mathfrak{R}_{jk} = (R^a{}_{bjk}), \tag{1.8}$$

obtaining

$$\mathfrak{R}_{jk} = \partial_j \Gamma_k - \partial_k \Gamma_j + [\Gamma_j, \Gamma_k], \tag{1.9}$$

where $[\Gamma_j, \Gamma_k]$ is the matrix commutator. Going further, we can define a connection 1-form Γ and a curvature 2-form Ω by

$$\Gamma = \sum_j \Gamma_j \, dx_j, \quad \Omega = \frac{1}{2} \sum_{j,k} \mathfrak{R}_{jk} \, dx_j \wedge dx_k, \tag{1.10}$$

and the formula (1.9) is equivalent to

$$\Omega = d\Gamma + \Gamma \wedge \Gamma. \tag{1.11}$$

Now suppose (g_{jk}) is C^1. We have from (1.6) that Γ is continuous, so, by (1.11), Ω is a distribution of the form

$$\Omega \in C^{-1}. \tag{1.12}$$

The result (1.12) holds whenever the metric tensor is C^1. In case $M \subset \mathbb{R}^n$ is a C^2 surface, we can show that the curvature is continuous by bringing in the generalized theorema egregium, relating the curvature to the second fundamental form. This is defined through the generalized Gauss map

$$\begin{aligned} &P : M \longrightarrow M(n, \mathbb{R}), \\ &P(x) = \text{orthogonal projection of } \mathbb{R}^n \text{ onto } T_x M, \end{aligned} \tag{1.13}$$

via

$$(D_X P)Y = \mathrm{II}(X, Y). \tag{1.14}$$

See §6.2 of [T8]. We have

$$\mathrm{II} : M \longrightarrow \mathrm{Hom}(TM \otimes TM, \nu M), \tag{1.15}$$

where νM denotes the normal bundle to M. The generalized theorema egregium says

$$\langle R(X,Y)Z, W\rangle = \langle \mathrm{II}(Y,Z), \mathrm{II}(X,W)\rangle - \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle. \tag{1.16}$$

See Proposition 6.2.12 of [T8]. Consequently, we have

Proposition 1.1. *If $M \subset \mathbb{R}^n$ is a C^2 surface, then P is smooth of class C^1, hence* II *is continuous, so (1.16) implies the Riemann tensor R is continuous.*

Corollary 1.2. *In the setting of Proposition 1.1, the Ricci tensor* Ric *of M is continuous.*

In local coordinates (see [T1], Appendix C, (3.23))

$$\mathrm{Ric}_{jk} = R^a{}_{jak}. \tag{1.17}$$

Using (1.6)–(1.7), one can derive the formula

$$\begin{aligned}\mathrm{Ric}_{jk} &= \tfrac{1}{2} g^{\ell m}[-\partial_\ell \partial_m g_{jk} - \partial_j \partial_k g_{\ell m} + \partial_k \partial_m g_{\ell j} + \partial_\ell \partial_j g_{km}] + M_{jk}(g, \nabla g) \\ &= -\tfrac{1}{2} g^{\ell m} \partial_\ell \partial_m g_{jk} + \tfrac{1}{2} g_{j\ell} \partial_k \lambda^\ell + \tfrac{1}{2} g_{k\ell} \partial_j \lambda^\ell + H_{jk}(g, \nabla g),\end{aligned} \tag{1.18}$$

where $M_{jk}(g, \nabla g)$ and $H_{jk}(g, \nabla g)$ are quadratic forms in ∇g, and

$$\lambda^\ell = g^{jk} \Gamma^\ell{}_{jk}. \tag{1.19}$$

Now the Laplace-Beltrami operator Δ on M (acting on real valued functions) is given by

$$\Delta u = g^{jk} u_{;j;k} = g^{jk} \partial_j \partial_k u - \lambda^\ell \partial_\ell u. \tag{1.20}$$

In particular, if $(x_1, \dots, x_k)$ are the local coordinate functions on a local coordinate patch,

$$\Delta x_\ell = -\lambda^\ell. \tag{1.21}$$

Definition. A local coordinate system $(x_1, \dots, x_k)$ on M is a system of harmonic coordinates provided

$$\Delta x_\ell = 0, \quad 1 \le \ell \le k.$$

Thus we see that, if we use local harmonic coordinates, the formula for the Ricci tensor vastly simplifies. See [T1], Chapter 14, (4.94).

Proposition 1.3. *In local harmonic coordinates,*

$$\begin{aligned}\mathrm{Ric}_{jk} &= -\tfrac{1}{2} g^{\ell m} \partial_\ell \partial_m g_{jk} + H_{jk}(g, \nabla g) \\ &= -\tfrac{1}{2} \partial_\ell g^{\ell m} \partial_m g_{jk} + Q_{jk}(g, \nabla g),\end{aligned} \tag{1.22}$$

where $Q_{jk}(g, \nabla g)$ is also a quadratic form in ∇g.

The key to further investigations is to turn this around, and write

$$\partial_\ell g^{\ell m} \partial_m g_{jk} = -2\, \mathrm{Ric}_{jk} + 2 Q_{jk}(g, \nabla g). \tag{1.23}$$

This is a quasilinear elliptic PDE for the metric tensor. Elliptic theory will reveal further important properties, given properties of the Ricci tensor (such as boundedness).

The structure of the rest of this note is as follows. In §2 we record the classical result that if the metric tensor of M is class C^r with $r > 0$, there exist harmonic coordinates. We proceed to analyze the equation (1.23) for the metric tensor. We bring in the notion of symbol smoothing to treat this quasilinear elliptic equation, and show that, if a priori the metric tensor is C^1 and the Ricci tensor is bounded, then, in local harmonic coordinates,

$$g_{jk} = -2E^{\#} \operatorname{Ric}_{jk}, \quad \text{mod} \quad H^{s,p}, \ \forall\, s < 3, \ p < \infty, \tag{1.24}$$

where $E^{\#}$ is a pseudodifferential operator with symbol in the Hörmander class $S^{-2}_{1,\delta}$, $\delta < 1$. The condition that Ric_{jk} be continuous is an endpoint case for pseudodifferential operator estimates. It turns out that (1.24) implies, not $g_{jk} \in C^2$, but

$$\nabla^2 g_{jk} \in \text{bmo}, \tag{1.25}$$

a result we discuss in §3. One significant consequence, treated in §3, is that if M is a Riemannian manifold with bounded Ricci tensor, then, in harmonic coordinates, ∇g_{jk} has a log-Lipschitz modulus of continuity. We discuss in §4 how this leads to a well defined geodesic flow on T^*M.

In §§5–6 we look at the wave equation

$$\partial_t^2 u + a\partial_t u - \Delta u = 0, \tag{1.26}$$

on $\mathbb{R} \times M$, and present results of [T7] on propagation of singularities and energy decay.

In §5 we describe propagation of singularities results in case M is a compact Riemannian manifold with bounded Ricci tensor and $a \in C^{r-1}$ for some $r > 1$. In §6 we assume in addition that $a \geq 0$ on M, and

$$a(x) \geq a_0 > 0 \quad \text{on} \quad U, \tag{1.27}$$

where $U \subset M$ is an open set. We discuss how, if U satisfies the control condition

$$\begin{aligned} &\text{there exists } T_0 < \infty \text{ such that each geodesic} \\ &\text{on } M \text{ of length } T_0 \text{ intersects } U, \end{aligned} \tag{1.28}$$

then solutions to (1.26) with initial data $u(0) \in H^1(M)$, $\partial_t u(0) \in L^2(M)$ have a uniform rate of exponential energy decay.

2. Harmonic coordinates and other matters involving elliptic PDE

Here is a classical result on the existence of harmonic coordinates.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^k$ be open, carrying a metric tensor (g_{jk}), smooth of class C^r, $0 < r < 1$. Take $p \in \Omega$. Then there is a neighborhood $\mathcal{O}$ of p and there are harmonic functions u_ℓ on $\mathcal{O}$, satisfying*

$$\partial_j a^{jk} \partial_k u_\ell = 0, \quad 1 \le \ell \le n, \quad u_\ell \in C^{r+1}, \tag{2.1}$$

with

$$a^{jk} = g^{1/2} g^{jk}, \quad g = \det(g_{jk}), \tag{2.1A}$$

and forming a coordinate system on $\mathcal{O}$. Furthermore, the metric tensor in the new coordinate system is also of class C^r.

For a proof, see Proposition 9.1 in Chapter 3 of [T2]. We complement this with the following.

Proposition 2.2. *In the setting of Proposition 2.1, assume $g_{jk} \in C^1$. Then, on $\mathcal{O}$,*

$$u_\ell \in H^{2,p}, \quad \forall\, p < \infty. \tag{2.2}$$

This is a special case of Proposition 9.4 in Chapter 3 of [T2]. One consequence, observed in Proposition 9.5, is that, in the new coordinate system,

$$g_{jk} \in H^{1,p}, \quad \forall\, p < \infty. \tag{2.3}$$

The regularity conclusions follow from certain elliptic regularity results. While we skip these arguments here, we will delve into the proofs of other elliptic regularity results shortly.

At this point we are ready to investigate regularity for (g_{jk}) when it satisfies (1.23), i.e.,

$$\partial_\ell g^{\ell m} \partial_m g_{jk} = -2 \operatorname{Ric}_{jk} + 2 Q_{jk}(g, \nabla g), \tag{2.4}$$

where $Q_{jk}(g, \nabla g)$ is a quadratic form in ∇g, and we have the information that (2.3) holds. In particular, $g_{jk} \in C^r$ for all $r < 1$. Also

$$f_{jk} = 2 Q_{jk}(g, \nabla g) \in L^p, \quad \forall\, p < \infty, \tag{2.5}$$

hence, if $\mathrm{Ric}_{jk} \in L^\infty$,

$$\partial_\ell g^{\ell m} \partial_m g_{jk} = \tilde{f}_{jk} \in L^p, \quad \forall\, p < \infty. \tag{2.6}$$

To proceed, we write

$$A_\ell(x, D) = g^{\ell m} \partial_m \in OPC^r S^1_{1,0}, \tag{2.7}$$

and bring in symbol smoothing, to write

$$A_\ell(x, \xi) = A_\ell^\#(x, \xi) + A_\ell^b(x, \xi), \tag{2.8}$$

where we pick $\delta \in (0, 1)$ and set

$$A_\ell^\#(x, \xi) = \sum_{i=0}^\infty J_{\varepsilon_i} A_\ell(x, \xi)\, \psi_i(\xi), \quad \varepsilon_i = 2^{-i\delta}. \tag{2.9}$$

Here $J_\varepsilon = \varphi(\varepsilon D_x)$ is a Friedrichs mollifier (in the x variables) and $\{\psi_i\}$ is a Littlewood-Paley partition of unity, satisfying $\operatorname{supp} \psi_i \subset \{\xi \in \mathbb{R}^k : \langle \xi \rangle \sim 2^i\}$. This process of symbol smoothing is treated in Chapter 13, §9 of [T1]. As shown there, we have

$$A_\ell^\#(x, \xi) \in S^1_{1,\delta}, \quad A_\ell^b(x, \xi) \in C^r S^{1-r\delta}_{1,\delta}, \tag{2.10}$$

where

$$p(x, \xi) \in S^m_{1,\delta} \Leftrightarrow |D_\xi^\beta D_\xi^\alpha p(x, \xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m+\delta|\beta|}, \tag{2.11}$$

and the more general symbol class $C^r S^m_{1,\delta}$ has a somewhat parallel definition, given in (9.1)–(9.2), Chapter 13, of [T1]. We have

$$p(x, \xi) \in S^m_{1,\delta} \Rightarrow p(x, D) : H^{s+m,p} \to H^{s,p}, \quad \forall\, s \in \mathbb{R},\ p \in (1, \infty), \tag{2.11A}$$

and, as shown in Proposition 9.10, Chapter 13 of [T1],

$$\begin{gathered} p(x, \xi) \in C^r S^m_{1,\delta} \Rightarrow p(x, D) : H^{s+m,p} \to H^{s,p}, \text{ for} \\ p \in (1, \infty), \quad -(1 - \delta) r < s < r. \end{gathered} \tag{2.11B}$$

Consequently,

$$A_\ell^\#(x, D) : H^{s+1,p} \longrightarrow H^{s,p}, \quad \forall\, s \in \mathbb{R},\ p \in (1, \infty), \tag{2.12}$$

and

$$\begin{gathered} A_\ell^b(x, D) : H^{s+1-r\delta,p} \longrightarrow H^{s,p}, \text{ for} \\ p \in (1, \infty), \quad -(1 - \delta) r < s < r. \end{gathered} \tag{2.13}$$

Now we can write (2.4) as

$$L^{\#} g_{jk} = -\partial_\ell A^b_\ell g_{jk} - 2\,\mathrm{Ric}_{jk} + f_{jk}, \tag{2.14}$$

(as before, using the summation convention) where

$$\begin{aligned} L^{\#} = \partial_\ell A^{\#}_\ell &\in OPS^2_{1,\delta} \text{ is elliptic,} \\ \text{with parametrix } E^{\#} &\in OPS^{-2}_{1,\delta}. \end{aligned} \tag{2.15}$$

Hence, mod C^∞,

$$g_{jk} = -E^{\#}\partial_\ell A^b_\ell g_{jk} - E^{\#}(2\,\mathrm{Ric}_{jk} - f_{jk}). \tag{2.16}$$

Thus

$$\begin{aligned} g_{jk} \in H^{1,p} &\Rightarrow A^b_\ell g_{jk} \in H^{r\delta,p} \\ &\Rightarrow E^{\#}\partial_\ell A^b_\ell g_{jk} \in H^{1+r\delta,p}, \end{aligned} \tag{2.17}$$

by (2.13) with $s = r\delta$. Since, by (2.5),

$$E^{\#} f_{jk} \in H^{2,p}, \tag{2.18}$$

we see from (2.16) that

$$g_{jk} \in H^{1+r\delta,p} + H^{2,p}, \quad \forall\, p < \infty,\ r, \delta < 1. \tag{2.19}$$

In other words,

$$g_{jk} \in H^{r,p}, \quad \forall\, r < 2,\ p < \infty, \tag{2.20}$$

which is an improvement on (2.3). Consequently, $g_{jk} \in C^r$ for all $r < 2$, and

$$f_{jk} = 2Q_{jk}(g, \nabla g) \in C^{r-1}, \quad \forall\, r < 2. \tag{2.22}$$

We now make a second pass through the PDE (2.6). In the symbol decomposition (2.8), we now have

$$A^b_\ell(x,\xi) \in C^r S^{1-r\delta}_{1,\delta}, \quad \forall\, r < 2. \tag{2.23}$$

Thus g_{jk} satisfies (2.16) with

$$E^{\#}\partial_\ell A^b_\ell g_{jk} \in H^{s,p}, \quad \forall\, s < 3,\ p \in (1,\infty), \tag{2.24}$$

so

$$g_{jk} = -2E^{\#}\,\mathrm{Ric}_{jk}, \quad \mathrm{mod}\ \ H^{s,p}, \quad \forall\, s < 3,\ p < \infty. \tag{2.25}$$

Now the mapping properties given in (2.11A) do not extend to the endpoint case $p = \infty$. Rather, one has, for $\delta \in [0,1)$,

$$p(x,\xi) \in S^0_{1,\delta} \Longrightarrow p(x,D) : L^\infty \to \mathrm{bmo}, \tag{2.26}$$

where bmo denotes the John-Nirenberg space, introduced in [JN] and developed into a major tool in analysis in [FS] (more precisely, bmo denotes the local version, introduced in [G]). See the next section for further discussion of (2.26).

Given (2.25), it follows from (2.26) that

$$\mathrm{Ric}_{jk} \in L^\infty \Longrightarrow \nabla^2 g_{jk} \in \mathrm{bmo}\,. \tag{2.27}$$

We formally state the conclusion.

Proposition 2.3. *Let M be a k-dimensional C^2 manifold, with a C^1 metric tensor. Assume its Ricci tensor is bounded. Then M has a system of harmonic coordinates, and in such a coordinate system*

$$\nabla^2 g_{jk} \in \text{bmo}\,. \tag{2.28}$$

Corollary 2.4. *Let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold, smooth of class C^2. Then M has a system of harmonic coordinates, and in each such coordinate system (2.28) holds.*

Remark 1. As stated in Proposition 2.1, for a manifold with a C^1 metric tensor, one obtains local harmonic coordinates that have the regularity $u_\ell \in H^{2,p}$, for all $p < \infty$, with respect to the original coordinates. One does not expect to have $u_\ell \in C^2$ in general. However, as shown in Proposition 5.1 of [T4], if $M \subset \mathbb{R}^n$ is a k-dimensional submanifold that is of class C^2, then in fact

$$u_\ell \in C^2. \tag{2.29}$$

This follows from the regularity (2.28) of the metric tensor in harmonic coordinates, combined with Theorem 2.1 of [T4], relating regularity of metric tensors and isometries.

Remark 2. Once one has local harmonic coordinates on M in which the metric tensor has the regularity (2.28), then on coordinate overlaps the various harmonic functions satisfy a PDE of the form (2.1) with $a^{jk} \in H^{2,p}$, for all $p < \infty$, hence, as noted in (10.25) in Chapter 3 of [T2], it follows from Proposition 1.13 in Chapter 3 of [T2] that (with respect to the new coordinates)

$$u_\ell \in H^{3,p}, \quad \forall\, p < \infty, \tag{2.30}$$

so in these harmonic coordinates M has the regularity structure given by (2.30). Thus we see that, in the setting of Corollary 2.4, M gets an intrinsic structure a good bit smoother than class C^2!

Remark 3. When $\dim M = 2$, Proposition 2.1 yields isothermal coordinates, giving M the structure of a Riemann surface (C^∞ smooth), uniquely determined by its metric tensor. Of course, in these new coordinates, the metric tensor need not be any smoother than guaranteed by results given above. If the Ricci tensor (scalar curvature for $k = 2$) in the original coordinates is rough, you're stuck.

3. bmo-Sobolev spaces

For a function f on $\mathbb{R}^k$, we define the BMO-seminorm

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{V(Q)} \int_Q |f - f_Q|\, dx, \tag{3.1}$$

the sup taken over all k-dimensional cubes in $\mathbb{R}^k$, where $V(Q)$ is the volume of Q and f_Q denotes the mean value of f on Q. We then define the bmo-norm

$$\|f\|_{\mathrm{bmo}} = \|f\|_{\mathrm{BMO}} + \|\Psi_0(D)f\|_{L^\infty}, \tag{3.2}$$

with $\Psi_0 \in C_0^\infty(\mathbb{R}^k)$, $\Psi_0(0) \neq 0$. The space BMO was introduced in [JN], and developed further in [FS], particularly with the duality

$$\mathfrak{H}^1(\mathbb{R}^k)^* = \mathrm{BMO}(\mathbb{R}^k), \tag{3.3}$$

and its counterpart, proved in [G],

$$\mathfrak{h}^1(\mathbb{R}^k)^* = \mathrm{bmo}(\mathbb{R}^k). \tag{3.4}$$

For definitions of the Hardy spaces $\mathfrak{H}^1$ and $\mathfrak{h}^1$, see Chapter 1, §2 of [T2]. The local spaces $\mathfrak{h}^1$ and bmo are well suited for presentation on compact manifolds. See also [T6] for a study on a larger class of complete Riemannian manifolds.

One of the major results of [FS] involved the study of classical singular integral operators on Hardy spaces and BMO. Going further, it is shown in (2.73) and (2.87) in Chapter 1 of [T2] that

$$\begin{aligned} p(x,\xi) \in S^0_{1,\delta},\ \delta < 1 \Rightarrow p(x,D) &: \mathfrak{h}^1(\mathbb{R}^k) \to \mathfrak{h}^1(\mathbb{R}^k), \\ p(x,D) &: \mathrm{bmo}(\mathbb{R}^k) \to \mathrm{bmo}(\mathbb{R}^k), \end{aligned} \tag{3.5}$$

the latter result following by duality from the former. Note that the later result in (3.5) is a bit stronger than (2.26).

Now we can define bmo-Sobolev spaces on $\mathbb{R}^k$,

$$\mathrm{bmo}^s(\mathbb{R}^k) = (1 - \Delta_0)^{-s/2}\, \mathrm{bmo}(\mathbb{R}^k), \tag{3.6}$$

where $\Delta_0 = \partial_1^2 + \cdots + \partial_k^2$, and obtain from (3.5) that

$$p(x,\xi) \in S^m_{1,\delta},\ \delta < 1 \Rightarrow p(x,D) : \mathrm{bmo}^{s+m}(\mathbb{R}^k) \to \mathrm{bmo}^s(\mathbb{R}^k),\ \forall\, m, s \in \mathbb{R}. \tag{3.7}$$

We can restate Proposition 2.3 as follows.

Proposition 3.1. *Let M be a C^2 manifold with C^1 metric tensor. Assume its Ricci tensor is bounded. Then M has a system of harmonic coordinates, and in such a coordinate system*

$$g_{jk} \in \mathrm{bmo}^2 . \tag{3.8}$$

It is useful to relate bmo-Sobolev spaces to Zygmund spaces, defined as follows. For $s \in \mathbb{R}$, we define the C^s_*-norm

$$\|f\|_{C^s_*} = \sup_{k\ge 0} \langle k\rangle^s \|\psi_k(D) f\|_{L^\infty}, \tag{3.9}$$

where $\{\psi_k : k \ge 0\}$ is the Littlewood-Paley partition of unity introduced in (2.9). Parallel to (2.11A) and (3.7), we have

$$p(x,\xi) \in S^m_{1,\delta},\ \delta < 1 \Rightarrow p(x,D) : C^{s+m}_*(\mathbb{R}^k) \to C^s_*(\mathbb{R}^k), \quad \forall\, m, s \in \mathbb{R}. \tag{3.10}$$

See Corollary 9.2 in Chapter 13 of [T1]. In particular, parallel to (3.6),

$$C^s_*(\mathbb{R}^k) = (1-\Delta_0)^{-s/2} C^0_*(\mathbb{R}^k). \tag{3.11}$$

One can show from the definition (3.1)–(3.2) above that

$$\|\psi_k(D) f\|_{L^\infty} \le C \|f\|_{\mathrm{bmo}}, \tag{3.12}$$

hence

$$\mathrm{bmo}(\mathbb{R}^k) \subset C^0_*(\mathbb{R}^k), \tag{3.13}$$

and consequently, for each $s \in \mathbb{R}$,

$$\mathrm{bmo}^s(\mathbb{R}^k) \subset C^s_*(\mathbb{R}^k). \tag{3.14}$$

Elements of $C^1_*(\mathbb{R}^k)$ are not quite Lipschitz. Instead, they have a log-Lipschitz modulus of continuity:

$$|f(x+y) - f(x)| \le C|y| \, \log \frac{1}{|y|} \, \|f\|_{C^1_*}, \quad \text{for} \ \ x, y \in \mathbb{R}^k,\ |y| \le \frac{1}{2}. \tag{3.15}$$

For a proof of this, see (1.22) in Chapter 1 of [T2], or (2.63) in Chapter 17 of [T1].

4. Geodesic flows on manifolds with bounded Ricci tensor

The geodesic equation can be written in Hamiltonian form,

$$\frac{dx_\ell}{dt} = \frac{\partial E}{\partial \xi_\ell}, \quad \frac{d\xi_\ell}{dt} = -\frac{\partial E}{\partial x_\ell}, \quad E(x,\xi) = \frac{1}{2} g^{jk}(x)\xi_j\xi_k, \tag{4.1}$$

that is,

$$\frac{dx_\ell}{dt} = g^{j\ell}(x)\xi_j, \quad \frac{d\xi_\ell}{dt} = -\frac{\partial g^{jk}}{\partial x_\ell}\xi_j\xi_k. \tag{4.2}$$

If one merely has $g_{jk} \in C^1$, the vector field arising in (4.2) is merely continuous (in x). In such a case one does have local local existence of solutions to (4.2), satisfying given initial conditions (see Proposition A.1 in Chapter 1 of [T1]), but not necessarily uniqueness. On the other hand, if we know the Ricci tensor is bounded, then in local harmonic coordinates we have

$$g^{jk} \in \mathrm{bmo}^2, \quad \text{hence} \ \frac{\partial}{\partial x_\ell} g^{jk} \in \mathrm{bmo}^1 . \tag{4.3}$$

Thus, while the vector field arising in (4.2) might not quite be Lipschitz, it does have the log-Lipschitz modulus of continuity, given by (3.15). In such a case, we can use Osgood's theorem, which says that a system of ODE of the form

$$\frac{dy}{dt} = F(y), \quad u(0) = y_0, \tag{4.4}$$

has a unique solution, depending continuously on y_0, provided F has a modulus of continuity ω satisfying

$$\int_0^1 \frac{ds}{\omega(s)} = \infty. \tag{4.5}$$

See Proposition A.2 in Chapter 1 of [T1]. This holds for

$$\omega(s) = s \log\frac{1}{s}, \quad 0 < s \le \frac{1}{e}. \tag{4.6}$$

Hence if M is a compact Riemannian manifold with bounded Ricci tensor (and $g_{jk} \in C^1$ in some coordinate system), then we have a global geodesic flow on T^*M.

Osgood's theorem is established via Gronwall's inequality. It is shown that if $|F(x_1) - F(x_2)| \le \kappa\omega(|x_1 - x_2|)$, the flow $\mathcal{F}^t$ generated by F satisfies

$$|\mathcal{F}^t x_1 - \mathcal{F}^t x_2| \le \vartheta(|x_1 - x_2|, t), \tag{4.7}$$

where $\vartheta(a,t)$ is defined by

$$\int_a^{\vartheta(a,t)} \frac{ds}{\omega(s)} = \kappa t. \tag{4.8}$$

In particular, if $\omega(s)$ is given by (4.6), then

$$\vartheta(a,t) = a^{\exp(-\kappa t)}, \tag{4.9}$$

so

$$|\mathcal{F}^t x_1 - \mathcal{F}^t x_2| \le |x_1 - x_2|^{\exp(-\kappa t)}. \tag{4.10}$$

See §2.5 of [A-T] for examples of metric tensors on $\mathbb{R}^2$ satisfying

$$g_{jk} \in \bigcap_{p<\infty} H^{2,p}, \tag{4.11}$$

and such that ∇g_{jk} has a modulus of continuity only slightly worse than log-Lipschitz, with the property that there is branching of geodesics, hence no well defined geodesic flow. A further investigation of geodesic branching, with some unanswered questions, is given in [T5].

5. Propagation of singularities for wave equations on rough manifolds

For use in the next section, we want to describe propagation of singularities results (or rather propagation of smoothness) for solutions $u = u(t, x)$ to wave equations of the form

$$\partial_t^2 u + a\partial_t u - \Delta u = 0, \tag{5.1}$$

where Δ is the Laplace operator on a Riemannian manifold with somewhat rough metric tensor, and $a = a(x)$ also has limited smoothness. In §3 of [T7] we found it convenient to treat solutions to equations of the form

$$\partial_j A^{jk} \partial_k u + \partial_j (b^j u) = 0, \tag{5.2}$$

where A^{jk} and b^j have limited regularity. To make (5.2) directly applicable to (5.1), we write (5.1) in local coordinates as

$$\partial_t g^{1/2} \partial_t u - \partial_j g^{1/2} g^{jk} \partial_k u + \partial_t (g^{1/2} a u) = 0, \tag{5.2A}$$

with $t = x_0$.

The following result is Proposition 3.2 of [T7].

Proposition 5.1. *Take $r \in (1, 2)$. Assume $\Omega \subset \mathbb{R}^k$ is open and*

$$u \in H^{\sigma-(r\delta-1)}_{\text{loc}}(\Omega) \tag{5.3}$$

solves (5.2) on Ω, where $A^{jk} = A^{kj}$ are real valued and satisfy

$$|\nabla A^{jk}(x) - \nabla A^{jk}(y)| \le C|x-y| \log \frac{1}{|x-y|}, \tag{5.4}$$

for $|x - y|$ small, and $b^j \in C^{r-1}(\Omega)$ are real valued. Assume

$$\delta \in (0, 1),\ \ \delta r > 1,\ \ \textit{and}\ -(1-\delta)(r-1) < \sigma < r - 1. \tag{5.5}$$

Assume $\mathcal{O} \subset T^*\Omega \setminus 0$ is a conic open set and

$$u \in H^{\sigma}_{\text{mcl}}(\mathcal{O}). \tag{5.6}$$

Take

$$p_1(x, \xi) = A^{jk}(x)\xi_j\xi_k|\xi|^{-1}, \tag{5.7}$$

and let

$$(x_0,\xi_0)\in\mathcal{O},\quad p_1(x_0,\xi_0)=0. \tag{5.8}$$

Let γ be the orbit of the Hamiltonian vector field H_{p_0} through (x_0,ξ_0).

Then there is a conic neighborhood Γ of γ such that

$$u\in H^\sigma_{\mathrm{mcl}}(\Gamma). \tag{5.9}$$

In particular, this conclusion holds for

$$\sigma=0. \tag{5.10}$$

Remark 1. We characterize (5.6) as follows. By definition, if $\mathcal{O}$ is an open conic subset of $T^*\Omega\setminus 0$, a distribution u on Ω belongs to $H^\sigma_{\mathrm{mcl}}(\mathcal{O})$ provided $\varphi(x,D)u\in H^\sigma$ for each $\varphi(x,D)\in OPS^0_{1,0}$ with total symbol supported in $\mathcal{O}$.

Remark 2. Microlocal elliptic regularity applies on

$$\mathcal{N}=\{(x,\xi)\in T^*\Omega\setminus 0: p_1(x,\xi)\neq 0\}. \tag{5.11}$$

Given that (5.2) holds, $A^{jk}\in C^r,\ b^j\in C^{r-1}$, $1<r<2$, we have

$$u\in H^{1+\sigma}_{\mathrm{mcl}}(\mathcal{N}). \tag{5.12}$$

The case $a\equiv 0$ of Proposition 5.1 was established in Chapter 3, §11 of [T2]. Bringing in nontrivial a led to additional arguments in [T7], particularly in allowing $a\in C^{r-1}$ instead of making the stricter hypothesis $a\in C^r$. The analysis leading to the microlocal propagation result was patterned after the positive commutator argument pioneered in [H]. It also involved a symbol smoothing, of the form

$$P=\partial_j A_j^{\#}+\partial_j A_j^b, \tag{5.13}$$

somewhat parallel to that in (2.8)–(2.10), except that here the operator P is not elliptic. As in [H], implementation of the positive commutator method involves a use of a sharp Garding inequality:

Lemma 5.2. *Let $q(x,\xi)\in C^sS^m_{1,0}$ be scalar and satisfy $q(x,\xi)\geq -C_0$. Then, for all $u\in C_0^\infty$,*

$$\mathrm{Re}(q(x,D)u,u)\geq -C_1\|u\|^2_{L^2}, \tag{5.14}$$

provided

$$s>0,\quad m\leq\frac{2s}{2+s}. \tag{5.15}$$

This result, in turn, is Proposition 2.4.A of [T0]. The proof given there makes use of a symbol smoothing, $q=q^{\#}+q^b$, and an application of the Fefferman-Phong inequality to $q^{\#}(x,D)$.

6. Wave decay on manifolds with bounded Ricci tensor

The paper [T7] examined solutions to wave equations with dissipation,

$$Lu = \partial_t^2 u + a(x)\partial_t u - \Delta u = 0, \tag{6.1}$$

on $\mathbb{R} \times M$, when M is a compact, connected Riemannian manifold with Laplace operator Δ, and $a \geq 0$, and (following [RT]) sought conditions that guarantee exponential decay of the energy

$$E(u(t)) = \frac{1}{2} \int\limits_M \bigl\{ |\nabla_x u(t,x)|^2 + |u_t(t,x)|^2 \bigr\}\, dV(x), \tag{6.2}$$

as $t \nearrow +\infty$, given

$$u(0) = f \in H^1(M), \quad \partial_t u(0) = g \in L^2(M). \tag{6.3}$$

This initial value problem has a unique solution $u \in C(\mathbb{R}, H^1(M)) \cap C^1(\mathbb{R}, L^2(M))$, and we have the dissipation identity

$$\frac{d}{dt} E(u(t)) = -\int\limits_M a(x) |\partial_t u(t,x)|^2\, dV(x). \tag{6.4}$$

It is fairly easy to show that $E(u(t))$ decays exponentially as $t \nearrow +\infty$ for each (f,g) as in (6.3), provided M has a continuous metric tensor and $a \in L^\infty(M)$ satisfies $a(x) \geq a_0 > 0$ for all $x \in M$. It is of interest to assume instead that

$$a(x) \geq a_0 > 0, \quad \forall\, x \in U, \tag{6.5}$$

where U is some open subset of M, and see when this condition implies exponential energy decay. This was treated in [RT] when M has a smooth metric tensor and $a \in C^\infty(M)$ is ≥ 0 on M and satisfies (6.5). Then [RT] showed one has exponential energy decay provided the following condition holds:

(6.6) **Control condition:** There exists $T_0 < \infty$ such that each geodesic in M of length T_0 intersects U.

The necessity of such a condition follows from work of [Ral]. In outline, the argument of [RT] goes as follows. First, propagation of singularity results of [H], applied to $\partial_t u$, which also solves (6.1), yield, for some $T_1 > T_0$,

$$\begin{aligned} &\int_0^{T_1} \int\limits_M |\partial_t u(s,x)|^2\, dV(x)\, ds \\ &\leq C \int_0^{T_1} \int\limits_U |\partial_t u(s,x)|^2\, dV(x)\, ds + C \|u_t\|^2_{H^{-1}([0,T_1]\times M)}. \end{aligned} \tag{6.7}$$

Then an argument incorporating functional analysis and unique continuation allows one to drop the last term on the right side of (6.7), after perhaps enlarging T_1, and perhaps also expanding U slightly. Then, via (6.4), one obtains

$$E(u(T_1)) \le E(u(0)) - C_1 \|\partial_t u\|^2_{L^2([0,T_1]\times M)}, \tag{6.8}$$

for all solutions to (6.1) and (6.3), with C_1 independent of f and g. From here, a further argument, pursued in greater generality in §2 of [T7], allows one to pass to

$$E(u(T_1)) \le E(u(0)) - C_2 \int_0^{T_1} E(u(s))\, ds, \tag{6.9}$$

and since $E(u(s)) \searrow$, by (6.4), this implies

$$E(u(T_1)) \le (1 + C_2 T_1)^{-1} E(u(0)), \tag{6.10}$$

hence

$$E(u(kT_1)) \le (1 + C_2 T_1)^{-k} E(u(0)), \tag{6.11}$$

yielding exponential energy decay.

The primary result of [T7] is the following.

Proposition 6.1. *Let M be a compact manifold with metric tensor satisfying*

$$|\nabla g_{jk}(x) - \nabla g_{jk}(y)| \le C|x-y| \log \frac{1}{|x-y|}, \tag{6.12}$$

for $|x-y|$ small, and assume that $a \ge 0$ on M, that

$$a \in C^{r-1}(M), \tag{6.13}$$

for some $r > 1$, and that a satisfies (6.5). If the control condition (6.6) holds, then there is a uniform exponential rate of decay of energy $E(u(t))$, for all solutions to (6.1) with initial data as in (6.3).

The proof of Proposition 6.1 follows the outline of the smooth case, with some necessary elaborations. To begin, one can take $T \ge T_0 + 2$ and use Proposition 5.1 to deduce that there exists $\sigma_0 > 0$ such that if $0 < \sigma < \sigma_0$ and $v \in H^{-\sigma}([0,T]\times M)$ solves $Lv = 0$, then, with U as in (6.5)–(6.6),

$$\begin{aligned} v \in H^{-\sigma}([0,T]\times M) &\cap L^2([0,T]\times U) \\ &\Longrightarrow v \in L^2([1,2]\times M), \end{aligned} \tag{6.14}$$

with associated estimate

$$\|v\|_{L^2([1,2]\times M)} \le C\|v\|_{L^2([0,T]\times U)} + C\|v\|_{H^{-\sigma}([0,T]\times M)}. \tag{6.15}$$

As shown in Appendix A of [T7], we have an estimate

$$\|v\|_{L^2([0,T]\times M)} \le C_T \|v\|_{L^2([1,2]\times M)}, \tag{6.16}$$

hence

$$\|v\|_{L^2([0,T]\times M)} \le C_T\|v\|_{L^2([0,T]\times U)} + C_T\|v\|_{H^{-\sigma}([0,T]\times M)}. \tag{6.17}$$

We apply this to $v = u_t$ and obtain a variant of (6.7). Then additional arguments allow one to drop the last term on the right side of (6.17), and proceed to analogues of (6.8)–(6.9), hence to (6.11).

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