Review of M. Taylor's "Partial Differential Equations, Vols. 1–3,"

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Let it be said at the outset that this is a very important collection of volumes written by one of the modern masters of his subject. Many mathematicians, from ambitious graduate students to researchers, will be learning aspects of the theory of partial differential equations from these books. But they are more than that; they contain a veritable Cours d'Analyse Moderne. Thus Chapter 1 contains the basic theory of vector fields, flows, Hamiltonian flows, geodesics, differential forms, and in particular symplectic forms. Chapter 3 on Fourier analysis contains an introduction to the theory of distributions, including temtered distributions, special functions, the finite Fourier transforms, including the fast algorithm for its evaluation. Chapter 4 describes the L^2 Sobolev spaces in $\mathbb{R}^n$ and on unbounded domains, imbedding theorems, and the use of complex interpolation theorems. Chapter 7 describes the classical pseudodifferential operators, their properties, and how they are used. Chapter 10 on index theory includes a discussion of Clifford algebras and spin manifolds. Chapter 11 on Brownian motion defines Wiener measure, derives the Feynman-Kac formula, discusses martingales, stochastic integrals, and stochastic differential equations. Chapter 12 on the $\overline{\partial}$-Neumann problem is a minicourse using PDE techniques on analytic functions of several complex variables. Chapter 13 reviews the L^p Sobolev spaces and derives the embedding theorems, including the Gagliardo-Nirenberg-Moser estimates and Trudinger's inequality. Paradifferential operators are discussed, as well as the Young measure and Hardy space.

Further background information is given in three appendices. Appendix A is a thorough review of functional analysis, including the theory of compact operators, and a proof of the basic facts about Hilbert-Schmidt and trace-class operators. A proof of Lidskii's important theorem is to be found in Chapter 9. The appendix concludes with the spectral theory of selfadjoint operators and the rudiments of semigroup theory.

Appendix B is a compendium of the basics of manifold theory, vector bundles, and Lie groups. The Campbell-Baker-Hausdorff formula is derived, and the basic facts of the representation of Lie groups and algebras is outlined, with special emphasis on the classical groups.

Appendix C is an introduction to differential geometry. Covariant differentiation is defined, as is the curvature of a connection, as well as the curvature tensor of a Riemannian manifold. First the classical, then a very general Gauss-Bonnet formula is derived.

The author's philosophy about the main subject is well summarized by the first paragraph of the introduction.

"Partial differential equations is a many-faceted subject. Created to describe the mechanical behavior of objects such as vibrating strings and blowing winds, it has developed into a body of material that interacts with many branches of mathematics, such as differential geometry, complex analysis, and harmonic analysis, as well as a ubiquitous factor in the description and elucidation of problems in mathematical physics."

The materials are organized according to these principles. Chapter 2 introduces the central linear PDEs of second order via a variational principle (Hamilton's principle). The wave equation and the equations of linear elasticity are derived by appropriate linearizations. The Laplace operator on Riemannian manifolds is defined and the associated wave equation is set up; uniqueness for the initial value problem and finite propagation speed is proved by energy estimates, derived from integration by parts. An extension to Lorentz manifolds is given. The chapter concludes with a discussion of the Hodge Laplacian on forms and Maxwell's equations.

Chapter 5 is devoted to the study of linear elliptic equations, regularity, and boundary value problems. Starting with the Dirichlet and Neumann problems for the Laplace operator, the treatment is extended to general equations and coercive boundary conditions. The Hodge decomposition is derived for manifolds with boundaries.

Chapter 6 deals with linear evolution equations; the heat equation, wave equation, and Maxwell's equations are discussed in detail. The Cauchy-Kowalewski theorem is proved and used as approximating procedure for solving the initial value problem for fairly general hyperbolic equations; convergence is proved using energy estimates. The geometric optics description of solutions of the wave equation is discussed in detail, including the formation of caustics, amply illustrated by figures.

In Chapter 7 pseudodifferential operators are used to define and prove microlocal regularity of solutions of elliptic equations and to construct a parametrix for elliptic and parabolic equations.

Chapter 8 shows that the Laplace operator, properly defined as a Friedrichs extension, is selfadjoint under various boundary conditions. Spectral asymptotics is discussed via the heat equation. Concrete examples are presented in loving detail: the spherical Laplacian, the Laplace operator on hyperbolic space, the harmonic oscillator, and the Euclidean Laplacian with a Coulomb potential. The chapter ends with a scherzo of special functions.

Chapter 9 discusses scatterings of solutions of the wave equation by obstacles. The Lax-Phillips semigroup is used to relate the poles of the scattering matrix to the large time asymptotic behavior of solutions of the wave equation. The uniqueness of the inverse scattering problem is proved for smooth scatterers, and an approximate inversion method is described.

In Chapter 10 the Dirac operator and its generalizations are introduced. The Atiyah-Singer formula for the index of an elliptic operator is derived and is specialized to give a Gauss-Bonnet formula the the Riemann-Roch theorem. Spin manifolds are spun.

In Chapter 11, probabilistic methods (Brownian motion, martingales, stochastic differential equations) are used to study diffusions.

Chapter 14 deals with nonlinear elliptic equations, from semilinear ones to quasilinear and fully nonlinear ones. Variational methods are introduced, as well as the Schauder degrees. The uniformization of Riemann surfaces is obtained as a byproduct. Minimal surfaces are discussed in Euclidean space and in Riemannian manifolds. The de Giorgi-Nash-Moser regularity theorem is proved as well as the Krylov-Safonov estimates. Completely nonlinear equations, in particular the Monge-Ampere equations, are discussed. There is more in this rich chapter than can be done justice to in a brief review.

Chapter 15 on nonlinear parabolic equations starts with existence and regularity results for semilinear parabolic equations; some of the existence theorems are used to show the existence of harmonic maps. Semilinear parabolic systems are studied as a model for reaction diffusion equations. The next topic is quasilinear parabolic equations; using energy estimates in Sobolev norms and an approximation procedure employing mollifiers, a short-time existence theorem is proved for smooth enough initial data. Global solutions exist if one can derive a global a priori bound; this was done by Nash under the assumption of uniform ellipticity of the spatial nonlinear second-order operator in the parabolic equation.

Chapter 16 is devoted to nonlinear hyperbolic equations, starting with quasilinear, symmetric and symmetrizable, hyperbolic systems. A short time existence theorem, in any number of space dimensions, is not hard to derive using higher order energy estimates and Sobolev imbedding theorems, provided the initial values are smooth enough. It is not to be expected that smooth solutions can be extended for all t, because of shock formation. As an application of the local theory Taylor presents Garabedian's beautiful reduction of the Cauchy-Kowalevsky theorem for nonlinear analytic initian value problems to the previous result for nonlinear symmetric hyperbolic equations.

Next the equations governing ideal incompressible fluids are derived; they are the conservation of mass, momentum, and energy, and form a symmetrizable hyperbolic system. A generalized solution of a conservation law satisfies the conservation law in its integrated form; that is the same as saying that the differential equations are satisfied in the sense of distributions. Such generalized solutions may contain discontinuities, along which the Rankine-Hugoniot conditions must be satisfied. Weak solutions are not uniquely determined by their initial values; the equation must be supplemented by so-called entropy conditions. For scalar equations such conditions were given by Kruzhkov; generalized solutions that satisfy Kruzhkov's conditions are L^1 contractive, which implies uniqueness for the initial value problem. Existence can be proved by the viscosity method.

For hyperbolic systems of conservation laws the situation is more complicated, even in one space variable. There are entropy conditions; for 2×2 systems there are enough entropies to show global existence of solutions, using Tartar's beautiful idea of compensated compactness. Another, earlier, way of constructing global solutions has been given by Glimm, using a method of random choice, and the decrease of an interaction function. This method, only alluded to in these volumes, can be extended to systems bigger than 2×2, especially in Robin Young's version. Many examples of generalized solutions are given, generously illustrated with pictures.

Alas, nothing is known about global existence of generalized solutions in more than one space variable.

Chapter 17 is about incompressible fluid, governed by the Euler equations in the absence of viscosity, and by Navier-Stokes for viscous flows. Short-time solutions in $\mathbb{R}^n$ or $\mathbb{T}^n$ can be constructed by solving the initial value problem for the mollified Euler equation; using higher energy estimates, the initial data are assumed to lie in H^k, $k > n/2 + 1$, and be divergence free. The Beale-Kato-Majda result is proved and is used to show that in two dimensions Euler flows exist for all time. These results are then extended to flows in bounded regions, whose normal component on the boundary is zero.

Next, the Navier-Stokes flows are studied on a compact Riemannian manifold. Short-time solutions with smooth initial data are constructed as limits of a mollified equation; this is precisely the method used by Leray to construct flows on $\mathbb{R}^3$. To prove convergence, parabolic estimates are used. The solution exists for all time if the initial data are small enough. Furthermore, solutions in a weak sense exist for all time; this was proved by Leray in $\mathbb{R}^3$ and by E. Hopf for bounded domains. The smoothness and uniqueness of these Leray-Hopf solutions is one of the fascinating open problems of fluid dynamics.

The last chapter is on general relativity, an increasingly active research topic today. Taylor derives the Einstein field equations and describes in some detail the Schwarzschild solution. There is a brief discussion of the coupled Einstein-Maxwell equations, and of relativistic fluid dynamics, including gravitational collapse. The chapter ends with a study of the initial value problem.

Each section ends with a substantial list of problems; most of them are nonroutine extensions of material discussed in the text.

The exposition is clear throughout the book; to be sure, some parts are easier to absorb than others. This reviewer appreciated the author's willingness, after an invariant presentation of a topic, to relent and write out the equations in coordinates.

The presentation in these volumes is ahistorical, but in generalcredit is given where credit is due. There are some omissions: Friedrichs and Lewy are not acknowledged as the originators of the idea of using energy estimates to prove the existence of solutions of initial value problems for hyperbolic equations; nor is Schauder named as the first to push through this idea. In the chapter on the $\bar{\partial}$-Neumann problem one misses the names of Garabedian and Spencer, its inventors. In Chapter 8, on the asymptotics of the eigenvalues of the Laplace operator, Weyl and Courant are not named, and Carleman is not identified as the originator of the use of Tauberian theorems.

Among its predecessors this work is closest in spirit and organization to the two volumes of Courant-Hilbert. Both have the attractive feature that the individual chapters can be read separately.

To summarize: these volumes will be read by several generations of readers eager to learn the modern theory of partial differential equations of mathematical physics and the analysis in which this theory is rooted.

References

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