Manifolds Whose Weyl Spectral Asymptotics Have Small but not Tiny Remainders

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Abstract

A compact, n-dimensional Riemannian manifold M has Weyl spectral asymptotics with remainder $E_M(R)$, i.e., the spectral counting function satisfies

$$\mathcal{N}(\Delta_M, R) = C(M)R^n + E_M(R),$$

with $E_M(R) = o(R^n)$. Generally, one actually has $E_M(R) = O(R^{n-1})$, and one seeks conditions under which stronger estimates hold on the remainder. We produce n-dimensional manifolds whose Weyl remainders are $o(R^{n-1})$ but not $O(R^{n-1-\alpha})$ for any $\alpha > 0$.

Contents

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1 Introduction

If M is a compact, n-dimensional Riemannian manifold with Laplace operator Δ_M, then $L^2(M)$ has an orthonormal basis of eigenfunctions $\{u_j\}$, satisfying

$$\Delta_M u_j = -\lambda_j^2 u_j, \quad \lambda_j \nearrow \infty. \tag{1.1}$$

We define the spectral counting function by

$$\mathcal{N}(\Delta_M, R) = \#\{j : \lambda_j \le R\}. \tag{1.2}$$

For this, there is the Weyl asymptotic formula

$$\mathcal{N}(\Delta_M, R) = C(M)R^n + E_M(R), \tag{1.3}$$

with $E_M(R) = o(R^n)$, the Weyl remainder. A classical improvement of this estimate is

$$E_M(R) = O(R^{n-1}), \tag{1.4}$$

see [7]. Much work has been done to see when this estimate can be further improved. In [5] it is shown that one can take

$$E_M(R) = o(R^{n-1}) \tag{1.5}$$

if M has "not too many" closed geodesics. It was shown in [1] that, under certain geometric hypotheses involving no conjugate points, one can improve (1.5) to

$$E_M(R) = O(R^{n-1}/\log R). \tag{1.6}$$

The recent paper [3] obtained such an estimate in much greater generality. Going further, there are various examples for which one has

$$E_M(R) = O(R^{n-1-\alpha}), \tag{1.7}$$

for some $\alpha > 0$. We say M has spectral asymptotics with algebraically small Weyl remainder. The classical example for (1.7) is $M = \mathbb{T}^n$, the flat torus (the sharp value of α for which (1.7) holds is not known, cf. [2]). In [8], (1.7) is established for Cartesian products of spheres (with at least 2 factors). Other examples are studied in [13]–[14].

In this paper we produce examples of compact Riemannian manifolds M for which the remainder estimate (1.5) holds, but for which the stronger estimate (1.7) fails for each $\alpha > 0$.

Before describing our main results on this, we recall the classical cases for which the estimate (1.5) fails, namely the n-dimensional unit spheres

S^n in $\mathbb{R}^{n+1}$. In such cases, there are exact formulas for the eigenvalues of Δ_{S^n}, and for the dimensions of the eigenspaces, and these dimensions are seen to be sufficiently large that no improvement of (1.4) is possible. For later use, we describe this situation for S^2 in more detail. As is well known, $L^2(S^2)$ has an orthonormal basis $\{Y_k^\ell : k \in \mathbb{Z}^+, \ell \in \mathbb{Z}, |\ell| \le k\}$ (of "spherical harmonics"), satisfying

$$\Delta_{S^2} Y_k^\ell = -k(k+1) Y_k^\ell, \quad X Y_k^\ell = \ell Y_k^\ell, \quad k \in \mathbb{Z}^+,\ |\ell| \le k, \tag{1.8}$$

where

$$X = iY, \tag{1.9}$$

and Y is the vector field generating 2π-periodic rotation of $\mathbb{R}^3$ about the x_3-axis. In this case, the $-k(k+1)$-eigenspace of Δ_{S^2} has dimension $2k+1$, foreclosing the possibility of (1.5) holding.

To start with 2D examples, our construction of examples where (1.7) fails will involve taking $M = S^2$ as a manifold, but giving M a different metric tensor. The new metric tensor (g_{ij}) will match up with the standard metric (γ_{ij}) of S^2, to infinite order, at the equator $x_3 = 0$, but will differ from (γ_{ij}) off $x_3 = 0$. To show that (1.7) fails for such M, we will show that, for each $\rho \in (1/2, 1)$, the sequence of spaces

$$W_k(S^2) = W_k^\rho(S^2) = \operatorname{Span}\{Y_k^\ell : k - k^\rho \le \ell \le k\}, \quad k \in \mathbb{Z}^+, \tag{1.10}$$

yields *quasimodes* for Δ_M. One ingredient in this analysis is to examine how the functions $u \in W_k(S^2)$ concentrate on the equator $x_3 = 0$ as $k \to \infty$. In §2 we establish such concentration results. For $\nu \in \mathbb{N},\ s \ge 0$, we obtain

$$\begin{aligned} u \in W_k(S^2) \Rightarrow \|x_3^\nu u\|_{H^s(S^2)} &\le C\|u\|_{H^{s-\nu\delta/2}(S^2)} \\ &\le C k^{s-\nu\delta/2} \|u\|_{L^2}, \quad \delta = 1 - \rho, \end{aligned} \tag{1.11}$$

which is an effective concentration result when $\nu\delta/2 > s$.

We bring in tools from microlocal analysis to establish (1.11) and related estimates. In more detail, with $\rho \in (1/2, 1)$ and $\Lambda = (-\Delta_{S^2} + 1/4)^{1/2} - 1/2$, we set

$$F(\Lambda, X) = \varphi\big((\Lambda - X)\Lambda^{-\rho}\big), \tag{1.12}$$

where we pick

$$\varphi \in C_0^\infty(\mathbb{R}), \quad \varphi(\tau) = 1 \text{ for } |\tau| \le 1,\ 0 \text{ for } |\tau| \ge 2. \tag{1.13}$$

Results of [10] and [11], Chapter 12, imply that $F(\Lambda, X)$ is a pseudodifferential operator (of non-classical type):

$$F(\Lambda, X) \in OPS^0_{\rho,\delta}, \quad \text{principal symbol } f(x,\xi) = F(|\xi|_x, \langle Y(x), \xi\rangle), \tag{1.14}$$

leading on the one hand to

$$u = F(\Lambda, X)u, \quad \text{for} \ \ u \in W_k(S^2), \tag{1.15}$$

and on the other to

$$x_3^\nu F(\Lambda, X) \in OPS^{-\nu\delta/2}_{\rho,\delta}, \tag{1.16}$$

from which we deduce (1.11).

We take up concentration estimates of the eigenfunctions of the Laplace operator Δ_S on S^n for $n \geq 3$ in §3. In such a case, $L^2(S^n)$ is an orthogonal direct sum of eigenspaces

$$V_k(S^n) = \{u \in C^\infty(S^n) : \Delta_S u = -\mu_k^2 u\}, \quad \mu_k^2 = k^2 + (n-1)k. \tag{1.17}$$

Here we look at the joint spectrum of the commuting operators Δ_S and L, a second order differential operator that acts like the Laplace operator on $(n-1)$-spheres. (For $n = 2$, $L = -X^2$.) Instead of the pair (Λ, X), we take

$$(\Lambda, \Lambda_0), \quad \Lambda = (-\Delta_S)^{1/2}, \quad \Lambda_0 = -L\Lambda^{-1}, \tag{1.18}$$

and instead of $F(\Lambda, X)$, we use

$$G(\Lambda, \Lambda_0) = \varphi\big((\Lambda - \Lambda_0)\Lambda^{-\rho}\big), \tag{1.19}$$

as before, with $\rho \in (1/2, 1)$. Instead of (1.10), we take

$$W_k(S^n) = W_k^\rho(S^n) = \bigoplus_j \{\widetilde{V}_{kj}(S^n) : \mu_k^2 - \mu_k^{1+\rho} \leq \sigma_j^2 \leq \mu_k^2\}, \tag{1.20}$$

where

$$\widetilde{V}_{kj}(S^n) = \{u \in V_k(S^n) : Lu = -\sigma_j^2 u\}, \tag{1.21}$$

so

$$V_k(S^n) = \bigoplus_{j=0}^{k} \widetilde{V}_{kj}(S^n), \quad \sigma_j^2 = j(j + n - 2). \tag{1.22}$$

In place of (1.15)–(1.16), we have

$$u = G(\Lambda, \Lambda_0)u, \quad \text{for} \ \ u \in W_k(S^n), \tag{1.23}$$

and (again with $\delta = 1 - \rho$)

$$x_{n+1}^\nu G(\Lambda, \Lambda_0) \in OPS^{-\nu\delta/2}_{\rho,\delta}. \tag{1.24}$$

To use the spaces $W_k(S^n)$ in our search for manifolds M for which (1.7) fails, it is important to have a good lower bound on their dimensions. We show that

$$\dim W_k(S^n) \ge Ck^{-\delta} \dim V_k(S^n), \tag{1.25}$$

which is clear from (1.10) when $n = 2$. For $n \ge 3$, we get this from the isomorphism

$$\widetilde{V}_{kj}(S^n) \approx V_j(S^{n-1}), \quad 0 \le j \le k, \tag{1.26}$$

a result that can be restated in terms of an $SO(n)$-equivariant isomorphism

$$V_k(S^n) \approx \bigoplus_{\ell=0}^{k} V_\ell(S^{n-1}). \tag{1.27}$$

This is established in §3, with the help of a dimension count, done in Appendix A.

In §4 we introduce the following family of n-dimensional Riemannian manifolds. We take M to be S^n, endowed with a metric tensor (g_{ij}) that agrees with the standard metric tensor (γ_{ij}) of S^n to order ν on the equator (for some integer $\nu \ge 2$), i.e.,

$$g_{ij} = \gamma_{ij} + \sigma_{ij}, \quad \sigma_{ij} = O(x_{n+1}^{\nu}). \tag{1.28}$$

In such a case, we have from (1.16) and (1.24) that

$$(\Delta_S - \Delta_M)\mathcal{G} = Q \in OPS^{2-\nu\delta/2}_{\rho,\delta}, \tag{1.29}$$

where $\mathcal{G} = F(\Lambda, X)$ for $n = 2$, $\mathcal{G} = G(\Lambda, \Lambda_0)$ for $n \ge 3$. We deduce from (1.15) and (1.23) that

$$u \in W_k(S^n) \Longrightarrow (-\Delta_M - \mu_k^2)u = Qu, \tag{1.30}$$

and hence

$$u \in W_k(S^n) \Longrightarrow \|(\Lambda_M - \mu_k)u\|_{L^2} \le Ck^{-\sigma}\|u\|_{L^2}, \tag{1.31}$$

with

$$\Lambda_M = (-\Delta_M)^{1/2}, \quad \sigma = \frac{\nu\delta}{2} - 1. \tag{1.32}$$

Recall that we take $\rho \in (1/2, 1)$, $\delta = 1 - \rho$. If we pick ν sufficiently large, then $\sigma > 0$. The estimate (1.31) establishes that elements of $W_k(S^n)$ are quasimodes for Δ_M.

Using this set of quasimodes, we establish Proposition 4.2, which shows that for k sufficiently large, $\mu_k^2 = k(k+n-1)$, there is an orthonormal set

$$\{\psi_k^\ell : 1 \le \ell \le \dim W_k(S^n)\} \subset L^2(M), \tag{1.33}$$

of eigenfunctions of Δ_M, satisfying

$$\Lambda_M \psi_k^\ell = \mu_{k\ell} \psi_k^\ell, \quad |\mu_{k\ell} - \mu_k| \le C k^{-\sigma}. \tag{1.34}$$

Note that there are $\dim W_k(S^n)$ elements in this set, and we have the estimate (1.25).

This puts us in a position to show in §5 that, in such a situation, and with the hypothesis on ν strengthened to

$$\frac{\nu}{2} > 1 + \frac{1}{\delta}, \tag{1.35}$$

so $\sigma > \delta$ in (1.34), then, if the remainder estimate holds, we must have $\alpha \le 1 - \rho$. Taking $\rho \nearrow 1$ ($\delta \searrow 0$), we obtain in Theorem 5.2 the following result.

Theorem A. If M is an n-dimensional Riemannian manifold as described above, and if its metric tensor matches the standard metric tensor on S^n to infinite order at the equator, then the remainder estimate (1.7) in the Weyl asymptotic formula (1.3) cannot hold for any $\alpha > 0$.

Further tasks

It seems very likely that one can find Riemannian manifolds M of the sort described in Theorem 5.2, having the property that the set of closed geodesics has measure zero, so [5] implies $E_M(R) = o(R^{n-1})$.

Going further, it is intriguing to guess that some of these can be shown to satisfy the conditions in [3], yielding

$$E_M(R) = O(R^{n-1}/\log R).$$

We intend to look into this in future work.

2 Concentration of spherical harmonics on the equator of S^2

Here we take $\rho \in (1/2, 1)$, consider the family

$$W_k(S^2) = \operatorname{Span}\{Y_k^\ell : k - k^\rho \le \ell \le k\}, \tag{2.1}$$

and examine how elements of $W_k(S^2)$ concentrate on the equator $x_3 = 0$ of the sphere S^2, as $k \to \infty$. It is convenient to bring in the operator

$$\Lambda = \Bigl(-\Delta_{S^2} + \frac{1}{4}\Bigr)^{1/2} - \frac{1}{2}, \tag{2.2}$$

an elliptic, first-order pseudodifferential operator on S^2 (we write $\Lambda \in OPS^1(S^2)$). Note that

$$\Lambda Y_k^\ell = k Y_k^\ell, \quad X Y_k^\ell = \ell Y_k^\ell, \tag{2.3}$$

for $k \ge 0,\ |\ell| \le k$. We next set

$$F(\Lambda, X) = \varphi\bigl((\Lambda - X)\Lambda^{-\rho}\bigr), \tag{2.4}$$

where we pick

$$\varphi \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\varphi \subset [-2, 2], \quad \varphi(\tau) = 1 \text{ for } |\tau| \le 1. \tag{2.5}$$

For convenience, we also assume

$$\varphi \ge 0, \quad \varphi(\tau) \searrow \text{ for } \tau \ge 0. \tag{2.6}$$

Note that

$$Y_k^\ell \in W_k(S^2) \Longrightarrow F(\Lambda, X) Y_k^\ell = Y_k^\ell. \tag{2.7}$$

What makes (2.7) effective for concentration estimates comes from the analysis of $F(\Lambda, X)$ as a pseudodifferential operator. Indeed, for $\rho \in (0, 1]$, the function

$$F(\eta) = \varphi\bigl((\eta_1 - \eta_2)\eta_1^{-\rho}\bigr) \tag{2.8}$$

satisfies estimates

$$|D_\eta^\alpha F(\eta)| \le C_\alpha \langle \eta \rangle^{-\rho|\alpha|}, \quad \text{on } \{\eta : \eta_1 \ge 1,\ |\eta_2| \le \eta_1\}. \tag{2.9}$$

Hence, for $\rho \in (1/2, 1]$, one has

$$F(\Lambda, X) \in OPS^0_{\rho,\delta}(S^2), \quad \delta = 1 - \rho, \tag{2.10}$$

with principal symbol

$$f(x,\xi)=F(|\xi|_x,\langle Y(x),\xi\rangle),\quad \bmod\ S^{-(\rho-\delta)}_{\rho,\delta}. \tag{2.11}$$

The implication that, for $\rho\in(1/2,1]$,

$$(2.9)\Longrightarrow(2.10)-(2.11)$$

is established in [10], and in [11], Chapter 11, Theorem 1.3, with complements in (1.2)–(1.4) on p. 297, in the broader setting of $F(A_1,\dots,A_k)$, where A_j are commuting, self-adjoint operators in $OPS^1(M)$, satisfying the ellipticity condition

$$A_1^2+\cdots+A_k^2 \text{ is elliptic in } OPS^2(M). \tag{2.12}$$

This is also established in [9], for $\rho=1$, but here we need it for $1/2<\rho<1$.

The analysis in [10]–[11] involved representing $e^{iy\cdot A}$, for small $y\in\mathbb{R}^k$, as a family of Fourier integral operators,

$$e^{iy\cdot A}u(x)=(2\pi)^{-n/2}\int b(y,x,\xi)e^{i\varphi(y,x,\xi)}\hat{u}(\xi)\,d\xi, \tag{2.13}$$

modulo smoothing operators, and deducing that, if $F\in S^m_\rho(\mathbb{R}^k)$,

$$F(A)u(x)=(2\pi)^{-n/2}\int q(x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi, \tag{2.14}$$

modulo smoothing, where

$$q(x,\xi)e^{ix\cdot\xi}=F(D_y)\big[b(y,x,\xi)e^{i\varphi(y,x,\xi)}\big]\big|_{y=0},$$

to which a stationary phase analysis applies, yielding

$$\begin{gathered} q(x,\xi)\sim F(a(x,\xi))+\sum_{|\alpha|\ge 1}F^{(\alpha)}(a(x,\xi))\psi_\alpha(x,\xi),\\ \psi_\alpha(x,\xi)\in S^{[|\alpha|/2]}_{cl}, \end{gathered} \tag{2.15}$$

where $a(x,\xi)=(a_1(x,\xi),\dots,a_k(x,\xi))$ and $[z]$ denotes the greatest integer $\le z$. Compare [6], Theorem 2.16.

Returning to the setting of (2.4)–(2.5), we have the following.

Proposition 2.1 *Given* $\rho\in(1/2,1)$*, the operator* $F(\Lambda,X)$ *defined by* (2.4)–(2.5) *satisfies*

$$F(\Lambda,X)\in OPS^0_{\rho,\delta}(S^2),\quad \delta=1-\rho, \tag{2.16}$$

with principal symbol

$$f(x,\xi)=\varphi\big((1-\langle Y(x),\hat{\xi}\rangle)|\xi|_x^\delta\big),\quad \hat{\xi}=\frac{\xi}{|\xi|_x}. \tag{2.17}$$

To proceed, note that $\langle Y(x), \hat{\xi}\rangle \le |Y(x)|$, and hence

$$1 - \langle Y(x), \hat{\xi}\rangle \ge C x_3^2, \tag{2.18}$$

so (2.17) yields

$$|f(x,\xi)| \le \varphi(c x_3^2 |\xi|^\delta), \tag{2.19}$$

hence, for $M \in (0,\infty)$,

$$\bigl(x_3^2|\xi|^\delta\bigr)^M |f(x,\xi)| \le C_M. \tag{2.20}$$

This leads to the following.

Proposition 2.2 *In the setting of Proposition 2.1, we have, for each $\nu \in \mathbb{N}$,*

$$x_3^\nu F(\Lambda, X) \in OPS^{-\nu\delta/2}_{\rho,\delta}(S^2), \quad \delta = 1 - \rho. \tag{2.21}$$

In light of the Sobolev mapping property

$$P \in OPS^m_{\rho,\delta}(M) \Longrightarrow P : H^{s+m}(M) \to H^s(M), \tag{2.22}$$

valid for $0 \le \delta < \rho \le 1$, hence for $\delta = 1-\rho,\ \rho \in (1/2, 1)$, we have, for $\nu \in \mathbb{N}$,

$$\begin{aligned} u \in W_k(S^2) \Rightarrow \|x_3^\nu u\|_{H^s} &= \|x_3^\nu F(\Lambda, X) u\|_{H^s} \\ &\le C_\nu \|u\|_{H^{s-\nu\delta/2}} \\ &\le C_\nu k^{s-\nu\delta/2} \|u\|_{L^2}, \end{aligned} \tag{2.23}$$

as advertised in (1.11). Note that, by (2.1),

$$\dim W_k(S^2) \ge k^\rho. \tag{2.24}$$

3 Concentration of spherical harmonics on the equator of S^n

The Laplace operator Δ_S on S^n has eigenspaces

$$V_k = \{u \in L^2(S^n) : -\Delta_S u = \mu_k^2 u\}, \quad \mu_k^2 = k^2 + (n-1)k, \tag{3.1}$$

mutually orthogonal spaces of dimension

$$\dim V_k = \binom{k+n-1}{k} + \binom{k+n-2}{k-1}, \tag{3.2}$$

spanning $L^2(S^n)$. We want to analyze how certain elements of V_k concentrate on the equator

$$S^{n-1} = \{\omega \in S^n : \omega_{n+1} = 0\},$$

as $k \to \infty$, extending results of §2. To do this, we bring in the second order differential operator L on S^n, the image of the Laplace operator on $SO(n)$ under its action on $S^n \subset \mathbb{R}^{n+1}$, via rotation in the $(x_1, \dots, x_n)$-plane, normalized so that, for $u \in C^\infty(S^n)$,

$$Lu\big|_{S^{n-1}} = \Delta_{S^{n-1}}\big(u\big|_{S^{n-1}}\big). \tag{3.3}$$

The operators Δ_S and L commute and are self adjoint on $L^2(S^n)$. In case $n = 2$, $L = Y^2$. We can write

$$V_k = \bigoplus_\ell V_{k\ell}, \quad V_{k\ell} = \{u \in V_k : Lu = -\ell^2 u\}. \tag{3.4}$$

(Here ℓ runs over $\mathbb{R}^+$; it need not be an integer.) We will obtain estimates on how elements of $V_{k\ell}$ concentrate on the equator for ℓ sufficiently close to μ_k.

To proceed, we fix $\rho \in (1/2, 1)$, take $\varphi \in C_0^\infty(\mathbb{R})$, satisfying (2.5)–(2.6), and set

$$\begin{aligned} G(\Lambda, \Lambda_0) &= \varphi(-(\Delta_S - L)\Lambda^{-1}\Lambda^{-\rho}) \\ &= \varphi((\Lambda - \Lambda_0)\Lambda^{-\rho}), \end{aligned} \tag{3.5}$$

where

$$\Lambda = \sqrt{-\Delta_S}, \quad \Lambda_0 = -L\Lambda^{-1} \in OPS^1(S^n). \tag{3.6}$$

Parallel to (2.10), we have

$$G(\Lambda, \Lambda_0) \in OPS^0_{\rho,\delta}(S^n), \quad \delta = 1 - \rho. \tag{3.7}$$

Note that

$$
\begin{aligned}
u \in V_{k\ell} \Longrightarrow \Lambda_0 u &= \frac{\ell^2}{\mu_k} u \\
\Longrightarrow (\Lambda - \Lambda_0)\lambda^{-\rho} u &= \frac{\mu_k^2 - \ell^2}{\mu_k^{1+\rho}} u.
\end{aligned}
\tag{3.8}
$$

Hence, if we set

$$
W_k = \bigoplus_{\ell} \{V_{k\ell} : \mu_k^2 - \ell^2 \le \mu_k^{1+\rho}\},
\tag{3.9}
$$

we have

$$
u \in W_k \Longrightarrow G(\Lambda, \Lambda_0) u = u.
\tag{3.10}
$$

To apply (3.10) to estimate how elements of W_k concentrate on the equator, we aim to bring in arguments parallel to those provided to prove Propositions 2.1 and 2.2. First, parallel to (2.17), the operator $G(\Lambda, \Lambda_0)$ has principal symbol

$$
g(x, \xi) = \varphi((1 - \sigma_{-L}(x, \hat{\xi}))|\xi|_x^\delta), \quad \hat{\xi} = \frac{\xi}{|\xi|_x},
\tag{3.11}
$$

with complete symbol expansion derived from (2.15). Next, parallel to (2.18), we have

$$
\sigma_{-L}(x, \hat{\xi}) \le 1 - c x_{n+1}^2,
\tag{3.12}
$$

hence

$$
1 - \sigma_{-L}(x, \hat{\xi}) \ge c x_{n+1}^2,
\tag{3.13}
$$

and therefore

$$
|g(x, \xi)| \le \varphi(c x_{n+1}^2 |\xi|^\delta),
\tag{3.14}
$$

so, for $M \in (0, \infty)$,

$$
(x_{n+1}^2 |\xi|^\delta)^M |g(x, \xi)| \le C_M.
\tag{3.15}
$$

This leads to the following.

Proposition 3.1 *Given $\rho \in (1/2, 1)$, the operator $G(\Lambda, \Lambda_0)$, defined by (3.5)–(3.6), satisfies, for each $\nu \in \mathbb{Z}^+$,*

$$
x_{n+1}^\nu G(\Lambda, \Lambda_0) \in OPS_{\rho,\delta}^{-\nu\delta/2}(S^n), \quad \delta = 1 - \rho.
\tag{3.16}
$$

Having (3.16), we bring in (3.10) to deduce that, for $\nu \in \mathbb{N},\ s \in \mathbb{R}$,

$$
\begin{aligned}
u \in W_k \Rightarrow \|x_{n+1}^\nu u\|_{H^s} &= \|x_{n+1}^\nu G(\Lambda, \Lambda_0) u\|_{H^s} \\
&\le C_\nu \|u\|_{H^{s-\nu\delta/2}} \\
&\le C_\nu \mu_k^{s-\nu\delta/2} \|u\|_{L^2},
\end{aligned}
\tag{3.17}
$$

parallel to (2.23). As before, this estimate is particularly valuable for $\nu\delta/2 > s$.

At this point, it behooves us to establish a lower estimate on

$$\dim W_k, \tag{3.18}$$

extending the estimate (2.24), done for $n = 2$. We aim to establish an estimate of the form

$$\dim W_k \geq C(\dim V_k) k^{\rho-1}. \tag{3.19}$$

To tackle this, it is convenient to refine our notation a bit, relabeling V_k in (3.1) as

$$V_k(S^n) = \{u \in L^2(S^n) : \Delta_S u = -k(k+n-1)u\}, \tag{3.20}$$

and rewriting (3.4) as

$$V_k(S^n) = \bigoplus_{j=0}^{k} \widetilde{V}_{kj}(S^n), \tag{3.21}$$

where

$$\widetilde{V}_{kj}(S^n) = \{u \in V_k(S^n) : Lu = -j(j+n-2)u\}. \tag{3.22}$$

We also relabel W_k as $W_k(S^n)$, and, in place of (3.9), write

$$\begin{gathered} W_k(S^n) = \bigoplus_j \Big\{ \widetilde{V}_{kj}(S^n) : \mu_k^2 - \mu_k^{1+\rho} \leq \sigma_j^2 \leq \mu_k^2 \Big\}, \\ \mu_k^2 = k(k+n-1), \quad \sigma_j^2 = j(j+n-2). \end{gathered} \tag{3.23}$$

The following is key to our dimension estimate.

Proposition 3.2 *For $0 \leq j \leq k$, $n \geq 3$,*

$$\widetilde{V}_{kj}(S^n) \approx V_j(S^{n-1}). \tag{3.24}$$

Proof. Note that the natural action of $SO(n)$ on $L^2(S^n)$ leaves each space $\widetilde{V}_{kj}(S^n)$ in (3.21) invariant. In view of (3.22), we see that, for each $j \in \{0, \dots, k\}$, $\widetilde{V}_{kj}(S^n)$ is either 0 or a direct sum of spaces isomorphic to $V_j(S^{n-1})$. Furthermore, Proposition 2.4 of [15] implies that, if $\widetilde{V}_{kj}(S^n) \neq 0$, then

$$SO(n) \text{ acts irreducibly on } \widetilde{V}_{kj}(S^n). \tag{3.25}$$

Hence either (3.24) holds or $\widetilde{V}_{kj} = 0$.

At this point, we see that Proposition 3.2 is equivalent to the assertion that there is an $SO(n)$-equivariant isomorphism

$$V_k(S^n) \approx \bigoplus_{j=0}^{k} V_j(S^{n-1}), \tag{3.26}$$

and so far we know that the left side of (3.26) is isomorphic to an $SO(n)$-invariant linear subspace of the right side. Hence the proof of Proposition 3.2 is done if we show that

$$\dim V_k(S^n) = \sum_{j=0}^{k} \dim V_j(S^{n-1}). \tag{3.27}$$

This computation is carried out in Appendix A. □

To proceed toward a proof of (3.19), we have from (3.24) that

$$\dim W_k(S^n) = \sum_j \Big\{ \dim V_j(S^{n-1}) : \mu_k - \mu_k^{1+\rho} \le \sigma_j^2 \le \mu_k^2 \Big\}. \tag{3.28}$$

Note that the restriction on j (beyond $0 \le j \le k$) can be written

$$\mu_k \sqrt{1 - \mu_k^{\rho-1}} \le \sigma_j \le \mu_k, \tag{3.29}$$

so in light of (3.23), the number of summands in (3.28) is

$$\approx \frac{1}{2}\mu_k^\rho \approx \frac{1}{2}k^\rho, \tag{3.30}$$

for large k. We bring in the asymptotics

$$\dim V_k(S^n) \sim C_n k^{n-1}, \quad \text{as} \quad k \to \infty, \tag{3.31}$$

which follow from (3.2), and the variant

$$\dim V_j(S^{n-1}) \sim C_{n-1} j^{n-2}. \tag{3.32}$$

This leads to the estimate

$$\begin{aligned} \dim W_k(S^n) &\ge C k^{n-2} \cdot k^\rho \\ &\ge C \, \dim V_k(S^n) \, k^{\rho-1}, \end{aligned} \tag{3.33}$$

as asserted in (3.19).

Summarizing the main results of this section, we have the following.

Proposition 3.3 *Take* $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, $n \geq 2$. *For* $k \geq 1$, *there exist linear subspaces* $W_k(S^n) \subset V_k(S^n)$ *satisfying*

$$\dim W_k(S^n) \geq Ck^{-\delta} \dim V_k(S^n), \tag{3.34}$$

$$u \in W_k(S^n) \Longrightarrow G(\Lambda, \Lambda_0)u = u, \tag{3.35}$$

with $G(\Lambda, \Lambda_0)$ *as in* (3.5)–(3.7) *and* (3.16), *and, for* $\nu \in \mathbb{N}$, $s \in \mathbb{R}$,

$$u \in W_k(S^n) \Longrightarrow \|x_{n+1}^{\nu} u\|_{H^s} \leq C_{\nu} \mu_k^{s-\nu\delta/2} \|u\|_{L^2}. \tag{3.36}$$

(Recall that $\mu_k \sim k$.)

4 Elements of $W_k(S^n)$ as quasimodes for perturbed Laplace operators

As indicated in the introduction, we take the Riemannian manifold M to be S^n, endowed with a metric tensor that is a perturbation of the standard metric tensor of the unit sphere, and investigate how elements of $W_k(S^n)$ yield quasimodes for the Laplace-Beltrami operator Δ_M. We start by examining how Δ_S and Δ_M are related. The metric tensors (g_{ij}) of M and (γ_{ij}) of S^n are related by

$$g_{ij} = \gamma_{ij} + \sigma_{ij}, \quad \sigma_{ij}\big|_{x_{n+1}=0} = 0, \tag{4.1}$$

more precisely, we assume

$$\sigma_{ij} = O(x_{n+1}^{\nu}), \tag{4.2}$$

for some $\nu \in \mathbb{N}$ $(\nu \geq 2)$. Now we compare Laplace operators

$$\begin{aligned} \Delta_S u &= \gamma^{-1/2}\partial_i(\gamma^{1/2}\gamma^{ij}\partial_j u), \\ \Delta_M u &= g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j u). \end{aligned}$$

We obtain

$$\begin{aligned} -\Delta_M &= -\Delta_S + h^{ij}\partial_i\partial_j + h^j\partial_j, \\ h^{ij} &= O(x_{n+1}^{\nu}), \quad h^j = O(x_{n+1}^{\nu-1}). \end{aligned} \tag{4.3}$$

Consequently, by Proposition 2.2, with $n = 2$, $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and $F(\Lambda, X)$ as in (2.4)–(2.5),

$$(\Delta_S - \Delta_M)F(\Lambda, X) = Q \in OPS^{2-\nu\delta/2}_{\rho,\delta}. \tag{4.4}$$

Similarly, by Proposition 3.1, with $n \geq 3$, $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and $G(\Lambda, \Lambda_0)$ as in (3.5)–(3.6),

$$(\Delta_S - \Delta_M)G(\Lambda, \Lambda_0) = Q \in OPS^{2-\nu\delta/2}_{\rho,\delta}. \tag{4.5}$$

Thanks to (2.7) for $n = 2$, (3.35), for $n \geq 3$, we therefore have (with μ_k as in (3.1))

$$u \in W_k(S^n) \Longrightarrow (-\Delta_M - \mu_k^2)u = Qu. \tag{4.6}$$

Let us set

$$\Lambda_M = (-\Delta_M)^{1/2}, \text{ so } -\Delta_M - \mu_k^2 = (\Lambda_M + \mu_k)(\Lambda_M - \mu_k). \tag{4.7}$$

It follows that for $u \in W_k(S^n)$,

$$
\begin{aligned}
\|(\Lambda_M - \lambda_k)u\|_{L^2} &\le \mu_k^{-1}\|(-\Delta_M - \mu_k^2)u\|_{L^2} \\
&\le Ck^{-1}\|Qu\|_{L^2} \\
&\le ck^{-1}\|u\|_{H^{-(\nu\delta/2-2)}} \\
&\le Ck^{-(\nu\delta/2-1)}\|u\|_{L^2}.
\end{aligned} \tag{4.8}
$$

We record our quasimode estimate.

Proposition 4.1 *Take* $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, *and pick* ν *sufficiently large that*

$$\sigma = \frac{\nu\delta}{2} - 1 \tag{4.9}$$

is positive. Assume the metric tensor on M *satisfies* (4.1)–(4.2). *Then*

$$u \in W_k(S^n) \Longrightarrow \|(\Lambda_M - \mu_k)u\|_{L^2} \le Ck^{-\sigma}\|u\|_{L^2}. \tag{4.10}$$

We next show that there is a sequence of actual eigenvalues of Λ_M close to μ_k. To start, it follows directly from (4.10) that there exists $\psi_k^1 \in C^\infty(M)$ such that

$$\|\psi_k^1\|_{L^2(M)} = 1, \quad \Lambda_M\psi_k^1 = \mu_{k1}\psi_k^1, \quad |\mu_{k1} - \mu_k| \le Ck^{-\sigma}. \tag{4.11}$$

Of course, Λ_M need not leave $W_k(S^n)$ invariant, and we cannot say that ψ_k^1 is in, or even particularly close to, $W_k(S^n)$. Set

$$Z_1 = \operatorname{Span}\psi_k^1. \tag{4.12}$$

Then

$$(1 + \Lambda_M)^{-1} : Z_1^\perp \longrightarrow Z_1^\perp. \tag{4.13}$$

We have

$$\dim W_k(S^n) \ge 2 \Longrightarrow W_k(S^n) \cap Z_1^\perp \ne 0, \tag{4.14}$$

in which case

$$
\begin{aligned}
&\exists\, \psi_k^2 \in Z_1^\perp, \text{ with unit norm, such that} \\
&\Lambda_M\psi_k^2 = \mu_{k2}\psi_k^2, \quad |\mu_{k2} - \mu_k| \le Ck^{-\sigma}.
\end{aligned} \tag{4.15}
$$

Continue, producing an orthonormal set ψ_k^ℓ of smooth elements of $L^2(M)$, satisfying

$$\Lambda_M\psi_k^\ell = \mu_{k\ell}\psi_k^\ell, \quad |\mu_{k\ell} - \mu_k| \le Ck^{-\sigma}, \tag{4.16}$$

for $1 \leq \ell \leq L$, and set

$$Z_L = \operatorname{Span}(\psi_k^1, \dots, \psi_k^L), \tag{4.17}$$

so $(1+\Lambda_M)^{-1} : Z_L^\perp \to Z_L^\perp$. We have

$$\dim W_k(S^n) > L \Longrightarrow W_k(S^n) \cap Z_L^\perp \neq 0, \tag{4.18}$$

in which case

$$\begin{aligned} &\exists\, \psi_k^{L+1} \in Z_L^\perp, \quad \text{with unit norm, such that} \\ &\Lambda_M \psi_k^{L+1} = \mu_{k,L+1} \psi_k^{L+1}, \quad |\mu_{k,L+1} - \mu_k| \leq C k^{-\sigma}. \end{aligned} \tag{4.19}$$

We can do this right up to the point where

$$L = \dim W_k(S^n). \tag{4.20}$$

This construction leads to the following result on eigenvalues of Δ_M close to $-\mu_k^2$.

Proposition 4.2 *Keep the setting of Proposition 4.1, including having the metric tensor on M satisfying* (4.1)–(4.2)*. Then, for k sufficiently large, there exists an orthonormal set*

$$\{\psi_k^\ell : 1 \leq \ell \leq \dim W_k(S^n)\} \subset L^2(M) \tag{4.21}$$

of eigenfunctions of Δ_M, satisfying (4.16)*. Furthermore,*

$$\begin{aligned} \dim W_k(S^n) &\geq C k^{-\delta} \dim V_k(S^n) \\ &\geq C' k^{n-1-\delta}. \end{aligned} \tag{4.22}$$

5 Necessary condition for an algebraically small Weyl remainder

As in §4, M is a compact, n-dimensional Riemannian manifold, whose metric tensor is a perturbation of that of the standard sphere S^n, satisfying (4.1)–(4.2). We seek a necessary condition that

$$\mathcal{N}(\Delta_M, R) = C(M)R^n + O(R^{n-1-\alpha}), \tag{5.1}$$

for some $\alpha \in (0,1)$. Having this, we deduce a sufficient condition for (5.1) to fail for all $\alpha > 0$. Recall further details of this set-up. We pick $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and then take ν in (4.2) sufficiently large that

$$\sigma = \frac{\nu\delta}{2} - 1 \tag{5.2}$$

is positive.

To continue, if (5.1) holds for all (large) R, then, for $b \in [0,1]$,

$$\mathcal{N}(\Delta_M, R+b) - \mathcal{N}(\Delta_M, R-b) = 2nC(M)bR^{n-1} + O(R^{n-1-\alpha}). \tag{5.3}$$

Let us take $b = cR^{-\sigma}$, so

$$\begin{aligned} \mathcal{N}(\Delta_M, R + cR^{-\sigma}) &- \mathcal{N}(\Delta_M, R - cR^{-\sigma}) \\ &= 2ncC(m)R^{n-1-\sigma} + O(R^{n-1-\alpha}). \end{aligned} \tag{5.4}$$

Proposition 4.2 implies there exists $c \in (0, \infty)$ such that, if $R = \mu_k$, then

$$\mathcal{N}(\Delta_M, R + cR^{-\sigma}) - \mathcal{N}(\Delta_M, R - cR^{-\sigma}) \geq CR^{n-1-\delta}. \tag{5.5}$$

We deduce that (for $R = \mu_k$)

$$CR^{n-1-\delta} \leq 2ncC(M)R^{n-1-\sigma} + O(R^{n-1-\alpha}). \tag{5.6}$$

At this point, we strengthen our hypothesis on ν, from $\sigma > 0$ to

$$\sigma > \delta, \quad \text{i.e.,} \ \frac{\nu}{2} > \frac{\delta + 1}{\delta}. \tag{5.7}$$

With this arranged, we see that (5.6) implies

$$\alpha \leq \delta = 1 - \rho. \tag{5.8}$$

This establishes the following.

Proposition 5.1 *Let M be a compact, n-dimensional Riemannian manifold. Pick $\rho \in (1/2, 1)$, $\delta = 1 - \rho$, and assume $\nu \in \mathbb{N}$ satisfies* (5.7). *Then assume the metric tensor on M satisfies* (4.1)–(4.2)*, i.e., matches the standard metric tensor on S^n to order ν at the equator. In such a case, if the Weyl asymptotic formula* (5.1) *holds, we must have $\alpha \leq 1 - \rho$.*

From here, we have the following conclusion.

Theorem 5.2 *In the setting of Proposition 5.1, if the metric tensor on M matches the standard metric tensor on S^n to infinite order at the equator, then* (5.1) *cannot hold for any $\alpha > 0$.*

A Dimension counts

Work in §3 makes use of the $SO(n)$-equivariant isomorphism

$$V_k(S^n) \approx \bigoplus_{\ell=0}^{k} V_\ell(S^{n-1}), \tag{A.1}$$

where $V_k(S^n)$ denotes the $-k(k+n-1)$-eigenspace of the Laplace operator on S^n, and $V_\ell(S^{n-1})$ is similarly defined. As seen there, results on irreducibility of certain $SO(n)$ actions enable one to establish (A.1) once we have the identity

$$\dim V_k(S^n) = \sum_{\ell=0}^{k} \dim V_\ell(S^{n-1}). \tag{A.2}$$

We establish this here.

In preparation, we recall a standard approach to computing the left side of (A.2), using the isomorphism

$$V_k(S^n) \approx \mathcal{H}_k(\mathbb{R}^{n+1}), \tag{A.3}$$

the space of harmonic polynomials on $\mathbb{R}^{n+1}$, homogeneous of degree k, and the decomposition

$$\mathcal{P}_k(\mathbb{R}^{n+1}) = \mathcal{H}_k(\mathbb{R}^{n+1}) \oplus |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^{n+1}). \tag{A.4}$$

Here and below,

$$\begin{aligned}
&\mathcal{P}_k(\mathbb{R}^{n+1}) = \text{ space of polynomials on } \mathbb{R}^{n+1}, \text{ homogeneous of degree } k, \\
&\mathcal{P}^k(\mathbb{R}^{n+1}) = \text{ space of polynomials on } \mathbb{R}^{n+1}, \text{ of degree } \le k, \\
&d_k(n+1) = \dim \mathcal{P}_k(\mathbb{R}^{n+1}).
\end{aligned} \tag{A.5}$$

Note that

$$d_k(n+1) = \dim \mathcal{P}^k(\mathbb{R}^n) = d_k(n) + d_{k-1}(n) + \cdots + d_0(n), \tag{A.6}$$

with a similar result for $d_j(m)$, for other values of j and m.

Using (A.3)–(A.6) yields

$$\begin{aligned}
\dim V_k(S^n) &= d_k(n+1) - d_{k-2}(n+1) \\
&= d_k(n) + d_{k-1}(n).
\end{aligned} \tag{A.7}$$

Similarly,

$$\sum_{\ell=0}^{k} \dim V_\ell(S^{n-1}) = \sum_{\ell=0}^{k} \Big\{ d_\ell(n-1) + d_{\ell-1}(n-1) \Big\}. \tag{A.8}$$

On the other hand, (A.6) (with n replaced by $n-1$) gives

$$d_k(n) = \sum_{\ell=0}^{k} d_\ell(n-1), \tag{A.9}$$

and similarly we have

$$d_{k-1}(n) = \sum_{\ell=0}^{k-1} d_\ell(n-1) = \sum_{\ell=0}^{k} d_{\ell-1}(n-1). \tag{A.10}$$

Together, (A.7)–(A.10) yield the desired identity (A.2).

Remark. There is the classical computation

$$d_k(n) = \binom{k+n-1}{k}. \tag{A.11}$$

In light of this, the identity (A.9) is equivalent to

$$\sum_{\ell=0}^{k} \binom{\ell+m}{\ell} = \binom{k+m+1}{k} \tag{A.12}$$

(with $m = n-2$), which is sometimes given the whimsical label, the "hockey stick identity."

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