Differential Geometry

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Contents

Introduction

These are notes for a course in differential geometry, for students who had a course on manifold theory the previous semester, which in turn followed a course on the elementary differential geometry of curves and surfaces.

Section 1 recalls some basic concepts of elementary geometry, and extends them from surfaces in $\mathbb{R}^3$ to hypersurfaces in $\mathbb{R}^n$, and then to manifolds with Riemannian metrics, defining arc length and deriving the ODE for a geodesic. The ODE in general has a somewhat messy form; a more elegant form will be produced in §11, following some material on vector fields and related topics in §§2–10. These sections also contain a review of material from the previous course on manifold theory, such as differential forms and deRham cohomology.

The material in §§11–16, on geodesics, covariant derivatives, and curvature, is the heart of the course. We take the intrinsic definitions of these objects as fundamental, though the very important relations between curvature and the second fundamental form are studied in §16, in the general context of one Riemannian manifold imbedded in another (not necessarily Euclidean space).

In §§17–20 we cover the famous Gauss-Bonnet Theorem, and its higher dimensional extension, which involves a study of characteristic classes, certain deRham cohomology classes derived from curvature. An essential tool is the Chern-Weil theory of characteristic classes, developed in §19. This in turn is treated most conveniently in terms of a "principal bundle," a structure which is in a sense more fundamental than that of a vector bundle. Section 18 develops the basic theory of principal bundles.

Sections 21–24 study the Hodge theory, representing elements of deRham cohomology by harmonic forms. This leads to simple proofs of some fundamental results, such as Poincare duality and the Kunneth formula, for products of manifolds. Parts of this study make use of the theory of elliptic partial differential equations, not presented here, but contained in the author's book [T1].

Sections 25–28 study spinors and some applications. This is a situation in which it is particularly helpful to use concepts involving principal bundles, developed in §18.

Sections 29–31 are devoted to minimal surfaces, which to some extent are higher dimensional analogues of geodesics.

At the end are a number of appendices, on various background or auxiliary material useful for understanding the main body of the text. This includes definitions of metric and topological spaces, manifolds, vector bundles, and Lie groups, and proofs of some basic results, such as the inverse function theorem and the local existence of solutions to ODE. There are also sets of exercises, on determinants, the cross product, and trigonometric functions, intended to give the reader a fresh perspective of these elementary topics, which appear frequently in the study of differential geometry. In addition, the final handful of appendices deal with some special topics in differential geometry, which complement material in the main text but did not find space there.

We close this introduction with some comments on where this material comes from and where it is going. Most of the main body of the text (§§2–31) was adapted from material in differential geometry scattered through the three-volume text [T1]. This was augmented by the introductory material in §1, developed in an advanced calculus course. The material in §§1–10 has since been rewritten and appears, in more polished form, in [T4]. The text [T3] treats linear algebra, and contains further material on exterior algebra and Clifford algebra, as well as more detailed presentations of other linear algebra topics, such as given here in Appendices J and M. The theory of Lie groups, sketched here in Appendix H and used in a number of other sections, receives a fairly thorough treatment in [T2].

1. Surfaces, Riemannian metrics, and geodesics

Suppose S is a smooth hypersurface in $\mathbb{R}^n$. If $\gamma(t) = \big(x_1(t), \dots, x_n(t)\big),\ a \le t \le b$, is a smooth curve in S, its length is

$$L = \int_a^b \|\gamma'(t)\|\, dt. \tag{1.1}$$

where

$$\|\gamma'(t)\|^2 = \sum_{j=1}^n x_j'(t)^2. \tag{1.2}$$

A curve γ is said to be a geodesic if, for $|t_1 - t_2|$ sufficiently small, $t_j \in [a,b]$, the curve $\gamma(t),\ t_1 \le t \le t_2$ has the shortest length of all smooth curves in Ω from $\gamma(t_1)$ to $\gamma(t_2)$.

Our first goal is to derive an equation for geodesics. So let $\gamma_0(t)$ be a smooth curve in S $(a \le t \le b)$, joining p and q. Suppose $\gamma_s(t)$ is a smooth family of such curves. We look for a condition guaranteeing that $\gamma_0(t)$ has minimum length. Since the length of a curve is independent of its parametrization, we may as well suppose

$$\|\gamma_0'(t)\| = c_0,\ \text{ constant, for } a \le t \le b. \tag{1.3}$$

Let N denote a field of normal vectors to S. Note that, with $\partial_s\gamma_s(t) = (\partial/\partial s)\gamma_s(t)$,

$$V = \partial_s\gamma_s(t) \perp N. \tag{1.4}$$

Also, any vector field $V \perp N$ over the image of γ_0 can be obtained by some variation γ_s of γ_0, provided $V = 0$ at p and q. Recall we are assuming $\gamma_s(a) = p,\ \gamma_s(b) = q$. If $L(s)$ denotes the length of γ_s, we have

$$L(s) = \int_a^b \|\gamma_s'(t)\|\, dt, \tag{1.5}$$

and hence

$$\begin{aligned} L'(s) &= \frac{1}{2}\int_a^b \|\gamma_s'(t)\|^{-1}\partial_s\big(\gamma_s'(t), \gamma_s'(t)\big)\, dt \\ &= \frac{1}{c_0}\int_a^b \big(\partial_s\gamma_s'(t), \gamma_s'(t)\big)\, dt,\ \text{ at } s = 0. \end{aligned} \tag{1.6}$$

Using the identity

$$\frac{d}{dt}\big(\partial_s\gamma_s(t),\gamma_s'(t)\big)=\big(\partial_s\gamma_s'(t),\gamma_s'(t)\big)+\big(\partial_s\gamma_s(t),\gamma_s''(t)\big), \tag{1.7}$$

together with the fundamental theorem of calculus, in view of the fact that

$$\partial_s\gamma_s(t)=0 \text{ at } t=a \text{ and } b, \tag{1.8}$$

we have

$$L'(s)=-\frac{1}{c_0}\int_a^b \big(V(t),\gamma_s''(t)\big)\,dt,\ \text{ at } s=0. \tag{1.9}$$

Now, if γ_0 were a geodesic, we would have

$$L'(0)=0, \tag{1.10}$$

for all such variations. In other words, we must have $\gamma_0''(t)\perp V$ for all vector fields V tangent to S (and vanishing at p and q), and hence

$$\gamma_0''(t)\|N. \tag{1.11}$$

This vanishing of the tangential curvature of γ_0 is the geodesic equation for a hypersurface in $\mathbb{R}^n$.

We proceed to derive from (1.11) an ODE in standard form. Suppose S is defined locally by $u(x)=C,\ \nabla u\neq 0$. Then (1.11) is equivalent to

$$\gamma_0''(t)=K\nabla u(\gamma_0(t)) \tag{1.12}$$

for a scalar K which remains to be determined. But the condition that $u(\gamma_0(t))=C$ implies

$$\gamma_0'(t)\cdot\nabla u(\gamma_0(t))=0,$$

and differentiating this gives

$$\gamma_0''(t)\cdot\nabla u(\gamma_0(t))=-\gamma_0'(t)\cdot D^2u(\gamma_0(t))\cdot\gamma_0'(t) \tag{1.13}$$

where D^2u is the matrix of second order partial derivatives of u. Comparing (1.12) and (1.13) gives K, and we obtain the ODE

$$\gamma_0''(t)=-\Big|\nabla u\big(\gamma_0(t)\big)\Big|^{-2}\Big[\gamma_0'(t)\cdot D^2u\big(\gamma_0(t)\big)\cdot\gamma_0'(t)\Big]\nabla u\big(\gamma_0(t)\big) \tag{1.14}$$

for a geodesic γ_0 lying in S.

A smooth m-dimensional surface $M\subset\mathbb{R}^n$ is characterized by the following property. Given $p\in M$, there is a neighborhood U of p in M and a smooth map

$\varphi : \mathcal{O} \to U$, from an open set $\mathcal{O} \subset \mathbb{R}^m$ bijectively to U, with injective derivative at each point. Such a map φ is called a *coordinate chart* on M. We call $U \subset M$ a coordinate patch. If all such maps φ are smooth of class C^k, we say M is a surface of class C^k. In §7 we will define analogous notions of a C^k surface with boundary, and of a C^k surface with corners.

There is associated an $m \times m$ matrix $G(x) = \big(g_{jk}(x)\big)$ of functions on $\mathcal{O}$, defined in terms of the inner product of vectors tangent to M :

$$g_{jk}(x) = D\varphi(x)e_j \cdot D\varphi(x)e_k = (\partial_j \varphi) \cdot (\partial_k \varphi) = \sum_{\ell=1}^{n} \frac{\partial \varphi_\ell}{\partial x_j} \frac{\partial \varphi_\ell}{\partial x_k}, \tag{1.15}$$

where $\{e_j : 1 \le j \le m\}$ is the standard orthonormal basis of $\mathbb{R}^m$. Equivalently,

$$G(x) = D\varphi(x)^t \; D\varphi(x). \tag{1.16}$$

We call (g_{jk}) the metric tensor of M, on U. Note that this matrix is positive definite. From a coordinate-independent point of view, the metric tensor on M specifies inner products of vectors tangent to M, using the inner product of $\mathbb{R}^n$.

Suppose there is another coordinate chart $\psi : \Omega \to U$, and we compare (g_{jk}) with $H = (h_{jk})$, given by

$$h_{jk}(y) = D\psi(y)e_j \cdot D\psi(y)e_k, \quad \text{i.e., } H(y) = D\psi(y)^t \; D\psi(y). \tag{1.17}$$

Using the Implicit Function Theorem, we can write $\varphi = \psi \circ F$, where $F : \mathcal{O} \to \Omega$ is a diffeomorphism. See Fig. 1.1. (See the exercises for more on this.) By the chain rule, $D\varphi(x) = D\psi(y)DF(x)$, where $y = F(x)$. Consequently,

$$G(x) = DF(x)^t H(y) DF(x), \tag{1.18}$$

or equivalently,

$$g_{jk}(x) = \sum_{i,\ell} \frac{\partial F_i}{\partial x_j} \frac{\partial F_\ell}{\partial x_k} h_{i\ell}(y). \tag{1.19}$$

There is a more general concept, of a Riemannian manifold. Here, to begin, one has a smooth manifold M. A brief description of what a manifold is can be found in §E. A manifold comes equipped with a set of coordinate charts $\{U_\ell, \varphi_\ell\}$, where $\{U_\ell\}$ is an open cover of M and $\varphi_\ell : \mathcal{O}_\ell \to U_\ell$ is a homeomorphism from an open set $\mathcal{O}_\ell \subset \mathbb{R}^n$ onto U_ℓ. Furthermore, if $U_\ell \cap U_m = U_{\ell m} \neq \emptyset$, then $F_{\ell m} = \varphi_\ell^{-1} \circ \varphi_m$ is a C^∞ diffeomorphism of $\mathcal{O}_{\ell m} = \varphi_m^{-1}(U_{\ell m}) \subset \mathcal{O}_m$ onto $\varphi_\ell^{-1}(U_{\ell m}) \subset \mathcal{O}_\ell$. A Riemannian metric on such a smooth manifold M is a smooth inner product on tangent vectors to M. If U and V are vector fields on M, then, on U_ℓ, we can write

$$U = \sum u^j(x) \frac{\partial}{\partial x_j}, \quad V = \sum v^j(x) \frac{\partial}{\partial x_j}, \tag{1.20}$$

and the inner product has the form

$$\langle U, V\rangle = \sum_{j,k} g_{jk}(x)\, u^j(x) v^k(x), \tag{1.21}$$

where (g_{jk}) is a smooth positive definite matrix. With respect to the (U_m, φ_m)-coordinates, we have

$$U = \sum a^j(x) \frac{\partial}{\partial x_j}, \quad V = \sum b^j(x) \frac{\partial}{\partial x_j}, \tag{1.22}$$

where, with $F = \varphi_\ell^{-1} \varphi_m$,

$$a^j = \sum_k \frac{\partial F_j}{\partial x_k} u^k, \quad b^j = \sum_k \frac{\partial F_j}{\partial x_k} v^k. \tag{1.23}$$

Then

$$\langle U, V\rangle = \sum_{j,k} h_{jk}(x)\ a^j(x) b^j(x), \tag{1.24}$$

where (h_{jk}) is related to (g_{jk}) exactly as in (1.18)–(1.19). This transformation law makes the Riemannian metric a "tensor field" of type $(2,0)$. In §12 we will define tensor fields of type (j,k); see also §F.

Now, if $\gamma : [a,b] \to M$ is a smooth curve on a Riemannian manifold, we define the length to be

$$L = \int_a^b \|\gamma'(t)\|\ dt, \tag{1.25}$$

as in (1.1), where now $\gamma'(t) = T(t)$ is a tangent vector to M at $\gamma(t)$, and

$$\|\gamma'(t)\|^2 = \langle T(t), T(t)\rangle. \tag{1.26}$$

In a local coordinate patch, if $\gamma(t) = x(t) = \big(x_1(t), \dots, x_n(t)\big)$, we have

$$\|\gamma'(t)\|^2 = \sum_{j,k} g_{jk}\big(x(t)\big) \dot{x}_j(t) \dot{x}_k(t). \tag{1.27}$$

Now suppose $\gamma_s,\ a \le t \le b$, is a one-parameter family of curves on M, all with the same end-points, represented in a coordinate system by $x_s(t)$. We produce a calculation, similar to that in (1.6)-(1.9), of $L'(0)$ when $L(s)$ is given by

$$L(s) = \int_a^b \Big[g_{jk}\big(x_s(t)\big) \dot{x}_s^j(t)\ \dot{x}_s^k(t)\Big]^{\frac{1}{2}} dt. \tag{1.28}$$

We use the notation $T^j = \dot{x}_0^j(t)$, $V^j = (\partial/\partial s)x_s^j(t)|_{s=0}$. Calculating in a spirit similar to that of (1.6), we have (with $x = x_0$)

$$L'(0) = \frac{1}{c_0}\int_a^b \Big[g_{jk}\frac{\partial}{\partial s}\dot{x}_s^j(t)\big|_{s=0}T^k + \frac{1}{2}V^j\frac{\partial g_{k\ell}}{\partial x_j}T^kT^\ell\Big]\,dt. \tag{1.29}$$

Now, in analogy with (1.7), we can write

$$\frac{d}{dt}\Big(g_{jk}\big(x(t)\big)V^jT^k\Big) = g_{jk}\frac{\partial}{\partial s}\dot{x}_s^j(t)\big|_{s=0}T^k + g_{jk}V^j\ddot{x}^k(t) + T^\ell\frac{\partial g_{jk}}{\partial x_\ell}V^jT^k. \tag{1.30}$$

Thus, by the fundamental theorem of calculus,

$$L'(0) = -\frac{1}{c_0}\int_a^b \Big[g_{jk}V^j\ddot{x}^k + T^\ell\frac{\partial g_{jk}}{\partial x_\ell}V^jT^k - \frac{1}{2}V^j\frac{\partial g_{k\ell}}{\partial x_j}T^kT^\ell\Big]\,dt \tag{1.31}$$

and the stationary condition $L'(0) = 0$ for all variations of the form described before implies

$$g_{jk}\ddot{x}^k(t) = -\Big(\frac{\partial g_{jk}}{\partial x_\ell} - \frac{1}{2}\frac{\partial g_{k\ell}}{\partial x_j}\Big)\dot{x}^k\dot{x}^\ell. \tag{1.32}$$

This takes a more standard form if we symmetrize the quantity in parentheses with respect to k and ℓ. We get the system of ODE

$$\ddot{x}^\ell + \dot{x}^j\ \dot{x}^k\ \Gamma^\ell{}_{jk} = 0, \tag{1.33}$$

where

$$g_{k\ell}\Gamma^\ell{}_{ij} = \frac{1}{2}\Big[\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\Big]. \tag{1.34}$$

In §11 we will re-derive the geodesic equation on a Riemannian manifold, making fundamental use of the notion of a covariant derivative, which we introduce there, and relating it to the quantities $\Gamma^\ell{}_{ij}$ appearing above, called "Christoffel symbols." Before getting to that, we will devote §§2–10 to a study of vector fields and differential forms, useful for further development of differential geometry.

Exercises

1. Let $S \subset \mathbb{R}^3$ be a surface of revolution about the z-axis, given by

$$x^2 + y^2 = f(z),$$

where $f(z)$ is a smooth positive function of z. Write out the geodesic equation, in the form (1.14).

2. Parametrize the surface S of Exercise 1 by

$$\varphi(u,v) = \bigl(g(u)\ \cos v, g(u)\ \sin v, u\bigr), \tag{1.35}$$

where $g(u) = \sqrt{f(u)}$. Work out the geodesic equation in these coordinates, in the form (1.33). Note that the metric is given by

$$G(u,v) = \begin{pmatrix} 1+g'(u)^2 & 0 \\ 0 & g(u)^2 \end{pmatrix}. \tag{1.36}$$

In Exercises 3–4, let $M \subset \mathbb{R}^n$ be a smooth m-dimensional surface, such as discussed in (1.15)–(1.19). In particular, the maps $\varphi : \mathcal{O} \to U \subset M$ and $\psi : \Omega \to U$ are as described there. Suppose $\varphi(x_0) = p,\ \psi(y_0) = p$.

3. Set $T_pM = \operatorname{Range} D\varphi(x_0)$, and denote N_pM its orthogonal complement. Pick a linear isomorphism $A : \mathbb{R}^{n-m} \to N_pM$, and define

$$\Phi : \mathcal{O} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n, \quad \Phi(x,z) = \varphi(x) + Az.$$

Show that Φ is a diffeomorphism from some neighborhood of $(x_0, 0) \in \mathcal{O} \times \mathbb{R}^{n-m}$ onto a neighborhood of p in $\mathbb{R}^n$.
Hint. Show that $D\Phi(x_0, 0)$ is surjective, hence bijective.

4. Make a similar construction for ψ, producing a map $\Psi : \Omega \times \mathbb{R}^{n-m} \to \mathbb{R}^n$. Show that, for x close to x_0 and y close to y_0,

$$\Psi^{-1} \circ \Phi(x,0) = (F(x), 0), \quad \Phi^{-1} \circ \Psi(y,0) = (F^{-1}(y), 0).$$

Deduce that F, introduced after (1.17), is a diffeomorphism (in particular, is smooth).

2. Flows and vector fields

Let $U \subset \mathbb{R}^n$ be open. A vector field on U is a smooth map

$$X : U \longrightarrow \mathbb{R}^n. \tag{2.1}$$

Consider the corresponding ODE

$$\frac{dy}{dt} = X(y), \quad y(0) = x, \tag{2.2}$$

with $x \in U$. A curve $y(t)$ solving (2.2) is called an integral curve of the vector field X. It is also called an *orbit*. For fixed t, write

$$y = y(t, x) = \mathcal{F}_X^t(x). \tag{2.3}$$

The locally defined $\mathcal{F}_X^t$, mapping (a subdomain of) U to U, is called the *flow* generated by the vector field X.

The vector field X defines a differential operator on scalar functions, as follows:

$$\mathcal{L}_X f(x) = \lim_{h \to 0} h^{-1} \bigl[f(\mathcal{F}_X^h x) - f(x) \bigr] = \frac{d}{dt} f(\mathcal{F}_X^t x)\big|_{t=0}. \tag{2.4}$$

We also use the common notation

$$\mathcal{L}_X f(x) = Xf, \tag{2.5}$$

that is, we apply X to f as a first order differential operator.

Note that, if we apply the chain rule to (2.4) and use (2.2), we have

$$\mathcal{L}_X f(x) = X(x) \cdot \nabla f(x) = \sum a_j(x) \frac{\partial f}{\partial x_j}, \tag{2.6}$$

if $X = \sum a_j(x) e_j$, with $\{e_j\}$ the standard basis of $\mathbb{R}^n$. In particular, using the notation (2.5), we have

$$a_j(x) = X x_j. \tag{2.7}$$

In the notation (2.5),

$$X = \sum a_j(x) \frac{\partial}{\partial x_j}. \tag{2.8}$$

We note that X is a *derivation*, i.e., a map on $C^\infty(U)$, linear over $\mathbb{R}$, satisfying

$$X(fg) = (Xf)g + f(Xg). \tag{2.9}$$

Conversely, any derivation on $C^\infty(U)$ defines a vector field, i.e., has the form (2.8), as we now show.

Proposition 2.1. *If X is a derivation on $C^\infty(U)$, then X has the form (2.8).*

Proof. Set $a_j(x) = Xx_j$, $X^\# = \sum a_j(x)\partial/\partial x_j$, and $Y = X - X^\#$. Then Y is a derivation satisfying $Yx_j = 0$ for each j; we aim to show that $Yf = 0$ for all f. Note that, whenever Y is a derivation

$$1 \cdot 1 = 1 \Rightarrow Y \cdot 1 = 2Y \cdot 1 \Rightarrow Y \cdot 1 = 0,$$

i.e., Y annihilates constants. Thus in this case Y annihilates all polynomials of degree ≤ 1.

Now we show $Yf(p) = 0$ for all $p \in U$. Without loss of generality, we can suppose $p = 0$, the origin. Then, by (C.8), we can take $b_j(x) = \int_0^1 (\partial_j f)(tx)dt$, and write

$$f(x) = f(0) + \sum b_j(x)x_j.$$

It immediately follows that Yf vanishes at 0, so the proposition is proved.

If U is a manifold, it is natural to regard a vector field X as a section of the tangent bundle of U, as explained in Appendix F. Of course, the characterization given in Proposition 2.1 makes good invariant sense on a manifold.

A fundamental fact about vector fields is that they can be "straightened out" near points where they do not vanish. To see this, suppose a smooth vector field X is given on U such that, for a certain $p \in U$, $X(p) \neq 0$. Then near p there is a hypersurface M which is nowhere tangent to X. We can choose coordinates near p so that p is the origin and M is given by $\{x_n = 0\}$. Thus we can identify a point $x' \in \mathbb{R}^{n-1}$ near the origin with $x' \in M$. We can define a map

$$\mathcal{F} : M \times (-t_0, t_0) \longrightarrow U \tag{2.10}$$

by

$$\mathcal{F}(x', t) = \mathcal{F}_X^t(x'). \tag{2.11}$$

This is C^∞ and has surjective derivative, so by the Inverse Function Theorem is a local diffeomorphism. This defines a new coordinate system near p, in whch the flow generated by X has the form

$$\mathcal{F}_X^s(x', t) = (x', t + s). \tag{2.12}$$

If we denote the new coordinates by $(u_1, \dots, u_n)$, we see that the following result is established.

Theorem 2.2. *If X is a smooth vector field on U with $X(p) \neq 0$, then there exists a coordinate system $(u_1, \dots, u_n)$ centered at p (so $u_j(p) = 0$) with respect to which*

$$X = \frac{\partial}{\partial u_n}. \tag{2.13}$$

Exercises

1. Suppose $h(x,y)$ is homogeneous of degree 0, i.e., $h(rx,ry) = h(x,y)$, so $h(x,y) = k(x/y)$. Show that the ODE
$$\frac{dy}{dx} = h(x,y)$$
is changed to a separable ODE for $u = u(x)$, if $u = y/x$.

2. Using Exercise 1, discuss constructing the integral curves of a vector field

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y},$$

when $f(x,y)$ and $g(x,y)$ are homogeneous of degree a, i.e.,

$$f(rx,ry) = r^a\ f(x,y) \text{ for } r > 0,$$

and similarly for g.

3. Describe the integral curves of

$$(x^2+y^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

4. Describe the integral curves of

$$A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y},$$

when $A(x,y) = a_1x + a_2y + a_3,\ B(x,y) = b_1x + b_2y + b_3$.

5. Let $X = f(x,y)(\partial/\partial x) + g(x,y)(\partial/\partial y)$ be a vector field on a disc $\Omega \subset \mathbb{R}^2$. Suppose
div $X = 0$, i.e., $\partial f/\partial x + \partial g/\partial y = 0$. Show that a function $u(x,y)$ such that

$$\partial u/\partial x = g, \quad \partial u/\partial y = -f$$

is given by a line integral. Show that $Xu = 0$ and hence integrate X.
Reconsider this exercise after reading §6.

6. Find the integral curves of the vector field
$$X = (2xy + y^2 + 1)\frac{\partial}{\partial x} + (x^2 + 1 - y^2)\frac{\partial}{\partial y}.$$

7. Show that
$$\mathrm{div}(e^v X) = e^v(\mathrm{div}\ X + Xv).$$
Hence, if X is a vector field on $\Omega \subset \mathbb{R}^2$ as in Exercise 5, show that you can integrate X if you can construct a function $v(x, y)$ such that $Xv = -\mathrm{div}\ X$. Construct such v if either
$$(\mathrm{div}\ X)/f(x, y) = \varphi(x) \quad \text{or} \quad (\mathrm{div}\ X)/g(x, y) = \psi(y).$$
For now, we define $\mathrm{div}\ X = \partial X_1/\partial x_1 + \cdots + \partial X_n/\partial x_n$. See §8 for another definition.

8. Find the integral curves of the vector field
$$X = 2xy\frac{\partial}{\partial x} + (x^2 + y^2 - 1)\frac{\partial}{\partial y}.$$

Let X be a vector field on $\mathbb{R}^n$, with a critical point at 0, i.e., $X(0) = 0$. Suppose that, for $x \in \mathbb{R}^n$ near 0,
$$X(x) = Ax + R(x), \quad \|R(x)\| = O(\|x\|^2), \tag{2.14}$$
where A is an $n \times n$ matrix. We call Ax the linearization of X at 0.

9. Suppose all the eigenvalues of A have negative real part. Construct a quadratic polynomial $Q : \mathbb{R}^n \to [0, \infty)$, such that $Q(0) = 0$, $\big(\partial^2 Q/\partial x_j \partial x_k\big)$ is positive definite, and such that, for any integral curve $x(t)$ of X as in (2.14),
$$\frac{d}{dt}Q(x(t)) < 0 \ \text{ if } \ t \geq 0$$
provided $x(0) = x_0 (\neq 0)$ is close enough to 0. Deduce that, for small enough C, if $\|x_0\| \leq C$, then $x(t)$ exists for all $t \geq 0$ and $x(y) \to 0$ as $t \to \infty$.
Hint. Take $Q(x) = \langle x, x\rangle$, using Exercise 10 below.

10. Let A be an $n \times n$ matrix, all of whose eigenvalues λ_j have *negative* real part. Show there exists a Hermitian inner product $\langle , \rangle$ on $\mathbb{C}^n$ such that $\mathrm{Re}\ \langle Au, u\rangle < 0$ for nonzero $u \in \mathbb{C}^n$.

Hint. Put A in Jordan normal form, but with εs instead of 1s above the diagonal, where ε is small compared with $|\mathrm{Re}\ \lambda_j|$.

3. Lie brackets

If $F : V \to W$ is a diffeomorphism between two open domains in $\mathbb{R}^n$, or between two smooth manifolds, and Y is a vector field on W, we define a vector field $F_\# Y$ on V so that

$$\mathcal{F}^t_{F_\# Y} = F^{-1} \circ \mathcal{F}^t_Y \circ F, \tag{3.1}$$

or equivalently, by the chain rule,

$$F_\# Y(x) = \bigl(DF^{-1}\bigr)\bigl(F(x)\bigr)Y\bigl(F(x)\bigr). \tag{3.2}$$

In particular, if $U \subset \mathbb{R}^n$ is open and X is a vector field on U, defining a flow $\mathcal{F}^t$, then for a vector field Y, $\mathcal{F}^t_\# Y$ is defined on most of U, for $|t|$ small, and we can define the Lie derivative:

$$\mathcal{L}_X Y = \lim_{h \to 0} \ h^{-1}\bigl(\mathcal{F}^h_\# Y - Y\bigr) = \frac{d}{dt}\mathcal{F}^t_\# Y\big|_{t=0}, \tag{3.3}$$

as a vector field on U.

Another natural construction is the operator-theoretic bracket:

$$[X, Y] = XY - YX, \tag{3.4}$$

where the vector fields X and Y are regarded as first order differential operators on $C^\infty(U)$. One verifies that (3.4) defines a derivation on $C^\infty(U)$, hence a vector field on U. The basic elementary fact about the Lie bracket is the following.

Theorem 3.1. *If X and Y are smooth vector fields, then*

$$\mathcal{L}_X Y = [X, Y]. \tag{3.5}$$

Proof. Let us first verify the identity in the special case

$$X = \frac{\partial}{\partial x_1}, \quad Y = \sum b_j(x)\frac{\partial}{\partial x_j}.$$

Then $\mathcal{F}^t_\# Y = \sum b_j(x + te_1)\partial/\partial x_j$. Hence, in this case $\mathcal{L}_X Y = \sum(\partial b_j/\partial x_1)\partial/\partial x_j$, and a straightforward calculation shows this is also the formula for $[X, Y]$, in this case.

Now we verify (3.5) in general, at any point $x_0 \in U$. First, if X is nonvanishing at x_0, we can choose a local coordinate system so the example above gives the identity. By continuity, we get the identity (3.5) on the closure of the set of points x_0 where $X(x_0) \neq 0$. Finally, if x_0 has a neighborhood where $X = 0$, clearly $\mathcal{L}_X Y = 0$ and $[X, Y] = 0$ at x_0. This completes the proof.

Corollary 3.2. *If X and Y are smooth vector fields on U, then*

$$\frac{d}{dt}\mathcal{F}^t_{X\#}Y = \mathcal{F}^t_{X\#}[X,Y] \tag{3.6}$$

for all t.

Proof. Since locally $\mathcal{F}^{t+s}_X = \mathcal{F}^s_X\mathcal{F}^t_X$, we have the same identity for $\mathcal{F}^{t+s}_{X\#}$, which yields (3.6) upon taking the s-derivative.

We make some further comments about cases when one can explicitly integrate a vector field X in the plane, exploiting "symmetries" that might be apparent. In fact, suppose one has in hand a vector field Y such that

$$[X,Y] = 0. \tag{3.7}$$

By (3.6), this implies $\mathcal{F}^t_{Y\#}X = X$ for all t. Suppose one has an explicit hold on the flow generated by Y, so one can produce explicit local coordinates (u,v) with respect to which

$$Y = \frac{\partial}{\partial u}. \tag{3.8}$$

In this coordinate system, write $X = a(u,v)\partial/\partial u + b(u,v)\partial/\partial v$. The condition (3.7) implies $\partial a/\partial u = 0 = \partial b/\partial u$, so in fact we have

$$X = a(v)\frac{\partial}{\partial u} + b(v)\frac{\partial}{\partial v}. \tag{3.9}$$

Integral curves of (3.9) satisfy

$$u' = a(v), \quad v' = b(v) \tag{3.10}$$

and can be found explicitly in terms of integrals; one has

$$\int b(v)^{-1}\,dv = t + C_1, \tag{3.11}$$

and then

$$u = \int a(v(t))\,dt + C_2. \tag{3.12}$$

More generally than (3.7), we can suppose that, for some constant c,

$$[X,Y] = cX, \tag{3.13}$$

which by (3.6) is the same as

$$\mathcal{F}^t_{Y\#} X = e^{-ct} X. \tag{3.14}$$

An example would be

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}, \tag{3.15}$$

where f and g satisfy "homogeneity" conditions of the form

$$f(r^a x, r^b y) = r^{a-c} f(x,y), \quad g(r^a x, r^b y) = r^{b-c} g(x,y), \tag{3.16}$$

for $r > 0$; in such a case one can take explicitly

$$\mathcal{F}^t_Y(x,y) = (e^{at}x, e^{bt}y). \tag{3.17}$$

Now, if one again has (3.8) in a local coordinate system (u,v), then X must have the form

$$X = e^{cu}\Big[a(v)\frac{\partial}{\partial u} + b(v)\frac{\partial}{\partial v}\Big] \tag{3.18}$$

which can be explicitly integrated, since

$$u' = e^{cu}a(v),\ v' = e^{cu}b(v) \Longrightarrow \frac{du}{dv} = \frac{a(v)}{b(v)}. \tag{3.19}$$

The hypothesis (3.13) implies that the linear span (over $\mathbb{R}$) of X and Y is a two dimensional solvable Lie algebra. Sophus Lie devoted a good deal of effort to examining when one could use constructions of solvable Lie algebras of vector fields to explicitly integrate vector fields; his investigations led to his foundation of what is now called the theory of Lie groups.

Exercises

1. Verify that the bracket (3.4) satisfies the "Jacobi identity"

$$[X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z],$$

i.e.,

$$[\mathcal{L}_X, \mathcal{L}_Y]Z = \mathcal{L}_{[X,Y]}Z.$$

2. Find the integral curves of

$$X = (x+y^2)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

using (3.16).

3. Find the integral curves of

$$X = (x^2y + y^5)\frac{\partial}{\partial x} + (x^2 + xy^2 + y^4)\frac{\partial}{\partial y}.$$

4. Integration on Riemannian manifolds

As stated at the end of §1, a Riemannian metric on a smooth m-dimensional manifold M is a smooth inner product on tangent vectors. To a local coordinate chart $\varphi : \mathcal{O} \to U \subset M$, there is associated an $m \times m$ matrix $G(x) = \big(g_{jk}(x)\big)$ of functions on $\mathcal{O}$, satisfying

$$\langle U, V\rangle = \sum g_{jk}(x)\ u^j(x) v^k(x), \tag{4.1}$$

where, in this coordinate chart, $U = \sum u^j(x)\partial/\partial x_j$ and $V = \sum v^j(x)\partial/\partial x_j$. In particular, if M is a surface in $\mathbb{R}^n$, the induced Riemannian metric is given by

$$g_{jk}(x) = D\varphi(x)e_j \cdot D\varphi(x)e_k = \sum_{\ell=1}^{n} \frac{\partial \varphi_\ell}{\partial x_j}\frac{\partial \varphi_\ell}{\partial x_k}, \tag{4.2}$$

where $\{e_j : 1 \le j \le m\}$ is the standard orthonormal basis of $\mathbb{R}^m$. Equivalently,

$$G(x) = D\varphi(x)^t\ D\varphi(x). \tag{4.3}$$

Suppose there is another coordinate chart $\psi : \Omega \to U$, We can write $\varphi = \psi \circ F$, where $F : \mathcal{O} \to \Omega$ is a diffeomorphism. As noted in §1, if $H = (h_{jk})$ expresses the Riemannian metric in the second coordinate system, then

$$G(x) = DF(x)^t H(y) DF(x), \tag{4.4}$$

or equivalently,

$$g_{jk}(x) = \sum_{i,\ell} \frac{\partial F_i}{\partial x_j}\frac{\partial F_\ell}{\partial x_k} h_{i\ell}(y). \tag{4.5}$$

If $f : M \to \mathbb{R}$ is a continuous function supported on a coordinate chart U, we will define the volume integral by

$$\int_M f\ dV = \int_{\mathcal{O}} f \circ \varphi(x)\ \sqrt{g(x)}\ dx, \tag{4.6}$$

where

$$g(x) = \det G(x). \tag{4.7}$$

We need to know that this is independent of the choice of coordinate chart $\varphi : \mathcal{O} \to U$. Thus, if we use $\psi : \Omega \to U$ instead, we want to show that (4.6) is equal to

$\int_\Omega f\circ\psi(y)\,\sqrt{h(y)}\,dy$, where $h(y) = \operatorname{Det} H(y)$. Indeed, since $f\circ\psi\circ F = f\circ\varphi$, we can apply the change of variable formula of multi-variable calculus, to get

$$\int\limits_\Omega f\circ\psi(y)\,\sqrt{h(y)}\,dy = \int\limits_{\mathcal{O}} f\circ\varphi(x)\,\sqrt{h(F(x))}\,|\mathrm{Det}\ DF(x)|\,dx. \tag{4.8}$$

Now, (4.4) implies that

$$\sqrt{g(x)} = |\det\ DF(x)|\,\sqrt{h(y)}, \tag{4.9}$$

so the right side of (4.8) is seen to be equal to (4.6), and our surface integral is well defined, at least for f supported in a coordinate patch. More generally, if $f : M \to \mathbb{R}$ has compact support, write it as a finite sum of terms, each supported on a coordinate patch, and use (4.6) on each patch. If dim $M = 2$, we will tend to use dS rather than dV.

Let us consider some special cases. First, consider a curve in $\mathbb{R}^n$, say $\varphi : [a, b] \to \mathbb{R}^n$. Then $G(x)$ is a 1×1 matrix, namely $G(x) = |\varphi'(x)|^2$. If we denote the curve in $\mathbb{R}^n$ by γ, rather than M, the formula (4.6) becomes

$$\int\limits_\gamma f\,ds = \int_a^b f\circ\varphi(x)\,|\varphi'(x)|\,dx. \tag{4.10}$$

Next, let us consider a surface $M\subset\mathbb{R}^3$, with a coordinate chart $\varphi : \mathcal{O} \to U \subset M$. For f supported in U, an alternative way to write the surface integral is

$$\int\limits_M f\,dS = \int\limits_{\mathcal{O}} f\circ\varphi(x)\,|\partial_1\varphi\times\partial_2\varphi|\,dx_1dx_2, \tag{4.11}$$

where $u\times v$ is the cross product of vectors u and v in $\mathbb{R}^3$. To see this, we compare this integrand with the one in (4.6). In this case,

$$g = \det\begin{pmatrix}\partial_1\varphi\cdot\partial_1\varphi & \partial_1\varphi\cdot\partial_2\varphi\\ \partial_2\varphi\cdot\partial_1\varphi & \partial_2\varphi\cdot\partial_2\varphi\end{pmatrix} = |\partial_1\varphi|^2|\partial_2\varphi|^2 - (\partial_1\varphi\cdot\partial_2\varphi)^2. \tag{4.12}$$

Recall (see (J.16)) that $|u\times v| = |u|\,|v|\,|\sin\theta|$, where θ is the angle between u and v. Equivalently, since $u\cdot v = |u|\,|v|\,\cos\theta$,

$$|u\times v|^2 = |u|^2|v|^2\big(1-\cos^2\theta\big) = |u|^2|v|^2 - (u\cdot v)^2. \tag{4.13}$$

Thus we see that $|\partial_1\varphi\times\partial_2\varphi| = \sqrt{g}$, in this case, and (4.11) is equivalent to (4.6).

An important class of surfaces is the class of graphs of smooth functions. Let $u\in C^1(\Omega)$, for an open $\Omega\subset\mathbb{R}^{n-1}$, and let M be the graph of $z = u(x)$. The map

$\varphi(x) = \big(x, u(x)\big)$ provides a natural coordinate system, in which the metric tensor is given by

$$g_{jk}(x) = \delta_{jk} + \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}. \tag{4.14}$$

If u is C^1, we see that g_{jk} is continuous. To calculate $g = \operatorname{Det}(g_{jk})$, at a given point $p \in \Omega$, if $\nabla u(p) \neq 0$, rotate coordinates so that $\nabla u(p)$ is parallel to the x_1 axis. We see that

$$\sqrt{g} = \big(1 + |\nabla u|^2\big)^{1/2}. \tag{4.15}$$

In particular, the $(n-1)$-dimensional volume of the surface M is given by

$$V_{n-1}(M) = \int\limits_M dV = \int\limits_\Omega \big(1 + |\nabla u(x)|^2\big)^{1/2}\, dx. \tag{4.16}$$

Particularly important examples of surfaces are the unit spheres S^{n-1} in $\mathbb{R}^n$,

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Spherical polar coordinates on $\mathbb{R}^n$ are defined in terms of a smooth diffeomorphism

$$R : (0,\infty) \times S^{n-1} \longrightarrow \mathbb{R}^n \setminus 0, \quad R(r,\omega) = r\omega. \tag{4.17}$$

If $(h_{\ell m})$ denotes the metric tensor on S^{n-1} induced from its inclusion in $\mathbb{R}^n$, we see that the Euclidean metric tensor can be written

$$\big(e_{jk}\big) = \begin{pmatrix} 1 & \\ & r^2 h_{\ell m} \end{pmatrix}. \tag{4.18}$$

Thus

$$\sqrt{e} = r^{n-1}\sqrt{h}. \tag{4.19}$$

We therefore have the following result for integrating a function in spherical polar coordinates.

$$\int\limits_{\mathbb{R}^n} f(x)\, dx = \int\limits_{S^{n-1}} \Big[\int_0^\infty f(r\omega) r^{n-1}\, dr\Big]\, dS(\omega). \tag{4.20}$$

We next compute the $(n-1)$-dimensional area A_{n-1} of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, using (12.20) together with the computation

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}, \tag{4.21}$$

which will be established in Exercise 2 below. Note that, whenever $f(x) = \varphi(|x|)$, (4.20) yields

$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\,dx = A_{n-1} \int_0^\infty \varphi(r) r^{n-1}\,dr. \tag{4.22}$$

In particular, taking $\varphi(r) = e^{-r^2}$ and using (4.21), we have

$$\pi^{n/2} = A_{n-1} \int_0^\infty e^{-r^2} r^{n-1}\,dr = \frac{1}{2} A_{n-1} \int_0^\infty e^{-s} s^{n/2-1}\,ds, \tag{4.23}$$

where we used the substitution $s = r^2$ to get the last identity. We hence have

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{4.24}$$

where $\Gamma(z)$ is Euler's Gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\,ds. \tag{4.25}$$

We need to complement (4.24) with some results on $\Gamma(z)$ allowing a computation of $\Gamma(n/2)$ in terms of more familiar quantities. Of course, setting $z = 1$ in (4.25), we immediately get

$$\Gamma(1) = 1. \tag{4.26}$$

Also, setting $n = 1$ in (4.23), we have

$$\pi^{1/2} = 2 \int_0^\infty e^{-r^2}\,dr = \int_0^\infty e^{-s} s^{-1/2}\,ds,$$

or

$$\Gamma\Big(\frac{1}{2}\Big) = \pi^{1/2}. \tag{4.27}$$

We can proceed inductively from (4.26)–(4.27) to a formula for $\Gamma(n/2)$ for any $n \in \mathbb{Z}^+$, using the following.

Lemma 4.1. *For all $z > 0$,*

$$\Gamma(z+1) = z\Gamma(z). \tag{4.28}$$

Proof. We can write

$$\Gamma(z+1) = -\int_0^\infty \Big(\frac{d}{ds} e^{-s}\Big) s^z\,ds = \int_0^\infty e^{-s}\,\frac{d}{ds}\big(s^z\big)\,ds,$$

the last identity by integration by parts. The last expression here is seen to equal the right side of (4.28).

Consequently, for $k \in \mathbb{Z}^+$,

$$\Gamma(k) = (k-1)!, \quad \Gamma\Big(k+\frac{1}{2}\Big) = \Big(k-\frac{1}{2}\Big)\cdots\Big(\frac{1}{2}\Big)\pi^{1/2}. \tag{4.29}$$

Thus (4.24) can be rewritten

$$A_{2k-1} = \frac{2\pi^k}{(k-1)!}, \quad A_{2k} = \frac{2\pi^k}{\big(k-\frac{1}{2}\big)\cdots\big(\frac{1}{2}\big)}. \tag{4.30}$$

We discuss another important example of a smooth surface, in the space $M(n) \approx \mathbb{R}^{n^2}$ of real $n \times n$ matrices, namely $SO(n)$, the set of matrices $T \in M(n)$ satisfying $T^tT = I$ and $\det T > 0$ (hence $\det T = 1$). The exponential map Exp: $M(n) \to M(n)$ defined by $\mathrm{Exp}(A) = e^A$ has the property

$$\mathrm{Exp} : \mathrm{Skew}(n) \longrightarrow SO(n), \tag{4.31}$$

where $\mathrm{Skew}(n)$ is the set of skew-symmetric matrices in $M(n)$. Also $D\ \mathrm{Exp}(0)A = A$; hence

$$D\ \mathrm{Exp}(0) = \iota : \mathrm{Skew}(n) \hookrightarrow M(n). \tag{4.32}$$

It follows from the Inverse Function Theorem that there is a neighborhood $\mathcal{O}$ of 0 in $\mathrm{Skew}(n)$ which is mapped by Exp diffeomorphically onto a smooth surface $U \subset M(n)$, of dimension $m = n(n-1)/2$. Furthermore, U is a neighborhood of I in $SO(n)$. For general $T \in SO(n)$, we can define maps

$$\varphi_T : \mathcal{O} \longrightarrow SO(n), \quad \varphi_T(A) = T\ \mathrm{Exp}(A), \tag{4.33}$$

and obtain coordinate charts in $SO(n)$, which is consequently a smooth manifold of dimension $\frac{1}{2}n(n-1)$ in $M(n)$. Note that $SO(n)$ is a closed bounded subset of $M(n)$; hence it is compact.

We use the inner product on $M(n)$ computed componentwise; equivalently,

$$\langle A, B\rangle = \mathrm{Tr}\,(B^tA) = \mathrm{Tr}\,(BA^t). \tag{4.34}$$

This produces a metric tensor on $SO(n)$. The surface integral over $SO(n)$ has the following important invariance property.

Proposition 4.2. *Given $f \in C\big(SO(n)\big)$, if we set*

$$\rho_T f(X) = f(XT), \quad \lambda_T f(X) = f(TX), \tag{4.35}$$

for $T, X \in SO(n)$, we have

$$\int\limits_{SO(n)} \rho_T f \, dV = \int\limits_{SO(n)} \lambda_T f \, dV = \int\limits_{SO(n)} f \, dV. \tag{4.36}$$

Proof. Given $T \in SO(n)$, the maps $R_T, L_T : M(n) \to M(n)$ defined by $R_T(X) = XT$, $L_T(X) = TX$ are easily seen from (4.34) to be isometries. Thus they yield maps of $SO(n)$ to itself which preserve the metric tensor, proving (4.36).

Since $SO(n)$ is compact, its total volume $V\big(SO(n)\big) = \int_{SO(n)} 1 \, dV$ is finite. We define the integral with respect to "Haar measure"

$$\int\limits_{SO(n)} f(g) \, dg = \frac{1}{V\big(SO(n)\big)} \int\limits_{SO(n)} f \, dV. \tag{4.37}$$

This is used in many arguments involving "averaging over rotations." One example will arise in the proof of Proposition 9.5. Also compare the discussion of Haar measure in §H.

Exercises

1. Define $\varphi : [0, \theta] \to \mathbb{R}^2$ to be $\varphi(t) = (\cos t, \sin t)$. Show that, if $0 < \theta \le 2\pi$, the image of $[0, \theta]$ under φ is an arc of the unit circle, of length θ. Deduce that the unit circle in $\mathbb{R}^2$ has total length 2π.
Remark. Use the definition of π given in §L.
This length formula provided the original definition of π, in ancient Greek geometry.

2. Let I_n denote the left side of (4.21). Show that $I_n = I_1^n$. Show that

$$I_2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta = 2\pi \int_0^{\infty} e^{-r^2} r \, dr.$$

Use the substitution $s = r^2$ to show that $I_2 = \pi$. Hence deduce (4.21).

3. Compute the volume of the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$.
Hint. Apply (4.22) with $\varphi = \chi_{[0,1]}$.

4. Suppose M is a surface in $\mathbb{R}^n$ of dimension 2, and $\varphi : \mathcal{O} \to U \subset M$ is a coordinate chart, with $\mathcal{O} \subset \mathbb{R}^2$. Set $\varphi_{jk}(x) = \big(\varphi_j(x), \varphi_k(x)\big)$, so $\varphi_{jk} : \mathcal{O} \to \mathbb{R}^2$. Show that the formula (4.6) for the surface integral is equivalent to

$$\int\limits_M f\ dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{\sum_{j<k} \Big(\det\ D\varphi_{jk}(x)\Big)^2}\ dx.$$

Hint. Show that the quantity under $\sqrt{}$ is equal to (4.12).

5. If M is an m-dimensional surface, $\varphi : \mathcal{O} \to M \subset M$ a coordinate chart, for $J = (j_1, \dots, j_m)$ set

$$\varphi_J(x) = \big(\varphi_{j_1}(x), \dots, \varphi_{j_m}(x)\big), \quad \varphi_J : \mathcal{O} \to \mathbb{R}^m.$$

Show that the formula (4.6) is equivalent to

$$\int\limits_M f\ dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{\sum_{j_1<\cdots<j_m} \Big(\det\ D\varphi_J(x)\Big)^2}\ dx.$$

Hint. Reduce to the following. For fixed $x_0 \in \mathcal{O}$, the quantity under $\sqrt{}$ is equal to $g(x)$ at $x = x_0$, in the case $D\varphi(x_0) = \big(D\varphi_1(x_0), \dots, D\varphi_m(x_0), 0, \dots, 0\big)$. Reconsider this problem when working on the exercises for §5.

6. Let M be the graph in $\mathbb{R}^{n+1}$ of $x_{n+1} = u(x),\ x \in \mathcal{O} \subset \mathbb{R}^n$. Let N be the unit normal to M given by $N = \big(1 + |\nabla u|^2\big)^{-1/2}(-\nabla u, 1)$. Show that, for a continuous function $f : M \to \mathbb{R}^{n+1}$,

$$\int\limits_M f \cdot N\, dS = \int\limits_{\mathcal{O}} f\big(x, u(x)\big) \cdot \big(-\nabla u(x), 1\big)\, dx.$$

The left side is often denoted $\int_M f \cdot d\mathbf{S}$.

7. Writing the equation of the upper hemisphere in $\mathbb{R}^{n+1}$ as $x_{n+1} = \sqrt{1 - |x|^2},\ x \in B_n =$ unit ball in $\mathbb{R}^n$, show that the area of S^n satisfies

$$A_n = 2 \int\limits_{B_n} \big(1 - |x|^2\big)^{-1/2}\, dx.$$

Making use of (4.22), deduce that

$$A_n = \vartheta_n A_{n-1}, \quad \vartheta_n = 2\int_0^1 \frac{r^{n-1}}{\sqrt{1-r^2}}\,dr = 2\int_0^{\pi/2} \sin^{n-1}\theta\;d\theta.$$

Use this to re-derive (4.30).
Hint. Write $\sin^{n-1}\theta = \sin^{n-3}\theta(1-\cos^2\theta)$ and integrate by parts to show that

$$\vartheta_n = \frac{n-2}{n-1}\vartheta_{n-2}.$$

Show directly that $\vartheta_1 = \pi$ and $\vartheta_2 = 2$, and deduce that

$$\vartheta_n = \pi^{1/2}\Gamma\big(\tfrac{n}{2}\big)/\Gamma\big(\tfrac{n+1}{2}\big).$$

8. For $G = SO(n)$, define $\kappa : G \to G$ by $\kappa(g) = g^{-1}$. Show that κ is an isometry. Deduce that

$$\int\limits_G f(g)\,dg = \int\limits_G f(g^{-1})\,dg.$$

Hint. Use R_{g_0} and $L_{g_0^{-1}}$ to reduce the problem to showing that $D\kappa(I)$ is an isometry on $T_I G = \mathrm{Skew}(n)$. What is $D\kappa(I)$?

5. Differential forms

It is very desirable to be able to make constructions which depend as little as possible on a particular choice of coordinate system. The calculus of differential forms, whose study we now take up, is one convenient set of tools for this purpose.

We start with the notion of a 1-form. It is an object that gets integrated over a curve; formally, a 1-form on $\Omega \subset \mathbb{R}^n$ is written

$$\alpha = \sum_j a_j(x)\ dx_j. \tag{5.1}$$

If $\gamma : [a,b] \to \Omega$ is a smooth curve, we set

$$\int\limits_\gamma \alpha = \int_a^b \sum a_j\big(\gamma(t)\big)\gamma_j'(t)\ dt. \tag{5.2}$$

In other words,

$$\int\limits_\gamma \alpha = \int\limits_I \gamma^*\alpha \tag{5.3}$$

where $I = [a,b]$ and $\gamma^*\alpha = \sum_j a_j(\gamma(t))\gamma_j'(t)$ is the *pull-back* of α under the map γ. More generally, if $F : \mathcal{O} \to \Omega$ is a smooth map ($\mathcal{O} \subset \mathbb{R}^m$ open), the pull-back $F^*\alpha$ is a 1-form on $\mathcal{O}$ defined by

$$F^*\alpha = \sum_{j,k} a_j(F(y)) \frac{\partial F_j}{\partial y_k}\ dy_k. \tag{5.4}$$

The usual change of variable for integrals gives

$$\int\limits_\gamma \alpha = \int\limits_\sigma F^*\alpha \tag{5.5}$$

if γ is the curve $F \circ \sigma$.

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism, and

$$X = \sum b^j(x) \frac{\partial}{\partial x_j} \tag{5.6}$$

is a vector field on Ω, recall that we have the vector field on $\mathcal{O}$:

$$F_\# X(y) = \big(DF^{-1}(p)\big)X(p), \quad p = F(y). \tag{5.7}$$

If we define a pairing between 1-forms and vector fields on Ω by

$$\langle X, \alpha \rangle = \sum_j b^j(x) a_j(x) = b \cdot a, \tag{5.8}$$

a simple calculation gives

$$\langle F_\# X, F^* \alpha \rangle = \langle X, \alpha \rangle \circ F. \tag{5.9}$$

Thus, a 1-form on Ω is characterized at each point $p \in \Omega$ as a linear transformation of *vectors* at p to $\mathbb{R}$.

More generally, we can regard a k-form α on Ω as a k-multilinear map on vector fields:

$$\alpha(X_1, \dots, X_k) \in C^\infty(\Omega); \tag{5.10}$$

we impose the further condition of anti-symmetry:

$$\alpha(X_1, \dots, X_j, \dots, X_\ell, \dots, X_k) = -\alpha(X_1, \dots, X_\ell, \dots, X_j, \dots, X_k). \tag{5.11}$$

There is a special notation we use for k-forms. If $1 \le j_1 < \cdots < j_k \le n$, $j = (j_1, \dots, j_k)$, we set

$$\alpha = \sum_j a_j(x)\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} \tag{5.12}$$

where

$$a_j(x) = \alpha(\partial_{j_1}, \dots, \partial_{j_k}), \quad \partial_j = \frac{\partial}{\partial x_j}. \tag{5.13}$$

More generally, we assign meaning to (5.12) summed over all k-indices $(j_1, \dots, j_k)$, where we identify

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = (\operatorname{sgn} \sigma)\ dx_{j_{\sigma(1)}} \wedge \cdots \wedge dx_{j_{\sigma(k)}}, \tag{5.14}$$

σ being a permutation of $\{1, \dots, k\}$. If any $j_m = j_\ell$ $(m \ne \ell)$, then (5.14) vanishes. A common notation for the statement that α is a k-form on Ω is

$$\alpha \in \Lambda^k(\Omega). \tag{5.15}$$

In particular, we can write a 2-form β as

$$\beta = \sum b_{jk}(x)\ dx_j \wedge dx_k \tag{5.16}$$

and pick coefficients satisfying $b_{jk}(x) = -b_{kj}(x)$. According to (5.12)–(5.13), if we set $U = \sum u_j(x)\partial/\partial x_j$ and $V = \sum v_j(x)\partial/\partial x_j$, then

$$\beta(U, V) = 2\sum b_{jk}(x)u^j(x)v^k(x). \tag{5.17}$$

If b_{jk} is not required to be antisymmetric, one gets $\beta(U,V) = \sum(b_{jk} - b_{kj})u^j v^k$.

If $F : \mathcal{O} \to \Omega$ is a smooth map as above, we define the pull-back $F^*\alpha$ of a k-form α, given by (5.12), to be

$$F^*\alpha = \sum_j a_j\big(F(y)\big)(F^*dx_{j_1}) \wedge \cdots \wedge (F^*dx_{j_k}) \tag{5.18}$$

where

$$F^*dx_j = \sum_\ell \frac{\partial F_j}{\partial y_\ell}\, dy_\ell, \tag{5.19}$$

the algebraic computation in (5.18) being performed using the rule (5.14). Extending (5.9), if F is a diffeomorphism, we have

$$(F^*\alpha)(F_\# X_1, \dots, F_\# X_k) = \alpha(X_1, \dots, X_k) \circ F. \tag{5.20}$$

If $B = (b_{jk})$ is an $n \times n$ matrix, then, by (5.14),

$$\begin{aligned} &\Big(\sum_k b_{1k}dx_k\Big) \wedge \Big(\sum_k b_{2k}dx_k\Big) \wedge \cdots \wedge \Big(\sum_k b_{nk}dx_k\Big) \\ &\quad = \Big(\sum_\sigma (\text{sgn } \sigma) b_{1\sigma(1)}b_{2\sigma(2)}\cdots b_{n\sigma(n)}\Big) dx_1 \wedge \cdots \wedge dx_n \\ &\quad = \big(\det B\big) dx_1 \wedge \cdots \wedge dx_n, \end{aligned} \tag{5.21}$$

Hence, if $F : \mathcal{O} \to \Omega$ is a C^1 map between two domains of dimension n, and $\alpha = A(x)dx_1 \wedge \cdots \wedge dx_n$ is an n-form on Ω, then

$$F^*\alpha = \det DF(y)\ A(F(y))\ dy_1 \wedge \cdots \wedge dy_n. \tag{5.22}$$

Comparison with the change of variable formula for multiple integrals suggests that one has an intrinsic definition of $\int_\Omega \alpha$ when α is an n-form on Ω, $n = \dim\ \Omega$. To implement this, we need to take into account that $\det DF(y)$ rather than $|\det DF(y)|$ appears in (5.21). We say a smooth map $F : \mathcal{O} \to \Omega$ between two open subsets of $\mathbb{R}^n$ *preserves orientation* if $\det DF(y)$ is everywhere positive. The object called an "orientation" on Ω can be identified as an equivalence class of nowhere vanishing n-forms on Ω, two such forms being equivalent if one is a multiple of another by a positive function in $C^\infty(\Omega)$; the standard orientation on $\mathbb{R}^n$ is determined by $dx_1 \wedge \cdots \wedge dx_n$. If S is an n-dimensional surface in $\mathbb{R}^{n+k}$, an orientation on S can also be specified by a nowhere vanishing form $\omega \in \Lambda^n(S)$. If such a form exists, S is said to be orientable. The equivalence class of positive multiples $a(x)\omega$ is said to consist of "positive" forms. A smooth map $\psi : S \to M$ between oriented n-dimensional surfaces preserves orientation provided $\psi^*\sigma$ is positive on S whenever $\sigma \in \Lambda^n(M)$ is positive. If S is oriented, one can choose coordinate charts which are all orientation preserving. We mention that there exist surfaces which cannot be oriented, such as two-dimensional real projective space.

If $\mathcal{O}, \Omega$ are open in $\mathbb{R}^n$ and $F : \mathcal{O} \to \Omega$ is an orientation preserving diffeomorphism, we have

$$\int_{\mathcal{O}} F^*\alpha = \int_{\Omega} \alpha. \tag{5.23}$$

More generally, if S is an n-dimensional manifold with an orientation, say the image of an open set $\mathcal{O} \subset \mathbb{R}^n$ by $\varphi : \mathcal{O} \to S$, carrying the natural orientation of $\mathcal{O}$, we can set

$$\int_S \alpha = \int_{\mathcal{O}} \varphi^*\alpha \tag{5.24}$$

for an n-form α on S. If it takes several coordinate patches to cover S, define $\int_S \alpha$ by writing α as a sum of forms, each supported on one patch.

We need to show that this definition of $\int_S \alpha$ is independent of the choice of coordinate system on S (as long as the orientation of S is respected). Thus, suppose $\varphi : \mathcal{O} \to U \subset S$ and $\psi : \Omega \to U \subset S$ are both coordinate patches, so that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is an orientation-preserving diffeomorphism, as in Fig. 1.1. We need to check that, if α is an n-form on S, supported on U, then

$$\int_{\mathcal{O}} \varphi^*\alpha = \int_{\Omega} \psi^*\alpha. \tag{5.25}$$

To see this, first note that, for any form α of any degree,

$$\psi \circ F = \varphi \Longrightarrow \varphi^*\alpha = F^*\psi^*\alpha. \tag{5.26}$$

It suffices to check this for $\alpha = dx_j$. Then (5.14) gives $\psi^*\, dx_j = \sum (\partial\psi_j/\partial x_\ell) dx_\ell$, so

$$F^*\psi^*\, dx_j = \sum_{\ell,m} \frac{\partial F_\ell}{\partial x_m}\frac{\partial \psi_j}{\partial x_\ell}\, dx_m, \quad \varphi^*\, dx_j = \sum_m \frac{\partial \varphi_j}{\partial x_m}\, dx_m; \tag{5.27}$$

but the identity of these forms follows from the chain rule:

$$D\varphi = (D\psi)(DF) \Longrightarrow \frac{\partial \varphi_j}{\partial x_m} = \sum_\ell \frac{\partial \psi_j}{\partial x_\ell}\frac{\partial F_\ell}{\partial x_m}. \tag{5.28}$$

Now that we have (5.26), we see that the left side of (5.25) is equal to

$$\int_{\mathcal{O}} F^*(\psi^*\alpha), \tag{5.29}$$

which is equal to the right side of (5.25), by (5.23). Thus the integral of an n-form over an oriented n-dimensional surface is well defined.

Exercises

1. If $F : U_0 \to U_1$ and $G : U_1 \to U_2$ are smooth maps and $\alpha \in \Lambda^k(U_2)$, then (5.26) implies

$$(G \circ F)^*\alpha = F^*(G^*\alpha) \text{ in } \Lambda^k(U_0).$$

In the special case that $U_j = \mathbb{R}^n$ and F and G are linear maps, and $k = n$, show that this identity implies

$$\det\,(GF) = (\det\,F)(\det\,G).$$

Let $\Lambda^k\mathbb{R}^n$ denote the space of k-forms (5.12) with constant coefficients. One can show that $\dim_{\mathbb{R}}\Lambda^k\mathbb{R}^n = \binom{n}{k}$. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then T^* preserves this class of spaces; we denote the map by $\Lambda^kT^* : \Lambda^k\mathbb{R}^n \longrightarrow \Lambda^k\mathbb{R}^m$. Similarly, replacing T by T^* yields

$$\Lambda^kT : \Lambda^k\mathbb{R}^m \longrightarrow \Lambda^k\mathbb{R}^n.$$

2. Show that Λ^kT is uniquely characterized as a linear map from $\Lambda^k\mathbb{R}^m$ to $\Lambda^k\mathbb{R}^n$ which satisfies

$$(\Lambda^kT)(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k), \quad v_j \in \mathbb{R}^m.$$

3. If $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$, define an inner product on $\Lambda^k\mathbb{R}^n$ by declaring an orthonormal basis to be

$$\{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}.$$

Show that, if $\{u_1, \dots, u_n\}$ is any other orthonormal basis of $\mathbb{R}^n$, then the set $\{u_{j_1} \wedge \cdots \wedge u_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}$ is an orthonormal basis of $\Lambda^k\mathbb{R}^n$.

4. Let $\varphi : \mathcal{O} \to \mathbb{R}^n$ be smooth, with $\mathcal{O} \subset \mathbb{R}^m$ open. Show that, for each $x \in \mathcal{O}$,

$$\|\Lambda^m D\varphi(x)\omega\|^2 = \det\, D\varphi(x)^t D\varphi(x),$$

where $\omega = e_1 \wedge \cdots \wedge e_m$. Take another look at Exercise 5 of §4.

5. Verify the identity (5.20).

6. Products and exterior derivatives of forms

Having discussed the notion of a differential form as something to be integrated, we now consider some operations on forms. There is a *wedge product*, or exterior product, characterized as follows. If $\alpha \in \Lambda^k(\Omega)$ has the form (5.12) and if

$$\beta = \sum_i b_i(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Lambda^\ell(\Omega), \tag{6.1}$$

define

$$\alpha \wedge \beta = \sum_{j,i} a_j(x) b_i(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \tag{6.2}$$

in $\Lambda^{k+\ell}(\Omega)$. A special case of this arose in (5.18)–(5.21). We retain the equivalence (5.14). It follows easily that

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{6.3}$$

Also, one can show that

$$(\alpha \wedge \beta)(X_1, \dots, X_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) \alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \beta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}).$$

In addition, there is an *interior product* if $\alpha \in \Lambda^k(\Omega)$ with a vector field X on Ω, producing $\iota_X \alpha = \alpha \rfloor X \in \Lambda^{k-1}(\Omega)$, defined by

$$(\alpha \rfloor X)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}). \tag{6.4}$$

Consequently, if $\alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$, $\partial_i = \partial/\partial x_i$, then

$$\alpha \rfloor \partial_{j_\ell} = (-1)^{\ell-1} dx_{j_1} \wedge \cdots \wedge \widehat{dx}_{j_\ell} \wedge \cdots \wedge dx_{j_k} \tag{6.5}$$

where $\widehat{dx}_{j_\ell}$ denotes removing the factor dx_{j_ℓ}. Furthermore,

$$i \notin \{j_1, \dots, j_k\} \Longrightarrow \alpha \rfloor \partial_i = 0.$$

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism and α, β are forms and X a vector field on Ω, it is readily verified that

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta), \quad F^*(\alpha \rfloor X) = (F^*\alpha) \rfloor (F_\# X). \tag{6.6}$$

We make use of the operators $\wedge_k$ and ι_k on forms:

$$\wedge_k \alpha = dx_k \wedge \alpha, \quad \iota_k \alpha = \alpha \rfloor \partial_k. \tag{6.7}$$

There is the following useful *anticommutation relation*:

$$\wedge_k \iota_\ell + \iota_\ell \wedge_k = \delta_{k\ell}, \tag{6.8}$$

where $\delta_{k\ell}$ is 1 if $k = \ell$, 0 otherwise. This is a fairly straightforward consequence of (6.5). We also have

$$\wedge_j \wedge_k + \wedge_k \wedge_j = 0, \quad \iota_j \iota_k + \iota_k \iota_j = 0. \tag{6.9}$$

From (6.8)–(6.9) one says that the operators $\{\iota_j, \wedge_j : 1 \le j \le n\}$ generate a "Clifford algebra." For more on this, see §26.

Another important operator on forms is the *exterior derivative*:

$$d : \Lambda^k(\Omega) \longrightarrow \Lambda^{k+1}(\Omega), \tag{6.10}$$

defined as follows. If $\alpha \in \Lambda^k(\Omega)$ is given by (5.12), then

$$d\alpha = \sum_{j,\ell} \frac{\partial a_j}{\partial x_\ell} \, dx_\ell \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{6.11}$$

Equivalently,

$$d\alpha = \sum_{\ell=1}^{n} \partial_\ell \wedge_\ell \alpha \tag{6.12}$$

where $\partial_\ell = \partial/\partial x_\ell$ and $\wedge_\ell$ is given by (6.7). The antisymmetry $dx_m \wedge dx_\ell = -dx_\ell \wedge dx_m$, together with the identity $\partial^2 a_j/\partial x_\ell \partial x_m = \partial^2 a_j/\partial x_m \partial x_\ell$, implies

$$d(d\alpha) = 0, \tag{6.13}$$

for any differential form α. We also have a product rule:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \quad \alpha \in \Lambda^k(\Omega), \ \beta \in \Lambda^j(\Omega). \tag{6.14}$$

The exterior derivative has the following important property under pull-backs:

$$F^*(d\alpha) = dF^*\alpha, \tag{6.15}$$

if $\alpha \in \Lambda^k(\Omega)$ and $F : \mathcal{O} \to \Omega$ is a smooth map. To see this, extending (6.14) to a formula for $d(\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_\ell)$ and using this to apply d to $F^*\alpha$, we have

$$\begin{aligned} dF^*\alpha = &\sum_{j,\ell} \frac{\partial}{\partial x_\ell} \big(a_j \circ F(x)\big) \, dx_\ell \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big) \\ &+ \sum_{j,\nu} (\pm) a_j \big(F(x)\big) \big(F^* dx_{j_1}\big) \wedge \cdots \wedge d\big(F^* dx_{j_\nu}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big). \end{aligned} \tag{6.16}$$

Now

$$d\bigl(F^* dx_i\bigr) = \sum_{j,\ell} \frac{\partial^2 F_i}{\partial x_j \partial x_\ell}\ dx_j \wedge dx_\ell = 0,$$

so only the first sum in (6.16) contributes to $dF^*\alpha$. Meanwhile,

$$F^* d\alpha = \sum_{j,m} \frac{\partial a_j}{\partial x_m}\bigl(F(x)\bigr)\ (F^* dx_m) \wedge \bigl(F^* dx_{j_1}\bigr) \wedge \cdots \wedge \bigl(F^* dx_{j_k}\bigr), \tag{6.17}$$

so (6.15) follows from the identity

$$\sum_\ell \frac{\partial}{\partial x_\ell}\bigl(a_j \circ F(x)\bigr)\ dx_\ell = \sum_m \frac{\partial a_j}{\partial x_m}\bigl(F(x)\bigr)\ F^* dx_m, \tag{6.18}$$

which in turn follows from the chain rule.

If $d\alpha = 0$, we say α is *closed*; if $\alpha = d\beta$ for some $\beta \in \Lambda^{k-1}(\Omega)$, we say α is *exact.* Formula (6.13) implies that every exact form is closed. The converse is not always true globally. Consider the multi-valued angular coordinate θ on $\mathbb{R}^2 \setminus (0,0)$; $d\theta$ is a single valued closed form on $\mathbb{R}^2 \setminus (0,0)$ which is not globally exact. As we will see below, every closed form is locally exact.

First we introduce another important construction. If $\alpha \in \Lambda^k(\Omega)$ and X is a vector field on Ω, generating a flow $\mathcal{F}_X^t$, the *Lie derivative* $\mathcal{L}_X\alpha$ is defined to be

$$\mathcal{L}_X\alpha = \frac{d}{dt}\bigl(\mathcal{F}_X^t\bigr)^*\alpha|_{t=0}. \tag{6.19}$$

Note the formal similarity to the definition (3.2) of $\mathcal{L}_X Y$ for a vector field Y. Recall the formula (3.4) for $\mathcal{L}_X Y$. The following is not only a computationally convenient formula for $\mathcal{L}_X\alpha$, but also an identity of fundamental importance.

Proosition 6.1. *We have*

$$\mathcal{L}_X\alpha = d(\alpha\rfloor X) + (d\alpha)\rfloor X. \tag{6.20}$$

Proof. First we compare both sides in the special case $X = \partial/\partial x_\ell = \partial_\ell$. Note that

$$\bigl(\mathcal{F}_{\partial_\ell}^t\bigr)^*\alpha = \sum_j a_j(x + te_\ell)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

so

$$\mathcal{L}_{\partial_\ell}\alpha = \sum_j \frac{\partial a_j}{\partial x_\ell}\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} = \partial_\ell\alpha. \tag{6.21}$$

To evaluate the right side of (6.20) with $X = \partial_\ell$, use (6.12) to write this quantity as

$$d(\iota_\ell \alpha) + \iota_\ell d\alpha = \sum_{j=1}^{n} \bigl(\partial_j \wedge_j \iota_\ell + \iota_\ell \partial_j \wedge_j\bigr)\alpha. \tag{6.22}$$

Using the commutativity of ∂_j with $\wedge_j$ and with ι_ℓ, and the anticommutation relations (6.8), we see that the right side of (6.22) is $\partial_\ell \alpha$, which coincides with (6.21). Thus the proposition holds for $X = \partial/\partial x_\ell$.

Now we can prove the proposition in general, for a smooth vector field X on Ω. It is to be verified at each point $x_0 \in \Omega$. If $X(x_0) \neq 0$, choose a coordinate system about x_0 so $X = \partial/\partial x_1$ and use the calculation above. This shows that the desired identity holds on the set of points $\{x_0 \in \Omega : X(x_0) \neq 0\}$, and by continuity it holds on the closure of this set. However, if $x_0 \in \Omega$ has a neighborhood on which X vanishes, it is clear that $\mathcal{L}_X \alpha = 0$ near x_0 and also $\alpha \rfloor X$ and $d\alpha \rfloor X$ vanish near x_0. This completes the proof.

The identity (6.20) can furnish a formula for the exterior derivative in terms of Lie brackets, as follows. By (3.4) and (6.20) we have, for a k-form ω,

$$\bigl(\mathcal{L}_X \omega\bigr)(X_1, \dots, X_k) = X \cdot \omega(X_1, \dots, X_k) - \sum_j \omega(X_1, \dots, [X, X_j], \dots, X_k). \tag{6.23}$$

Now (6.20) can be rewritten as

$$\iota_X d\omega = \mathcal{L}_X \omega - d\iota_X \omega. \tag{6.24}$$

This implies

$$(d\omega)(X_0, X_1, \dots, X_k) = \bigl(\mathcal{L}_{X_0} \omega\bigr)(X_1, \dots, X_k) - \bigl(d\iota_{X_0} \omega\bigr)(X_1, \dots, X_k). \tag{6.25}$$

We can substitute (6.23) into the first term on the right in (6.25). In case ω is a 1-form, the last term is easily evaluated; we get

$$(d\omega)(X_0, X_1) = X_0 \cdot \omega(X_1) - X_1 \cdot \omega(X_0) - \omega([X_0, X_1]). \tag{6.26}$$

More generally, we can tackle the last term on the right side of (6.25) by the same method, using (6.24) with ω replaced by the $(k-1)$-form $\iota_{X_0} \omega$. In this way we inductively obtain the formula

$$\begin{aligned}(d\omega)(X_0, \dots, X_k) = &\sum_{\ell=0}^{k} (-1)^\ell X_\ell \cdot \omega(X_0, \dots, \widehat{X}_\ell, \dots, X_k) \\ &+ \sum_{0 \le \ell < j \le k} (-1)^{j+\ell} \omega([X_\ell, X_j], X_0, \dots, \widehat{X}_\ell, \dots, \widehat{X}_j, \dots, X_k).\end{aligned} \tag{6.27}$$

Note that from (6.19) and the property $\mathcal{F}_X^{s+t} = \mathcal{F}_X^s \mathcal{F}_X^t$ it easily follows that

$$\frac{d}{dt}\big(\mathcal{F}_X^t\big)^*\alpha = \mathcal{L}_X\big(\mathcal{F}_X^t\big)^*\alpha = \big(\mathcal{F}_X^t\big)^*\mathcal{L}_X\alpha. \tag{6.28}$$

It is useful to generalize this. Let F_t be any smooth family of diffeomorphisms from M *into* M. Define vector fields X_t on $F_t(M)$ by

$$\frac{d}{dt}F_t(x) = X_t(F_t(x)). \tag{6.29}$$

Then it easily follows that, for $\alpha \in \Lambda^k M$,

$$\frac{d}{dt}F_t^*\alpha = F_t^*\mathcal{L}_{X_t}\alpha = F_t^*\big[d(\alpha\rfloor X_t) + (d\alpha)\rfloor X_t\big]. \tag{6.30}$$

In particular, if α is *closed*, then, if F_t are diffeomorphisms for $0 \le t \le 1$,

$$F_1^*\alpha - F_0^*\alpha = d\beta, \quad \beta = \int_0^1 F_t^*(\alpha\rfloor X_t)\,dt. \tag{6.31}$$

Using this, we can prove the celebrated *Poincaré Lemma.*

Theorem 6.2. *If B is the unit ball in $\mathbb{R}^n$, centered at* 0, $\alpha \in \Lambda^k(B)$, $k > 0$, *and* $d\alpha = 0$, *then* $\alpha = d\beta$ *for some* $\beta \in \Lambda^{k-1}(B)$.

Proof. Consider the family of maps $F_t : B \to B$ given by $F_t(x) = tx$. For $0 < t \le 1$ these are diffeomorphisms, and the formula (6.30) applies. Note that

$$F_1^*\alpha = \alpha, \quad F_0^*\alpha = 0.$$

Now a simple limiting argument shows (6.31) remains valid, so $\alpha = d\beta$ with

$$\beta = \int_0^1 F_t^*(\alpha\rfloor V)t^{-1}\,dt \tag{6.32}$$

where $V = r\partial/\partial r = \sum x_j\partial/\partial x_j$. Since $F_0^* = 0$, the apparent singularity in the integrand is removable.

Since in the proof of the theorem we dealt with F_t such that F_0 was not a diffeomorphism, we are motivated to generalize (6.31) to the case where $F_t : M \to N$ is a smooth family of maps, not necessarily diffeomorphisms. Then (6.29) does not work to define X_t as a vector field, but we do have

$$\frac{d}{dt}F_t(x) = Z(t,x); \quad Z(t,x) \in T_{F_t(x)}N. \tag{6.33}$$

Now in (6.31) we see that

$$F^*(\alpha\rfloor X_t)(Y_1,\dots,Y_{k-1}) = \alpha\big(F_t(x)\big)\big(X_t, DF_t(x)Y_1,\dots,DF_t(x)Y_{k-1}\big),$$

and we can replace X_t by $Z(t,x)$. Hence, in this more general case, if α is closed, we can write

$$F_1^*\alpha - F_0^*\alpha = d\beta; \quad \beta = \int_0^1 \gamma_t\, dt \tag{6.34}$$

where, at $x \in M$,

$$\gamma_t(Y_1,\dots,Y_{k-1}) = \alpha\big(F_t(x)\big)\big(Z(t,x), DF_t(x)Y_1,\dots,DF_t(x)Y_{k-1}\big). \tag{6.35}$$

For an alternative approach to this homotopy invariance, see Exercise 6.

Exercises

1. If α is a k-form, verify the formula (6.14), i.e., $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^k\alpha\wedge d\beta$. If α is closed and β is exact, show that $\alpha\wedge\beta$ is exact.

2. Let F be a vector field on U, open in $\mathbb{R}^3$, $F = \sum_1^3 f_j(x)\partial/\partial x_j$. Consider the 1-form $\varphi = \sum_1^3 f_j(x)\, dx_j$. Show that $d\varphi$ and curl F are related in the following way:

$$\text{curl } F = \sum_1^3 g_j(x)\,\frac{\partial}{\partial x_j},$$
$$d\varphi = g_1(x)\, dx_2\wedge dx_3 + g_2(x)\, dx_3\wedge dx_1 + g_3(x)\, dx_1\wedge dx_2.$$

3. If F and φ are related as in Exercise 3, show that curl F is uniquely specified by the relation

$$d\varphi\wedge\alpha = \langle \text{curl } F, \alpha\rangle\omega$$

for all 1-forms α on $U\subset\mathbb{R}^3$, where $\omega = dx_1\wedge dx_2\wedge dx_3$ is the volume form. Show that curl F is also uniquely specified by

$$d\varphi = \omega\rfloor(\text{curl } F).$$

4. Let B be a ball in $\mathbb{R}^3$, F a smooth vector field on B. Show that

$$\exists\ u\in C^\infty(B) \text{ s.t. } F = \text{ grad } u \Longleftrightarrow \text{ curl } F = 0.$$

Hint. Compare $F =$ grad u with $\varphi = du$.

5. Let B be a ball in $\mathbb{R}^3$ and G a smooth vector field on B. Show that

$$\exists \text{ vector field } F \text{ s.t. } G = \text{ curl } F \Longleftrightarrow \text{ div } G = 0.$$

Hint. If $G = \sum_1^3 g_j(x)\,dx_j$, consider $\psi = g_1(x)\,dx_2 \wedge dx_3 + g_2(x)\,dx_3 \wedge dx_1 + g_3(x)\,dx_1 \wedge dx_2$. Compute $d\psi$.

6. Suppose $f_0, f_1 : X \to Y$ are smoothly homotopic maps, via $\Phi : X \times \mathbb{R} \to Y$, $\Phi(x,j) = f_j(x)$. Let $\alpha \in \Lambda^k(Y)$ be closed. Apply (6.31) to $\tilde{\alpha} = \Phi^*\alpha \in \Lambda^k(X\times\mathbb{R})$, with $F_t(x,s) = (x, s+t)$, to obtain $\tilde{\beta} \in \Lambda^{k-1}(X \times \mathbb{R})$ such that $F_1^*\tilde{\alpha} - \tilde{\alpha} = d\tilde{\beta}$, and from there produce $\beta \in \Lambda^{k-1}(X)$ such that $f_1^*\alpha - f_0^*\alpha = d\beta$.
Hint. Use $\beta = \iota^*\tilde{\beta}$, where $\iota(x) = (x, 0)$.

7. Verify the anticommutation relation (6.8). Show that, if ξ is a 1-form and we define $\wedge_\xi \alpha = \xi \wedge \alpha$, and X is a vector field, then (6.8) implies

$$\wedge_\xi \iota_X \alpha + \iota_X \wedge_\xi \alpha = \langle X, \xi\rangle \alpha,$$

where $\langle X, \xi\rangle$ is the dual pairing, as in (5.8).

For the next set of exercises, let Ω be a planar domain, $X = f(x,y)\partial/\partial x + g(x,y)\partial/\partial y$ a nonvanishing vector field on Ω. Consider the 1-form $\alpha = g(x,y)\,dx - f(x,y)\,dy$.

8. Let $\gamma : I \to \Omega$ be a smooth curve, $I = (a,b)$. Show that the image $C = \gamma(I)$ is the image of an integral curve of X if and only if $\gamma^*\alpha = 0$. Consequently, with slight abuse of notation, one describes the integral curves by $g\,dx - f\,dy = 0$.
If α is exact, i.e., $\alpha = du$, conclude the level curves of u are the integral curves of X.

9. A function φ is called an integrating factor if $\tilde{\alpha} = \varphi\alpha$ is exact, i.e., if $d(\varphi\alpha) = 0$, provided Ω is simply connected. Show that an integrating factor always exists, at least locally. Show that $\varphi = e^v$ is an integrating factor if and only if $Xv = -$ div X.
Reconsider Exercise 7 in §2.
Find an integrating factor for $\alpha = (x^2 + y^2 - 1)\,dx - 2xy\,dy$.

10. Let Y be a vector field which you know how to linearize (i.e., conjugate to $\partial/\partial x$) and suppose $\mathcal{L}_Y\alpha = 0$. Show how to construct an integrating factor for α. Treat the more general case $\mathcal{L}_X\alpha = c\alpha$ for some constant c. Compare the discussion in §3 of the situation where $[X,Y] = cX$.

7. The general Stokes formula

A basic result in the theory of differential forms is the generalized *Stokes formula*:

Proposition 7.1. *Given a compactly supported $(k-1)$-form β of class C^1 on an oriented k-dimensional manifold $\overline{M}$ (of class C^2) with boundary ∂M, with its natural orientation,*

$$\int_M d\beta = \int_{\partial M} \beta \tag{7.1}$$

The orientation induced on ∂M is uniquely determined by the following requirement. If

$$\overline{M} = \mathbb{R}^k_- = \{x \in \mathbb{R}^k : x_1 \leq 0\} \tag{7.2}$$

then $\partial M = \{(x_2, \dots, x_k)\}$ has the orientation determined by $dx_2 \wedge \cdots \wedge dx_k$.

Proof. Using a partition of unity and invariance of the integral and the exterior derivative under coordinate transformations, it suffices to prove this when $\overline{M}$ has the form (7.2). In that case, we will be able to deduce (7.1) from the fundamental theorem of calculus. Indeed, if

$$\beta = b_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k, \tag{7.3}$$

with $b_j(x)$ of bounded support, we have

$$d\beta = (-1)^{j-1} \frac{\partial b_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_k. \tag{7.4}$$

If $j > 1$, we have

$$\int_M d\beta = \int \Big\{ \int_{-\infty}^{\infty} \frac{\partial b_j}{\partial x_j}\, dx_j \Big\}\, dx' = 0, \tag{7.5}$$

and also $\kappa^* \beta = 0$, where $\kappa : \partial M \to \overline{M}$ is the inclusion. On the other hand, for $j = 1$, we have

$$\begin{aligned} \int_M d\beta &= \int \Big\{ \int_{-\infty}^{0} \frac{\partial b_1}{\partial x_1}\, dx_1 \Big\}\, dx_2 \cdots dx_k \\ &= \int b_1(0, x')\, dx' \\ &= \int_{\partial M} \beta. \end{aligned} \tag{7.6}$$

This proves Stokes' formula (7.1).

It is useful to allow singularities in ∂M. We say a point $p \in \overline{M}$ is a *corner* of dimension ν if there is a neighborhood $\overline{U}$ of p in $\overline{M}$ and a C^2 diffeomorphism of $\overline{U}$ onto a neighborhood of 0 in

$$K = \{x \in \mathbb{R}^k : x_j \le 0, \ \text{for } 1 \le j \le k - \nu\}, \tag{7.7}$$

where k is the dimension of M. If M is a C^2 manifold and every point $p \in \partial M$ is a corner (of some dimension), we say $\overline{M}$ is a C^2 manifold wih corners. In such a case, ∂M is a locally finite union of C^2 manifolds with corners. The following result extends Proposition 7.1.

Proposition 7.2. *If $\overline{M}$ is a C^2 manifold of dimension k, with corners, and β is a compactly supported $(k-1)$-form of class C^1 on $\overline{M}$, then (7.1) holds.*

Proof. It suffices to establish this when β is supported on a small neighborhood of a corner $p \in \partial M$, of the form $\overline{U}$ described above. Hence it suffices to show that (7.1) holds whenever β is a $(k-1)$-form of class C^1, with compact support on K in (7.7); and we can take β to have the form (7.3). Then, for $j > k - \nu$, (7.5) still holds, while, for $j \le k - \nu$, we have, as in (7.6),

$$\begin{aligned} \int_K d\beta &= (-1)^{j-1} \int \Big\{ \int_{-\infty}^0 \frac{\partial b_j}{\partial x_j}\, dx_j \Big\}\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\ &= (-1)^{j-1} \int b_j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_k)\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\ &= \int_{\partial K} \beta. \end{aligned} \tag{7.8}$$

The reason we required $\overline{M}$ to be a manifold of class C^2 (with corners) in Propositions 7.1 and 7.2 is the following. Due to the formulas (5.18)–(5.19) for a pull-back, if β is of class C^j and F is of class C^ℓ, then $F^*\beta$ is generally of class C^μ, with $\mu = \min(j, \ell - 1)$. Thus, if $j = \ell = 1$, $F^*\beta$ might be only of class C^0, so there is not a well-defined notion of a differential form of class C^1 on a C^1 manifold, though such a notion is well defined on a C^2 manifold. This problem can be overcome, and one can extend Propositions 7.1–7.2 to the case where $\overline{M}$ is a C^1 manifold (with corners), and β is a $(k-1)$-form with the property that both β and $d\beta$ are continuous. We will not go into the details. Substantially more sophisticated generalizations are given in [Fed].

Exercises

1. Consider the region $\overline{\Omega} \subset \mathbb{R}^2$ defined by

$$\overline{\Omega} = \{(x, y) : 0 \le y \le x^2, 0 \le x \le 1\}.$$

Show that the boundary points $(1, 0)$ and $(1, 1)$ are corners, but $(0, 0,)$ is *not* a corner. The boundary of $\overline{\Omega}$ is too sharp at $(0, 0)$ to be a corner; it is called a "cusp." Extend Proposition 7.2. to treat this region.

2. Suppose $U \subset \mathbb{R}^n$ is an open set with smooth boundary $M = \partial U$, and U has the standard orientation, determined by $dx_1 \wedge \cdots \wedge dx_n$. (See the paragraph above (5.23).) Let $\varphi \in C^1(\mathbb{R}^n)$ satisfy $\varphi(x) < 0$ for $x \in U$, $\varphi(x) > 0$ for $x \in \mathbb{R}^n \setminus \overline{U}$, and grad $\varphi(x) \neq 0$ for $x \in \partial U$, so grad φ points out of U. Show that the natural oriemtation on ∂U, as defined after Proposition 7.1, is the same as the following. The equivalence class of forms $\beta \in \Lambda^{n-1}(\partial U)$ defining the orientation on ∂U satisfies the property that $d\varphi \wedge \beta$ is a *positive* multiple of $dx_1 \wedge \cdots \wedge dx_n$, on ∂U.

3. Suppose $U = \{x \in \mathbb{R}^n : x_n < 0\}$. Show that the orientation on ∂U described above is that of $(-1)^{n-1}\ dx_1 \wedge \cdots \wedge dx_{n-1}$.

4. If $\omega \in \Lambda^k(\overline{M})$ determines the orientation of $\overline{M}$ and N is a vector field over ∂M, nowhere tangent to ∂M, pointing *out* of M, show that

$$\iota^*(\omega \rfloor N) \in \Lambda^{k-1}(\partial M)$$

determines the orientation of ∂M. Here, $\iota : \partial M \to \overline{M}$ is the inclusion.

8. The classical Gauss, Green, and Stokes formulas

The case of (7.1) where $S = \overline{\Omega}$ is a region in $\mathbb{R}^2$ with smooth boundary yields the classical Green Theorem. In this case, we have

$$\beta = f\,dx + g\,dy, \quad d\beta = \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)\,dx \wedge dy, \tag{8.1}$$

and hence (7.1) becomes the following

Proposition 8.1. *If $\overline{\Omega}$ is a region in $\mathbb{R}^2$ with smooth boundary, and f and g are smooth functions on $\overline{\Omega}$, which vanish outside some compact set in $\overline{\Omega}$, then*

$$\iint\limits_{\Omega} \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)\,dx\,dy = \int\limits_{\partial\Omega} (f\,dx + g\,dy). \tag{8.2}$$

Note that, if we have a vector field $X = X_1 \partial/\partial x + X_2 \partial/\partial y$ on $\overline{\Omega}$, then the integrand on the left side of (8.2) is

$$\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} = \text{ div } X, \tag{8.3}$$

provided $g = X_1,\ f = -X_2$. We obtain

$$\iint\limits_{\Omega} \text{ div } X\ dx\ dy = \int\limits_{\partial\Omega} \big(-X_2\ dx + X_1\ dy\big). \tag{8.4}$$

If $\partial\Omega$ is parametrized by arc-length, as $\gamma(s) = \big(x(s), y(s)\big)$, with orientation as defined for Proposition 15.1, then the unit normal ν, to $\partial\Omega$, pointing *out* of Ω, is given by $\nu(s) = \big(\dot{y}(s), -\dot{x}(s)\big)$, and (16.4) is equivalent to

$$\iint\limits_{\Omega} \text{ div } X\ dx\ dy = \int\limits_{\partial\Omega} \langle X, \nu\rangle\ ds. \tag{8.5}$$

This is a special case of Gauss' Divergence Theorem. We now derive a more general form of the Divergence Theorem. We begin with a definition of the divergence of a vector field on a surface M.

Let M be a region in $\mathbb{R}^n$, or an n-dimensional manifold, provided with a volume form $\omega_M \in \Lambda^n M$. Let X be a vector field on M. Then the divergence of X, denoted div X, is a function on M which measures the rate of change of the volume form under the flow generated by X. Thus it is defined by

$$\mathcal{L}_X \omega_M = (\text{div } X)\omega_M. \tag{8.6}$$

Here, $\mathcal{L}_X$ denotes the Lie derivative. In view of the general formula $\mathcal{L}_X\alpha = d\alpha\rfloor X + d(\alpha\rfloor X)$, derived in (6.20), since $d\omega = 0$ for any n-form ω on M, we have

$$(\mathrm{div}\ X)\omega_M = d(\omega_M\rfloor X). \tag{8.7}$$

If $M = \mathbb{R}^n$, with the standard volume element

$$\omega = dx_1 \wedge \cdots \wedge dx_n, \tag{8.8}$$

and if

$$X = \sum X^j(x)\frac{\partial}{\partial x_j}, \tag{8.9}$$

then

$$\omega\rfloor X = \sum_{j=1}^n (-1)^{j-1} X^j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{8.10}$$

Hence, in this case, (8.7) yields the familiar formula

$$\mathrm{div}\ X = \sum_{j=1}^n \partial_j X^j, \tag{8.11}$$

where we use the notation

$$\partial_j f = \frac{\partial f}{\partial x_j}. \tag{8.12}$$

Suppose now that M is endowed with a metric tensor $g_{jk}(x)$. Then M carries a natural volume element ω_M, determined by the condition that, if one has a coordinate system in which $g_{jk}(p_0) = \delta_{jk}$, then $\omega_M(p_0) = dx_1 \wedge \cdots \wedge dx_n$. This condition produces the following formula, in any oriented coordinate system:

$$\omega_M = \sqrt{g}\ dx_1 \wedge \cdots \wedge dx_n, \quad g = \det(g_{jk}); \tag{8.13}$$

compare (4.7).

We now compute div X when the volume element on M is given by (8.13). We have

$$\omega_M\rfloor X = \sum_j (-1)^{j-1} X^j \sqrt{g}\ dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \tag{8.14}$$

and hence

$$d(\omega_M\rfloor X) = \partial_j(\sqrt{g}X^j)\ dx_1 \wedge \cdots \wedge dx_n. \tag{8.15}$$

Here, as below, we use the summation convention. Hence the formula (8.7) gives

$$\text{div } X = g^{-1/2}\partial_j(g^{1/2}X^j). \tag{8.16}$$

We now derive the Divergence Theorem, as a consequence of Stokes' formula, which we recall is

$$\int_M d\alpha = \int_{\partial M} \alpha, \tag{8.17}$$

for an $(n-1)$-form on $\overline{M}$, assumed to be a smooth compact oriented manifold with boundary. If $\alpha = \omega_M \rfloor X$, formula (8.7) gives

$$\int_M (\text{div } X)\,\omega = \int_{\partial M} \omega_M \rfloor X. \tag{8.18}$$

This is one form of the Divergence Theorem. We will produce an alternative expression for the integrand on the right before stating the result formally.

Given that ω_M is the volume form for M determined by a Riemannian metric, we can write the interior product $\omega_M \rfloor X$ in terms of the volume element $\omega_{\partial M}$ on ∂M, with its induced Riemannian metric, as follows. Pick coordinates on M, centered at $p_0 \in \partial M$, such that ∂M is tangent to the hyperplane $\{x_n = 0\}$ at $p_0 = 0$, and such that $g_{jk}(p_0) = \delta_{jk}$. Then it is clear that, at p_0,

$$j^*(\omega_M \rfloor X) = \langle X, \nu\rangle\, \omega_{\partial M} \tag{8.19}$$

where ν is the unit vector normal to ∂M, pointing out of M and $j : \partial M \hookrightarrow M$ the natural inclusion. The two sides of (8.19), which are both defined in a coordinate independent fashion, are hence equal on ∂M, and the identity (8.18) becomes

$$\int_M (\text{div } X)\,\omega_M = \int_{\partial M} \langle X, \nu\rangle\, \omega_{\partial M}. \tag{8.20}$$

Finally, we adopt the following common notation: we denote the volume element on M by dV and that on ∂M by dS, obtaining the *Divergence Theorem*:

Theorem 8.2. *If $\overline{M}$ is a compact manifold with boundary, X a smooth vector field on $\overline{M}$, then*

$$\int_M (\mathit{div}\ X)\ dV = \int_{\partial M} \langle X, \nu\rangle\ dS, \tag{8.21}$$

where ν is the unit outward-pointing normal to ∂M.

The only point left to mention here is that M need not be orientable. Indeed, we can treat the integrals in (8.21) as surface integrals, as in §4, and note that all objects in (8.21) are independent of a choice of orientation. To prove the general case, just use a partition of unity supported on orientable pieces.

The definition of the divergence of a vector field given by (8.6), in terms of how the flow generated by the vector field magnifies or diminishes volumes, is a good geometrical characterization, explaining the use of the term "divergence."

We obtain some further integral identities. First, we apply (8.21) with X replaced by uX. We have the following "derivation" identity:

$$\text{div } uX = u \text{ div } X + \langle du, X\rangle = u \text{ div } X + Xu, \tag{8.22}$$

which follows easily from the formula (8.16). The Divergence Theorem immediately gives

$$\int\limits_M (\text{div } X)u \; dV + \int\limits_M Xu \; dV = \int\limits_{\partial M} \langle X, \nu\rangle u \; dS. \tag{8.23}$$

Replacing u by uv and using the derivation identity $X(uv) = (Xu)v + u(Xv)$, we have

$$\int\limits_M \bigl[(Xu)v + u(Xv)\bigr] dV = -\int\limits_M (\text{div } X)uv \; dV + \int\limits_{\partial M} \langle X, \nu\rangle uv \; dS. \tag{8.24}$$

It is very useful to apply (8.23) to a gradient vector field X. If v is a smooth function on M, grad v is a vector field satisfying

$$\langle \text{grad } v, Y\rangle = \langle Y, dv\rangle, \tag{8.25}$$

for any vector field Y, where the brackets on the left are given by the metric tensor on M and those on the right by the natural pairing of vector fields and 1-forms. Hence grad $v = X$ has components $X^j = g^{jk}\partial_k v$, where (g^{jk}) is the matrix inverse of (g_{jk}).

Applying div to grad v, we have the *Laplace operator*:

$$\Delta v = \text{ div grad } v = g^{-1/2}\partial_j\bigl(g^{jk} g^{1/2}\partial_k v\bigr). \tag{8.26}$$

When M is a region in $\mathbb{R}^n$ and we use the standard Euclidean metric, so div X is given by (8.11), we have the Laplace operator on Euclidean space:

$$\Delta v = \frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_n^2}. \tag{8.27}$$

Now, setting $X =$ grad v in (8.23), we have $Xu = \langle \text{grad } u, \text{grad } v\rangle$, and $\langle X, \nu\rangle = \langle \nu, \text{grad } v\rangle$, which we call the normal derivative of v, and denote $\partial v/\partial \nu$. Hence

$$\int_M u(\Delta v)\ dV = -\int_M \langle \text{grad } u, \text{grad } v\rangle\ dV + \int_{\partial M} u\frac{\partial v}{\partial \nu}\ dS. \tag{8.28}$$

If we interchange the roles of u and v and subtract, we have

$$\int_M u(\Delta v)\ dV = \int_M (\Delta u)v\ dV + \int_M \Big[u\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu}v\Big]\ dS. \tag{8.29}$$

Formulas (8.28)–(8.29) are also called Green formulas. We will make further use of them in §9.

We return to the Green formula (8.2), and give it another formulation. Consider a vector field $Z = (f, g, h)$ on a region in $\mathbb{R}^3$ containing the planar surface $U = \{(x, y, 0) : (x, y) \in \Omega\}$. If we form

$$\text{curl } Z = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{pmatrix} \tag{8.30}$$

we see that the integrand on the left side of (8.2) is the k-component of curl Z, so (8.2) can be written

$$\iint_U (\text{curl } Z)\cdot k\ dA = \int_{\partial U} (Z\cdot T)\ ds, \tag{8.31}$$

where T is the unit tangent vector to ∂U. To see how to extend this result, note that k is a unit *normal* field to the planar surface U.

To formulate and prove the extension of (8.31) to any compact oriented surface with boundary in $\mathbb{R}^3$, we use the relation between curl and exterior derivative discussed in Exercises 2–3 of §6. In particular, if we set

$$F = \sum_{j=1}^3 f_j(x)\frac{\partial}{\partial x_j}, \quad \varphi = \sum_{j=1}^3 f_j(x)\ dx_j, \tag{8.32}$$

then curl $F = \sum_1^3 g_j(x)\ \partial/\partial x_j$ where

$$d\varphi = g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2. \tag{8.33}$$

Now Suppose $\overline{M}$ is a smooth oriented $(n-1)$-dimensional surface with boundary in $\mathbb{R}^n$. Using the orientation of M, we pick a unit normal field N to M as follows. Take a smooth function v which vanishes on M but such that $\nabla v(x) \neq 0$ on M.

Thus ∇v is normal to M. Let $\sigma \in \Lambda^{n-1}(M)$ define the orientation of M. Then $dv \wedge \sigma = a(x)\ dx_1 \wedge \cdots \wedge dx_n$, where $a(x)$ is nonvanishing on M. For $x \in M$, we take $N(x) = \nabla v(x)/|\nabla v(x)|$ if $a(x) > 0$, and $N(x) = -\nabla v(x)/|\nabla v(x)|$ if $a(x) < 0$. We call N the "positive" unit normal field to the oriented surface M, in this case. Part of the motivation for this characterization of N is that, if $\Omega \subset \mathbb{R}^n$ is an open set with smooth boundary $M = \partial\Omega$, and we give M the induced orientation, as described in §7, then the positive normal field N just defined coincides with the unit normal field pointing out of Ω. Compare Exercises 2–3 of §7.

Now, if $G = (g_1, \dots, g_n)$ is a vector field defined near M, then

$$\int_M (N \cdot G)\ dS = \int_M \Big(\sum_{j=1}^n (-1)^{j-1} g_j(x)\, dx_1 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_n \Big). \tag{8.34}$$

This result follows from (8.19). When $n = 3$ and $G =$ curl F, we deduce from (8.32)–(8.33) that

$$\iint_M d\varphi = \iint_M (N \cdot \text{curl } F)\, dS. \tag{8.35}$$

Furthermore, in this case we have

$$\int_{\partial M} \varphi = \int_{\partial M} (F \cdot T)\, ds, \tag{8.36}$$

where T is the unit tangent vector to ∂M, specied as follows by the orientation of ∂M; if $\tau \in \Lambda^1(\partial M)$ defines the orientation of ∂M, then $\langle T, \tau \rangle > 0$ on ∂M. We call T the "forward" unit tangent vector field to the oriented curve ∂M. By the calculations above, we have the classical Stokes formula:

Proposition 8.3. *If $\overline{M}$ is a compact oriented surface with boundary in $\mathbb{R}^3$, and F is a C^1 vector field on a neighborhood of $\overline{M}$, then*

$$\iint_M (N \cdot \mathit{curl}\ F)\, dS = \int_{\partial M} (F \cdot T)\, ds, \tag{8.37}$$

where N is the positive unit normal field on M and T the forward unit tangent field to ∂M.

Exercises

1. Given a Hamiltonian vector field

$$H_f = \sum_{j=1}^{n} \Big[\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big],$$

calculate div H_f directly from (8.11).

2. Show that the identity (8.21) for div (uX) follows from (8.7) and

$$du \wedge (\omega \rfloor X) = (Xu)\omega.$$

Prove this identity, for any n-form ω on M^n. What happens if ω is replaced by a k-form, $k < n$?

3. Relate problem 2 to the calculations

$$\mathcal{L}_{uX}\alpha = u\mathcal{L}_X\alpha + du \wedge (\iota_X\alpha) \tag{8.38}$$

and

$$du \wedge (\iota_X\alpha) = -\iota_X(du \wedge \alpha) + (Xu)\alpha, \tag{8.39}$$

valid for any k-form α. The last identity follows from (6.8).

4. Show that

$$\text{div } [X, Y] = X(\text{div } Y) - Y(\text{div } X).$$

5. Show that, if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear *rotation*, then, for a C^1 vector field Z on $\mathbb{R}^3$,

$$F_\#(\text{curl } Z) = \text{curl}\,(F_\# Z). \tag{8.40}$$

6. Let $\overline{M}$ be the graph in $\mathbb{R}^3$ of a smooth function, $z = u(x, y),\ (x, y) \in \mathcal{O} \subset \mathbb{R}^2$, a bounded region with smooth boundary (maybe with corners). Show that

$$\begin{aligned}\int_M (\text{curl } F \cdot N)\, dS = \iint_{\mathcal{O}} \Big[\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\Big(-\frac{\partial u}{\partial x}\Big) &+ \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\Big(-\frac{\partial u}{\partial y}\Big) \\ &+ \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\Big]\, dx\, dy,\end{aligned} \tag{8.41}$$

where $\partial F_j/\partial x$ and $\partial F_j/\partial y$ are evaluated at $\big(x, y, u(x, y)\big)$. Show that

$$\int_{\partial M} (F \cdot T)\, ds = \int_{\partial \mathcal{O}} \Big(\widetilde{F}_1 + \widetilde{F}_3 \frac{\partial u}{\partial x}\Big)\, dx + \Big(\widetilde{F}_2 + \widetilde{F}_3 \frac{\partial u}{\partial y}\Big)\, dy, \tag{8.42}$$

where $\widetilde{F}_j(x, y) = F_j\big(x, y, u(x, y)\big)$. Apply Green's Theorem, with $f = \widetilde{F}_1 + \widetilde{F}_3(\partial u/\partial x)$, $g = \widetilde{F}_2 + \widetilde{F}_3(\partial u/\partial y)$, to show that the right sides of (8.41) and (8.42) are equal, hence proving Stokes' Theorem in this case.

7. Let $M \subset \mathbb{R}^n$ be the graph of a function $x_n = u(x')$, $x' = (x_1, \dots, x_{n-1})$. If

$$\beta = \sum_{j=1}^{n} (-1)^{j-1} g_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

as in (8.34), and $\varphi(x') = \big(x', u(x')\big)$, show that

$$\begin{aligned} \varphi^* \beta &= (-1)^n \left[\sum_{j=1}^{n-1} g_j\big(x', u(x')\big) \frac{\partial u}{\partial x_j} - g_n\big(x', u(x')\big) \right] dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= (-1)^{n-1} G \cdot (-\nabla u, 1)\ dx_1 \wedge \cdots \wedge dx_{n-1}, \end{aligned}$$

where $G = (g_1, \dots, g_n)$, and verify the identity (8.34) in this case.
Hint. For the last part, recall Exercises 2–3 of §7, regarding the orientation of M.

8. Let S be a smooth oriented 2-dimensional surface in $\mathbb{R}^3$, and M an open subset of S, with smooth boundary; see Fig. 8.1. Let N be the upward unit normal field to S, defined by its orientation. For $x \in \partial M$, let $\nu(x)$ be the unit vector, tangent to M, normal to ∂M, and pointing out of M, and let T be the forward unit tangent vector field to ∂M. Show that, on ∂M,

$$N \times \nu = T, \quad \nu \times T = N.$$

9. If M is an oriented $(n-1)$-dimensional surface in $\mathbb{R}^n$, with positive unit normal field N, show that the volume form ω_M on M is given by $\omega_M = \omega \rfloor N$, where $\omega = dx_1 \wedge \cdots \wedge dx_n$ is the standard volume form on $\mathbb{R}^n$. Deduce that the volume form on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is given by

$$\omega_{S^{n-1}} = \sum_{j=1}^{n-1} (-1)^j x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

if S^{n-1} inherits the standard orientation as the boundary of the unit ball.

8B. Symbols, and a more general Green-Stokes formula

Let P be a differential operator of order m on a manifold M; P could operate on sections of a vector bundle. In local coordinates, P has the form

$$Pu(x) = \sum_{|\alpha|\le m} p_\alpha(x) D^\alpha u(x), \tag{8B.1}$$

where $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n},\ D_j = (1/i)\partial/\partial x_j$. The coefficients $p_\alpha(x)$ could be matrix valued. The homogeneous polynomial in $\xi \in \mathbb{R}^n$ ($n = \dim M$),

$$p_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha \tag{8B.2}$$

is called the principal symbol (or just the symbol) of P. We want to give an intrinsic characterization, which will show that $p_m(x,\xi)$ is well defined on the cotangent bundle of M. For a smooth function ψ, a simple calculation, using the product rule and chain rule of differentiation, gives

$$P\big(u(x)e^{i\lambda\psi}\big) = \big[p_m(x,d\psi)u(x)\lambda^m + r(x,\lambda)\big]e^{i\lambda\psi}, \tag{8B.3}$$

where $r(x,\lambda)$ is a polynomial of degree $\le m-1$ in λ. In (8B.3), $p_m(x,d\psi)$ is evaluated by substituting $\xi = (\partial\psi/\partial x_1,\dots,\partial\psi/\partial x_n)$ into (9.2). Thus the formula

$$p_m(x,d\psi)u(x) = \lim_{\lambda\to\infty} \lambda^{-m} e^{-i\lambda\psi} P\big(u(x)e^{i\lambda\psi}\big) \tag{8B.4}$$

provides an intrinsic characteristization of the symbol of P as a function on T^*M. We also use the notation

$$\sigma_P(x,\xi) = p_m(x,\xi). \tag{8B.5}$$

If

$$P : C^\infty(M,E_0) \longrightarrow C^\infty(M,E_1) \tag{8B.6}$$

where E_0 and E_1 are smooth vector bundles over M, then, for each $(x,\xi)\in T^*M$,

$$p_m(x,\xi) : E_{0x} \longrightarrow E_{1x} \tag{8B.7}$$

is a linear map between fibers. It is easy to verify that, if P_2 is a second differential operator, mapping $C^\infty(M,E_1)$ to $C^\infty(M,E_2)$, then

$$\sigma_{P_2P}(x,\xi) = \sigma_{P_2}(x,\xi)\sigma_P(x,\xi). \tag{8B.8}$$

If M has a Riemannian metric, and the vector bundles E_j have metrics, then the formal adjoint P^t of a differential operator of order m like (8B.6) is a differential operator of order m :

$$P^t : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_0),$$

defined by the condition that

$$(Pu, v) = (u, P^t v) \tag{8B.9}$$

if u and v are smooth compactly supported sections of the bundles E_0 and E_1. If u and v are supported on a coordinate patch $\mathcal{O}$ on M, over which E_j are trivialized, so u and v have components u^σ, v^σ, and if the metrics on E_0 and E_1 are denoted $h_{\sigma\delta}$, $\tilde{h}_{\sigma\delta}$, respectively, while the Riemannian metric is g_{jk}, then we have

$$(Pu, v) = \int_{\mathcal{O}} \tilde{h}_{\sigma\delta}(x) \big(Pu\big)^\sigma \, \overline{v}^\delta \, \sqrt{g(x)} \, dx. \tag{8B.10}$$

Substituting (8B.1) and integrating by parts produces an expression for P^t, of the form

$$P^t v(x) = \sum_{|\alpha| \le m} p_\alpha^t(x) D^\alpha v(x). \tag{8B.11}$$

In particular, one sees that the principal symbol of P^t is given by

$$\sigma_{P^t}(x, \xi) = \sigma_P(x, \xi)^t. \tag{8B.12}$$

Compare the specific formula (from (8.24))

$$X^t = -X - (\operatorname{div} X),$$

for the formal adjoint of a real vector field, which has a purely imaginary symbol.

Now suppose M is a compact smooth manifold with smooth boundary. We want to obtain a generalization of formula (8.24), i.e.,

$$(Xu, v) - (u, X^t v) = \int_{\partial M} \langle \nu, X \rangle u \overline{v} \; dS, \tag{8B.13}$$

to the case where P is a general *first order* differential operator, acting on sections of a vector bundle as in (8B.6). Using a partition of unity, we can suppose u and v are supported in a coordinate patch $\mathcal{O}$ in M. If the patch is disjoint from ∂M, then of course (8B.9) holds. Otherwise, suppose $\mathcal{O}$ is a patch in $\mathbb{R}^n_+$. If the first order operator P has the form

$$Pu = \sum_{j=1}^{n} a_j(x) \frac{\partial u}{\partial x_j} + b(x)u, \tag{8B.14}$$

then

$$\int_{\mathcal{O}} \langle Pu, v\rangle \sqrt{g}\ dx = \int_{\mathcal{O}} \Big[\sum_{j=1}^{n} \langle a_j(x) \frac{\partial u}{\partial x_j}, v\rangle + \langle b(x)u, v\rangle\Big] \sqrt{g}\ dx. \tag{8B.15}$$

If we apply the fundamental theorem of calculus, the only boundary integral comes from the term involving $\partial u/\partial x_n$. Thus we have

$$\int_{\mathcal{O}} \langle Pu, v\rangle \sqrt{g}\, dx = \int_{\mathcal{O}} \langle u, P^t v\rangle \sqrt{g}\, dx - \int_{\mathbb{R}^{n-1}} \langle a_n(x', 0)u, v\rangle \sqrt{g(x', 0)}\, dx', \tag{8B.16}$$

where $dx' = dx_1 \cdots dx_{n-1}$. If we pick the coordinate patch so that $\partial/\partial x_n$ is the unit inward normal at ∂M, then $\sqrt{g(x', 0)}\, dx'$ is the volume element on ∂M, and we are ready to establish:

Proposition 8B.1**.** *If M is a smooth compact manifold with boundary and P is a first order differential operator (acting on sections of a vector bundle), then*

$$(Pu, v) - (u, P^t v) = \frac{1}{i} \int_{\partial M} \langle \sigma_P(x, \nu)u, v\rangle\, dS, \tag{8B.17}$$

Proof. The formula (8B.17), which arose via a choice of local coordinate chart, is invariant, and hence valid independently of choices.

As in (8B.13), ν denotes the *outward* pointing unit normal to ∂M; we use the Riemannian metric on M to identify tangent vectors and cotangent vectors.

We will see an important application of (8B.17) in §21, where we consider the Laplace operator on k-forms.

Exercises

1. Consider the divergence operator acting on (complex valued) vector fields:

$$\operatorname{div} : C^\infty(\Omega, \mathbb{C}^n) \longrightarrow C^\infty(\Omega), \quad \Omega \subset \mathbb{R}^n.$$

Show that its symbol is

$$\sigma_{\text{div}}(x,\xi)v = i\langle v,\xi\rangle.$$

2. Consider the gradient operator acting on (complex valued) functions:

$$\text{grad} : C^\infty(\Omega) \longrightarrow C^\infty(\Omega,\mathbb{C}^n), \quad \Omega \subset \mathbb{R}^n.$$

Show that its symbol is

$$\sigma_{\text{grad}}(x,\xi) = i\xi.$$

3. Consider the operator

$$L = \text{grad div} : C^\infty(\Omega,\mathbb{C}^n) \longrightarrow C^\infty(\Omega,\mathbb{C}^n).$$

Show that its symbol is

$$\sigma_L(x,\xi) = -|\xi|^2 P_\xi,$$

where $P_\xi \in \text{End}(\mathbb{C}^n)$ is the orthogonal projection onto the (complex) linear span of ξ.

4. Generalize Exercises 1–3 to the case of a Riemannian manifold.

9. Topological applications of differential forms

Differential forms are a fundamental tool in calculus. In addition, they have important applications to topology. We give a few here, starting with simple proofs of some important topological results of Brouwer.

Proposition 9.1. *There is no continuous retraction $\varphi : B \to S^{n-1}$ of the closed unit ball B in $\mathbb{R}^n$ onto its boundary S^{n-1}.*

In fact, it is just as easy to prove the following more general result. The approach we use is adapted from [Kan].

Proposition 9.2. *If $\overline{M}$ is a compact oriented manifold with nonempty boundary ∂M, there is no continuous retraction $\varphi : \overline{M} \to \partial M$.*

Proof. A retraction φ satisfies $\varphi \circ j(x) = x$ where $j : \partial M \hookrightarrow \overline{M}$ is the natural inclusion. By a simple approximation, if there were a continuous retraction there would be a smooth one, so we can suppose φ is smooth.

Pick $\omega \in \Lambda^{n-1}(\partial M)$ to be the volume form on ∂M, endowed with some Riemannian metric ($n = \dim M$), so $\int_{\partial M} \omega > 0$. Now apply Stokes' theorem to $\alpha = \varphi^* \omega$. If φ is a retraction, $j^* \varphi^* \omega = \omega$, so we have

$$\int\limits_{\partial M} \omega = \int\limits_{M} d\varphi^* \omega. \tag{9.1}$$

But $d\varphi^* \omega = \varphi^* d\omega = 0$, so the integral (9.1) is zero. This is a contradiction, so there can be no retraction.

A simple consequence of this is the famous Brouwer Fixed Point Theorem.

Theorem 9.3. *If $F : B \to B$ is a continuous map on the closed unit ball in $\mathbb{R}^n$, then F has a fixed point.*

Proof. We are claiming that $F(x) = x$ for some $x \in B$. If not, define $\varphi(x)$ to be the endpoint of the ray from $F(x)$ to x, continued until it hits $\partial B = S^{n-1}$. It is clear that φ would be a retraction, contradicting Proposition 9.1.

We next show that an even dimensional sphere cannot have a smooth nonvanishing vector field.

Proposition 9.4. *There is no smooth nonvanishing vector field on S^n if $n = 2k$ is even.*

Proof. If X were such a vector field, we could arrange it to have unit length, so we would have $X : S^n \to S^n$ with $X(v) \perp v$ for $v \in S^n \subset \mathbb{R}^{n+1}$. Thus there is a unique unit speed geodesic γ_v from v to $X(v)$, of length $\pi/2$. Define a smooth family of maps $F_t : S^n \to S^n$ by $F_t(v) = \gamma_v(t)$. Thus $F_0(v) = v,\ F_{\pi/2}(v) = X(v)$, and $F_\pi = A$ would be the *antipodal map*, $A(v) = -v$. By (6.34), we deduce that $A^*\omega - \omega = d\beta$ is exact, where ω is the volume form on S^n. Hence, by Stokes' theorem,

$$\int_{S^n} A^*\omega = \int_{S^n} \omega. \tag{9.2}$$

On the other hand, it is straightforward that $A^*\omega = (-1)^{n+1}\omega$, so (9.2) is possible only when n is odd.

Note that an important ingredient in the proof of both Proposition 9.2 and Proposition 9.4 is the existence of n-forms on a compact oriented n-dimensional manifold M which are not exact (though of course they are closed). We next establish the following important counterpoint to the Poincaré lemma.

Proposition 9.5. *If M is a compact connected oriented manifold of dimension n and $\alpha \in \Lambda^n M$, then $\alpha = d\beta$ for some $\beta \in \Lambda^{n-1}(M)$ if and only if*

$$\int_M \alpha = 0. \tag{9.3}$$

We have already discussed the necessity of (9.3). To prove the sufficiency, we first look at the case $M = S^n$.

In that case, any n-form α is of the form $a(x)\omega,\ a \in C^\infty(S^n),\ \omega$ the volume form on S^n, with its standard metric. The group $G = SO(n+1)$ of rotations of $\mathbb{R}^{n+1}$ acts as a transitive group of isometries on S^n. In §4 we constructed the integral of functions over $SO(n+1)$, with respect to Haar measure. (See also §H.)

As noted in §4, we have the map $\text{Exp} : \text{Skew}(n+1) \to SO(n+1)$, giving a diffeomorphism from a ball $\mathcal{O}$ about 0 in $\text{Skew}(n+1)$ onto an open set $U \subset SO(n+1) = G$, a neighborhood of the identity. Since G is compact, we can pick a finite number of elements $\xi_j \in G$ such that the open sets $U_j = \{\xi_j g : g \in U\}$ cover G. Pick $\eta_j \in \text{Skew}(n+1)$ such that $\text{Exp}\,\eta_j = \xi_j$. Define $\Phi_{jt} : U_j \to G$ for $0 \le t \le 1$ by

$$\Phi_{jt}\big(\xi_j\ \text{Exp}(A)\big) = (\text{Exp}\ t\eta_j)(\text{Exp}\ tA), \quad A \in \mathcal{O}. \tag{9.4}$$

Now partition G into subsets Ω_j, each of whose boundaries has content zero, such that $\Omega_j \subset U_j$. If $g \in \Omega_j$, set $g(t) = \Phi_{jt}(g)$. This family of elements of $SO(n+1)$ defines a family of maps $F_{gt} : S^n \to S^n$. Now, as in (6.31) we have,

$$\alpha = g^*\alpha - d\kappa_g(\alpha), \quad \kappa_g(\alpha) = \int_0^1 F_{gt}^*(\alpha \rfloor X_{gt})\,dt, \tag{9.5}$$

for each $g \in SO(n+1)$, where X_{gt} is the family of vector fields on S^n generated by F_{gt}, as in (6.29). Therefore,

$$\alpha = \int_G g^* \alpha \, dg - d \int_G \kappa_g(\alpha) \, dg. \tag{9.6}$$

Now the first term on the right is equal to $\overline{\alpha}\omega$, where $\overline{\alpha} = \int a(g \cdot x) dg$ is a constant; in fact, the constant is

$$\overline{\alpha} = \frac{1}{\text{Vol } S^n} \int_{S^n} \alpha. \tag{9.7}$$

Thus in this case (9.3) is precisely what serves to make (9.6) a representation of α as an exact form. This finishes the case $M = S^n$.

For a general compact, oriented, connected M, proceed as follows. Cover M with open sets $\mathcal{O}_1, \dots, \mathcal{O}_K$ such that each $\overline{\mathcal{O}}_j$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$. Set $U_1 = \mathcal{O}_1$, and inductively enlarge each $\mathcal{O}_j$ to U_j, so that $\overline{U}_j$ is also diffeomorphic to the closed ball, and such that $U_{j+1} \cap U_j \neq \emptyset$, $1 \leq j < K$. You can do this by drawing a simple curve from $\overline{\mathcal{O}}_{j+1}$ to a point in U_j and thickening it. Pick a smooth partition of unity φ_j, subordinate to this cover.

Given $\alpha \in \Lambda^n M$, satisfying (9.3), take $\tilde{\alpha}_j = \varphi_j \alpha$. Most likely $\int \tilde{\alpha}_1 = c_1 \neq 0$, so take $\sigma_1 \in \Lambda^n M$, with compact support in $U_1 \cap U_2$, such that $\int \sigma_1 = c_1$. Set $\alpha_1 = \tilde{\alpha}_1 - \sigma_1$, and redefine $\tilde{\alpha}_2$ to be the old $\tilde{\alpha}_2$ plus σ_1. Make a similar construction using $\int \tilde{\alpha}_2 = c_2$, and continue. When you are done, you have

$$\alpha = \alpha_1 + \cdots + \alpha_K, \tag{9.8}$$

with α_j compactly supported in U_j. By construction,

$$\int \alpha_j = 0 \tag{9.9}$$

for $1 \leq j < K$. But then (9.3) implies $\int \alpha_K = 0$ too.

Now pick $p \in S^n$ and define smooth maps

$$\psi_j : M \longrightarrow S^n \tag{9.10}$$

which map U_j diffeomorphically onto $S^n \setminus p$, and map $M \setminus U_j$ to p. There is a unique $v_j \in \Lambda^n S^n$, with compact support in $S^n \setminus p$, such that $\psi^* v_j = \alpha_j$. Clearly

$$\int_{S^n} v_j = 0,$$

so by the case $M = S^n$ of Proposition 9.5 already established, we know that $v_j = dw_j$ for some $w_j \in \Lambda^{n-1}S^n$, and then

$$\alpha_j = d\beta_j, \quad \beta_j = \psi_j^* w_j. \tag{9.11}$$

This concludes the proof.

We can sharpen and extend some of the topological results given above, using the notion of the degree of a map between compact oriented manifolds. Let X and Y be compact oriented n-dimensional manifolds. We want to define the degree of a smooth map $F : X \to Y$. To do this, assume Y is connected. We pick $\omega \in \Lambda^n Y$ such that

$$\int_Y \omega = 1. \tag{9.12}$$

We want to define

$$\operatorname{Deg}(F) = \int_X F^*\omega. \tag{9.13}$$

The following result shows that Deg (F) is indeed well defined by this formula. The key argument is an application of Proposition 9.5.

Lemma 9.6. *The quantity (9.13) is independent of the choice of ω, as long as (9.12) holds.*

Proof. Pick $\omega_1 \in \Lambda^n Y$ satisfying $\int_Y \omega_1 = 1$, so $\int_Y (\omega - \omega_1) = 0$. By Proposition 9.5, this implies

$$\omega - \omega_1 = d\alpha, \text{ for some } \alpha \in \Lambda^{n-1}Y. \tag{9.14}$$

Thus

$$\int_X F^*\omega - \int_X F^*\omega_1 = \int_X dF^*\alpha = 0, \tag{9.15}$$

and the lemma is proved.

The following is a most basic property.

Proposition 9.7. *If F_0 and F_1 are homotopic, then Deg $(F_0) =$ Deg (F_1).*

Proof. As noted in Exercise 6 of §6, if F_0 and F_1 are homotopic, then $F_0^*\omega - F_1^*\omega$ is exact, say $d\beta$, and of course $\int_X d\beta = 0$.

We next give an alternative formula for the degree of a map, which is very useful in many applications. A point $y_0 \in Y$ is called a regular value of F, provided that, for each $x \in X$ satisfying $F(x) = y_0$, $DF(x) : T_xX \to T_{y_0}Y$ is an isomorphism. The easy case of Sard's Theorem, discussed in Appendix O, implies that *most* points in Y are regular. Endow X with a volume element ω_X, and similarly endow Y with ω_Y. If $DF(x)$ is invertible, define $JF(x) \in \mathbb{R} \setminus 0$ by $F^*(\omega_Y) = JF(x)\omega_X$. Clearly the *sign* of $JF(x)$, i.e., sgn $JF(x) = \pm 1$, is independent of choices of ω_X and ω_Y, as long as they determine the given orientations of X and Y.

Proposition 9.8. *If y_0 is a regular value of F, then*

$$\textit{Deg}\,(F) = \sum \{sgn\ JF(x_j) : F(x_j) = y_0\}. \tag{9.16}$$

Proof. Pick $\omega \in \Lambda^n Y$, satisfying (9.12), with support in a small neighborhood of y_0. Then $F^*\omega$ will be a sum $\sum \omega_j$, with ω_j supported in a small neighborhood of x_j, and $\int \omega_j = \pm 1$ as sgn $JF(x_j) = \pm 1$.

The following result, which extends Proposition 9.7, is a powerful tool in degree theory.

Proposition 9.9. *Let $\overline{M}$ be a compact oriented manifold with boundary, dim $M = n+1$. Given a smooth map $F : \overline{M} \to Y$, let $f = F\big|_{\partial M} : \partial M \to Y$. Then*

$$\textit{Deg}\,(f) = 0.$$

Proof. Applying Stokes' Theorem to $\alpha = F^*\omega$, we have

$$\int_{\partial M} f^*\omega = \int_M dF^*\omega.$$

But $dF^*\omega = F^*d\omega$, and $d\omega = 0$ if dim $Y = n$, so we are done.

An easy corollary of this is Brouwer's no-retraction theorem. Compare the proof of Proposition 9.2. (In fact, note how similar are the proofs of Propositions 9.2 and 9.9.)

Corollary 9.10. *If $\overline{M}$ is a compact oriented manifold with nonempty boundary ∂M, then there is no smooth retraction $\varphi : \overline{M} \to \partial M$.*

Proof. Without loss of generality, we can assume $\overline{M}$ is connected. If there were a retraction, then $\partial M = \varphi(\overline{M})$ must also be connected, so Proposition 9.9 applies. But then we would have, for the map $id. = \varphi\big|_{\partial M}$, the contradiction that its degree is both zero and 1.

For another application of degree theory, let X be a compact smooth oriented hypersurface in $\mathbb{R}^{n+1}$, and set $\Omega = \mathbb{R}^{n+1} \setminus X$. (Assume $n \geq 1$.) Given $p \in \Omega$, define

$$F_p : X \longrightarrow S^n, \quad F_p(x) = \frac{x-p}{|x-p|}. \tag{9.17}$$

It is clear that Deg (F_p) is constant on each connected component of Ω. It is also easy to see that, when p crosses X, Deg (F_p) jumps by ± 1. Thus Ω has at least two connected components. This is most of the smooth case of the Jordan-Brouwer separation theorem:

Theorem 9.11. *If X is a smooth compact oriented hypersurface of $\mathbb{R}^{n+1}$, which is connected, then $\Omega = \mathbb{R}^{n+1} \setminus X$ has exactly 2 connected components.*

Proof. X being oriented, it has a smooth global normal vector field. Use this to separate a small collar neighborhood $\mathcal{C}$ of X into 2 pieces; $\mathcal{C} \setminus X = \mathcal{C}_\prime \cup \mathcal{C}_1$. The collar $\mathcal{C}$ is diffeomorphic to $[-1,1] \times X$, and each $\mathcal{C}_j$ is clearly connected. It suffices to show that any connected component $\mathcal{O}$ of Ω intersects either $\mathcal{C}_0$ or $\mathcal{C}_1$. Take $p \in \partial\mathcal{O}$. If $p \notin X$, then $p \in \Omega$, which is open, so p cannot be a boundary point of any component of Ω. Thus $\partial\mathcal{O} \subset X$, so $\mathcal{O}$ must intersect a $\mathcal{C}_j$. This completes the proof.

Let us note that, of the two components of Ω, exactly one is unbounded, say Ω_0, and the other is bounded; call it Ω_1. Then we claim that, if X is given the orientation it gets as $\partial\Omega_1$,

$$p \in \Omega_j \Longrightarrow \operatorname{Deg}(F_p) = j. \tag{9.18}$$

Indeed, for p very far from X, $F_p : X \to S^n$ is not onto, so its degree is 0. And when p crosses X, from Ω_0 to Ω_1, the degree jumps by $+1$.

For a simple closed curve in $\mathbb{R}^2$, this result is the smooth case of the Jordan curve theorem. That special case of the argument given above can be found in [Sto].

We remark that, with a bit more work, one can show that any compact smooth hypersurface in $\mathbb{R}^{n+1}$ is orientable. A proof will be sketched in §24.

The next application of degree theory is useful in the study of closed orbits of planar vector fields. Let C be a simple smooth closed curve in $\mathbb{R}^2$, parametrized by arc-length, of total length L. Say C is given by $x = \gamma(t),\ \gamma(t+L) = \gamma(t)$. Then we have a unit tangent field to C, $T(\gamma(t)) = \gamma'(t)$, defining

$$T : C \longrightarrow S^1. \tag{9.19}$$

Proposition 9.12. *For T given by (9.19), we have*

$$\mathit{Deg}\,(T) = 1. \tag{9.20}$$

Proof. Pick a tangent line ℓ to C such that C lies on one side of ℓ, as in Fig. 9.1. Without changing $\operatorname{Deg}(T)$, you can flatten out C a little, so it intersects ℓ along a line segment, from $\gamma(L_0)$ to $\gamma(L) = \gamma(0)$, where we take $L_0 = L - 2\varepsilon,\ L_1 = L - \varepsilon$.

Now T is close to the map $T_s : C \to S^1$, given by

$$T_s\big(\gamma(t)\big) = \frac{\gamma(t+s) - \gamma(t)}{|\gamma(t+s) - \gamma(t)|}, \tag{9.21}$$

for any $s > 0$ small enough; hence T and such T_s are homotopic; hence T and T_s are homotopic for *all* $s \in (0, L)$. Furthermore, we can even let $s = s(t)$ be any continuous function $s : [0, L] \to (0, L)$, such that $s(0) = s(L)$. In particular, T is homotopic to the map $V : C \to S^1$, obtained from (9.21) by taking

$$s(t) = L_1 - t, \text{ for } t \in [0, L_0],$$

and $s(t)$ going monotonically from $L_1 - L_0$ to L_1 for $t \in [L_0, L]$. Note that

$$V\big(\gamma(t)\big) = \frac{\gamma(L_1) - \gamma(t)}{|\gamma(L_1) - \gamma(t)|}, \quad 0 \le t \le L_0.$$

The parts of V over the ranges $0 \le t \le L_0$ and $L_0 \le t \le L$, respectively, are illustrated in Figures 9.1 and 9.2. We see that V maps the segment of C from $\gamma(0)$ to $\gamma(L_0)$ into the lower half of the circle S^1, and it maps the segment of C from $\gamma(L_0)$ to $\gamma(L)$ into the upper half of the circle S^1. Therefore V (hence T) is homotopic to a one-to-one map of C onto S^1, preserving orientation, and (9.20) is proved.

The material of this section can be cast in the language of deRham cohomology, which we now define. Let M be a smooth manifold. A smooth k-form u is said to be *exact* if $u = dv$ for some smooth $(k-1)$-form v, and *closed* if $du = 0$. Since $d^2 = 0$, every exact form is closed:

$$\mathcal{E}^k(M) \subset \mathcal{C}^k(M), \tag{9.22}$$

where $\mathcal{E}^k(M)$ and $\mathcal{C}^k(M)$ denote respectively the spaces of exact and closed k-forms. The deRham cohomology groups are defined as quotient spaces:

$$\mathcal{H}^k(M) = \mathcal{C}^k(M)/\mathcal{E}^k(M). \tag{9.23}$$

There are no (-1)-forms, so $\mathcal{E}^0(M) = 0$. A 0-form is a real-valued function, and it is closed if and only if it is constant on each connected component of M, so

$$\mathcal{H}^0(M) \approx \mathbb{R}^\nu, \quad \nu = \# \text{ connected components of } M. \tag{9.24}$$

An immediate consequence of Proposition 9.5 is the following:

Proposition 9.13. *If M is a compact connected oriented manifold of dimension n, then*

$$\mathcal{H}^n(M) \approx \mathbb{R}. \tag{9.25}$$

Via the pull-back of forms, a smooth map $F : X \to Y$ between two manifolds induces maps on cohomology:

$$F^* : \mathcal{H}^j(Y) \longrightarrow \mathcal{H}^j(X). \tag{9.26}$$

If X and Y are both compact, connected, and oriented, and of dimension n, then $F^* : \mathcal{H}^n(Y) \to \mathcal{H}^n(X)$, and, via the isomorphisms $\mathcal{H}^n(X) \approx \mathbb{R} \approx \mathcal{H}^n(Y)$, this map is simply multiplication by Deg F.

DeRham cohomology plays an important role in material we develop later, particularly in the theory of characteristic classes, in §§19–20. Also, the Hodge theory of §§21–24 provides some useful tools in the study of deRham cohomology.

Exercises

1. Show that the identity map $I : X \to X$ has degree 1.

2. Show that, if $F : X \to Y$ is not onto, then $\mathrm{Deg}(F) = 0$.

3. If $A : S^n \to S^n$ is the antipodal map, show that $\mathrm{Deg}(A) = (-1)^{n-1}$.

4. Show that the homotopy invariance property given in Proposition 9.7 can be deduced as a corollary of Proposition 9.9.
Hint. Take $\overline{M} = X \times [0, 1]$.

5. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. Show that, if we identify $S^2 \approx \mathbb{C} \cup \{\infty\}$, then $p : \mathbb{C} \to \mathbb{C}$ has a unique continuous extension $\widetilde{p} : S^2 \to S^2$, with $\widetilde{p}(\infty) = \infty$. Show that

$$\mathrm{Deg}\, \widetilde{p} = n.$$

Deduce that $\widetilde{p} : S^2 \to S^2$ is onto, and hence that $p : \mathbb{C} \to \mathbb{C}$ is onto. In particular, each nonconstant polynomial in z has a complex root. This result is the Fundamental Theorem of Algebra.
Hint. For z large, set $\zeta = 1/z$ and consider

$$\varphi(\zeta) = \frac{1}{p(1/\zeta)} = \frac{\zeta^n}{1 + a_{n-1}\zeta + \cdots + a_0\zeta^n} = g(\zeta)^n,$$

with

$$g(\zeta) = \zeta + b\zeta^2 + \cdots .$$

Show that, for $w \in \mathbb{C}$, $|w|$ small, $\varphi^{-1}(w)$ consists of n points.

6. Suppose X, Y, and Z are compact and oriented, with Y and Z connected. Let $f : X \to Y$ and $g : Y \to Z$ be smooth. Show that

$$\mathrm{Deg}\,(g \circ f) = \mathrm{Deg}\,(g)\,\mathrm{Deg}\,(f).$$

10. Critical points and index of a vector field

A critical point of a vector field V is a point where V vanishes. Let V be a vector field defined on a neighborhood $\mathcal{O}$ of $p \in \mathbb{R}^n$, with a single critical point, at p. Then, for any small ball B_r about p, $B_r \subset \mathcal{O}$, we have a map

$$V_r : \partial B_r \to S^{n-1}, \quad V_r(x) = \frac{V(x)}{|V(x)|}. \tag{10.1}$$

The degree of this map is called the *index* of V at p, denoted $\mathrm{ind}_p(V)$; it is clearly independent of r. If V has a finite number of critical points, then the index of V is defined to be

$$\mathrm{Index}\,(V) = \sum \mathrm{ind}_{p_j}(V). \tag{10.2}$$

If $\psi : \mathcal{O} \to \mathcal{O}'$ is an orientation preserving diffeomorphism, taking p to p and V to W, then we claim

$$\mathrm{ind}_p(V) = \mathrm{ind}_p(W). \tag{10.3}$$

In fact, $D\psi(p)$ is an element of $GL(n, \mathbb{R})$ with positive determinant, so it is homotopic to the identity, and from this it readily follows that V_r and W_r are homotopic maps of $\partial B_r \to S^{n-1}$. Thus one has a well defined notion of the index of a vector field with a finite number of critical points on any oriented manifold M.

A vector field V on $\mathcal{O} \subset \mathbb{R}^n$ is said to have a non-degenerate critical point at p provided $DV(p)$ is a nonsingular $n \times n$ matrix. The following formula is convenient.

Proposition 10.1**.** *If V has a nondegenerate critical point at p, then*

$$ind_p(V) = \ sgn\ det\ DV(p). \tag{10.4}$$

Proof. If p is a nondegenerate critical point, and we set $\psi(x) = DV(p)x$, $\psi_r(x) = \psi(x)/|\psi(x)|$, for $x \in \partial B_r$, it is readily verified that ψ_r and V_r are homotopic, for r small. The fact that $\mathrm{Deg}(\psi_r)$ is given by the right side of (10.4) is an easy consequence of Proposition 9.8.

The following is an important global relation between index and degree.

Proposition 10.2**.** *Let $\overline{\Omega}$ be a smooth bounded region in $\mathbb{R}^{n+1}$. Let V be a vector field on $\overline{\Omega}$, with a finite number of critical points p_j, all in the interior Ω. Define $F : \partial\Omega \to S^n$ by $F(x) = V(x)/|V(x)|$. Then*

$$Index\,(V) = Deg\,(F). \tag{10.5}$$

Proof. If we apply Proposition 9.9 to $\overline{M} = \overline{\Omega} \setminus \bigcup_j B_\varepsilon(p_j)$, we see that $\mathrm{Deg}(F)$ is equal to the sum of degrees of the maps of $\partial B_\varepsilon(p_j)$ to S^n, which gives (10.5).

Next we look at a process of producing vector fields in higher dimensional spaces from vector fields in lower dimensional spaces.

Proposition 10.3. *Let W be a vector field on $\mathbb{R}^n$, vanishing only at* 0. *Define a vector field V on $\mathbb{R}^{n+k}$ by $V(x,y) = \big(W(x), y\big)$. Then V vanishes only at* $(0,0)$. *Then we have*

$$ind_0 W = ind_{(0,0)} V. \tag{10.6}$$

Proof. If we use Proposition 9.8 to compute degrees of maps, and choose $y_0 \in S^{n-1} \subset S^{n+k-1}$, a regular value of W_r, and hence also for V_r, this identity follows.

We turn to a more sophisticated variation. Let M be a compact oriented n dimensional surface in $\mathbb{R}^{n+k}$, W a (tangent) vector field on M with a finite number of critical points p_j. Let $\overline{\Omega}$ be a small tubular neighborhood of X, $\pi : \overline{\Omega} \to X$ mapping $z \in \overline{\Omega}$ to the nearest point in M. Let $\varphi(z) = \operatorname{dist}(z, X)^2$. Now define a vector field V on $\overline{\Omega}$ by

$$V(z) = W(\pi(z)) + \nabla\varphi(z). \tag{10.7}$$

Proposition 10.4. *If $F : \partial\Omega \to S^{n+k-1}$ is given by $F(z) = V(z)/|V(z)|$, then*

$$Deg\,(F) = Index\,(W). \tag{10.8}$$

Proof. We see that all the critical points of V are points in M which are critical for W, and, as in Proposition 10.3, Index (W) = Index (V). But Proposition 10.2 implies Index (V) = Deg (F).

Since $\varphi(z)$ is increasing as one goes away from M, it is clear that, for $z \in \partial\Omega$, $V(z)$ points *out of* $\overline{\Omega}$, provided it is a sufficiently small tubular neighborhood of M. Thus $F : \partial\Omega \to S^{n+k-1}$ is homotopic to the *Gauss map*

$$N : \partial\Omega \longrightarrow S^{n+k-1}, \tag{10.9}$$

given by the outward pointing normal. This immediately gives:

Corollary 10.5. *Let M be a compact oriented surface in $\mathbb{R}^{n+k}$, $\overline{\Omega}$ a small tubular neighborhood of M, and $N : \partial\Omega \to S^{n+k-1}$ the Gauss map. If W is a vector field on M with a finite number of critical points, then*

$$Index\,(W) = Deg\,(N). \tag{10.10}$$

Clearly the right side of (10.10) is independent of the choice of W. Thus any two vector fields on M with a finite number of critical points have the same index, i.e., Index (W) is an invariant of M. This invariant is denoted

$$\operatorname{Index}(W) = \chi(M), \tag{10.11}$$

and is called the Euler characteristic of M. See the exercises for more results on $\chi(M)$.

Exercises

A nondegenerate critical point p of a vector field V is said to be a source if the real parts of the eigenvalues of $DV(p)$ are all positive, a sink if they are all negative, and a saddle if they are all either positive or negative, and there exist some of each sign. Such a critical point is called a center if all orbits of V close to p are closed orbits, which stay near p; this requires all the eigenvalues of $DV(p)$ to be purely imaginary.

In Exerises 1–3, V is a vector field on a region $\Omega \subset \mathbb{R}^2$.

1. Let V have a nondegenerate critical point at p. Show that

$$\begin{aligned}
p \text{ saddle} &\Longrightarrow \operatorname{ind}_p(V) = -1 \\
p \text{ source} &\Longrightarrow \operatorname{ind}_p(V) = 1 \\
p \text{ sink} &\Longrightarrow \operatorname{ind}_p(V) = 1 \\
p \text{ center} &\Longrightarrow \operatorname{ind}_p(V) = 1
\end{aligned}$$

2. If V has a closed orbit γ, show that the map $T : \gamma \to S^1,\ T(x) = V(x)/|V(x)|$, has degree $+1$.
Hint. Use Proposition 9.12.

3. If V has a closed orbit γ whose inside $\mathcal{O}$ is contained in Ω, show that V must have at least one critical point in $\mathcal{O}$, and that the sum of the indices of such critical points must be $+1$.
Hint. Use Proposition 10.2.
If V has exactly one critical point in $\mathcal{O}$, show that it cannot be a saddle.

4. Let M be a compact oriented 2-dimensional manifold. Given a triangulation of M, within each triangle construct a vector field, vanishing at 7 points as illustrated in Fig. 10.1, with the vertices as attractors, the center as a repeller, and the midpoints of each side as saddle points. Fit these together to produce a smooth vector field X on M. Show directly that

$$\operatorname{Index}(X) = V - E + F, \tag{10.12}$$

where

$$V = \# \text{ vertices}, \; E = \# \text{ edges}, \; F = \# \text{ faces},$$

in the triangulation.

5. More generally, construct a vector field on an n-simplex so that, when a compact oriented n-dimensional surface M is triangulated into simplices, one produces a vector field X on M such that

$$\text{Index}\,(X) = \sum_{j=0}^{n} (-1)^j \nu_j, \tag{10.13}$$

where ν_j is the number of j-simplices in the triangulation, i.e., $\nu_0 = \#$ vertices, $\nu_1 = \#$ edges, $\dots, \nu_n = \#$ of n-simplices.
See Fig. 10.2 for a picture of a 3-simplex, with its faces (i.e., 2-simplices), edges, and vertices labelled.

The right side of (10.13) is one definition of $\chi(M)$. As we have seen that the left side of (10.13) is independent of the choice of X, it follows that the right side is independent of the choice of triangulation.

6. Let M be the sphere S^n, which is homeomorphic to the boundary of an $(n+1)$-simplex. Computing the right side of (10.13), show that

$$\chi(S^n) = 2 \text{ if } n \text{ even}, \quad 0 \text{ if } n \text{ odd}. \tag{10.14}$$

Conclude that, if n is even, there is no smooth nowhere vanishing vector field on S^n. (This re-proves Proposition 9.4.)

7. Consider the vector field R on S^2 generating rotation about an axis. Show that R has two critical points, at the "poles." Classify the critical points, compute Index(R), and compare the $n = 2$ case of (10.14). Do the same for a vector field with a source at the north pole and a sink at the south pole. Generalize from S^2 to S^n.

8. Assume $M^n \subset \mathbb{R}^{n+1}$; let M be oriented as the boundary of a bounded domain $\overline{\Omega}$. If $G : M \to S^n$ denotes the Gauss map of M to its outward normal, show that

$$\big(1 + (-1)^n\big)\text{Deg } G = \text{ Deg } N = \chi(M). \tag{10.15}$$

In particular,

$$n \ \text{ even} \Longrightarrow \operatorname{Deg} G = \frac{1}{2}\chi(M). \tag{10.16}$$

Hint. Note that the manifold $\partial\Omega$ in (10.9) consists essentially of 2 copies of M, with opposite orientations.

9. Show that the computation of the index of a vector field X on a manifold M is independent of orientation, and that $\operatorname{Index}(X)$ can be defined when M is not orientable.

10. Comparing $\operatorname{Index}(X)$ and $\operatorname{Index}(-X)$, show that, for a compact smooth manifold M,

$$\dim M \ \text{ odd} \Longrightarrow \chi(M) = 0.$$

11. Let M, $\overline{\Omega}$, and G be as in Exercise 8. Show that there exists a smooth vector field W on $\overline{\Omega}$, with only isolated critical points, such that $W = G$ on $\partial\Omega$. Hence $\operatorname{Deg} G = \operatorname{Index} W$. Show that W gives rise to a vector field $\widetilde{W}$ on the double $\widetilde{\Omega}$ of $\overline{\Omega}$ such that $\operatorname{Index} \widetilde{W} = (1 + (-1)^{n+1}) \operatorname{Index} W$. Deduce that

$$n \ \text{ odd} \Longrightarrow \operatorname{Deg} G = \frac{1}{2}\chi(\widetilde{\Omega}). \tag{10.17}$$

Remark. Material in §§23–24 will identify the right side of (10.17) with $\chi(\overline{\Omega})$ (when n is odd). Furthermore, it will be seen that $\chi(\partial\Omega) = 2\chi(\overline{\Omega})$ when n is even. Hence, whether n is even or odd, one has

$$\operatorname{Deg} G = \chi(\overline{\Omega}). \tag{10.18}$$

12. Let $\overline{\Omega} \subset \mathbb{R}^4 \subset S^4$ be a tubular neighborhood of a diffeomorphic image of $\mathbb{T}^2$ in $\mathbb{R}^4$, let $M = \partial\Omega$, and define $G : M \to S^3$ as in Exercise 8. We have

$$\chi(\overline{\Omega}) = 0, \quad \operatorname{Deg} G = 0.$$

Consider $\overline{\Omega}_2 = S^4 \setminus \Omega$. It follows from material in §24 (cf. (24.8)) that

$$\chi(\overline{\Omega}_2) = 2.$$

Use this to produce a region $\overline{\Omega}_1 \subset \mathbb{R}^4$ diffeomorphic to $\overline{\Omega}_2$, with boundary $\partial\Omega_1$ diffeomorphic to M, and Gauss map

$$G_1 : \partial\Omega_1 \to S^3, \quad \operatorname{Deg} G_1 = 2.$$

Note the contrast with (10.16), which shows that $\operatorname{Deg} G$ is a differential topological invariant of M, when $\dim M$ is even.

13. Let M be a compact 2-dimensional surface. Let $\widetilde{M}$ be obtained by attaching a handle to M. Show that

$$\chi(\widetilde{M}) = \chi(M) - 2.$$

Hint. Take a vector field X on M as in Exercise 4. Attach one end of the handle to a circle about a source of X and the other to a circle about a sink of X. Produce a vector field $\widetilde{X}$ on $\widetilde{M}$ with all the singularities of X, *except* that one source and one sink are deleted. See Fig. 10.3.

11. Geodesics on Riemannian manifolds

We want to re-do the derivation of §1 of the ODE for a geodesic on a Riemannian manifold M. As before, let $\gamma_s(t)$ be a one parameter family of curves satisfying $\gamma_s(a) = p$, $\gamma_s(b) = q$, and (1.3). Then

$$V = \partial_s \gamma_s(t)|_{s=0} \tag{11.1}$$

is a vector field defined on the curve $\gamma_0(t)$, vanishing at p and q, and a general vector field of this sort could be obtained by a variation $\gamma_s(t)$. Let

$$T = \gamma_s'(t). \tag{11.2}$$

With the notation of (1.1), we have, parallel to (1.6),

$$\begin{aligned} L'(s) &= \int_a^b V\langle T, T\rangle^{1/2}\, dt \\ &= \frac{1}{2c_0}\int_a^b V\langle T, T\rangle\, dt, \quad \text{at } s = 0, \end{aligned} \tag{11.3}$$

assuming γ_0 has constant speed c_0, as in (1.3). Now we need a generalization of $(\partial/\partial s)\gamma_s'(t)$, and of the formula (1.7). One natural approach involves the notion of a *covariant derivative.*

If X and Y are vector fields on M, the covariant derivative $\nabla_X Y$ is a vector field on M. The following properties are to hold: we assume $\nabla_X Y$ is additive in both X and Y, that

$$\nabla_{fX} Y = f\nabla_X Y, \tag{11.4}$$

for $f \in C^\infty(M)$, and

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y, \tag{11.5}$$

i.e., ∇_X acts as a derivation. The operator ∇_X is required to have the following relation to the Riemannian metric:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{11.6}$$

One further property, called the "zero torsion condition," will uniquely specify ∇ :

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{11.7}$$

If these properties hold, one says ∇ is a "Levi-Civita connection." We have the following existence result.

Proposition 11.1. *A Riemannian metric has associated a unique Levi-Civita connection, given by*

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle. \tag{11.8}$$

Proof. To obtain the formula (11.8), cyclically permute $X,\ Y,\ Z$ in (11.6) and take the appropriate alternating sum, using (11.7) to cancel out all terms involving ∇ but two copies of $\langle \nabla_X Y, Z\rangle$. This derives the formula and establishes uniqueness. On the other hand, if (11.8) is taken as the definition of $\nabla_X Y$, then verification of the properties (11.4)–(11.7) is a routine exercise.

For example, if you interchange the roles of Y and Z in (11.8), and add to it the resulting formula for $2\langle Y, \nabla_X Z\rangle$, you get cancellation of all terms on the right side except $X\langle Y, Z\rangle + X\langle Z, Y\rangle$; this gives (11.6).

We can resume our analysis of (11.3), which becomes

$$L'(s) = \frac{1}{c_0}\int_a^b \langle \nabla_V T, T\rangle\, dt, \quad \text{at } s = 0. \tag{11.9}$$

Since $\partial/\partial s$ and $\partial/\partial t$ commute, we have $[V, T] = 0$ on γ_0, and (11.7) implies

$$L'(s) = \frac{1}{c_0}\int_a^b \langle \nabla_T V, T\rangle\, dt \ \text{ at } s = 0. \tag{11.10}$$

The replacement for (1.7) is

$$T\langle V, T\rangle = \langle \nabla_T V, T\rangle + \langle V, \nabla_T T\rangle, \tag{11.11}$$

so, by the fundamental theorem of calculus,

$$L'(0) = -\frac{1}{c_0}\int_a^b \langle V, \nabla_T T\rangle\, dt. \tag{11.12}$$

If this is to vanish for all smooth vector fields over γ_0, vanishing at p and q, we must have

$$\nabla_T T = 0. \tag{11.13}$$

This is the geodesic equation for a general Riemannian metric.

If the Riemannian metric takes the form $g_{jk}(x)$, in a coordinate chart, and ∇ is the corresponding Levi-Civita connection, the *Christoffel symbols* ${\Gamma^k}_{ij}$ are defined by

$$\nabla_{\partial_i}\partial_j = \sum_k {\Gamma^k}_{ij}\partial_k, \tag{11.14}$$

where $\partial_k = \partial/\partial x_k$. The formula (11.8) implies

$$g_{k\ell}\Gamma^\ell{}_{ij} = \frac{1}{2}\Big[\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\Big]. \tag{11.15}$$

Note the coincidence with (1.34). We can rewrite the geodesic equation (11.13) for $\gamma_0(t) = x(t)$ as follows. With $x = (x_1, \dots, x_n)$, $T = (\dot{x}^1, \dots, \dot{x}^n)$, we have

$$0 = \sum_\ell \nabla_T\big(\dot{x}^\ell \partial_\ell\big) = \sum_\ell \big[\ddot{x}^\ell \partial_\ell + \dot{x}^\ell \nabla_T \partial_\ell\big]. \tag{11.16}$$

In view of (11.14), this becomes

$$\ddot{x}^\ell + \dot{x}^j\ \dot{x}^k\ \Gamma^\ell{}_{jk} = 0 \tag{11.17}$$

(with the summation convention), just as in (1.33). The standard existence and uniqueness theory applies to this system of second order ODE. We will call any smooth curve satisfying the equation (11.13), or equivalently (11.17), a geodesic. Shortly we will verify that such a curve is indeed locally length minimizing. Note that if $T = \gamma'(t)$, then $T\langle T, T\rangle = 2\langle \nabla_T T, T\rangle$, so if (11.13) holds $\gamma(t)$ automatically has constant speed.

For a given $p \in M$, the exponential map

$$\mathrm{Exp}_p : U \longrightarrow M, \tag{11.18}$$

is defined on a neighborhood U of $0 \in T_pM$ by

$$\mathrm{Exp}_p(v) = \gamma_v(1) \tag{11.19}$$

where $\gamma_v(t)$ is the unique constant speed geodesic satisfying

$$\gamma_v(0) = p, \quad \gamma_v'(0) = v. \tag{11.20}$$

Note that $\mathrm{Exp}_p(tv) = \gamma_v(t)$. It is clear that Exp_p is well defined and C^∞ on a sufficiently small neighborhood U of $0 \in T_pM$, and its derivative at 0 is the identity. Thus, perhaps shrinking U, we have that Exp_p is a diffeomorphism of U onto a neighborhood $\mathcal{O}$ of p in M. This provides what is called an exponential coordinate system, or a normal coordinate system. Clearly the geodesics through p are the lines through the origin in this coordinate system. We claim that, in this coordinate system,

$$\Gamma^\ell{}_{jk}(p) = 0. \tag{11.21}$$

Indeed, since the line through the origin in any direction $a\partial_j + b\partial_k$ is a geodesic, we have

$$\nabla_{(a\partial_j + b\partial_k)}(a\partial_j + b\partial_k) = 0 \ \text{ at } p, \tag{11.22}$$

for all $a, b \in \mathbb{R}$, and all j, k. This implies

$$\nabla_{\partial_j} \partial_k = 0 \text{ at } p \text{ for all } j, k, \tag{11.23}$$

which implies (11.21). We note that (11.21) implies $\partial g_{jk}/\partial x_\ell = 0$ at p, in this exponential coordinate system. In fact, a simple manipulation of (11.15) gives

$$\frac{\partial g_{jk}}{\partial x_\ell} = g_{mk} \Gamma^m{}_{j\ell} + g_{mj} \Gamma^m{}_{k\ell}. \tag{11.24}$$

As a consequence, a number of calculations in differential geometry can be simplified by working in exponential coordinate systems.

We now establish a result, known as the *Gauss Lemma*, which implies that a geodesic is locally length minimizing. For a small, let $\Sigma_a = \{v \in \mathbb{R}^n : \|v\| = a\}$, and let $S_a = \operatorname{Exp}_p(\Sigma_a)$.

Proposition 11.2. *Any unit speed geodesic through p hitting S_a at $t = a$ is orthogonal to S_a.*

Proof. If $\gamma_0(t)$ is a unit speed geodesic, $\gamma_0(0) = p$, $\gamma_0(a) = q \in S_a$, and $V \in T_q M$ is tangent to S_a, there is a smooth family of unit speed geodesics, $\gamma_s(t)$, such that $\gamma_s(0) = p$ and $(\partial/\partial s)\gamma_s(a)\big|_{s=0} = V$. Using (11.10)–(11.11) for this family, with $0 \le t \le a$, since $L(s)$ is constant, we have

$$0 = \int_0^a T\langle V, T\rangle \, dt = \langle V, \gamma_0'(a)\rangle, \tag{11.25}$$

which proves the proposition.

Corollary 11.3. *Suppose $Exp_p : B_a \to M$ is a diffeomorphism of $B_a = \{v \in T_pM : |v| \le a\}$ onto its image $\mathcal{B}$. Then, for each $q \in \mathcal{B}$, $q = Exp_p(w)$, the curve $\gamma(t) = Exp_p(tw)$, $0 \le t \le 1$, is the unique shortest path from p to q.*

Proof. We can assume $|w| = a$. Let $\sigma : [0, 1] \to M$ be another constant speed path from p to q, say $|\sigma'(t)| = b$. We can assume $\sigma(t) \in \mathcal{B}$ for all $t \in [0, 1]$; otherwise restrict σ to $[0, \beta]$ where $\beta = \inf\{t : \sigma(t) \in \partial\mathcal{B}\}$ and the argument below will show this segment has length $\ge a$.

For all t such that $\sigma(t) \in \mathcal{B} \setminus p$, we can write $\sigma(t) = \operatorname{Exp}_p(r(t)\omega(t))$, for uniquely determined $\omega(t)$ in the unit sphere of T_pM, and $r(t) \in (0, a]$ If we pull the metric tensor of M back to B_a, we have

$$|\sigma'(t)|^2 = r'(t)^2 + r(t)^2 |\omega'(t)|^2,$$

by the Gauss lemma. Hence

$$\begin{aligned} b = \ell(\sigma) &= \int_0^1 |\sigma'(t)| \, dt \\ &= \frac{1}{b} \int_0^1 |\sigma'(t)|^2 \, dt \\ &\ge \frac{1}{b} \int_0^1 r'(t)^2 \, dt. \end{aligned} \tag{11.26}$$

Cauchy's inequality yields

$$\int_0^1 |r'(t)|\,dt \le \Bigl(\int_0^1 r'(t)^2\,dt\Bigr)^{1/2},$$

so the last quantity in (11.26) is $\ge a^2/b$. This implies $b \ge a$, with equality only if $|\omega'(t)| = 0$ for all t, so the corollary is proven.

The following is a useful converse.

Proposition 11.4. *Let $\gamma : [0,1] \to M$ be a constant speed Lipschitz curve from p to q that is absolutely length minimizing. Then γ is a smooth curve satisfying the geodesic equation.*

Proof. We make use of the following fact, which will be established below. Namely, there exists $a > 0$ such that, for each point $x \in \gamma$, $\mathrm{Exp}_x : B_a \to M$ is a diffeomorphism of B_a onto its image (and a is *independent* of $x \in \gamma$).

So choose $t_0 \in [0,1]$ and consider $x_0 = \gamma(t_0)$. The hypothesis implies that γ must be a length minimizing curve from x_0 to $\gamma(t)$, for all $t \in [0,1]$ By Corollary 11.3, $\gamma(t)$ coincides with a geodesic for $t \in [t_0, t_0+\alpha]$ and for $t \in [t_0-\beta, t_0]$, where $t_0+\alpha = \min(t_0+a, 1)$ and $t_0-\beta = \max(t_0-a, 0)$. We need only show that, if $t_0 \in (0,1)$, these two geodesic segments fit together smoothly, i.e., that γ is smooth in a neighborhood of t_0.

To see this, pick $\varepsilon > 0$ such that $\varepsilon < \min(t_0, a)$, and consider $t_1 = t_0 - \varepsilon$. The same argument as above applied to this case shows that γ coincides with a smooth geodesic on a segment including t_0 in its interior, so we are done.

The asserted lower bound on a follows from compactness plus the following observation. Given $p \in M$, there is a neighborhood $\mathcal{O}$ of $(p,0)$ in TM on which

$$\mathcal{E} : \mathcal{O} \longrightarrow M, \quad \mathcal{E}(x,v) = \mathrm{Exp}_x(v), \quad (v \in T_xM) \tag{11.27}$$

is defined. Let us set

$$\mathcal{F}(x,v) = \bigl(x, \mathrm{Exp}_x(v)\bigr), \quad \mathcal{F} : \mathcal{O} \longrightarrow M \times M. \tag{11.28}$$

We readily compute that

$$D\mathcal{F}(p,0) = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

as a map on $T_pM \oplus T_pM$, where we use Exp_p to identify $T_{(p,0)}T_pM \approx T_pM \oplus T_pM \approx T_{(p,p)}(M \times M)$. Hence the inverse function theorem implies that $\mathcal{F}$ is a diffeomorphism from a neighborhood of $(p,0)$ in TM onto a neighborhood of (p,p) in $M \times M$.

Let us remark that, though a geodesic is locally length minimizing, it need not be globally length minimizing. There are many simple examples of this, which we leave to the reader to produce.

In addition to length, another quantity associated with a smooth curve $\gamma : [a, b] \to M$ is *energy*:

$$E = \frac{1}{2} \int_a^b \langle T, T \rangle \, dt, \tag{11.29}$$

which differs from the arclength integral in that the integrand here is $\langle T, T \rangle$, rather than the square root of this quantity. If one has a family γ_s of curves, with variation (11.1) and with fixed endpoints, then

$$E'(0) = \frac{1}{2} \int_a^b V \langle T, T \rangle \, dt. \tag{11.30}$$

This is just like the formula (11.3) for $L'(0)$, except for the factor $1/2c_0$ in (11.3), but to get (11.30) we do *not* need to assume that the curve γ_0 has constant speed. Now the computations (11.9)–(11.12) have a parallel here:

$$E'(0) = -\int_a^b \langle V, \nabla_T T \rangle \, dt. \tag{11.31}$$

Hence the stationary condition for the energy functional (11.29) is $\nabla_T T = 0$, which coincides with the geodesic equation (11.13).

Exercises

1. Verify that the definition of ∇_X given by (11.8) does indeed provide a Levi-Civita connection, having properties (11.4)–(11.7).

2. Let G be a compact Lie group, with a bi-invariant Riemannian metric (such as $SO(n)$, discussed in §4). Show that any unit speed geodesic γ on G with $\gamma(0) = e$, the identity element, is a subgroup of G, i.e., $\gamma(s + t) = \gamma(s)\gamma(t)$. Hence the two notions of the exponential map on $T_e G$ coincide.
Hint. Given $p = \gamma(t_0)$, define $R_p : G \to G$ by $R_p(g) = pg^{-1}p$. Show that this is an isometry of G that fixes p and leaves γ invariant, though reversing its direction. Deduce from this that $p^2 = \gamma(2t_0)$.

3. Let M be a connected Riemannian manifold. Define $d(p, q)$ to be the infimum of lengths of smooth curves from p to q. Show this makes M a metric space.

4. Let M be a connected Riemannian manifold which, with the metric of Exercise 3, is compact. Show that any $p, q \in M$ can be joined by a geodesic of length $d(p, q)$. *Hint.* Let $\gamma_k : [0, 1] \to M,\ \gamma_k(0) = p,\ \gamma_k(1) = q$, be constant speed curves of lengths $\ell_k \to d(p, q)$. Use Ascoli's theorem to produce a Lipschitz curve of length $d(p, q)$ as a uniform limit of a subsequence of these.

Exercises 5–7 deal with an extension of the result of Exercise 4, known as the Hopf-Rinow theorem. We assume that M is a connected Riemannian manifold which is "geodesically complete," i.e., for each $x \in M,\ v \in T_x M,\ \mathrm{Exp}_x(tv)$ is defined for all $t \in \mathbb{R}$. The theorem states that for such a manifold, any two points can be joined by a geodesic.

Fix $p, q \in M$, and set $r = d(p, q)$. Assume $a > 0$ and $\mathrm{Exp}_p : B_a \to \mathcal{B}$ is a diffeomorphism.

5. Show that there exists $p_1 \in \partial\mathcal{B}$ such that $d(p_1, q) = \inf \{d(y, q) : y \in \partial\mathcal{B}\}$. Show that

$$d(p_1, q) = r - a.$$

Let $\gamma : \mathbb{R} \to M$ be the unit speed geodesic such that $\gamma(0) = p,\ \gamma(a) = p_1$.

6. Assume $\mathrm{Exp}_{p_1} : B_{a_1} \to \mathcal{B}_1$ is a diffeomorphism. Pick $p_2 \in \partial\mathcal{B}_1$ such that $d(p_2, q) = \inf \{d(y, q) : y \in \partial\mathcal{B}_1\}$. Show that $d(p_2, q) = r - a - a_1$ and that $d(p, p_2) = a + a_1$. Then show that

$$p_2 = \gamma(a + a_1).$$

Hint. Recall Proposition 11.4.

7. Show that, for all $t \in [0, r]$,

$$d(\gamma(t), q) = r - t.$$

Hint. Look at the set of $t \in [0, r]$ for which the conclusion is true.

8. Given a connected Riemannian manifold M, show that M is geodesically complete if and only if it is complete as a metric space, with metric defined as in Exercise 3.

12. The covariant derivative and divergence of tensor fields

The covariant derivative of a vector field on a Riemannian manifold was introduced in §11, in connection with the study of geodesics. We will briefly recall this concept here and relate the divergence of a vector field to the covariant derivative, before generalizing these notions to apply to more general tensor fields. A still more general setting for covariant derivatives is discussed in §13.

If X and Y are vector fields on a Riemannian manifold M, then $\nabla_X Y$ is a vector field on M, the covariant derivative of Y with respect to X. We have the properties

$$\nabla_{(fX)} Y = f \nabla_X Y \tag{12.1}$$

and

$$\nabla_X (fY) = f \nabla_X Y + (Xf) Y, \tag{12.2}$$

the latter being the *derivation property.* Also, ∇ is related to the metric on M by

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \tag{12.3}$$

where $\langle X, Y\rangle = g_{jk} X^j Y^k$ is the inner product on tangent vectors. The Levi-Civita connection on M is uniquely specified by (12.1)–(12.3) and the torsion-free property:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{12.4}$$

There is the explicit defining formula (derived already in (11.8)):

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle, \end{aligned} \tag{12.5}$$

which follows from cyclically permuting X, Y, and Z in (12.3) and combining the results, exploiting (12.4) to cancel out all covariant derivatives but one. Another way of writing this is the following. If

$$X = X^k \partial_k, \quad \partial_k = \frac{\partial}{\partial x_k} \quad \text{(summation convention)}, \tag{12.6}$$

then

$$\nabla_{\partial_j} X = X^k{}_{;j}\ \partial_k \tag{12.7}$$

with

$$X^k{}_{;j} = \partial_j X^k + \sum_\ell \Gamma^k{}_{\ell j} X^\ell \tag{12.8}$$

where the "connection coefficients" are given by the formula

$$\Gamma^{\ell}{}_{jk} = \frac{1}{2} g^{\ell\mu} \Big[\frac{\partial g_{j\mu}}{\partial x_k} + \frac{\partial g_{k\mu}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_\mu} \Big], \tag{12.9}$$

equivalent to (12.5). We also recall that $\partial g_{k\mu}/\partial x_j$ can be recovered from $\Gamma^{\ell}{}_{jk}$:

$$\frac{\partial g_{k\mu}}{\partial x_j} = g_{\ell\mu}\Gamma^{\ell}{}_{jk} + g_{\ell k}\Gamma^{\ell}{}_{j\mu}. \tag{12.10}$$

The divergence of a vector field has an important expression in terms of the covariant derivative.

Proposition 12.1. *Given a vector field X with components X^k as in (12.6),*

$$\textit{div}\ X = X^j{}_{;j}. \tag{12.11}$$

Proof. This can be deduced from our previous formula for div X,

$$\begin{aligned} \text{div } X &= g^{-1/2}\partial_j(g^{1/2}X^j) \\ &= \partial_j X^j + (\partial_j \log g^{1/2})X^j. \end{aligned} \tag{12.12}$$

One way to see this is the following. We can think of ∇X as defining a tensor field of type $(1, 1)$:

$$(\nabla X)(Y) = \nabla_Y X. \tag{12.13}$$

Then the right side of (12.11) is the trace of such a tensor field:

$$X^j{}_{;j} = \text{ Tr } \nabla X. \tag{12.14}$$

This is clearly defined independently of any choice of coordinate system. If we choose an exponential coordinate system centered at a point $p \in M$, then $g_{jk}(p) = \delta_{jk}$ and $\partial g_{jk}/\partial x_\ell = 0$ at p, so (12.12) gives div $X = \partial_j X^j$ at p, in this coordinate system, while the right side of (12.11) is equal to $\partial_j X^j + \Gamma^j{}_{\ell j} X^\ell = \partial_j X^j$ at p. This proves the identity (12.11).

The covariant derivative can be applied to forms, and other tensors, by requiring ∇ to be a derivation. On scalar functions, set

$$\nabla_X u = Xu. \tag{12.15}$$

For a 1-form α, $\nabla_X \alpha$ is characterized by the identity

$$\langle Y, \nabla_X \alpha \rangle = X\langle Y, \alpha \rangle - \langle \nabla_X Y, \alpha \rangle. \tag{12.16}$$

Denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M, and by $\Lambda^1(M)$ the space of smooth 1-forms; each of these is a module over $C^\infty(M)$. Generally, a tensor field of type (k, j) defines a map (with j factors of $\mathfrak{X}(M)$ and k of $\Lambda^1(M)$)

$$F : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \times \Lambda^1(M) \times \cdots \times \Lambda^1(M) \longrightarrow C^\infty(M), \tag{12.17}$$

which is linear in each factor, over the ring $C^\infty(M)$. A vector field is of type $(1, 0)$ and a 1-form is of type $(0, 1)$. The covariant derivative $\nabla_X F$ is a tensor of the same type, defined by

$$\begin{aligned}(\nabla_X F)(Y_1, \ldots, Y_j, \alpha_1, \ldots, \alpha_k) = X \cdot \big(F(Y_1, \ldots, Y_j, \alpha_1, \ldots, \alpha_k)\big)& \\ - \sum_{\ell=1}^{j} F(Y_1, \ldots, \nabla_X Y_\ell, \ldots, Y_j, \alpha_1, \ldots, \alpha_k)& \\ - \sum_{\ell=1}^{k} F(Y_1, \ldots, Y_j, \alpha_1, \ldots, \nabla_X \alpha_\ell, \ldots, \alpha_k),&\end{aligned} \tag{12.18}$$

where $\nabla_X \alpha_\ell$ is uniquely defined by (12.16). We can naturally consider ∇F as a tensor field of type $(k, j + 1)$:

$$(\nabla F)(X, Y_1, \ldots, Y_j, \alpha_1, \ldots, \alpha_k) = (\nabla_X F)(Y_1, \ldots, Y_j, \alpha_1, \ldots, \alpha_k). \tag{12.19}$$

For example, if Z is a vector field, ∇Z is a vector field of type $(1, 1)$, as already anticipated in (12.13). Hence it makes sense to consider the tensor field $\nabla(\nabla Z)$, of type $(1, 2)$. For vector fields X and Y, we define the *Hessian* $\nabla^2_{(X,Y)} Z$ to be the vector field characterized by

$$\langle \nabla^2_{(X,Y)} Z, \alpha \rangle = (\nabla \nabla Z)(X, Y, \alpha). \tag{12.20}$$

Since, by (12.19), if $F = \nabla Z$, we have

$$F(Y, \alpha) = \langle \nabla_Y Z, \alpha \rangle, \tag{12.21}$$

and, by (12.18),

$$(\nabla_X F)(Y, \alpha) = X \cdot \big(F(Y, \alpha)\big) - F(\nabla_X Y, \alpha) - F(Y, \nabla_X \alpha), \tag{12.22}$$

it follows by substituting (12.21) into (12.22) and using (12.16), that

$$\nabla^2_{(X,Y)} Z = \nabla_X \nabla_Y Z - \nabla_{(\nabla_X Y)} Z; \tag{12.23}$$

this is a useful formula for the Hessian of a vector field.

More generally, for any tensor field F, of type (j,k), the Hessian $\nabla^2_{(X,Y)}F$, also of type (j,k), is defined in terms of the tensor field $\nabla^2 F = \nabla(\nabla F)$, of type $(j,k+2)$, by the same type of formula as (12.20), and we have

$$\nabla^2_{(X,Y)}F = \nabla_X(\nabla_Y F) - \nabla_{(\nabla_X Y)}F, \tag{12.24}$$

by an argument similar to that for (12.23).

The metric tensor g is of type $(0,2)$, and the identity (12.3) is equivalent to

$$\nabla_X g = 0 \tag{12.25}$$

for all vector fields X, i.e., to $\nabla g = 0$. In index notation, this means

$$g_{jk;\ell} = 0, \ \text{ or equivalently, } g^{jk}{}_{;\ell} = 0. \tag{12.26}$$

We also note that the zero torsion condition (12.4) implies

$$u_{;j;k} = u_{;k;j} \tag{12.27}$$

when u is a smooth scalar function, with second covariant derivative $\nabla\nabla u$, a tensor field of type $(0,2)$. It turns out that analogous second order derivatives of a vector field differ by a term arising from the curvature tensor; this point is discussed in §§13–15.

We have seen an expression for the divergence of a vector field in terms of the covariant derivative. We can use this latter characterization to provide a general notion of divergence of a tensor field. If T is a tensor field of type (k,j), with components

$$T_\alpha{}^\beta = T_{\alpha_1\cdots\alpha_j}{}^{\beta_1\cdots\beta_k} = T(\partial_{\alpha_1},\dots,\partial_{\alpha_j}, dx_{\beta_1},\dots,dx_{\beta_k}) \tag{12.28}$$

in a given coordinate system, then div T is a tensor field of type $(k-1,j)$, with components

$$T_{\alpha_1\cdots\alpha_j}{}^{\beta_1\cdots\beta_{k-1}\ell}{}_{;\ell}. \tag{12.29}$$

In view of the special role played by the last index, the divergence of a tensor field T is mainly interesting when T has some symmetry property.

In view of (12.11), we know that a vector field X generates a volume preserving flow if and only if $X^j{}_{;j} = 0$. Complementing this, we investigate the condition that the flow generated by X consists of isometries, i.e., the flow leaves the metric g invariant, or equivalently

$$\mathcal{L}_X g = 0. \tag{12.30}$$

For vector fields U, V, we have

$$
(12.31)\qquad \begin{aligned}
(\mathcal{L}_X g)(U,V) &= -\langle \mathcal{L}_X U, V\rangle - \langle U, \mathcal{L}_X V\rangle + X\langle U,V\rangle \\
&= \langle \nabla_X U - \mathcal{L}_X U, V\rangle + \langle U, \nabla_X V - \mathcal{L}_X V\rangle \\
&= \langle \nabla_U X, V\rangle + \langle U, \nabla_V X\rangle
\end{aligned}
$$

where the first identity follows from the derivation property of $\mathcal{L}_X$, the second from the metric property (12.3) expressing $X\langle U,V\rangle$ in terms of covariant derivatives, and the third from the zero torsion condition (12.4). If U and V are coordinate vector fields $\partial_j = \partial/\partial x_j$, we can write this identity as

$$
(12.32)\qquad (\mathcal{L}_X g)(\partial_j, \partial_k) = g_{k\ell} X^{\ell}{}_{;j} + g_{j\ell} X^{\ell}{}_{;k}.
$$

Thus X generates a group of isometries (one says X is a *Killing field*) if and only if

$$
(12.33)\qquad g_{k\ell} X^{\ell}{}_{;j} + g_{j\ell} X^{\ell}{}_{;k} = 0.
$$

This takes a slightly shorter form for the covariant field

$$
(12.34)\qquad X_j = g_{jk} X^k.
$$

We state formally the consequence, which follows immediately from (12.33) and the vanishing of the covariant derivatives of the metric tensor.

Proposition 12.2. *X is a Killing vector field if and only if*

$$
(12.35)\qquad X_{k;j} + X_{j;k} = 0.
$$

Generally, half the quantity on the left side of (12.35) is called the *deformation tensor* of X. If we denote by ξ the 1-form $\xi = \sum X_j \, dx_j$, the deformation tensor is the *symmetric part* of $\nabla\xi$, a tensor field of type $(0,2)$. It is also useful to identify the antisymmetric part, which is naturally regarded as a 2-form.

Proposition 12.3. *We have*

$$
(12.36)\qquad d\xi = \frac{1}{2}\sum_{j,k} (X_{j;k} - X_{k;j})\, dx_k \wedge dx_j.
$$

Proof. By definition,

$$
(12.37)\qquad d\xi = \frac{1}{2}\sum_{j,k} (\partial_k X_j - \partial_j X_k)\, dx_k \wedge dx_j,
$$

and the identity with the right side of (12.36) follows from the symmetry $\Gamma^{\ell}{}_{jk} = \Gamma^{\ell}{}_{kj}$.

There is a useful generalization of the concept of a Killing field, namely a *conformal* Killing field, which is a vector field X whose flow consists of conformal diffeomorphisms of M, i.e., preserves the metric tensor up to a scalar factor:

$$\mathcal{F}_X^{t*} g = \alpha(t,x) g \Longleftrightarrow \mathcal{L}_X g = \lambda(x) g. \tag{12.38}$$

Note that the trace of $\mathcal{L}_X g$ is $2 \operatorname{div} X$, by (12.32), so the last identity in (12.38) is equivalent to $\mathcal{L}_X g = (2/n)(\operatorname{div} X) g$, or, with $(1/2)\mathcal{L}_X g = \operatorname{Def} X$,

$$\operatorname{Def} X - \frac{1}{n}(\operatorname{div} X) g = 0 \tag{12.39}$$

is the equation of a conformal Killing field.

To end this section, we note that concepts developed so far for Riemannian manifolds, i.e., manifolds with positive definite metric tensors, have extensions to indefinite metric tensors, including *Lorentz* metrics.

A Riemannian metric tensor produces a symmetric isomorphism

$$G : T_x M \longrightarrow T_x^* M, \tag{12.40}$$

which is positive. More generally, a symmetric isomorphism (12.40) corrsponds to a nondegenerate metric tensor. Such a tensor has a well defined signature (j,k), $j+k = n = \dim M$; at each $x \in M$, $T_x M$ has a basis $\{e_1, \dots, e_n\}$ of mutualy orthogonal vectors such that $\langle e_1, e_1\rangle = \cdots = \langle e_j, e_j\rangle = 1$, while $\langle e_{j+1}, e_{j+1}\rangle = \cdots = \langle e_n, e_n\rangle = -1$. If $j = 1$ (or $k = 1$), we say M has a Lorentz metric.

Exercises

1. Let φ be a tensor field of type $(0,k)$ on a Riemannian manifold, endowed with its Levi-Civita connection. Show that

$$(\mathcal{L}_X \varphi - \nabla_X \varphi)(U_1, \dots, U_k) = \sum_j \varphi(U_1, \dots, \nabla_{U_j} X, \dots, U_k).$$

How does this generalize (12.31)?

2. Recall the formula (6.27), when ω is a k-form:

$$\begin{aligned}(d\omega)(X_0, \dots, X_k) = &\sum_{j=0}^{k} (-1)^j X_j \cdot \omega(X_0, \dots, \widehat{X}_j, \dots, X_k) \\ &+ \sum_{0 \le \ell < j \le k} (-1)^{j+\ell} \omega([X_\ell, X_j], X_0, \dots, \widehat{X}_\ell, \dots, \widehat{X}_j, \dots, X_k).\end{aligned}$$

Show that the last double sum can be replaced by

$$-\sum_{\ell<j}(-1)^j\omega(X_0,\dots,\nabla_{X_j}X_\ell,\dots,\widehat{X}_j,\dots,X_k)$$
$$-\sum_{\ell>j}(-1)^j\omega(X_0,\dots,\widehat{X}_j,\dots,\nabla_{X_j}X_\ell,\dots,X_k).$$

3. Using Exercise 2 and the expansion of $(\nabla_{X_j}\omega)(X_0,\dots,\widehat{X}_j,\dots,X_k)$ via the derivation property, show that

$$(d\omega)(X_0,\dots,X_k)=\sum_{j=0}^{k}(-1)^j(\nabla_{X_j}\omega)(X_0,\dots,\widehat{X}_j,\dots,X_k). \tag{12.41}$$

Note that this generalizes Proposition 12.3.

4. Prove the identity

$$\partial_j \log\sqrt{g}=\sum_\ell \Gamma^\ell{}_{\ell j}.$$

Use either the identity (12.11), involving the divergence, or the formula (12.9) for $\Gamma^\ell{}_{jk}$. Which is easier?

5. Show that the characterization (12.17) of a tensor field of type (k,j) is equivalent to the condition that F be a section of the vector bundle $\big(\otimes^j T^*\big)\otimes\big(\otimes^k T\big)$, or equivalently, of the bundle $\mathrm{Hom}\,(\otimes^j T,\otimes^k T)$. Think of other variants.

6. The operation $X_j=g_{jk}X^k$ is called *lowering indices.* It produces a 1-form (section of T^*M) from a vector field (section of TM), implementing the isomorphism (12.38). Similarly one can raise indices, e.g.,

$$Y^j=g^{jk}Y_k,$$

producing a vector field from a 1-form, i.e., implementing the inverse isomorphism. Define more general operations raising and lowering indices, passing from tensor fields of type (j,k) to other tensor fields, of type (ℓ,m), with $\ell+m=j+k$. One says these tensor fields are associated to each other via the metric tensor.

7. Using (12.16) show that, if $\alpha = a_k(x)\,dx_k$ (summation convention) then $\nabla_{\partial_j}\alpha = a_{k;j}\,dx_k$ with

$$a_{k;j} = \partial_j a_k - \sum_\ell \Gamma^\ell{}_{kj} a_\ell.$$

Compare (12.8). Use this to verify that (12.36) and (12.37) are equal. Work out a corresponding formula for $\nabla_{\partial_\ell} T$ when T is a tensor field of type (k,j), as in (12.28). Show

$$\begin{aligned} T_{\alpha_1\cdots\alpha_j}{}^{\beta_1\cdots\beta_k}{}_{;\ell} = \partial_\ell T_{\alpha_1\cdots\alpha_j}{}^{\beta_1\cdots\beta_k} &+ \sum_{i,s} T_{\alpha_1\cdots\alpha_j}{}^{\beta_1\cdots i\cdots\beta_k}\,\Gamma^{\beta_s}{}_{i\ell} \\ &- \sum_{i,t} T_{\alpha_1\cdots i\cdots\alpha_j}{}^{\beta_1\cdots\beta_k}\,\Gamma^i{}_{\alpha_t\ell}. \end{aligned}$$

8. Using the formula (12.23) for the Hessian, show that, for vector fields X, Y, Z on M,

$$\big(\nabla^2_{(X,Y)} - \nabla^2_{(Y,X)}\big)Z = \big([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\big)Z.$$

Denoting this by $R(X,Y,Z)$, show that it is linear in each of its three arguments over the ring $C^\infty(M)$, e.g., $R(X,Y,fZ) = f\,R(X,Y,Z)$ for $f \in C^\infty(M)$. Discussion of $R(X,Y,Z)$ as the *curvature tensor* is given in §15.

9. Verify (12.24). For a function u, to show that $\nabla^2_{(X,Y)}u = \nabla^2_{(Y,X)}u$, use the special case

$$\nabla^2_{(X,Y)}u = XYu - (\nabla_X Y)\cdot u$$

of (12.24). Note that this is an invariant formulation of (12.27). Show that

$$\nabla^2_{(X,Y)}u = \frac{1}{2}(\mathcal{L}_V g)(X,Y), \quad V = \text{ grad } u.$$

10. Let ω be the *volume form* of an oriented Riemannian manifold M. Show that $\nabla_X\omega = 0$ for all vector fields X.
Hint. To obtain the identity at x_0, use a normal coordinate system centered at x_0.

11. Let X be a vector field on a Riemannian manifold M. Show that the formal adjoint of ∇_X, acting on vector fields, is

$$\nabla_X^* Y = -\nabla_X Y - (\operatorname{div} X)Y. \tag{12.42}$$

12. Show that the formal adjoint of $\mathcal{L}_X$, acting on vector fields, is

$$\mathcal{L}_X^* Y = -\mathcal{L}_X Y - (\operatorname{div} X)Y - 2\ \mathrm{Def}\,(X)Y, \tag{12.43}$$

where Def(X) is a tensor field of type $(1,1)$, given by

$$\frac{1}{2}(\mathcal{L}_X g)(Z,Y) = g(Z, \mathrm{Def}\,(X)Y), \tag{12.44}$$

g being the metric tensor.

13. With div defined by (12.29) for tensor fields, show that

$$\operatorname{div}(X \otimes Y) = (\operatorname{div} Y)X + \nabla_Y X. \tag{12.45}$$

14. If we define Def : $C^\infty(M,T) \to C^\infty(M, S^2T^*)$ by Def$\,(X) = (1/2)\mathcal{L}_X g$, show that

$$\mathrm{Def}^*\, u = -\operatorname{div} u,$$

where $(\operatorname{div}\ u)^j = u^{jk}{}_{;k}$, as in (12.29).

15. If $\gamma(s)$ is a unit speed geodesic on a Riemannian manifold M, $\gamma'(s) = T(s)$, and X is a vector field on M, show that

$$\frac{d}{ds}\langle T(s), X(\gamma(s))\rangle = \frac{1}{2}\big(\mathcal{L}_X g\big)(T,T). \tag{12.46}$$

Deduce that, if X is a Killing field, then $\langle T, X\rangle$ is constant on γ.
Hint. Show that the left side of (12.46) is equal to $\langle T, \nabla_T X\rangle$.

16. Let M be the surface of revolution $x^2 + y^2 = g(z)^2$, discussed in Exercises 1–2 of §1, with $ds^2 = \big(1 + g'(u)^2\big)du^2 + g(u)^2 dv^2$. Show that $\partial/\partial v$ is a Killing field. Deduce that a unit speed geodesic $\big(x(t), y(t), z(t)\big)$ on M satisfies

$$\dot z(t) = \pm \frac{1}{g(z)}\sqrt{\frac{g(z)^2 - c^2}{1 + g'(z)^2}}. \tag{12.47}$$

Hint. Representing the path as $(u(t), v(t))$, write out $\langle T, \partial/\partial v\rangle$ and $\langle T, T\rangle$, and eliminate $\dot v(t)$.

17. In the setting of Exercise 16, suppose $c > \inf\, g(u)$. Show that a unit speed geodesic on which $\langle T, \partial/\partial v\rangle = c$ must be confined to the region $\{(x,y,z) : g(z) \ge c\}$.

13. Covariant derivatives and curvature on general vector bundles

Let $E \to M$ be a vector bundle, either real or complex. A covariant derivative, or connection, on E is a map

$$\nabla_X : C^\infty(M,E) \longrightarrow C^\infty(M,E) \tag{13.1}$$

assigned to each vector field X on M, satisfying the following four conditions:

$$\nabla_X(u+v) = \nabla_X u + \nabla_X v, \tag{13.2}$$

$$\nabla_{(fX+Y)} u = f\nabla_X u + \nabla_Y u, \tag{13.3}$$

$$\nabla_X(fu) = f\nabla_X u + (Xf)u, \tag{13.4}$$

where u, v are sections of E, f is a smooth scalar function. The examples which arose in §§11–12 are the Levi-Civita connection on a Riemannian manifold, in which case E is the tangent bundle, and associated connections on tensor bundles.

One general construction of connections is the following. Let F be a vector space, with an inner product; we have the trivial bundle $M \times F$. Let E be a subbundle of this trivial bundle; for each $x \in M$, let P_x be the orthogonal projection of F on $E_x \subset F$. Any $u \in C^\infty(M,E)$ can be regarded as a function from M to F, and for a vector field X, we can apply X componentwise to any function on M with values in F; call this action $u \mapsto D_X u$. Then a connection on M is given by

$$\nabla_X u(x) = P_x D_X u(x). \tag{13.5}$$

If M is imbedded in a Euclidean space $\mathbb{R}^N$, then $T_x M$ is naturally identified with a linear subspace of $\mathbb{R}^N$ for each $x \in M$. In this case one can verify that the connection defined by (13.5) coincides with the Levi-Civita connection, where M is given the metric induced from its imbedding in $\mathbb{R}^N$. See Corollary 16.2 for a proof.

Generally, a connection defines the notion of "parallel transport" along a curve γ in M. A section u of E over γ is obtained from $u(\gamma(t_0))$ by parallel transport if it satisfies $\nabla_T u = 0$ on γ, where $T = \dot{\gamma}(t)$.

Formulas for covariant derivatives, involving indices, are produced in terms of a choice of *local frame* for E, i.e., a set $e_\alpha,\ 1 \le \alpha \le K$, of sections of E over an open set U which forms a basis of E_x for each $x \in U$; $K = \dim E_x$. Given such a local frame, a smooth section u of E over U is specified by

$$u = u^\alpha e_\alpha \quad \text{(summation convention)}. \tag{13.6}$$

If $\partial_j = \partial/\partial x_j$ in a coordinate system on U, we set

$$\nabla_{\partial_j} u = u^\alpha{}_{;j} e_\alpha = (\partial_j u^\alpha + u^\beta \Gamma^\alpha{}_{\beta j}) e_\alpha, \tag{13.7}$$

the connection coefficients $\Gamma^\alpha{}_{\beta j}$ being defined by

$$\nabla_{\partial_j} e_\beta = \Gamma^\alpha{}_{\beta j} e_\alpha. \tag{13.8}$$

A vector bundle $E \to M$ may have an inner product on its fibers. In that case, a connection on E is called a *metric connection* provided that

$$X\langle u, v\rangle = \langle \nabla_X u, v\rangle + \langle u, \nabla_X v\rangle \tag{13.9}$$

for any vector field X and smooth sections u, v of E.

The curvature of a connection is defined by

$$R(X,Y)u = [\nabla_X, \nabla_Y]u - \nabla_{[X,Y]}u \tag{13.10}$$

where X and Y are vector fields and u is a section of E. It is easy to verify that (13.10) is linear in X, Y, and u, over $C^\infty(M)$. With respect to local coordinates, giving $\partial_j = \partial/\partial x_j$, and a local frame $\{e_\alpha\}$ on E, as in (13.6), we define the components $R^\alpha{}_{\beta jk}$ of the curvature by

$$R(\partial_j, \partial_k)e_\beta = R^\alpha{}_{\beta jk} e_\alpha, \tag{13.11}$$

as usual, using the summation convention. Since ∂_j and ∂_k commute, $R(\partial_j, \partial_k)e_\beta = [\nabla_{\partial_j}, \nabla_{\partial_k}]e_\beta$. Applying the formulas (13.7)–(13.8), we can express the components of R in terms of the connection coefficients. The formula is seen to be

$$R^\alpha{}_{\beta jk} = \partial_j \Gamma^\alpha{}_{\beta k} - \partial_k \Gamma^\alpha{}_{\beta j} + \Gamma^\alpha{}_{\gamma j}\Gamma^\gamma{}_{\beta k} - \Gamma^\alpha{}_{\gamma k}\Gamma^\gamma{}_{\beta j}. \tag{13.12}$$

The formulas (13.7) and (13.12) can be written in a shorter form, as follows. Given a choice of local frame $\{e_\alpha : 1 \le \alpha \le K\}$, we can define $K \times K$ matrices $\Gamma_j = (\Gamma^\alpha{}_{\beta j})$ and $\mathfrak{R}_{jk} = (R^\alpha{}_{\beta jk})$. Then (13.7) can be written as

$$\nabla_{\partial_j} u = \partial_j u + \Gamma_j u,$$

and (13.12) is equivalent to

$$\mathfrak{R}_{jk} = \partial_j \Gamma_k - \partial_k \Gamma_j + [\Gamma_j, \Gamma_k]. \tag{13.13}$$

Note that $\mathfrak{R}_{jk}$ is antisymmetric in j and k. Now we can define a "connection 1-form" Γ and a "curvature 2-form" Ω by

$$\Gamma = \sum_j \Gamma_j \, dx_j, \quad \Omega = \frac{1}{2}\sum_{j,k} \mathfrak{R}_{jk} \, dx_j \wedge dx_k. \tag{13.14}$$

Then we can write (13.7) as

$$\nabla_X u = Xu + \Gamma(X)u,$$

and the formula (13.12) is equivalent to

$$\Omega = d\Gamma + \Gamma \wedge \Gamma. \tag{13.15}$$

The curvature has symmetries, which we record here, for the case of general vector bundles. The Riemann curvature tensor, associated with the Levi-Civita connection, has additional symmetries, which will be described in §15.

Proposition 13.1. *For any connection ∇ on $E \to M$, we have*

$$R(X,Y)u = -R(Y,X)u. \tag{13.16}$$

If ∇ is a metric connection, then

$$\langle R(X,Y)u, v\rangle = -\langle u, R(X,Y)v\rangle. \tag{13.17}$$

Proof. (13.16) is obvious from the definition (13.10); this is equivalent to the antisymmetry of $R^\alpha{}_{\beta jk}$ in j and k noted above. If ∇ is a metric connection, we can use (13.9) to deduce

$$\begin{aligned} 0 &= (XY - YX - [X,Y])\langle u, v\rangle \\ &= \langle R(X,Y)u, v\rangle + \langle u, R(X,Y)v\rangle, \end{aligned}$$

which gives (13.17).

Next we record the following implication of a connection having *zero* curvature. A section u of E is said to be *parallel* if $\nabla_X u = 0$ for all vector fields X.

Proposition 13.2. *If $E \to M$ has a connection ∇ whose curvature is zero, then any $p \in M$ has a neighborhood U on which there is a frame $\{e_\alpha\}$ for E consisting of parallel sections: $\nabla_X e_\alpha = 0$ for all X.*

Proof. If U is a coordinate neighborhood, e_α is parallel provided $\nabla_{\partial_j} e_\alpha = 0$ for $j = 1, \dots, n = \dim M$. The condition that $R = 0$ is equivalent to the condition that the operators ∇_{∂_j} all *commute* with each other, for $1 \le j \le n$. Consequently, Frobenius' Theorem (see §I) allows us to solve the system of equations

$$\nabla_{\partial_j} e_\alpha = 0, \quad j = 1, \dots, n, \tag{13.18}$$

on a neighborhood of p, with e_α *prescribed* at the point p. If we pick $e_\alpha(p)$, $1 \le \alpha \le K$, to be a basis of E_p, then $e_\alpha(x)$, $1 \le \alpha \le K$, will be linearly independent in E_x for x close to p, so the local frame of parallel sections is constructed.

It is useful to record several formulas which result from choosing a local frame $\{e_\alpha\}$ by parallel translation along rays through a point $p \in M$, the origin in some coordinate system $(x_1, \dots, x_n)$, so

$$\nabla_{r\partial/\partial r} e_\alpha = 0, \quad 1 \le \alpha \le K. \tag{13.19}$$

It is important to know that each e_α is smooth. To see this, let $\{f_\alpha\}$ be an arbitrary smooth local frame field such that $f_\alpha(p) = e_\alpha(p)$; say $\nabla_{\partial_j} f_\beta = \widetilde{\Gamma}^\alpha{}_{\beta j} f_\alpha$. If we write $e_\alpha(x) = a^\beta{}_\alpha(x) f_\beta(x)$ and compute $\nabla_T e_\alpha(tx)$ when $T = \gamma'(t)$, $\gamma(t) = tx$, we obtain

$$\nabla_T e_\alpha(tx) = \Big[\frac{d}{dt} a^\beta{}_\alpha(tx) + a^\gamma{}_\alpha(tx)\, x_j\, \widetilde{\Gamma}^\beta{}_{\gamma j}(tx)\Big] f_\beta(tx),$$

and this vanishes for all x, i.e., (13.19) is satisfied, if and only if

$$\frac{d}{dt}a^\beta{}_\alpha(tx) - a^\gamma{}_\alpha(tx)\,x_j\,\widetilde{\Gamma}^\beta{}_{\gamma j}(tx) = 0,$$

for all β. The initial condition is $a^\beta{}_\alpha(0) = \delta^\beta{}_\alpha$. Now, let us replace $a^\beta{}_\alpha(tx)$ by $a^\beta{}_\alpha(x,t)$, and consider the initial value problem

$$\frac{\partial}{\partial t}a^\beta{}_\alpha(x,t) + \widetilde{\Gamma}^\beta{}_{\gamma j}(tx)\,x_j\,a^\gamma{}_\alpha(x,t) = 0, \quad a^\beta{}_\alpha(x,0) = \delta^\beta{}_\alpha.$$

It is seen that, for any $c > 0$, $a^\beta{}_\alpha(cx, t/c)$ also satisfies this problem, so $a^\beta{}_\alpha(cx, t/c) = a^\beta{}_\alpha(x,t)$. This implies that $a^\beta{}_\alpha(x) = a^\beta_\alpha(x,1)$. Hence we have

$$e_\alpha(x) = a^\beta{}_\alpha(x,1) f_\beta(x).$$

This makes the smooth dependence of $e_\alpha(x)$ on x manifest.

Now (13.19) means that $\sum x_j \nabla_{\partial_j} e_\alpha = 0$. Consequently the connection coefficients (13.8) satisfy

$$x_1\Gamma^\alpha{}_{\beta 1} + \cdots + x_n\Gamma^\alpha{}_{\beta n} = 0. \tag{13.20}$$

Differentiation with respect to x_j gives

$$\Gamma^\alpha{}_{\beta j} = -x_1\partial_j\Gamma^\alpha{}_{\beta 1} - \cdots - x_n\partial_j\Gamma^\alpha{}_{\beta n}. \tag{13.21}$$

In particular,

$$\Gamma^\alpha{}_{\beta j}(p) = 0. \tag{13.22}$$

Comparison of (13.21) with

$$\Gamma^\alpha{}_{\beta j} = x_1\partial_1\Gamma^\alpha{}_{\beta j}(p) + \cdots + x_n\partial_n\Gamma^\alpha{}_{\beta j}(p) + O(|x|^2) \tag{13.23}$$

gives

$$\partial_k\Gamma^\alpha{}_{\beta j} = -\partial_j\Gamma^\alpha{}_{\beta k} \text{ at } p. \tag{13.24}$$

Consequently the formula (13.12) for curvature becomes

$$R^\alpha{}_{\beta jk} = 2\,\partial_j\Gamma^\alpha{}_{\beta k} \text{ at } p, \tag{13.25}$$

with respect to such a local frame. Note that, near p, (13.12) gives

$$R^\alpha{}_{\beta jk} = \partial_j\Gamma^\alpha{}_{\beta k} - \partial_k\Gamma^\alpha{}_{\beta j} + O(|x|^2). \tag{13.26}$$

Given vector bundles $E_j \to M$ with connections ∇^j, there is a natural covariant derivative on the tensor product bundle $E_1 \otimes E_2 \to M$, defined by the derivation property

$$\nabla_X(u \otimes v) = (\nabla^1_X u) \otimes v + u \otimes (\nabla^2_X v). \tag{13.27}$$

Also, if A is a section of $\mathrm{Hom}(E_1, E_2)$, the formula

$$(\nabla^\#_X A)u = \nabla^2_X(Au) - A(\nabla^1_X v) \tag{13.28}$$

defines a connection on $\mathrm{Hom}(E_1, E_2)$.

Regarding the curvature tensor R as a section of $(\otimes^2 T^*) \otimes \ \mathrm{End}(E)$ is natural in view of the linearity properties of R given after (13.10). Thus if $E \to M$ has a connection with curvature R, and if M also has a Riemannian metric, yielding a connection on T^*M, then we can consider $\nabla_X R$. The following, known as *Bianchi's identity*, is an important result involving the covariant derivative of R.

Proposition 13.3. *For any connection on $E \to M$, the curvature satisfies*

$$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0, \tag{13.29}$$

or equivalently

$$R^\alpha{}_{\beta ij;k} + R^\alpha{}_{\beta jk;i} + R^\alpha{}_{\beta ki;j} = 0. \tag{13.30}$$

Proof. Pick any $p \in M$. Choose normal coordinates centered at p and choose a local frame field for E by radial parallel translation, as above. Then, by (13.22) and (13.26),

$$R^\alpha{}_{\beta ij;k} = \partial_k \partial_i \Gamma^\alpha{}_{\beta j} - \partial_k \partial_j \Gamma^\alpha{}_{\beta i} \quad \text{at } p. \tag{13.31}$$

Cyclically permuting (i,j,k) here and summing clearly gives 0, proving the proposition.

Note that we can regard a connection on E as defining an operator

$$\nabla : C^\infty(M,E) \longrightarrow C^\infty(M, T^* \otimes E), \tag{13.32}$$

in view of the linear dependence of ∇_X on X. If M has a Riemannian metric and E a Hermitian metric, it is natural to study the adjoint operator

$$\nabla^* : C^\infty(M, T^* \otimes E) \longrightarrow C^\infty(M,E) \tag{13.33}$$

If u and v are sections of E, ξ a section of T^*, we have

$$\begin{aligned} \big(v, \nabla^*(\xi \otimes u)\big) &= (\nabla v, \xi \otimes u) \\ &= (\nabla_X v, u) \\ &= (v, \nabla_X^* u) \end{aligned} \tag{13.34}$$

where X is the vector field corresponding to ξ via the Riemannian metric. Using the divergence theorem we can establish:

Proposition 13.4. *If E has a metric connection, then*

$$\begin{aligned} \nabla^*(\xi \otimes u) &= \nabla_X^* u \\ &= -\nabla_X u - (\mathit{div}\ X) u, \end{aligned} \tag{13.35}$$

Proof. The first identity follows from (13.34) and does not require E to have a metric connection. If E does have a metric connection, integrating

$$\langle \nabla_X v, u \rangle = -\langle v, \nabla_X u \rangle + X \langle v, u \rangle$$

and using the identity

$$\int_M Xf\, dV = -\int_M (\operatorname{div} X) f\, dV, \quad f \in C_0^\infty(M), \tag{13.36}$$

which is a special case of (8.23), we have the second identity in (13.35). This completes the proof.

Exercises

1. If ∇ and $\widetilde{\nabla}$ are two connections on a vector bundle $E \to M$, show that

$$\nabla_X u = \widetilde{\nabla}_X u + C(X, u) \tag{13.37}$$

where C is a smooth section of $\operatorname{Hom}(T \otimes E, E) \approx T^* \otimes \operatorname{End}(E)$. Show that conversely, if C is such a section and ∇ a connection, then (13.37) defines $\widetilde{\nabla}$ as a connection.

2. If ∇ and $\widetilde{\nabla}$ are related as in Exercise 1, show that their curvatures R and $\widetilde{R}$ are related by

$$(R - \widetilde{R})(X, Y)u = [C_X, \widetilde{\nabla}_Y]u - [C_Y, \widetilde{\nabla}_X]u - C_{[X,Y]}u + [C_X, C_Y]u, \tag{13.38}$$

where C_X is the section of $\operatorname{End}(E)$ defined by $C_X u = C(X, u)$.

In Exercises 3–5, let $P(x)$, $x \in M$, be a smooth family of projections on a vector space F, with range E_x, forming a vector bundle $E \to M$; E gets a natural connection via (13.5).

3. Let $\gamma : I \to M$ be a smooth curve through $x_0 \in M$. Show that parallel transport of $u(x_0) \in E_{x_0}$ along I is characterized by the following (with $P'(t) = dP(\gamma(t))/dt$, etc.):

$$\frac{du}{dt} = P'(t)u. \tag{13.39}$$

Hint. One needs to see that, if (13.39) holds, then $u(t) = P(t)u(t)$ and $P(t)u'(t) = 0$, granted $u(0) \in E_{x_0}$. Show that (13.39) implies

$$\frac{d}{dt}\big(I - P(t)\big)u(t) = -P(t)u'(t) = -P'(t)\big(I - P(t)\big)u(t),$$

making use of $PP' = P'(I-P)$. Given $(I-P(0))u(0) = 0$, deduce that $(I - P(t))u(t) = 0$ for all t, and complete the argument.

4. If each $P(x)$ is an orthogonal projection of the inner product space F onto E_x, show that you get a metric connection.
Hint. Show that $du/dt \perp u(\gamma(t))$ via $P'P = (I-P)P'$.
Suppose E_x has codimension one in F; pick unit $N_x \perp E_x$ and write $P_x u = u - \langle u, N_x\rangle N_x$. Show that (13.39) becomes (with $T = \gamma'(t)$)

$$\frac{du}{dt} = -\langle u, D_T N\rangle N. \tag{13.40}$$

5. In what sense can $\Gamma = -dP\ P = -(I-P)dP$ be considered the connection 1-form, as in (13.13)? Show that the curvature form (13.15) is given by

$$\Omega = P\ dP \wedge dP\ P. \tag{13.41}$$

For more on this, see (16.50)–(16.53).

6. Show that the formula

$$d\Omega = \Omega \wedge \Gamma - \Gamma \wedge \Omega \tag{13.42}$$

follows from (13.15). Relate this to the Bianchi identity. Compare (14.13) in the next section.

7. Let $E \to M$ be a vector bundle with connection ∇, with two local frame fields $\{e_\alpha\}$ and $\{f_\alpha\}$, defined over $U \subset M$. Suppose

$$f_\alpha(x) = g^\beta{}_\alpha(x) e_\beta(x), \quad e_\alpha(x) = h^\beta{}_\alpha(x) f_\beta(x);$$

note that $g^\beta{}_\gamma(x) h^\gamma{}_\alpha(x) = \delta^\beta{}_\alpha$. Let $\Gamma^\alpha{}_{\beta j}$ be the connection coefficients for the frame field $\{e_\alpha\}$, as in (13.7)–(13.8), and let $\widetilde{\Gamma}^\alpha{}_{\beta j}$ be the connection coefficients for the frame field $\{f_\alpha\}$. Show that

$$\widetilde{\Gamma}^\alpha{}_{\beta j} = h^\alpha{}_\mu \Gamma^\mu{}_{\gamma j} g^\gamma{}_\beta + h^\alpha{}_\gamma (\partial_j g^\gamma{}_\beta). \tag{13.43}$$

8. Let $E \to M$ be a complex vector bundle. Show that E has a Hermitian inner product. Given such an inner product, show that there exists a metric connection on E.

9. Let $\gamma_s(t)$ be a 1-parameter family of curves on M. Let $E \to M$ be a vactor bundle with connection. Say $T = \gamma_s'(t)$, $W = \partial_s \gamma_s(t)$. Assume u_s is a section of E over γ_s obtained by parallel translation, so $\nabla_T u = 0$. Show that $w = \nabla_W u$ satisfies

$$\nabla_T w + R(W, T)u = 0. \tag{13.44}$$

Hint. Start with $\nabla_W \nabla_T u = 0$ and use $[W, T] = 0$.
See Exercises 10–11 of §15 for further results, when $E = TM$.

10. In the setting of Exercise 9, suppose ∇ has curvature zero. Assume M is simply connected. Take $p, q \in M$. Show that, if $\xi_p \in E_p$ is given, γ is a path from p to q, and ξ is defined along γ by parallel translation, then $\xi_q \in E_q$ is independent of the choice of γ. Use this to give another proof of Proposition 13.2.

14. Second covariant derivatives and covariant-exterior derivatives

Let M be a Riemannian manifold, with Levi-Civita connection, and let $E \rightarrow M$ be a vector bundle with connection. In §13 we saw that the covariant derivative acting on sections of E yields an operator

$$\nabla : C^\infty(M, E) \longrightarrow C^\infty(M, T^* \otimes E). \tag{14.1}$$

Now on $T^* \otimes E$ we have the product connection, defined by (13.27), yielding

$$\nabla : C^\infty(T^* \otimes E) \longrightarrow C^\infty(M, T^* \otimes T^* \otimes E). \tag{14.2}$$

If we compose (14.1) and (14.2), we get a second order differential operator called the *Hessian*:

$$\nabla^2 : C^\infty(M, E) \longrightarrow C^\infty(T^* \otimes T^* \otimes E). \tag{14.3}$$

If u is a section of E, X and Y vector fields, (14.3) defines $\nabla^2_{X,Y}$ as a section of E; using the derivation properties we have the formula

$$\nabla^2_{X,Y} u = \nabla_X \nabla_Y u - \nabla_{(\nabla_X Y)} u. \tag{14.4}$$

Note that the antisymmetric part is given by the curvature of the connection on E :

$$\nabla^2_{X,Y} u - \nabla^2_{Y,X} u = R(X, Y) u. \tag{14.5}$$

Now the metric tensor on M gives a linear map $T^* \otimes T^* \rightarrow \mathbb{R}$, hence a linear bundle map $\gamma : T^* \otimes T^* \otimes E \rightarrow E$. We can consider the composition of this with ∇^2 in (14.3):

$$\gamma \circ \nabla^2 : C^\infty(M, E) \longrightarrow C^\infty(M, E). \tag{14.6}$$

We want to compare $\gamma \circ \nabla^2$ and $\nabla^* \nabla$, in the case when E has a Hermitian metric and a metric connection.

Proposition 14.1. *If ∇ is a metric connection on E, then*

$$\nabla^* \nabla = -\gamma \circ \nabla^2 \quad \text{on } C^\infty(M, E). \tag{14.7}$$

Proof. Pick a local orthonormal frame of vector fields $\{e_j\}$, with dual frame $\{v_j\}$. Then for $u \in C^\infty(M, E)$, $\nabla u = \sum v_j \otimes \nabla_{e_j} u$, so (13.35) implies

$$\nabla^* \nabla u = \sum \bigl[-\nabla_{e_j} \nabla_{e_j} u - (\operatorname{div} e_j) u\bigr]. \tag{14.8}$$

Using (14.4), we have

$$\nabla^*\nabla u = -\sum \nabla^2_{e_j,e_j} u - \sum \big[\nabla_{\nabla_{e_j} e_j} u + (\operatorname{div} e_j)\nabla_{e_j} u\big]. \tag{14.9}$$

The first term on the right is equal to $-\gamma \circ \nabla^2 u$. Now, given $p \in M$, if we choose the local frame $\{e_j\}$ such that $\nabla_{e_j} e_k = 0$ at p, the rest of the right side vanishes at p. This establishes the identity (14.7).

We next define a "covariant exterior derivative" operator

$$d^\nabla : C^\infty(M, \Lambda^k T^* \otimes E) \longrightarrow C^\infty(M, \Lambda^{k+1} T^* \otimes E) \tag{14.10}$$

as follows. For $k = 0$, $d^\nabla = \nabla$, given by (14.1), and we require

$$d^\nabla(\beta \wedge u) = (d\beta) \wedge u - \beta \wedge d^\nabla u \tag{14.11}$$

whenever β is a 1-form and u a section of $\Lambda^k T^* \otimes E$. The operator d^∇ is also called the "gauge exterior derivative." Unlike the case of the ordinary exterior derivative,

$$d^\nabla \circ d^\nabla : C^\infty(M, \Lambda^k T^* \otimes E) \longrightarrow C^\infty(M, \Lambda^{k+2} T^* \otimes E)$$

is not necessarily zero, but rather

$$d^\nabla d^\nabla u = \Omega \wedge u, \tag{14.12}$$

where Ω is the curvature, and we use the antisymmetry (13.16) to regard Ω as a section of $\Lambda^2 T^* \otimes \operatorname{End}(E)$, as in (13.15). Verification of (14.12) is a straightforward calculation; (14.5) is in fact the special case of this, for $k = 0$.

The following is an alternative form of Bianchi's identity (13.29):

$$d^\nabla \Omega = 0, \tag{14.13}$$

where the left side is a priori a section of $\Lambda^3 T^* \otimes \operatorname{End}(E)$. This can also be deduced from (14.12), the associative law $d^\nabla(d^\nabla d^\nabla) = (d^\nabla d^\nabla) d^\nabla$, and the natural derivation property generalizing (14.11):

$$d^\nabla(A \wedge u) = (d^\nabla A) \wedge u + (-1)^j A \wedge d^\nabla u, \tag{14.14}$$

where u is a section of $\Lambda^k T^* \otimes E$ and A a section of $\Lambda^j T^* \otimes \operatorname{End}(E)$.

Exercises

1. Let $E \to M$ be a vector bundle with connection ∇, $u \in C^\infty(M, E)$. Fix $p \in M$. Show that, if $\nabla u(p) = 0$, then $\nabla^2_{X,Y} u(p)$ is *independent* of the choice of connection on M.

2. In particular, Exercise 1 applies to the trivial bundle $\mathbb{R} \times M$, with trivial flat connection, for which $\nabla_X u = \langle X, du \rangle = Xu$. Thus, if $u \in C^\infty(M)$ is real valued and $du(p) = 0$, then $D^2 u(p)$ is well defined as a symmetric bilinear form on T_pM. If, in a coordinate system, $X = \sum X_j \partial/\partial x_j$, $Y = \sum Y_j \partial/\partial x_j$, show that

$$D^2_{X,Y} u(p) = \sum \frac{\partial^2 u}{\partial x_j \partial x_k}(p)\ X_j Y_k. \tag{14.15}$$

Show that this invariance fails if $du(p) \neq 0$.

3. If u is a smooth section of $\Lambda^k T^* \otimes E$, show that
(14.16)

$$\begin{aligned} d^\nabla u(X_0, \dots, X_k) = &\sum_j (-1)^j \nabla_{X_j} u(X_0, \dots, \widehat{X}_j, \dots, X_k) \\ &+ \sum_{j<\ell} (-1)^{j+\ell} u([X_j, X_\ell], X_0, \dots, \widehat{X}_j, \dots, \widehat{X}_\ell, \dots, X_k). \end{aligned}$$

Compare formula (6.27) and Exercises 2–3 in §12.

4. Use (14.16) to verify the identity (14.12), i.e., $d^\nabla d^\nabla u = \Omega \wedge u$.

5. If ∇ and $\widetilde{\nabla}$ are connections on $E \to M$, related by $\nabla_X u = \widetilde{\nabla}_X u + C(X, u)$, $C \in C^\infty(M, T^* \otimes \mathrm{End}(E))$, with curvatures R and $\widetilde{R}$, and curvature forms Ω and $\widetilde{\Omega}$, show that

$$\Omega - \widetilde{\Omega} = d^\nabla C + C \wedge C. \tag{14.17}$$

Here the wedge product of two sections of $T^* \otimes \mathrm{End}(E)$ is a section of $\Lambda^2 T^* \otimes \mathrm{End}(E)$, produced in a natural fashion, as in (13.15). Show that (14.17) is equivalent to (13.38).

15. The curvature tensor of a Riemannian manifold

The Levi-Civita connection, which was introduced in §11, is a metric connection on the tangent bundle TM of a manifold M with a Riemannian metric, uniquely specified among all such by the zero-torsion condition

$$\nabla_Y X - \nabla_X Y = [Y, X]. \tag{15.1}$$

We recall the defining formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle, \end{aligned} \tag{15.2}$$

derived in (11.8). Thus, in a local coordinate system with the naturally associated frame field on the tangent bundle, the connection coefficients (13.8) are given by

$$\Gamma^\ell{}_{jk} = \frac{1}{2} g^{\ell\mu} \Big[\frac{\partial g_{j\mu}}{\partial x_k} + \frac{\partial g_{k\mu}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_\mu} \Big], \tag{15.3}$$

The associated curvature tensor is the Riemann curvature tensor:

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z. \tag{15.4}$$

In a local coordinate system such as discussed above, the expression for the Riemann curvature is a special case of (13.12), i.e.,

$$R^j{}_{k\ell m} = \partial_\ell \Gamma^j{}_{km} - \partial_m \Gamma^j{}_{k\ell} + \Gamma^j{}_{\nu\ell}\Gamma^\nu{}_{km} - \Gamma^j{}_{\nu m}\Gamma^\nu{}_{k\ell}. \tag{15.5}$$

Consequently we have an expression of the form

$$R^j{}_{k\ell m} = L\Big(g_{\alpha\beta}, \frac{\partial^2 g_{\gamma\delta}}{\partial x_\mu \partial x_\nu}\Big) + Q\Big(g_{\alpha\beta}, \frac{\partial g_{\gamma\delta}}{\partial x_\mu}\Big), \tag{15.6}$$

where L is linear in the second order derivatives of $g_{\alpha\beta}(x)$ and Q is quadratic in the first derivatives of $g_{\alpha\beta}(x)$, each with coefficients depending on $g_{\alpha\beta}(x)$.

Building on Proposition 13.2, we have the following result on metrics whose Riemannian curvature is zero.

Proposition 15.1. *If (M, g) is a Riemannian manifold whose curvature tensor vanishes, then the metric g is flat, i.e., there is a coordinate system about each $p \in M$ in which $g_{jk}(x)$ is constant.*

Proof. It follows from Proposition 13.2 that on a neighborhood U of p there are parallel vector fields $V_{(j)},\ j=1,\dots,n=\dim M$, i.e., in a given coordinate system

$$\nabla_{\partial_k} V_{(j)} = 0, \quad 1 \le j,k \le n, \tag{15.7}$$

such that $V_{(j)}(p)$ form a basis of T_pM. Let $v_{(j)}$ be the 1-forms associated to $V_{(j)}$ by the metric g, so

$$v_{(j)}(X) = g(X, V_{(j)}) \tag{15.8}$$

for all vector fields X. Hence

$$\nabla_{\partial_k} v_{(j)} = 0, \quad 1 \le j,k \le n. \tag{15.9}$$

We have $v_{(j)} = \sum v^k_{(j)} dx_k$ with $v^k_{(j)} = v_{(j)}(\partial_k) = \langle \partial_k, v_{(j)} \rangle$. The zero torsion condition (15.1), in concert with (15.8), gives

$$\partial_\ell \langle v_{(j)}, \partial_k \rangle - \partial_k \langle v_{(j)}, \partial_\ell \rangle = \langle v_{(j)}, \nabla_{\partial_\ell} \partial_k \rangle - \langle v_{(j)}, \nabla_{\partial_k} \partial_\ell \rangle = 0, \tag{15.10}$$

which is equivalent to

$$dv_{(j)} = 0, \quad j = 1, \dots, n. \tag{15.11}$$

(This also follows from (12.36).) Hence, locally, there exist functions $x_j,\ j = 1, \dots, n$, such that

$$v_{(j)} = dx_j. \tag{15.12}$$

The functions $(x_1, \dots, x_n)$ give a coordinate system near p. In this coordinate system the inverse of the matrix $\big(g_{jk}(x)\big)$ has entries $g^{jk}(x) = \langle dx_j, dx_k \rangle$. Now, by (13.9),

$$\partial_\ell g^{jk}(x) = \langle \nabla_{\partial_\ell} dx_j, dx_k \rangle + \langle dx_j, \nabla_{\partial_\ell} dx_k \rangle = 0, \tag{15.13}$$

so the proof is complete.

We have seen in Proposition 13.1 that R has the following symmetries:

$$R(X,Y) = -R(Y,X), \tag{15.14}$$

$$\langle R(X,Y)Z, W \rangle = -\langle Z, R(X,Y)W \rangle, \tag{15.15}$$

i.e., in terms of

$$R_{jk\ell m} = \langle R(\partial_\ell, \partial_m)\partial_k, \partial_j \rangle \tag{15.16}$$

we have

$$R_{jk\ell m} = -R_{jkm\ell} \tag{15.17}$$

and

$$R_{jk\ell m} = -R_{kj\ell m}. \tag{15.18}$$

The Riemann tensor has additional symmetries:

Proposition 15.2. *The Riemann tensor satisfies*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{15.19}$$

and

$$\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle, \tag{15.20}$$

i.e.,

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0 \tag{15.21}$$

and

$$R_{ijk\ell} = R_{k\ell ij}. \tag{15.22}$$

Proof. Plugging in the definition of each of the three terms of (15.19), one gets a sum which is seen to cancel out by virtue of the zero torsion condition (15.1). This gives (15.19) and hence (15.21). The identity (15.22) is an automatic consequence of (15.17), (15.18), and (15.21), by elementary algebraic manipulations, which we leave as an exercise, to complete the proof. Also, (15.22) follows from (15.50) below. (In fact, so does (15.21).)

The identity (15.19) is sometimes called Bianchi's first identity, (13.29) then being called Bianchi's second identity.

There are contractions of the Riemann tensor which are important. The *Ricci tensor* is defined by

$$\mathrm{Ric}_{jk} = R^i{}_{jik} = g^{\ell m} R_{\ell jmk}, \tag{15.23}$$

the summation convention being understood. By (15.22), this is symmetric in j,k. We can also raise indices:

$$\mathrm{Ric}^j{}_k = g^{j\ell}\,\mathrm{Ric}_{\ell k}; \quad \mathrm{Ric}^{jk} = g^{k\ell}\,\mathrm{Ric}^j{}_\ell. \tag{15.24}$$

Contracting again defines the *scalar curvature*:

$$S = \mathrm{Ric}^j{}_j. \tag{15.25}$$

As we will see below, the special nature of $R_{ijk\ell}$ for dim $M = 2$ implies

$$\mathrm{Ric}_{jk} = \frac{1}{2} S g_{jk}, \quad \text{if } \dim M = 2. \tag{15.26}$$

The Bianchi identity (13.29) yields an important identity for the Ricci tensor. Specializing (13.30) to $\alpha = i,\ \beta = j$ and raising the second index gives

$$R^{ij}{}_{ij;k} + R^{ij}{}_{jk;i} + R^{ij}{}_{ki;j} = 0, \tag{15.27}$$

i.e., $S_{;k} - \mathrm{Ric}^i{}_{k;i} - \mathrm{Ric}^j{}_{k;j} = 0$, or

$$S_{;k} = 2\ \mathrm{Ric}^j{}_{k;j}. \tag{15.28}$$

This is called the Ricci identity. An equivalent form is

$$\mathrm{Ric}^{jk}{}_{;j} = \frac{1}{2}(S\ g^{jk})_{;j}. \tag{15.29}$$

The identity in this form leads us naturally to a tensor known as the *Einstein tensor*:

$$G^{jk} = \mathrm{Ric}^{jk} - \frac{1}{2}S\ g^{jk}. \tag{15.30}$$

The Ricci identity is equivalent to

$$G^{jk}{}_{;j} = 0. \tag{15.31}$$

As shown in §12, this means the Einstein tensor has *zero divergence*. This fact plays an important role in Einstein's equation for the gravitational field. Note that, by (15.26), the Einstein tensor always vanishes when $\dim M = 2$. On the other hand, the identity (15.31) has the following implication when $\dim M > 2$.

Proposition 15.3. *If dim $M = n > 2$ and the Ricci tensor is a scalar multiple of the metric tensor, the factor necessarily being $1/n$ times the scalar curvature:*

$$Ric_{jk} = \frac{1}{n} S g_{jk}, \tag{15.32}$$

then S must be a constant.

Proof. (15.32) is equivalent to

$$G^{jk} = \Bigl(\frac{1}{n} - \frac{1}{2}\Bigr) S g^{jk}.$$

By (15.31) and the fact that the covariant derivative of the metric tensor is 0, we have

$$0 = \Bigl(\frac{1}{n} - \frac{1}{2}\Bigr) S_{;k} g^{jk},$$

or $S_{;k} = 0$, which proves the proposition.

We now make some comments on curvature of Riemannian manifolds M of dimension 2. In general $R^{jk}{}_{\ell m}$ are components of a section of $\mathrm{End}(\Lambda^2 T^*)$. When $\dim M = 2$, such a section is naturally identified with a scalar. In this case, each component $R^{jk}{}_{\ell m}$ is either 0 or $\pm$

$$R^{12}{}_{12} = R^{21}{}_{21} = K, \tag{15.33}$$

where K is a scalar function on M called the *Gauss curvature* of M, when dim $M = 2$. We can write $R_{1212} = g_{1j}g_{2k}R^{jk}{}_{12} = (g_{11}g_{22} - g_{12}g_{21})R^{12}{}_{12}$. Hence we also have

$$R_{1212} = R_{2121} = gK, \quad g = \det(g_{jk}). \tag{15.34}$$

Suppose we pick normal coordinates centered at $p \in M$, so $g_{jk}(p) = \delta_{jk}$. We see that, if dim $M = 2$,

$$\mathrm{Ric}_{jk}(p) = R_{1j1k} + R_{2j2k}.$$

Now, the first term on the right is zero unless $j = k = 2$, and the second term is zero unless $j = k = 1$. Hence, $\mathrm{Ric}_{jk}(p) = K(p)\delta_{jk}$, in normal coordinates, so in arbitrary coordinates

$$\mathrm{Ric}_{jk} = Kg_{jk}; \ \text{ hence } \ K = \frac{1}{2}S, \ \text{if dim } M = 2. \tag{15.35}$$

Explicit formulas for K when M is a surface in $\mathbb{R}^3$ are given by (16.22) and (16.29), in the next section. (See also Exercises 2 and 5–7 below.) The following is a fundamental calculation of the Gauss curvature of a 2-dimensional surface whose metric tensor is expressed in *orthogonal coordinates*:

$$ds^2 = E(x)\, dx_1^2 + G(x)\, dx_2^2. \tag{15.36}$$

Proposition 15.4. *Suppose dim $M = 2$ and the metric is given in coordinates by (15.36). Then the Gauss curvature $k(x)$ is given by*

$$k(x) = -\frac{1}{2\sqrt{EG}}\left[\partial_1\Big(\frac{\partial_1 G}{\sqrt{EG}}\Big) + \partial_2\Big(\frac{\partial_2 E}{\sqrt{EG}}\Big)\right]. \tag{15.37}$$

To establish (15.37), one can first compute that

$$\Gamma_1 = \big(\Gamma^j{}_{k1}\big) = \frac{1}{2}\begin{pmatrix} \partial_1 E/E & \partial_2 E/E \\ -\partial_2 E/G & \partial_1 G/G \end{pmatrix}, \quad \Gamma_2 = \big(\Gamma^j{}_{k2}\big) = \frac{1}{2}\begin{pmatrix} \partial_2 E/E & -\partial_1 G/E \\ \partial_1 G/G & \partial_2 G/G \end{pmatrix}.$$

Then, computing $\mathfrak{R}_{12} = (R^j{}_{k12}) = \partial_1\Gamma_2 - \partial_2\Gamma_1 + \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1$, we have

$$\begin{aligned} R^1{}_{212} &= -\frac{1}{2}\partial_1\Big(\frac{\partial_1 G}{E}\Big) - \frac{1}{2}\partial_2\Big(\frac{\partial_2 E}{E}\Big) \\ &\qquad + \frac{1}{4}\Big(-\frac{\partial_1 E}{E}\frac{\partial_1 G}{E} + \frac{\partial_2 E}{E}\frac{\partial_2 G}{G}\Big) - \frac{1}{4}\Big(\frac{\partial_2 E}{E}\frac{\partial_2 E}{E} - \frac{\partial_1 G}{E}\frac{\partial_1 G}{G}\Big). \end{aligned} \tag{15.38}$$

Now $R_{1212} = E\ R^1{}_{212}$ in this case, and (15.34) yields

$$k(x) = \frac{1}{EG}\ R_{1212} = \frac{1}{G}\ R^1{}_{212}. \tag{15.39}$$

If we divide (15.38) by G, then in the resulting formula for $k(x)$ interchange E and G, and ∂_1 and ∂_2, and sum the two formulas for $k(x)$, we get

$$k(x) = -\frac{1}{4}\left[\frac{1}{G}\partial_1\Big(\frac{\partial_1 G}{E}\Big) + \frac{1}{E}\partial_1\Big(\frac{\partial_1 G}{G}\Big)\right] - \frac{1}{4}\left[\frac{1}{E}\partial_2\Big(\frac{\partial_2 E}{G}\Big) + \frac{1}{G}\partial_2\Big(\frac{\partial_2 E}{E}\Big)\right],$$

which is easily transformed into (15.37).

If $E = G = e^{2v}$, we obtain a formula for the Gauss curvature of a surface whose metric is a conformal multiple of the flat metric:

Corollary 15.5. *Suppose dim $M = 2$ and the metric is given in coordinates by*

$$g_{jk}(x) = e^{2v}\delta_{jk}, \tag{15.40}$$

for a smooth v. Then the Gauss curvature $k(x)$ is given by

$$k(x) = -(\Delta_0 v)e^{-2v}, \tag{15.41}$$

where Δ_0 is the flat Laplacian in these coordinates:

$$\Delta_0 v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}. \tag{15.42}$$

For an alternative formulation of (15.41), note that the Laplace operator for the metric g_{jk} is given by

$$\Delta f = g^{-1/2}\partial_j(g^{jk}g^{1/2}\partial_k f),$$

and in the case (15.40), $g^{jk} = e^{-2v}\delta^{jk}$ and $g^{1/2} = e^{2v}$, so we have

$$\Delta f = e^{-2v}\Delta_0 f, \tag{15.43}$$

and hence (15.41) is equivalent to

$$k(x) = -\Delta v. \tag{15.44}$$

The comparison of the Gauss curvature of two surfaces which are conformally equivalent is a source of a number of interesting results. The following generalization of Corollary 15.5 is useful.

Proposition 15.6. *Let M be a two dimensional manifold with metric g, whose Gauss curvature is $k(x)$. Suppose there is a conformally related metric*

$$g' = e^{2u}g. \tag{15.45}$$

Then the Gauss curvature $K(x)$ of g' is given by

$$K(x) = \big(-\Delta u + k(x)\big)e^{-2u}, \tag{15.46}$$

where Δ is the Laplace operator for the metric g.

Proof. We will use Corollary 15.5 as a tool in this proof. It is shown §N that (M, g) is locally conformally flat, so we can assume without loss of generality that (15.40) holds; hence $k(x)$ is given by (15.41). Then

$$(g')_{jk} = e^{2w}\delta_{jk}, \quad w = u + v, \tag{15.47}$$

and (15.41) gives

$$K(x) = -(\Delta_0 w)e^{-2w} = \big[-(\Delta_0 u)e^{-2v} - (\Delta_0 v)e^{-2v}\big]e^{-2u}. \tag{15.48}$$

By (15.43) we have $(\Delta_0 u)e^{-2v} = \Delta u$, and applying (15.41) for $k(x)$ gives (15.46).

We end this section with a study of $\partial_j \partial_k g_{\ell m}(p_0)$ when one uses a geodesic normal coordinate system centered at p_0. We know from §11 that in such a coordinate system $\Gamma^{\ell}{}_{jk}(p_0) = 0$ and hence $\partial_j g_{k\ell}(p_0) = 0$. Thus, in such a coordinate system, we have

$$R^j{}_{k\ell m}(p_0) = \partial_\ell \Gamma^j{}_{km}(p_0) - \partial_m \Gamma^j{}_{k\ell}(p_0), \tag{15.49}$$

and hence (15.3) yields

$$R_{jk\ell m}(p_0) = \frac{1}{2}\Big(\partial_j\partial_m g_{k\ell} + \partial_k\partial_\ell g_{jm} - \partial_j\partial_\ell g_{km} - \partial_k\partial_m g_{j\ell}\Big). \tag{15.50}$$

In light of the complexity of this formula, the following may be somewhat surprising. Namely, as Riemann showed, one has

$$\partial_j\partial_k g_{\ell m}(p_0) = -\frac{1}{3}R_{\ell jmk} - \frac{1}{3}R_{\ell kmj}. \tag{15.51}$$

This is related to the existence of non-obvious symmetries at the center of a geodesic normal coordinate system, such as $\partial_j\partial_k g_{\ell m}(p_0) = \partial_\ell\partial_m g_{jk}(p_0)$. To prove (15.51), by polarization it suffices to establish

$$\partial_j^2 g_{\ell\ell}(p_0) = -\frac{2}{3}R_{\ell j\ell j}, \quad \forall\ j,\ell \tag{15.52}$$

Proving this is a 2-dimensional problem, since (by (15.50)) both sides of the asserted identity in (15.52) are unchanged if M is replaced by the image under Exp_p of the 2-dimensional linear span of ∂_j and ∂_ℓ. All one needs to show is that, if $\dim M = 2$,

$$\partial_1^2 g_{22}(p_0) = -\frac{2}{3}K(p_0) \ \text{ and } \ \partial_1^2 g_{11}(p_0) = 0, \tag{15.53}$$

where $K(p_0)$ is the Gauss curvature of M at p_0. Of these, the second part is trivial, since $g_{11}(x) = 1$ on the horizontal line through p_0. To establish the first part of (15.53), it is convenient to use geodesic *polar* coordinates, (r,θ), in which

$$ds^2 = dr^2 + G(r,\theta)\,d\theta^2. \tag{15.54}$$

By comparison, in normal coordinates (x_1, x_2), along the x_1-axis, we have

$$g_{22}(s,0) = s^{-2}G(s,0). \tag{15.55}$$

Since $\partial_s g_{22}(s,0) = 0$ at $s = 0$, we have $G(s,0)/s^2 = 1+O(s^2)$. This sort of behavior holds along any ray through the origin, so we have

$$G(r,\theta) = r^2 H(r,\theta), \quad H(r,\theta) = 1 + O(r^2).$$

Now, for the metric (15.54), the formula (15.37) implies that the Gauss curvature is

$$K = -\frac{1}{2G}\partial_r^2 G + \frac{1}{4G^2}(\partial_r G)^2 = -\frac{H_r}{rH} - \frac{H_{rr}}{2H} + \frac{H_r^2}{4H^2}, \tag{15.56}$$

so at the center

$$K(p_0) = -H_{rr} - \frac{1}{2}H_{rr} = -\frac{3}{2}H_{rr}.$$

On the other hand, since we have seen that $g_{22}(s,0) = G(s,0)/s^2 = H(s,0)$, the rest of the identity (15.53) is established.

Exercises

Exercises 1–3 below concern the problem of producing 2-dimensional surfaces with constant Gauss curvature.

1. For a 2-dimensional Riemannian manifold M, take geodesic polar coordinates, so the metric is

$$ds^2 = dr^2 + G(r,\theta)\,d\theta^2.$$

Use formula (15.55) for the Gauss curvature, to deduce that

$$K = -\frac{\partial_r^2\sqrt{G}}{\sqrt{G}}.$$

Hence, if $K = -1$, then

$$\partial_r^2\sqrt{G} = \sqrt{G}.$$

Show that

$$\sqrt{G}(0,\theta) = 0, \quad \partial_r\sqrt{G}(0,\theta) = 1,$$

and deduce that $\sqrt{G}(r,\theta) = \varphi(r)$ is the unique solution to

$$\varphi''(r) - \varphi(r) = 0, \quad \varphi(0) = 0,\ \varphi'(0) = 1.$$

Deduce that

$$G(r,\theta) = \ \sinh^2\, r.$$

Use this computation to deduce that any two surfaces with Gauss curvature -1 are locally isometric.

2. Suppose M is a surface of revolution in $\mathbb{R}^3$, of the form

$$x^2 + y^2 = g(z)^2.$$

If it is parametrized by $x = g(u)\ \cos v,\ y = g(u)\ \sin v,\ z = u$, then

$$ds^2 = \bigl(1 + g'(u)^2\bigr)\, du^2 + g(u)^2\, dv^2.$$

Deduce from (15.37) that

$$K = -\frac{g''(u)}{g(u)\bigl(1 + g'(u)^2\bigr)^2}.$$

Hence, if $K = -1$,

$$g''(u) = g(u)\bigl(1 + g'(u)^2\bigr)^2.$$

Note that a sphere of radius R is given by such a formula with $g(u) = \sqrt{R^2 - u^2}$. Compute K in this case.

2A. Suppose instead that M is a surface of revolution, described in the form

$$z = f\bigl(\sqrt{x^2 + y^2}\bigr).$$

If it is parametrized by $x = u\ \cos v,\ y = u\ \sin v,\ z = f(u)$, then

$$ds^2 = \bigl(1 + f'(u)^2\bigr)\, du^2 + u^2\, dv^2.$$

Show that

$$K = -\frac{1}{u\sqrt{1 + f'(u)^2}}\, \frac{d}{du}\left(\frac{1}{\sqrt{1 + f'(u)^2}}\right) = -\frac{\varphi'(u)}{2u}, \quad \varphi(u) = \frac{1}{1 + f'(u)^2}.$$

Thus deduce that

$$K = -1 \Rightarrow \varphi(u) = u^2 + c \Rightarrow f(u) = \int \sqrt{\frac{1}{u^2 + c} - 1}\ du.$$

We note that this is an elliptic integral, for most values of c. Show that, for $c = 0$, you get

$$f(u) = \sqrt{1 - u^2} - \frac{1}{2}\log\bigl(1 + \sqrt{1 - u^2}\bigr) + \frac{1}{2}\log\bigl(1 - \sqrt{1 - u^2}\bigr).$$

3. Suppose M is a region in $\mathbb{R}^2$ whose metric tensor is a conformal multiple of the standard flat metric

$$g_{jk} = E(x)\delta_{jk} = e^{2v}\ \delta_{jk}.$$

Suppose $E = E(r),\ v = v(r)$. Deduce from (15.37) and (15.41) that

$$K = -\frac{1}{2E^2}\Bigl(E''(r) + \frac{1}{r}E'(r)\Bigr) + \frac{1}{2E^3}E'(r)^2 = -\Bigl(v''(r) + \frac{1}{r}v'(r)\Bigr)e^{-2v}.$$

Hence, if $K = -1$,

$$v''(r) + \frac{1}{r}v'(r) = e^{2v}.$$

Compute K when

$$g_{jk} = \frac{4}{(1-r^2)^2}\ \delta_{jk}.$$

4. Show that, whenever $g_{jk}(x)$ satisfies, at some point $p_0,\ g_{jk}(p_0) = \delta_{jk},\ \partial_\ell g_{jk}(p_0) = 0$, then (15.50) holds at p_0. If dim $M = 2$, deduce that

$$K(p_0) = -\frac{1}{2}\bigl(\partial_1^2 g_{22} + \partial_2^2 g_{11} - 2\partial_1\partial_2 g_{12}\bigr). \tag{15.57}$$

5. Suppose $M \subset \mathbb{R}^3$ is the graph of

$$x_3 = f(x_1, x_2),$$

so, using the natural (x_1, x_2) coordinates on M,

$$ds^2 = (1 + f_1^2)\,dx_1^2 + 2f_1 f_2\,dx_1\,dx_2 + (1 + f_2^2)\,dx_2^2,$$

where $f_j = \partial_j f$. Show that, if $\nabla f(0) = 0$, then Exercise 4 applies, so

$$\nabla f(0) = 0 \Longrightarrow K(0) = f_{11}f_{22} - f_{12}^2. \tag{15.58}$$

Compare the derivation of (16.22) in the next section.

6. If $M \subset \mathbb{R}^3$ is the surface of Exercise 5, then the *Gauss map* $N : M \to S^2$ is given by

$$N\bigl(x, f(x)\bigr) = \frac{(-f_1, -f_2, 1)}{\sqrt{1 + f_1^2 + f_2^2}}.$$

Show that, if $\nabla f(0) = 0$, then, at $p_0 = \big(0, f(0)\big)$, $DN(p_0) : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$DN(p_0) = -\begin{pmatrix} \partial_1^2 f(0) & \partial_1\partial_2 f(0) \\ \partial_2\partial_1 f(0) & \partial_2^2 f(0) \end{pmatrix}. \tag{15.59}$$

Here, $T_{p_0}M$ and $T_{(0,0,1)}S^2$ are both identified with the (x_1, x_2) plane. Deduce from Exercise 5 that

$$K(p_0) = \det DN(p_0).$$

7. Deduce from Exercise 6 that, whenever M is a smooth surface in $\mathbb{R}^3$, with Gauss map $N : M \to S^2$, then, with $DN(x) : T_xM \to T_{N(x)}S^2$,

$$K(x) = \det DN(x), \quad \forall\, x \in M. \tag{15.60}$$

Hint. Given $x \in M$, rotate coordinates so that T_xM is parallel to the (x_1, x_2) plane. This result is Gauss' Theorema Egregium for surfaces in $\mathbb{R}^3$. See Theorem 16.4 for a more general formulation; see also (16.35), and Exercises 5, 8, 9, and 14 of §16.

8. Recall from §11, that if $\gamma_s(t)$ is a family of curves $\gamma_s : [a, b] \to M$ satisfying $\gamma_s(a) = p$, $\gamma_s(b) = q$, and if $E(s) = (1/2)\int_a^b \langle T, T\rangle\, dt$, $T = \gamma_s'(t)$, then, with $V = (\partial/\partial s)\gamma_s(t)|_{s=0}$, $E'(s) = -\int_a^b \langle V, \nabla_T T\rangle\, dt$, leading to the stationary condition for E that $\nabla_T T = 0$, which is the geodesic equation. Now suppose $\gamma_{r,s}(t)$ is a 2-parameter family of curves, $\gamma_{r,s}(a) = p$, $\gamma_{r,s}(b) = q$. Let $V = (\partial/\partial s)\gamma_{r,s}(t)|_{0,0}$, $W = (\partial/\partial r)\gamma_{r,s}(t)|_{0,0}$. Show that

$$\frac{\partial^2}{\partial s\partial r}E(0,0) = \int_a^b \Big[\langle R(W,T)V, T\rangle + \langle \nabla_T V, \nabla_T W\rangle - \langle \nabla_W V, \nabla_T T\rangle\Big]\, dt. \tag{15.61}$$

Note that the last term in the integral vanishes if $\gamma_{0,0}$ is a geodesic. Show that, since V vanishes at the endpoints of $\gamma_{0,0}$, the middle term in the integrand above can be replaced by $-\langle V, \nabla_T\nabla_T W\rangle$.

9. If Z is a Killing field, generating an isometry on M (as in §12), show that

$$Z_{j;k;\ell} = R^m{}_{\ell kj}\, Z_m.$$

Hint. From Killing's equation $Z_{j;k} + Z_{k;j} = 0$, derive $Z_{j;k;\ell} = -Z_{k;\ell;j} - R^m{}_{k\ell j}\, Z_m$. Iterate this process 2 more times, going through the cyclic permutation of (j, k, ℓ). Use Bianchi's first identity. Note that the identity desired is equivalent to

$$\nabla^2_{(X,Y)}Z = R(Y,Z)X, \quad \text{if } Z \text{ is a Killing field.}$$

10. Derive the following equation of Jacobi for a variation of geodesics. If $\gamma_s(t)$ is a one-parameter family of geodesics (of speed $\sigma(s)$, independent of t), $T = \gamma_s'(t)$, and $W = \partial_s \gamma_s$, then

$$\nabla_T \nabla_T W = R(T, W)T.$$

A vector field W satisfying this equation is called a Jacobi field.
Hint. Start with $0 = \nabla_W \nabla_T T$, and use $[T, W] = 0$.
Compare the conclusion of Exercise 8.

11. Suppose $v \in T_pM$, $|v| = 1$, and Exp_p is defined near tv, $0 \le t \le a$. Show that

$$D\,\mathrm{Exp}_p(tv)w = \frac{1}{t} J_w(t),$$

where $J_a(t)$ is the Jacobi field along $\gamma(t) = \mathrm{Exp}_p(tv)$ such that

$$J_w(0) = 0, \quad \nabla_T J_w(0) = w.$$

See Appendix X for more material on Jacobi fields.

12. Raising the second index of $R^j{}_{k\ell m}$, you obtain $R^{jk}{}_{\ell m}$, the coordinate expression for $\mathcal{R}$, which can be regarded as a section of $\mathrm{End}(\Lambda^2 T)$.
Suppose $M = X \times Y$ with a product Riemannian metric, and associated curvatures $\mathcal{R}, \mathcal{R}_X, \mathcal{R}_Y$. Using the splitting

$$\Lambda^2(V \oplus W) = \Lambda^2 V \oplus \big(\Lambda^1 V \otimes \Lambda^1 W\big) \oplus \Lambda^2 W,$$

write $\mathcal{R}$ as a 3×3 block matrix. Show that

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_Y \end{pmatrix}$$

In Exercises 13–14, let X, Y, Z, etc., belong to the space $\mathfrak{g}$ of left invariant vector fields on a Lie group G, assumed to have a bi-invariant Riemannian metric. (Compact Lie groups have these.)

13. Show that any (constant speed) geodesic γ on G with $\gamma(0) = e$, the identity element, is a subgroup of G, i.e., $\gamma(s + t) = \gamma(s)\gamma(t)$. Deduce that $\nabla_X X = 0$ for

$X \in \mathfrak{g}$.
Hint. The first part reiterates Exercise 2 of §11.
Show that

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

Hint. $0 = \nabla_X X = \nabla_Y Y = \nabla_{(X+Y)}(X + Y)$.
This identity is called the Maurer-Cartan structure equation.

14. Show that

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z], \quad \text{and} \quad \langle R(X, Y)Z, W\rangle = -\frac{1}{4}\langle [X, Y], [Z, W]\rangle.$$

15. If $E \to M$ is a vector bundle with connection $\widetilde{\nabla}$, and $\nabla = \widetilde{\nabla} + C$, as in Exercises 1–2 of §13, and M has Levi-Civita connection D, so that $\mathrm{Hom}(T \otimes E, E)$ acquires a connection from D and $\widetilde{\nabla}$, which we'll also denote $\widetilde{\nabla}$, show that (13.38) is equivalent to

$$(R - \widetilde{R})(X, Y)u = (\widetilde{\nabla}_X C)(Y, u) - (\widetilde{\nabla}_Y C)(X, u) + [C_X, C_Y]u. \tag{15.62}$$

This is a general form of the "Palatini identity."

16. If g is a metric tensor and h a symmetric second order tensor field, consider the family of metric tensors $g_\tau = g + \tau h$, for τ close to zero, yielding the Levi-Civita connections

$$\nabla^\tau = \nabla + C(\tau),$$

where $\nabla = \nabla^0$. If $C' = C'(0)$, show that

$$\langle C'(X, Y), Z\rangle = \frac{1}{2}(\nabla_X h)(Y, Z) + \frac{1}{2}(\nabla_Y h)(X, Z) - \frac{1}{2}(\nabla_Z h)(X, Y). \tag{15.63}$$

Hint. Use (15.2).

17. Let $R(\tau)$ be the Riemann curvature tensor of g_τ, and set $R' = R'(0)$. Show that (15.62) yields

$$R'(X, Y)Z = (\nabla_X C')(Y, Z) - (\nabla_Y C')(X, Z). \tag{15.64}$$

Using (15.63), show that

(15.65)

$$2\langle R'(X,Y)Z,W\rangle = (\nabla^2_{Y,W}h)(X,Z) + (\nabla^2_{X,Z}h)(Y,W) - (\nabla^2_{X,W}h)(Y,Z) \\ - (\nabla^2_{Y,Z}h)(X,W) + h\big(R(X,Y)Z,W\big) + h\big(R(X,Y)W,Z\big).$$

Hint. Use the derivation property of the covariant derivative to obtain a formula for $\nabla_X C'$ from (15.63).

18. Show that

(15.66)

$$\begin{aligned} 6\langle R(X,Y)Z,W\rangle = {} & \tilde{K}(X+W,Y+Z) - \tilde{K}(Y+W,X+Z) \\ & - \tilde{K}(X,Y+Z) - \tilde{K}(Y,X+W) - \tilde{K}(Z,X+W) - \tilde{K}(W,Y+Z) \\ & + \tilde{K}(X,Y+W) + \tilde{K}(Y,Z+W) + \tilde{K}(Z,Y+W) + \tilde{K}(W,X+Z) \\ & + \tilde{K}(X,Z) + \tilde{K}(Y,W) - \tilde{K}(X,Y) - \tilde{K}(Y,Z), \end{aligned}$$

where

$$\tilde{K}(X,Y) = \langle R(X,Y)Y,X\rangle. \tag{15.67}$$

See (16.34) for an interpretation of the right side of (15.67).

19. Using (15.51), show that, in exponential coordinates centered at p, $g = \det\,(g_{jk})$ satisfies, for $|x|$ small,

$$g(x) = 1 - \frac{1}{3}\sum_{\ell,m} \mathrm{Ric}_{\ell m}(p)x_\ell x_m + O\big(|x|^3\big). \tag{15.68}$$

Deduce that, if A_{n-1} = area of $S^{n-1} \subset \mathbb{R}^n$ and V_n = volume of unit ball in $\mathbb{R}^n$, then, for r small,

$$V\big(B_r(p)\big) = \Big(V_n - \frac{A_{n-1}}{6n(n+2)}S(p)r^2 + O(r^3)\Big)r^n. \tag{15.69}$$

20. Recall that it was stated that (15.17) + (15.18) + (15.21) $\Rightarrow$ (15.22). Relate this to the following assertion about the symmetric group S_4, acting on $\{1,2,3,4\}$. Let e denote the identity element of S_4. Consider the cycles $\alpha = (1\ 2)$, $\beta = (3\ 4)$, $\gamma = (2\ 3\ 4)$. Let $\mathcal{I}$ be the left ideal in the group algebra $\mathbb{R}(S_4)$ generated by $\alpha+e$, $\beta+e$, and $\gamma^2+\gamma+e$.

Assertion. $(1\ 3)(2\ 4) - e \in \mathcal{I}$.

16. Geometry of submanifolds and subbundles

Let M be a Riemannian manifold, of dimension n, and let S be a submanifold, of dimension k, with the induced metric tensor. M has a Levi-Civita connection ∇ and Riemann tensor R. Denote by ∇^0 and R_S the connection and curvature of S. We aim to relate these objects. The *second fundamental form* is defined by

$$II(X,Y) = \nabla_X Y - \nabla^0_X Y \tag{16.1}$$

for X and Y tangent to S. Note that II is linear in X and in Y over $C^\infty(S)$. Also, by the torsion free condition,

$$II(X,Y) = II(Y,X). \tag{16.2}$$

Proposition 16.1. *$II(X,Y)$ is normal to S at each point.*

Proof. If X, Y, Z are tangent to S, we have

$$\langle \nabla_X Y, Z\rangle - \langle \nabla^0_X Y, Z\rangle = -\langle Y, \nabla_X Z\rangle + X\langle Y,Z\rangle + \langle Y, \nabla^0_X Z\rangle - X\langle Y, Z\rangle,$$

and making the obvious cancellation we obtain

$$\langle II(X,Y), Z\rangle = -\langle Y, II(X,Z)\rangle. \tag{16.3}$$

Using (16.2), we have

$$\langle II(X,Y), Z\rangle = -\langle Y, II(Z,X)\rangle, \tag{16.4}$$

i.e., the trilinear form given by the left side changes sign under a cyclic permutation of its arguments. Since three such permutations produce the original form, the left side of (16.4) must equal its own negative, hence be 0. This proves the proposition.

Denote by $\nu(S)$ the bundle of normal vectors to S, i.e., the normal bundle of S. It follows that II is a section of $\mathrm{Hom}(TS \otimes TS, \nu(S))$.

Corollary 16.2. *For X and Y tangent to S, $\nabla^0_X Y$ is the tangential projection on TS of $\nabla_X Y$.*

Let ξ be normal to S. We have a linear map, called the Weingarten map,

$$A_\xi : T_pS \longrightarrow T_pS \tag{16.5}$$

uniquely defined by

$$\langle A_\xi X, Y\rangle = \langle \xi, II(X,Y)\rangle. \tag{16.6}$$

We also define the section A of $\mathrm{Hom}(\nu(S) \otimes TS, TS)$ by

$$A(\xi, X) = A_\xi X. \tag{16.7}$$

We define a connection on $\nu(S)$ as follows; if ξ is a section of $\nu(S)$, set

$$\nabla^1_X \xi = P^\perp \nabla_X \xi,$$

where $P^\perp(x)$ is the orthogonal projection of T_xM onto $\nu_x(S)$. The following identity is called the *Weingarten formula.*

Proposition 16.3. *If ξ is a section of $\nu(S)$,*

$$\nabla^1_X \xi = \nabla_X \xi + A_\xi X. \tag{16.8}$$

Proof. It suffices to show that $\nabla_X \xi + A_\xi X$ is normal to S. In fact, if Y is tangent to S,

$$\begin{aligned}\langle \nabla_X \xi, Y\rangle + \langle A_\xi X, Y\rangle &= X\langle \xi, Y\rangle - \langle \xi, \nabla_X Y\rangle + \langle \xi, II(X,Y)\rangle \\ &= 0 - \langle \xi, \nabla^0_X Y\rangle \\ &= 0,\end{aligned}$$

which proves the proposition.

An equivalent statement is that, for X tangent to S, ξ normal to S,

$$\nabla_X \xi = \nabla^1_X \xi - A_\xi X \tag{16.9}$$

is an orthogonal decomposition, into components normal and tangent to S, respectively. Sometimes this is taken as the definition of A_ξ, or equivalently, by (16.6), of the second fundamental form.

In the special case that S is a hypersurface of M, i.e., dim $M =$ dim $S + 1$, if $\xi = N$ is a smooth unit normal field to S, we see that, for X tangent to S,

$$\langle \nabla_X N, N\rangle = \frac{1}{2} X\langle N, N\rangle = 0,$$

so $\nabla^1_X N = 0$ in this case, and (16.9) takes the form

$$\nabla_X N = -A_N X, \tag{16.10}$$

the classical form of the Weingarten formula. In some texts, A_N is called the *shape operator.*

We now compare the tensors R and R_S. Let X, Y, Z be tangent to S. Applications of (16.1) and (16.8) yield

$$\begin{aligned}\nabla_X \nabla_Y Z &= \nabla_X \big(\nabla^0_Y Z + II(Y,Z)\big) \\ &= \nabla^0_X \nabla^0_Y Z + II(X, \nabla^0_Y Z) - A_{II(Y,Z)} X + \nabla^1_X II(Y,Z).\end{aligned} \tag{16.11}$$

Reversing X and Y, we have

$$\nabla_Y \nabla_X Z = \nabla^0_Y \nabla^0_X Z + II(Y, \nabla^0_X Z) - A_{II(X,Z)} Y + \nabla^1_Y II(X,Z).$$

Also,

$$\nabla_{[X,Y]} Z = \nabla^0_{[X,Y]} Z + II([X,Y], Z). \tag{16.12}$$

From (16.11)–(16.12) we obtain the important identity

$$\begin{aligned}(R - R_S)(X,Y)Z = \big\{&II(X, \nabla^0_Y Z) - II(Y, \nabla^0_X Z) - II([X,Y], Z) \\ &+ \nabla^1_X II(Y,Z) - \nabla^1_Y II(X,Z)\big\} \\ &- \big\{A_{II(Y,Z)} X - A_{II(X,Z)} Y\big\}.\end{aligned} \tag{16.13}$$

Here, the quantity in the first set of braces $\{\quad\}$ is *normal* to S and the quantity in the second pair of braces is *tangent* to S. The identity (16.13) is called the *Gauss-Codazzi* equation. A restatement of the identity for the tangential components is the following, known as *Gauss' Theorema Egregium.*

Theorem 16.4. *For X, Y, Z, W tangent to S,*

$$\langle (R - R_S)(X,Y)Z, W\rangle = \langle II(Y,W), II(X,Z)\rangle - \langle II(X,W), II(Y,Z)\rangle. \tag{16.14}$$

The normal component of the identity (16.13) is specifically Codazzi's equation. It takes a shorter form in case S has codimension 1 in M. In that case, choose a unit normal field N, and let

$$II(X,Y) = \widetilde{II}(X,Y)N; \tag{16.15}$$

$\widetilde{II}$ is a tensor field of type $(0,2)$ on S. Then Codazzi's equation is equivalent to

$$\langle R(X,Y)Z, N\rangle = (\nabla^0_X \widetilde{II})(Y,Z) - (\nabla^0_Y \widetilde{II})(X,Z), \tag{16.16}$$

for X, Y, Z tangent to S, since of course $R_S(X,Y)Z$ is tangent to S.

In the classical case, where S is a hypersurface in flat Euclidean space, $R = 0$, and Codazzi's equation becomes

$$(\nabla^0_Y \widetilde{II})(X,Z) - (\nabla^0_X \widetilde{II})(Y,Z) = 0, \tag{16.17}$$

i.e., $\nabla^0 \widetilde{II}$ is a symmetric tensor field of type $(0,3)$. In this case, from the identity $\widetilde{II}_{jk;\ell} = \widetilde{II}_{\ell k;j}$ we deduce $A_j{}^k{}_{;k} = A_k{}^k{}_{;j} = (\mathrm{Tr}\ A)_{;j}$ where $A = A_N$ is the Weingarten map. Equivalently,

$$\mathrm{div}\ A = d(\mathrm{Tr}\ A). \tag{16.18}$$

An application of the Codazzi equation to minimal surfaces can be found in the exercises after §29. See Exercise 10 below for an application to umbilic surfaces.

It is useful to note the following characterization of the second fundamental form for a hypersurface M in $\mathbb{R}^n$. Translating and rotating coordinates, we can move a specific point $p \in M$ to the origin in $\mathbb{R}^n$ and suppose M is given locally by

$$x_n = f(x'), \quad \nabla f(0) = 0,$$

where $x' = (x_1, \dots, x_{n-1})$. We can then identify the tangent space of M at p with $\mathbb{R}^{n-1}$. Take $N = (0, \dots, 0, 1)$ in (16.15).

Proposition 16.5. *The second fundamental form of M at p is given by the Hessian of f :*

$$\widetilde{II}(X,Y) = \sum_{j,k=1}^{n-1} \frac{\partial^2 f}{\partial x_j \partial x_k}(0)\ X_j Y_k. \tag{16.19}$$

Proof. From (16.9) we have, for any ξ normal to M,

$$\langle II(X,Y),\xi\rangle = -\langle \nabla_X \xi, Y\rangle \tag{16.20}$$

where ∇ is the flat connection on $\mathbb{R}^n$. Taking

$$\xi = (-\partial_1 f, \dots, -\partial_{n-1} f, 1) \tag{16.21}$$

gives the desired formula.

If S is a surface in $\mathbb{R}^3$, given locally by $x_3 = f(x_1, x_2)$ with $\nabla f(0) = 0$, then the Gauss curvature of S at the origin is seen by (16.14) and (16.19) to equal

$$\det\Bigl(\frac{\partial^2 f(0)}{\partial x_j \partial x_k}\Bigr). \tag{16.22}$$

Recall the different derivation of this identity in (15.58). Consider the example of the unit sphere in $\mathbb{R}^3$, centered at $(0,0,1)$. Then the "south pole" lies at the origin, near which S^2 is given by

$$x_3 = 1 - \bigl(1 - x_1^2 - x_2^2\bigr)^{1/2}. \tag{16.23}$$

In this case (16.22) implies that the Gauss curvature K is equal to 1 at the south pole. Of course, by symmetry it follows that $K = 1$ everywhere on the unit sphere S^2.

Besides providing a good conception of the second fundamental form of a hypersurface in $\mathbb{R}^n$, Proposition 16.5 leads to useful formulas for computation, one of which we will give below, in (16.29). First, we give a more invariant reformulation of Proposition 16.5. Suppose the hypersurface M in $\mathbb{R}^n$ is given by

$$u(x) = c, \tag{16.24}$$

with $\nabla u \neq 0$ on M. Then we can use the computation (16.20) with $\xi = \operatorname{grad} u$ to obtain

$$\langle II(X,Y), \operatorname{grad} u\rangle = -(D^2u)(X,Y) \tag{16.25}$$

where D^2u is the Hessian of u; we can think of $(D^2u)(X,Y)$ as $Y \cdot (D^2u)X$, where D^2u is the $n \times n$ matrix of second order partial derivatives of u. In other words,

$$\widetilde{II}(X,Y) = -|\operatorname{grad} u|^{-1}(D^2u)(X,Y), \tag{16.26}$$

for X, Y tangent to M, provided we take N to be a *positive* multiple of grad u.

In particular, if M is a two dimensional surface in $\mathbb{R}^3$ given by (16.24), then the Gauss curvature at $p \in M$ is given by

$$K(p) = |\operatorname{grad} u|^{-2} \det{(D^2u)}|_{T_pM}, \tag{16.27}$$

where $D^2u|_{T_pM}$ denotes the restriction of the quadratic form D^2u to the tangent space T_pM, producing a linear transformation on T_pM via the metric on T_pM. With this calculation we can derive the following formula, extending (16.22).

Proposition 16.6. *If $M \subset \mathbb{R}^3$ is given by*

$$x_3 = f(x_1, x_2), \tag{16.28}$$

then, at $p = (x', f(x')) \in M$, the Gauss curvature is given by

$$K(p) = \bigl(1 + |\nabla f(x')|^2\bigr)^{-2} \ det\Bigl(\frac{\partial^2 f}{\partial x_j \partial x_k}\Bigr). \tag{16.29}$$

Proof. We can apply (16.27) with $u(x) = f(x_1, x_2) - x_3$. Note that $|\nabla u|^2 = 1 + |\nabla f(x')|^2$ and

$$D^2 u = \begin{pmatrix} D^2 f & 0 \\ 0 & 0 \end{pmatrix}. \tag{16.30}$$

Noting that a basis of T_pM is given by $(1, 0, \partial_1 f) = v_1, \ (0, 1, \partial_2 f) = v_2$, we readily obtain

$$\det\, D^2 u|_{T_pM} = \frac{\det\bigl(v_j \cdot (D^2 u) v_k\bigr)}{\det(v_j \cdot v_k)} = \bigl(1 + |\nabla f(x')|^2\bigr)^{-1} \det\, D^2 f, \tag{16.31}$$

which yields (16.29).

If you apply Proposition 16.6 to the case (16.23) of a hemisphere of unit radius, the calculation that $K = 1$ everywhere is easily verified. The formula (16.29) gives rise to interesting problems in nonlinear PDE, some of which are studied in Chapter 14 of [T1].

We now define the *sectional curvature* of a Riemannian manifold M. Given $p \in M$, let Π be a 2-plane in T_pM, $\Sigma = \ \mathrm{Exp}_p(\Pi)$. The sectonal curvature of M at p is

$$K_p(\Pi) = \text{ Gauss curvature of } \Sigma \text{ at } p. \tag{16.32}$$

If U and V form an orthonormal basis of $T_p\Sigma = \Pi$, then by definition of Gauss curvature,

$$K_p(\Pi) = \langle R_\Sigma(U, V)V, U\rangle. \tag{16.33}$$

We have the following more direct formula for the sectional curvature.

Proposition 16.7. *With U and V as above, R the Riemann tensor of M,*

$$K_p(\Pi) = \langle R(U, V)V, U\rangle. \tag{16.34}$$

Proof. It suffices to show that the second fundamental form of Σ *vanishes* at p. Since $II(X, Y)$ is symmetric, it suffices to show that $II(X, X) = 0$ for each $X \in T_pM$.

So pick a geodesic γ in M such that $\gamma(0) = p,\ \gamma'(0) = X$. Then $\gamma \subset \Sigma$, and γ must also be a geodesic in S, so

$$\nabla_T T = \nabla^0_T T, \quad T = \gamma'(t),$$

which implies $II(X, X) = 0$. This proves (16.34).

Note that, if $S \subset M$ has codimension 1, $p \in S$, and $\Pi \subset T_pS$, then, by (16.14),

$$K^S_p(\Pi) - K^M_p(\Pi) = \det \begin{pmatrix} \widetilde{II}(U,U) & \widetilde{II}(U,V) \\ \widetilde{II}(V,U) & \widetilde{II}(V,V) \end{pmatrix}. \tag{16.35}$$

Note how this is a direct generalization of (15.60).

The results above comparing connections and curvatures of a Riemannian manifold and a submanifold are special cases of more general results on subbundles, which arise in a number of interesting situations. Let E be a vector bundle over a manifold M, with an inner product, and a metric connection ∇. Let $E_0 \to M$ be a subbundle. For each $x \in M$, let P_x be the orthogonal projection of E_x onto E_{0x}. Set

$$\nabla^0_X u(x) = P_x \nabla_X u(x), \tag{16.36}$$

when u is a section of E_0. Note that, for scalar f,

$$\begin{aligned} \nabla^0_X fu(x) &= P_x\big(f\nabla_X u(x) + (Xf)u\big) \\ &= fP_x\nabla_X u(x) + (Xf)u(x), \end{aligned}$$

provided u is a section of E_0, so $P_x u(x) = u(x)$. This shows that (16.36) defines a connection on E_0. Since $\langle \nabla^0_X u, v\rangle = \langle \nabla_X u, v\rangle$ for sections u, v of E_0, it is clear that ∇^0 is also a metric connection. Similarly, if E_1 is the orthogonal bundle, a subbundle of E, a metric connection on E_1 is given by

$$\nabla^1_X v(x) = (I - P_x)\nabla_X v(x), \tag{16.37}$$

for a section v of E_1.

It is useful to treat ∇^0 and ∇^1 on an equal footing, so we define a new connection $\widetilde{\nabla}$ on E, also a metric connection, by

$$\widetilde{\nabla} = \nabla^0 \oplus \nabla^1. \tag{16.38}$$

Then there is the relation

$$\nabla_X = \widetilde{\nabla}_X + C_X \tag{16.39}$$

where

$$C_X = \begin{pmatrix} 0 & II_X^1 \\ II_X^0 & 0 \end{pmatrix} \tag{16.40}$$

Here, $II_X^0 : E_0 \to E_1$ is the second fundamental form of $E_0 \subset E$, and $II_X^1 : E_1 \to E_0$ is the second fundamental form of $E_1 \subset E$. We also set $II^j(X, u) = II_X^j u$. In this context, the Weingarten formula has the form

$$C_X^t = -C_X, \quad \text{i.e.,} \quad II_X^1 = -\big(II_X^0\big)^t. \tag{16.41}$$

Indeed, for any two connections related by (16.39), with $C \in \operatorname{Hom}(TM \otimes E, E)$, if ∇ and $\widetilde{\nabla}$ are both metric connections, the first part of (16.41) holds.

We remark that, when γ is a curve in a Riemannian manifold M, and for $p \in \gamma$, $E_p = T_pM$, $E_{0p} = T_p\gamma$, $E_{1p} = \nu(\gamma)$, the normal space, and if ∇ is the Levi-Civita connection on M, then $\widetilde{\nabla}$ is sometimes called the Fermi-Walker connection on γ. One also (especially) considers a timelike curve in a Lorentz manifold.

Let us also remark that, if we *start* with metric connections ∇^j on E_j, then form $\widetilde{\nabla}$ on E by (16.38), and then define ∇ on E by (16.39), provided that (16.40) holds, it follows that ∇ is also a metric connection on E, and ∇^j are recovered by (16.36) and (16.37).

In general, for any two connections ∇ and $\widetilde{\nabla}$, related by (16.39) for some $\operatorname{End}(E)$ valued 1-form C, we have the following relation between their curvature tensors R and $\widetilde{R}$, already anticipated in Exercise 2 of §13:

$$(R - \widetilde{R})(X, Y)u = \big\{[C_X, \widetilde{\nabla}_Y] - [C_Y, \widetilde{\nabla}_X] - C_{[X,Y]}\big\}u + [C_X, C_Y]u. \tag{16.42}$$

In case $\widetilde{\nabla} = \nabla^0 \oplus \nabla^1$ on $E = E_0 \oplus E_1$, and ∇ has the form (16.39), where C_X exchanges E_0 and E_1, it follows that the operator in brackets $\{\quad\}$ on the right side of (16.42) exchanges sections of E_0 and E_1, while the last operator $[C_X, C_Y]$ leaves invariant the sections of E_0 and E_1. In such a case these two components express respectively the Codazzi identity and Gauss' Theorema Egregium.

We will expand these formulas, writing $R(X, Y), \widetilde{R}(X, Y) \in \operatorname{End}(E_0 \oplus E_1)$ in the block matrix forms

$$R = \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} R_0 & 0 \\ 0 & R_1 \end{pmatrix}. \tag{16.43}$$

Then Gauss' equations become

$$\begin{aligned} (R_{00} - R_0)(X, Y)u &= II_X^1 II_Y^0 u - II_Y^1 II_X^0 u, \\ (R_{11} - R_1)(X, Y)u &= II_X^0 II_Y^1 u - II_Y^0 II_X^1 u, \end{aligned} \tag{16.44}$$

for a section u of E_0 or E_1, respectively. Equivalently, if v is also a section of E_0 or E_1, respectively,

$$\begin{aligned} \langle (R_{00}-R_0)(X,Y)u,v\rangle &= \langle II^0_X u, II^0_Y v\rangle - \langle II^0_Y u, II^0_X v\rangle \\ \langle (R_{11}-R_1)(X,Y)u,v\rangle &= \langle II^1_X u, II^1_Y v\rangle - \langle II^1_Y u, II^1_X v\rangle. \end{aligned} \tag{16.45}$$

The second part of (16.45) is also called the Ricci equation (not to be confused with the Ricci identity (15.28)).

Codazzi's equations become

$$\begin{aligned} R_{10}(X,Y)u &= II^0_X\nabla^0_Y u - II^0_Y\nabla^0_X u - II^0_{[X,Y]}u + \nabla^1_X II^0_Y u - \nabla^1_Y II^0_X u, \\ R_{01}(X,Y)u &= II^1_X\nabla^1_Y u - II^1_Y\nabla^1_X u - II^1_{[X,Y]}u + \nabla^0_X II^1_Y u - \nabla^0_Y II^1_X u, \end{aligned} \tag{16.46}$$

for sections u of E_0 and E_1, respectively. If we take the inner product of the first equation in (16.46) with a section v of E_1, we get

(16.47)
$$\begin{aligned} \langle R_{10}(X,Y)u,v\rangle = &-\langle \nabla^0_Y u, II^1_X v\rangle + \langle \nabla^0_X u, II^1_Y v\rangle - \langle II^0_{[X,Y]}u,v\rangle \\ &+\langle II^0_X u, \nabla^1_Y v\rangle - \langle II^0_Y u, \nabla^1_X v\rangle + X\langle II^0_Y u,v\rangle - Y\langle II^0_X u,v\rangle, \end{aligned}$$

using the metric property of ∇^0 and ∇^1, and the antisymmetry (16.40). If we perform a similar calculation for the second part of (16.46), in light of the fact that $R_{10}(X,Y)^t = -R_{01}(X,Y)$, we see that these two parts are equivalent, so we need retain only one of them. Furthermore, we can rewrite the first equation in (16.46) as follows. Form a connection on $\mathrm{Hom}(TM\oplus E_0, E_1)$ via the connections ∇^j on E_j and a Levi-Civita connection ∇^M on TM, via the natural derivation property, i.e.,

$$(\widetilde{\nabla}_X II^0)(Y,u) = \nabla^1_X II^0_Y u - II^0_Y\nabla^0_X u - II^0(\nabla^M_X Y, u). \tag{16.48}$$

Then (16.46) is equivalent to

$$R_{10}(X,Y)u = (\widetilde{\nabla}_X II^0)(Y,u) - (\widetilde{\nabla}_Y II^0)(X,u). \tag{16.49}$$

One case of interest is when E_1 is the trivial bundle $E_1 = M\times\mathbb{R}$, with one-dimensional fibre. For example, E_1 could be the normal bundle of a codimension one surface in $\mathbb{R}^n$. In this case, it is clear that both sides of the last half of (16.45) are tautologically zero, so Ricci's equation has no content in this case.

As a parenthetical comment, suppose E is a trivial bundle $E = M\times\mathbb{R}^n$, with complementary subbundles E_j, having metric connections constructed as in (16.36)–(16.37), from the trivial connection D on E, defined by componentwise differentiation, so

$$\nabla^0_X u = PD_X u, \quad \nabla^1_X u = (I-P)D_X u, \tag{16.50}$$

for sections of E_0 and E_1, respectively. There is the following alternative approach to curvature formulas. For $\widetilde{\nabla} = \nabla^0 \oplus \nabla^1$, we have

$$\widetilde{\nabla}_X u = D_X u + (D_X P)(I - 2P)u. \tag{16.51}$$

Note that, with respect to a choice of basis of $\mathbb{R}^n$ as a global frame field on $M \times \mathbb{R}^n$, we have the connection 1-form (13.14) given by

$$\Gamma = dP(I - 2P). \tag{16.52}$$

Since $dP = dP\,P + P\,dP$, we have $dP\,P = (I-P)dP$. Thus $\Gamma = P\,dP(I-P) - (I-P)dP\,P$ casts $\Gamma = -C$ in the form (16.40). We obtain directly from the formula $\Omega = d\Gamma + \Gamma \wedge \Gamma$, derived in (16.15), that the curvature of $\widetilde{\nabla}$ is given by

$$\Omega = dP \wedge dP = P\,dP \wedge dP\,P + (I - P)\,dP \wedge dP\,(I - P), \tag{16.53}$$

the last identity showing the respective curvatures of E_0 and E_1. Compare Exercise 5 of §13.

Our next goal is to invert the process above. That is, rather than starting with a flat bundle $E = M \times \mathbb{R}^n$ and obtaining connections on subbundles and second fundamental forms, we want to start with bundles $E_j \to M$, $j = 1, 2$, with metric connections ∇^j, and proposed second fundamental forms II^j, sections of $\operatorname{Hom}(TM \otimes E_j, E_{j'})$, and then obtain a flat connection ∇ on E via (16.38)–(16.40). Of course, we assume II^0 and II^1 are related by (16.41), so (16.39) makes ∇ a metric connection. Thus, according to equations (16.45) and (16.49), the connection ∇ is flat if and only if, for all sections u, v of E_0,

$$\begin{aligned} (\widetilde{\nabla}_X II^0)(Y, u) - (\widetilde{\nabla}_Y II^0)(X, u) &= 0 \\ \langle II^0_Y u, II^0_X v\rangle - \langle II^0_X u, II^0_Y v\rangle &= \langle R_0(X, Y)u, v\rangle, \end{aligned} \tag{16.54}$$

and, for all sections u, v of E_1,

$$\langle II^1_Y u, II^1_X v\rangle - \langle II^1_X u, II^1_Y v\rangle = \langle R_1(X, Y)u, v\rangle. \tag{16.55}$$

If these conditions are satisfied, then E will have a global frame field of sections $e_1, \dots, e_n$, such that $\nabla e_j = 0$, at least provided M is simply connected. Then, for each $p \in M$, we have an isometric isomorphism

$$J(p) : E_p \longrightarrow \mathbb{R}^n, \tag{16.56}$$

by expanding elements of E_p in terms of the basis $\{e_j(p)\}$. Thus $E_0 \subset E$ is carried by $J(p)$ to a family of linear subspaces $J(p)E_{0p} = V_p \subset \mathbb{R}^n$, with orthogonal complements $J(p)E_{1p} = N_p \subset \mathbb{R}^n$.

We now specialize to the case $E_0 = TM$, where M is an m-dimensional Riemannian manifold, with its Levi-Civita connection; E_1 is an auxiliary bundle over M, with metric connection ∇^1. We will assume M is simply connected. The following result is sometimes called the Fundamental Theorem of Surface Theory.

Theorem 16.8. *Let II^0 be a section of $Hom(TM \otimes TM, E_1)$, and set $II^1_X = -(II^0_X)^t$. Make the symmetry hypothesis*

$$II^0(X, Y) = II^0(Y, X). \tag{16.57}$$

Assume the equations (16.54)–(16.55) are satisfied, producing a trivialization of $E = E_0 \oplus E_1$, described by (16.56). Then there is an isometric immersion

$$X : M \longrightarrow \mathbb{R}^n, \tag{16.58}$$

and a natural identification of E_1 with the normal bundle of $S = X(M) \subset \mathbb{R}^n$, such that the second fundamental form of S is given by II^0.

To get this, we will construct the map (16.58) so that

$$DX(p) = J(p)\big|_{TM}, \tag{16.59}$$

for all $p \in M$. To see how to get this, consider one of the n components of J, $J_\nu(p) : E_p \to \mathbb{R}$. In fact, $J_\nu u = \langle e_\nu, u\rangle$. Let $\beta_\nu(p) = J_\nu(p)\big|_{T_pM}$; thus β_ν is a 1-form on M.

Lemma 16.9. *Each β_ν is closed, i.e., $d\beta_\nu = 0$.*

Proof. For vector fields X and Y on M, we have

$$\begin{aligned} d\beta_\nu(X, Y) &= X \cdot \beta_\nu(Y) - Y \cdot \beta_\nu(X) - \beta_\nu([X, Y]) \\ &= X \cdot \beta_\nu(Y) - Y \cdot \beta_\nu(X) - \beta_\nu(\nabla^0_X Y - \nabla^0_Y X). \end{aligned} \tag{16.60}$$

Using $\nabla_X = \nabla^0_X + II^0_X$ on sections of $E_0 = TM$, we see that this is equal to

$$\begin{aligned} &X \cdot J_\nu(Y) - Y \cdot J_\nu(X) - J_\nu(\nabla_X Y - \nabla_Y X) + J_\nu(II^0_X Y - II^0_Y X) \\ &= (\nabla_X J_\nu)Y - (\nabla_Y J_\nu)X + J_\nu(II^0_X Y - II^0_Y X). \end{aligned}$$

By construction, $\nabla_X J_\nu = 0$, while (16.57) says $II^0_X Y - II^0_Y X = 0$. Thus $d\beta_\nu = 0$.

Consequently, as long as M is simply connected, we can write $\beta_\nu = dx_\nu$ for some functions $x_\nu \in C^\infty(M)$; define (16.58) by $X(p) = (x_1(p), \dots, x_\nu(p))$. Thus (16.59) holds, so X is an isometric mapping. Furthermore it is clear that $J(p)$ maps E_{1p} precisely isometrically onto the normal space $N_p \subset \mathbb{R}^n$ to $S = X(M)$ at $X(p)$, displaying II^0 as the second fundamental form of S. Thus Theorem 16.8 is established.

Let us specialize Theorem 16.8 to the case where dim $M = n - 1$, so the fibers of E_1 are one-dimensional. As mentioned above, the Ricci identity (16.55) has no content in that case. We have the following special case of the Fundamental Theorem of Surface Theory.

Proposition 16.10. *Let M be an $(n-1)$-dimensional Riemann manifold; assume M is simply connected. Let there be given a symmetric tensor field $\widetilde{II}$, of type (0,2). Assume the following Gauss-Codazzi equations hold:*

$$
\begin{aligned}
\widetilde{II}(Y,Z)\widetilde{II}(X,W) - \widetilde{II}(X,Z)\widetilde{II}(Y,W) &= \langle R^M(X,Y)Z,W\rangle, \\
\big(\nabla^M_X \widetilde{II}\big)(Y,Z) - \big(\nabla^M_Y \widetilde{II}\big)(X,Z) &= 0,
\end{aligned}
\tag{16.61}
$$

where ∇^M is the Levi-Civita connection of M and R^M its Riemann curvature tensor. Then there is an isometric immersion $X : M \to \mathbb{R}^n$, such that the second fundamental form of $S = X(M) \subset \mathbb{R}^n$ is given by $\widetilde{II}$.

Exercises

1. Let $S \subset M$, with respective Levi-Civita connections ∇^0, ∇, respective Riemann tensors R_s, R, etc., as in the text. Let $\gamma_{s,t} : [a,b] \to S$ be a 2-parameter family of curves. One can also regard $\gamma_{s,t} : [a,b] \to M$. Apply the formula (15.61) for the second variation of energy in these two contexts and compare the results, to produce another proof of Gauss' formula (16.14) for $\langle (R - R_S)(X,Y)Z,W\rangle$ when X,Y,Z,W are all tangent to S.
Hint. Show that $\langle \nabla_X Z, \nabla_X W\rangle - \langle \nabla^0_X Z, \nabla^0_X W\rangle = \langle II(X,Z), II(X,W)\rangle$.

2. Let S be a surface in $\mathbb{R}^3$ given by $(u,v) \mapsto X(u,v) \in \mathbb{R}^3$, for $(u,v) \in \mathcal{O} \subset \mathbb{R}^2$. Set $\xi = X_u \times X_v$, so a unit normal field to S is given by $N = \xi/|\xi|$. Let $H = (h_{jk})$ be the 2×2 matrix with entries $h_{jk} = \widetilde{II}(\partial_j X, \partial_k X)$, where $\partial_1 X = X_u,\ \partial_2 X = X_v$. Show that

$$
H = \frac{1}{|\xi|}\begin{pmatrix} \xi\cdot X_{uu} & \xi\cdot X_{uv} \\ \xi\cdot X_{vu} & \xi\cdot X_{vv}\end{pmatrix} = \frac{1}{|\xi|}\widetilde{A}_\xi,
$$

(the last identity defining $\widetilde{A}_\xi$). Deduce that the Gauss curvature of S is given by

$$
K = \frac{\det H}{\det G} = \frac{1}{|\xi|^2}\,\frac{\det \widetilde{A}_\xi}{\det G}, \qquad G = \begin{pmatrix} X_u\cdot X_u & X_u\cdot X_v \\ X_v\cdot X_u & X_v\cdot X_v\end{pmatrix}.
$$

Hint. By the Weingarten formula (16.9), $\langle A_\xi U, V\rangle = -\langle D_U \xi, V\rangle$, where D is the standard flat connection on $\mathbb{R}^3$. Hence the upper left entry of H is $1/|\xi|$ times

$$
-X_u\cdot \partial_u(X_u\times X_v) = -X_u\cdot (X_{uu}\times X_v) = X_{uu}\cdot (X_u\times X_v).
$$

3. Let γ be a curve in $\mathbb{R}^3$, parametrized by arc length. Recall the Frenet apparatus, sketched in §K. At $p = \gamma(t)$, $T = \gamma'(t)$ spans $T_p\gamma$ and, if the curvature κ of γ is nonzero, unit vectors N and B span the normal bundle $\nu_p(\gamma)$, satisfying the system of ODE

$$\begin{aligned} T' &= \qquad \kappa N \\ N' &= -\kappa T \qquad + \tau B \\ B' &= \qquad - \tau N \end{aligned} \tag{16.62}$$

and furthermore, $B = T \times N,\ T = N \times B,\ N = B \times T$. Let ∇ denote the standard flat connection on $\mathbb{R}^3$, and $\nabla^0,\ \nabla^1$ the connections induced on $T(\gamma)$ and $\nu(\gamma)$, as in (16.1), (16.8). Show that

$$II(T,T) = \kappa N, \tag{16.63}$$

and that

$$\nabla^1_T N = \tau B, \quad \nabla^1_T B = -\tau N. \tag{16.64}$$

Compute the right side of the Weingarten formula

$$\nabla_T \xi = \nabla^1_T \xi + A_\xi T, \quad \xi = N, B, \tag{16.65}$$

and show that (16.63)–(16.65) is *equivalent* to (16.62).

4. Let $S \subset \mathbb{R}^3$ be a surface, with connection ∇^S, second fundamental form II^S, unit normal ν. Let γ be the curve of Problem 3, and suppose γ is a curve in S. Show that

$$\begin{aligned} II^S(T,T) &= \kappa \langle N, \nu \rangle \nu \\ &= \kappa N - \nabla^S_T T. \end{aligned}$$

If A_ν denotes the Weingarten map of S, as in (16.5), show that

$$\gamma \ \text{ geodesic on } S \Longrightarrow A_\nu T = \kappa T - \tau B \text{ and } N = \nu.$$

5. Use Theorem 16.4 to show that the Gauss curvature K of a surface $S \subset \mathbb{R}^3$ is equal to $\det A_\nu$. Use the symmetry of A_ν to show that each T_pS has an orthonormal basis T_1, T_2 such that $A_\nu T_j = \kappa_j T_j$, and $K = \kappa_1 \kappa_2$. An eigenvector of A_ν is called a direction of principal curvature (and the eigenvalues are called principal curvatures). Show that $T \in T_pS$ is a direction of principal curvature if and only if the geodesic through p in direction T has vanishing torsion τ at p.

6. Suppose M has the property that each sectional curvature $K_p(\Pi)$ is equal to K_p, independent of Π. Show that

$$\mathcal{R} = K_p I \text{ in } \mathrm{End}(\Lambda^2 T_p),$$

where $\mathcal{R}$ is as in Exercise 11 of §15. Show that K_p is constant, on each connected component of M, if $\dim M \geq 3$.
Hint. To do the last part, use Proposition 15.3.

7. Show that the formula (16.42) for $R - \widetilde{R}$ is equivalent to the formula (14.17). (This reiterates Exercise 5 of §14.) Also, relate (16.44) and (16.49) to (15.54).

Let M be a compact oriented hypersurface in $\mathbb{R}^n$. Let

$$N : M \to S^{n-1}$$

be given by the outward pointing normal. This is called the Gauss map.

8. If $n = 3$, show that $N^*\omega_0 = K\omega$, where $\omega_0,\ \omega$ are the area forms of S^2 and M, respectively, and K the Gauss curvature of M. Note that the degree of the Gauss map is

$$\mathrm{Deg}\,(N) = \frac{1}{4\pi} \int\limits_M N^*\omega_0.$$

See §9 for basic material on degrees of maps.

9. For general n, show that $N^*\omega_0 = J\omega$ with

$$J = (-1)^{n-1} \det A_N,$$

where ω, ω_0 are the volume forms and $A_N : T_pM \to T_pM$ is the Weingarten map (16.5). Consequently

$$\mathrm{Deg}\,(N) = \frac{(-1)^{n-1}}{A_{n-1}} \int\limits_M \bigl(\det A_N\bigr)\, dV, \tag{16.66}$$

where A_{n-1} is the area of S^{n-1}.
Hint. There is a natural identification of T_pM and $T_q(S^{n-1})$, as linear subspaces of $\mathbb{R}^n$, if $q = N(p)$. Show that the Weingarten formula gives

$$DN(p) = -A_N \in \operatorname{End}(T_pM) \approx \mathcal{L}(T_pM, T_qS^{n-1}). \tag{16.67}$$

10. Let S be a hypersurface in $\mathbb{R}^n$, with second fundamental form $\widetilde{II}$, as in (16.15). Suppose $\widetilde{II}$ is proportional to the metric tensor, $\widetilde{II} = \lambda(x)g$. Show that λ is constant, provided S is connected. (Assume $n \geq 3$.)
Hint. Use the Codazzi equation (16.17), plus the fact that $\nabla^0 g = 0$.
Alternative. Use (16.18) to get $d\lambda = (n-1)d\lambda$.

11. When S is a hypersurface in $\mathbb{R}^n$, a point p where $\widetilde{II} = \lambda g$ is called an umbilic point. If every point on S is umbilic, show that S has constant sectional curvature λ^2.
Hint. Apply Gauss' Theorema Egregium, in the form (16.14).

12. Let $S \subset \mathbb{R}^n$ be a k dimensional submanifold ($k < n$), with induced metric g and second fundamental form II. Let ξ be a section of the normal bundle $\nu(S)$. Consider the one parameter family of maps $S \to \mathbb{R}^n$,

$$\varphi_\tau(x) = x + \tau\xi(x), \quad x \in S, \ \tau \in (-\varepsilon, \varepsilon). \tag{16.68}$$

Let g_τ be the family of Riemannian metrics induced on S. Show that

$$\frac{d}{d\tau} g_\tau(X,Y)\big|_{\tau=0} = -2\langle \xi, II(X,Y)\rangle. \tag{16.69}$$

More generally, if $S \subset M$ is a submanifold, consider the one parameter family of submanifolds given by

$$\varphi_\tau(x) = \operatorname{Exp}_x\bigl(\tau\xi(x)\bigr), \quad x \in S, \ \tau \in (-\varepsilon, \varepsilon), \tag{16.70}$$

where $\operatorname{Exp}_x : T_xM \to M$ is the exponential map, determined by the Riemannian metric on M. Show that (16.69) holds in this more general case.

13. Let $M_1 \subset M_2 \subset M_3$ be Riemannian manifolds of dimension $n_1 < n_2 < n_3$, with induced metrics. For $j < k$, denote by II^{jk} the second fundamental form of

$M_j \subset M_k$, and A^{jk} the associated Weingarten map. For $x \in M_j$, denote by N_x^{jk} the orthogonal complement of T_xM_j in T_xM_k, and ${}^{jk}\nabla^1$ the natural connection on $N^{jk}(M_j)$. Let X and Y be tangent to M_1 and ξ be a section of $N^{12}(M_1)$. Show that

$$A_\xi^{12}X = A_\xi^{13}X.$$

Show that

$${}^{13}\nabla_X^1\xi = {}^{12}\nabla_X^1\xi + II^{23}(X,\xi), \text{ orthogonal decomposition,}$$

and that

$$II^{13}(X,Y) = II^{12}(X,Y) + II^{23}(X,Y), \text{ orthogonal decomposition.}$$

Relate this to Exercises 3–5 when $n_j = j$.

14. If $S \subset M$ has codimension 1, and Weingarten map $A : T_pS \to T_pS$, show that the Gauss equation (16.14) gives

$$\big\langle (R - R_S)(X,Y)Z, W\big\rangle = \big\langle (\Lambda^2 A)(X \wedge Y), Z \wedge W\big\rangle, \quad X,Y,Z,W \in T_pS. \tag{16.71}$$

Show that (with N a unit normal to S) the scalar curvatures of M and S are related by

$$S_M - S_S = -2 \ \mathrm{Tr}\ \Lambda^2 A + 2\,\mathrm{Ric}_M(N,N). \tag{16.72}$$

15. With the Ricci tensor Ric given by (16.23) and the sectional curvature $K_p(\Pi)$ by (16.32), show that, for $X \in T_pM$, of norm 1, if Ξ denotes the orthogonal complement of X in T_pM, then

$$\mathrm{Ric}(X,X) = \frac{n-1}{\mathrm{vol}\ S_p(\Xi)} \int\limits_{S_p(\Xi)} K_p(U,X)\ dV(U),$$

where $S_p(\Xi)$ is the unit sphere in Ξ, $n =$ dim M, and $K_p(U,X) = K_p(\Pi)$ where Π is the linear span of U and X. Show that the scalar curvature at p is given by

$$S = \frac{n(n-1)}{\mathrm{vol}\ G_2} \int\limits_{G_2} K_p(\Pi)\ dV(\Pi),$$

where G_2 is the space of 2-planes in T_pM. (For more on this space and other Grassmannians, see Appendix V.)

Special exercises on surfaces in $\mathbb{R}^3$

In Exercises 16–19, let M be an oriented surface in $\mathbb{R}^3$. Let $\{E_1, E_2\}$ be a (local) oriented orthonormal frame field of tangent vectors to M, and let $N = E_3 = E_1 \times E_2$. Define 1-forms ω_{jk} on M by

$$\omega_{jk}(X) = \langle D_X E_j, E_k \rangle, \tag{16.73}$$

where X is a tangent vector field to M and D is the standard flat connection on $\mathbb{R}^3$. Let ∇^0 denote the natural connection on M, R^0 its curvature tensor, A_N the Weingarten map.

16. Show that $\omega_{jk} = -\omega_{kj}$ and

$$A_N X = \omega_{13}(X)E_1 + \omega_{23}(X)E_2. \tag{16.74}$$

17. Differentiating $\omega_{12}(X) = \langle D_X E_1, E_2 \rangle$, etc., show that

$$d\omega_{12}(X, Y) = \omega_{13}(X)\omega_{32}(Y) - \omega_{13}(Y)\omega_{32}(X). \tag{16.75}$$

Hint. Use (6.26) to evaluate $d\omega_{12}(X, Y)$. Show that $\langle D_X E_1, D_Y E_2 \rangle = \omega_{13}(X)\omega_{23}(Y)$.

Note in particular from (16.74) that we obtain

$$d\omega_{12}(E_1, E_2) = -\det A_N. \tag{16.76}$$

18. Note that $\omega_{12}(X) = \langle \nabla^0_X E_1, E_2 \rangle$. Show that

$$d\omega_{12}(X, Y) = \langle R^0(X, Y)E_1, E_2 \rangle. \tag{16.77}$$

Deduce that

$$d\omega_{12} = -K\,\alpha, \tag{16.78}$$

where K is the Gauss curvature of M and α its area form. Comparing this with Exercise 16, deduce another proof of Gauss' Theorema Egregium, in this case.

19. Show that

$$\begin{aligned} d\omega_{13}(X, Y) &= \omega_{12}(X)\omega_{23}(Y) - \omega_{12}(Y)\omega_{23}(X), \\ d\omega_{23}(X, Y) &= \omega_{21}(X)\omega_{13}(Y) - \omega_{21}(Y)\omega_{13}(X), \end{aligned} \tag{16.79}$$

Hint. See Exercise 17. This time show that $\langle D_X E_1, D_Y E_2\rangle = \omega_{12}(Y)\omega_{23}(X)$. The equations (16.75) and (16.79) can be written as

$$d\omega_{12} = \omega_{13}\wedge\omega_{32},\quad d\omega_{13} = \omega_{12}\wedge\omega_{23},\quad d\omega_{23} = \omega_{21}\wedge\omega_{13}. \tag{16.80}$$

For the connection between (16.79) and Codazzi's equation, see Exercise 24.

Back to higher dimensions

In Exercises 20–26, let M be an $(n-1)$-dimensional oriented hypersurface in $\mathbb{R}^n$ and let $\{E_1,\dots,E_n\}$ be an oriented orthonormal (local) frame field defined over M, such that $\{E_1(x),\dots,E_{n-1}(x)\}$ is an oriented orthonormal basis of T_xM. Define 1-forms ω_{jk} on M again by (16.73). Let $P(x):\mathbb{R}^n\to T_xM$ denote the orthogonal projection, as in (16.50)–(16.53), specialized to the case at hand. Denote the curvature form of M by Ω.

20. Show that $\omega_{jk} = -\omega_{kj}$ and that

$$A_N X = \sum_{j=1}^{n-1} \omega_{jn}(X)E_j. \tag{16.81}$$

21. Show that

$$\begin{aligned}(D_XP)E_j &= \omega_{jn}(X)E_n,\quad 1\le j<n,\\ (D_XP)E_n &= A_N X.\end{aligned} \tag{16.82}$$

22. For X and Y tangent to M, one can write (16.53) as

$$\Omega(X,Y) = (D_XP)(D_YP) - (D_YP)(D_XP) = [D_XP, D_YP]. \tag{16.83}$$

Deduce that, for $1\le j<n$,

$$\Omega(X,Y)E_j = \omega_{jn}(Y)A_NX - \omega_{jn}(X)A_NY.$$

Using (16.81), deduce that, for $1\le j,k<n$,

$$\langle\Omega(X,Y)E_j,E_k\rangle = \langle A_NY,E_j\rangle\langle A_NX,E_k\rangle - \langle A_NX,E_j\rangle\langle A_NY,E_k\rangle, \tag{16.84}$$

yet again establishing Gauss' Theorema Egregium in this case.

23. Show that

$$d\omega_{jk}(X,Y) = \sum_\ell \bigl\{\omega_{j\ell}(X)\omega_{\ell k}(Y) - \omega_{j\ell}(Y)\omega_{\ell k}(X)\bigr\}. \tag{16.85}$$

Hint. Follow those of Exercises 17 and 19. This time, show that

$$\langle D_X E_j, D_Y E_k\rangle = \sum_\ell \omega_{j\ell}(X)\omega_{k\ell}(Y).$$

Note that (16.85) is equivalent to

$$d\omega_{jk} = \sum_\ell \omega_{j\ell} \wedge \omega_{\ell k}. \tag{16.86}$$

These equations are called Cartan's structure equations.

24. Show that the Codazzi equation (16.17) is equivalent to

$$\nabla^0_X(A_N Y) - \nabla^0_Y(A_N X) - A_N([X,Y]) = 0. \tag{16.87}$$

25. Show that (16.87) also follows from the case $k = n$ of (16.86), i.e., from

$$d\omega_{jn} = \sum_\ell \omega_{j\ell} \wedge \omega_{\ell n}. \tag{16.88}$$

Hint. Start with

$$\nabla^0_X(A_N Y) - \nabla^0_Y(A_N X) = \sum_j \bigl\{\nabla^0_X(\omega_{jn}(Y)E_j) - \nabla^0_Y(\omega_{jn}(X)E_j)\bigr\},$$

and use the derivation identity. As in Exercise 17, use (6.26) on $d\omega_{jn}(X,Y)$. Deduce that the left side of (16.87) is equal to

$$\sum_j \bigl\{\omega_{jn}(Y)\nabla^0_X E_j - \omega_{jn}(X)\nabla^0_Y E_j\bigr\} + \sum_j d\omega_{jn}(X,Y)E_j. \tag{16.89}$$

Show that $\nabla^0_X E_j = \sum_{k<n} \omega_{jk}(X)E_k$. Then work on (16.89), using (16.88).

26. Let $X(t)$ be a tangent vector field to M along a curve $\gamma(t)$ given by parallel translation, so $X(t) \in T_{\gamma(t)}M \subset \mathbb{R}^n$. Show that

$$\frac{dX}{dt} = II(X,T), \tag{16.90}$$

where $T = \gamma'(t)$.
Hint. Look at (13.40).
Generalize this result to higher codimension.
Hint. Look at (16.1). Note the difference in perspective, connected of course by (16.10).

17. The Gauss-Bonnet theorem for surfaces

If M is a compact oriented Riemannian manifold of dimension 2, the Gauss-Bonnet theorem says that, if K is the Gauss curvature,

$$\int_M K\,dV = 2\pi\,\chi(M). \tag{17.1}$$

There is an associated formula if M has a boundary. There are a number of significant variants of this, involving for example the index of a vector field. We present several proofs of the Gauss-Bonnet theorem and some of its variants here.

We begin with an estimate on the effect of parallel translation about a small closed piecewise smooth curve. This first result holds for a general vector bundle $E \to M$ with connection ∇ and curvature

$$\Omega = \frac{1}{2}\,R^{\alpha}{}_{\beta jk}\,dx_j \wedge dx_k,$$

with no restriction on $\dim M$.

Proposition 17.1. *If γ is a closed piecewise smooth loop on M, parametrized by arc length for $0 \le t \le b$, $\gamma(b) = \gamma(0)$, and if $u(t)$ is a section of E over γ defined by parallel transport, i.e., $\nabla_T u = 0$, $T = \dot{\gamma}$, then*

$$u^{\alpha}(b) - u^{\alpha}(0) = -\frac{1}{2}\sum_{j,k,\beta} R^{\alpha}{}_{\beta jk}\Big(\int_A dx_j \wedge dx_k\Big)u^{\beta}(0) + O(b^3), \tag{17.2}$$

where A is an oriented 2-surface in M with $\partial A = \gamma$, and $u^{\alpha}(t)$ are the components of u with respect to a local frame.

Proof. If we put a coordinate system on a neighborhood of $p = \gamma(0) \in M$, and choose a frame field for E, then parallel transport is defined by

$$\frac{du^{\alpha}}{dt} = -\Gamma^{\alpha}{}_{\beta k}\,u^{\beta}\frac{dx_k}{dt}. \tag{17.3}$$

As usual, we use the summation convention. Thus

$$u^{\alpha}(t) = u^{\alpha}(0) - \int_0^t \Gamma^{\alpha}{}_{\beta k}(\gamma(s))u^{\beta}(s)\frac{dx_k}{ds}\;ds. \tag{17.4}$$

We hence have

$$u^{\alpha}(t) = u^{\alpha}(0) - \Gamma^{\alpha}{}_{\beta k}(p)u^{\beta}(0)(x_k - p_k) + O(t^2). \tag{17.5}$$

We can solve (17.3) up to $O(t^3)$ if we use

$$\Gamma^\alpha{}_{\beta j}(x) = \Gamma^\alpha{}_{\beta j}(p) + (x_k - p_k)\partial_k \Gamma^\alpha{}_{\beta j} + O\big(|x-p|^2\big). \tag{17.6}$$

Hence

$$\begin{aligned} u^\alpha(t) = u^\alpha(0) - \int_0^t \big[\Gamma^\alpha{}_{\beta k}(p) + (x_j - p_j)\partial_j \Gamma^\alpha{}_{\beta k}(p)\big] \\ \cdot \big[u^\beta(0) - \Gamma^\beta{}_{\gamma\ell}(p) u^\gamma(0)(x_\ell - p_\ell)\big] \frac{dx_k}{ds}\, ds + O(t^3). \end{aligned} \tag{17.7}$$

If $\gamma(b) = \gamma(0)$, we get

$$\begin{aligned} u^\alpha(b) = u^\alpha(0) - \int_0^b x_j \, dx_k \, \big(\partial_j \Gamma^\alpha{}_{\beta k}\big) u^\beta(0) \\ + \int_0^b x_j \, dx_k \, \Gamma^\alpha{}_{\beta k} \Gamma^\beta{}_{\gamma j} u^\gamma(0) + O(b^3), \end{aligned} \tag{17.8}$$

the components of Γ and their first derivatives being evaluated at p. Now Stokes' theorem gives

$$\int\limits_\gamma x_j \, dx_k = \int\limits_A dx_j \wedge dx_k,$$

so

$$u^\alpha(b) - u^\alpha(0) = \Big[-\partial_j \Gamma^\alpha{}_{\beta k} + \Gamma^\alpha{}_{\gamma k} \Gamma^\gamma{}_{\beta j}\Big] \int\limits_A dx_j \wedge dx_k \; u^\beta(0) + O(b^3). \tag{17.9}$$

Recall that the curvature is given by

$$\Omega = d\Gamma + \Gamma \wedge \Gamma,$$

i.e.,

$$R^\alpha{}_{\beta jk} = \partial_j \Gamma^\alpha{}_{\beta k} - \partial_k \Gamma^\alpha{}_{\beta j} + \Gamma^\alpha{}_{\gamma j} \Gamma^\gamma{}_{\beta k} - \Gamma^\alpha{}_{\gamma k} \Gamma^\gamma{}_{\beta j}. \tag{17.10}$$

Now the right side of (17.10) is the antisymmetrization, with respect to j and k, of the quantity in brackets in (17.9). Since $\int_A dx_j \wedge dx_k$ is antisymmetric in j and k, we get the desired formula (17.2).

In particular, if dim $M = 2$, then we can write the End(E) valued 2-form Ω as

$$\Omega = \mathcal{R}\mu \tag{17.11}$$

where μ is the volume form on M and $\mathcal{R}$ is a smooth section of End(E) over M. If E has an inner product and ∇ is a metric connection, then $\mathcal{R}$ is skew adjoint. If

γ is a geodesic triangle which is "fat" in the sense that none of its angles is small, (17.2) implies

$$u(b) - u(0) = -\mathcal{R}u(0)(\text{Area } A) + O\big((\text{Area } A)^{3/2}\big). \tag{17.12}$$

If we specialize further, to oriented 2-dimensional M with $E = TM$, possessing the Levi-Civita connection of a Riemannian metric, then we take $J : T_pM \to T_pM$, to be counterclockwise rotation by $90°$, which defines an almost complex structure on M. Up to a scalar this is the unique skew-adjoint operator on T_pM, and, by (15.34),

$$\mathcal{R}u = -K\,Ju, \quad u \in T_pM, \tag{17.13}$$

where K is the Gauss curvature of M at p. Thus, in this case, (17.12) becomes

$$u(b) - u(0) = K\,Ju(0)(\text{Area } A) + O\big((\text{Area } A)^{3/2}\big). \tag{17.14}$$

On the other hand, if a tangent vector $X_0 \in T_pM$ undergoes parallel transport around a geodesic triangle, the action produced on T_pM is easily seen to be a rotation in T_pM through an angle that depends on the angle defect of the triangle. The argument can be seen by looking at Fig. 17.1. We see that the angle from X_0 to X_3 is

$$(\pi + \alpha) - (2\pi - \beta - \gamma - \xi) - \xi = \alpha + \beta + \gamma - \pi. \tag{17.15}$$

In this case, formula (17.14) implies

$$\alpha + \beta + \gamma - \pi = \int K\,dV + O\big((\text{Area } A)^{3/2}\big). \tag{17.16}$$

We can now use a simple analytical argument to sharpen this up to the following celebrated formula of Gauss.

Theorem 17.2. *If A is a geodesic triangle in M^2, with angles α, β, γ, then*

$$\alpha + \beta + \gamma - \pi = \int_A K\ dV. \tag{17.17}$$

Proof. Break up the geodesic triangle A into N^2 little geodesic triangles, each of diameter $O(N^{-1})$, area $O(N^{-2})$. Since the angle defects are *additive*, the estimate (17.17) implies

$$\begin{aligned} \alpha + \beta + \gamma - \pi &= \int_A K\,dV + N^2 O\big((N^{-2})^{3/2}\big) \\ &= \int_A K\,dV + O(N^{-1}), \end{aligned} \tag{17.18}$$

and passing to the limit as $N \to \infty$ gives (17.17).

Note that any region which is a contractible geodesic polygon can be divided into geodesic triangles. If a contractible region $\mathcal{O} \subset M$ with smooth boundary is approximated by geodesic polygons, a straightforward limit process yields the Gauss-Bonnet formula

$$\int_{\mathcal{O}} K \, dV + \int_{\partial\mathcal{O}} \kappa \, ds = 2\pi, \tag{17.19}$$

where κ is the geodesic curvature of $\partial\Omega$. We leave the details to the reader. Another proof will be given at the end of this section.

If M is a compact oriented 2-dimensional manifold without boundary, we can partition M into geodesic triangles. Suppose the triangulation of M so produced has

$$F \text{ faces (triangles)}, \ E \text{ edges}, \ V \text{ vertices}. \tag{17.20}$$

If the angles of the jth triangle are $\alpha_j, \beta_j, \gamma_j$, then clearly summing all the angles produces $2\pi V$. On the other hand, (17.17) applied to the jth triangle, and summed over j, yields

$$\sum_j (\alpha_j + \beta_j + \gamma_j) = \pi F + \int_M K \, dV. \tag{17.21}$$

Hence $\int_M K \, dV = (2V - F)\pi$. Since in this case all the faces are triangles, counting each triangle three times will count each edge twice, so $3F = 2E$. Thus we obtain

$$\int_M K \, dV = 2\pi(V - E + F). \tag{17.22}$$

This is equivalent to (17.1), in view of Euler's formula (see (10.11)–(10.12)),

$$\chi(M) = V - E + F. \tag{17.23}$$

We now derive a variant of (17.1) when M is described in another fashion. Namely, suppose M is diffeomorphic to a sphere with g handles attached. The number g is called the genus of the surface. The case $g = 2$ is illustrated in Fig. 17.2. We claim that

$$\int_M K \, dV = 4\pi(1 - g) \tag{17.24}$$

in this case. By virtue of (17.22), this is equivalent to the identity

$$2 - 2g = V - E + F = \chi(M). \tag{17.25}$$

Direct proofs of this are possible; see Exercise 13 of §10. Here we will provide a proof of (17.24), based on the fact that

$$\int_M K\,dV = C(M) \tag{17.26}$$

depends only on M, not on the metric imposed. This follows from (17.22), by forgetting the interpretation of the right side. Another argument yielding (17.26) can be found in Appendix P. The point we want to make is, given (17.26), i.e., the independence of choice of metric, we can work out what $C(M)$ is, as follows.

First, choosing the standard metric on S^2, for which $K = 1$ and Area $S^2 = 4\pi$, we have

$$\int_{S^2} K\,dV = 4\pi. \tag{17.27}$$

Now suppose M is obtained by adding g handles to S^2. Since we can alter the metric on M at will, make sure it coincides with the metric of a sphere near a great circle, in a neighborhood of each circle where a handle is attached to the main body A, as illustrated in Fig. 17.2. If we imagine adding two hemispherical caps to each handle H_j, rather than attaching it to A, we turn each H_j into a new sphere, so by (17.27) we have

$$4\pi = \int_{H_j \cup \text{ caps}} K\ dV = \int_{H_j} K\ dV + \int_{\text{caps}} K\ dV. \tag{17.28}$$

Since the caps fit together to form a sphere, we have $\int_{\text{caps}} K\ dV = 4\pi$, so for each j,

$$\int_{H_j} K\ dV = 0, \tag{17.29}$$

provided that M has a metric such as described above. Similarly, if we add $2g$ caps to the main body A, we get a new sphere, so

$$4\pi = \int_{A \cup \text{ caps}} K\ dV = \int_A K\ dV + 2g(2\pi), \tag{17.30}$$

or

$$\int_A K\ dV = 2\pi(2-2g). \tag{17.31}$$

Together (17.29) and (17.31) yield (17.24), and we get the identity (17.25) for free.

We now give another perspective on Gauss' formula, directly dealing with the fact that TM can be treated as a complex line bundle, when M is an oriented Riemannian manifold of dimension 2. We will produce a variant of Proposition 17.1 which has no remainder term, and which hence produces (17.16) with no remainder, directly, so Theorem 17.2 follows without the additional argument given above. The result is the following; again dim M is unrestricted.

Proposition 17.3. *Let $E \to M$ be a complex line bundle. Let γ be a piecewise smooth closed loop in M, with $\gamma(0) = \gamma(b) = p$, bounding an oriented surface A. Let ∇ be a connection on E, with curvature Ω. If $u(t)$ is a section of E over γ defined by parallel translation, then*

$$u(b) = \Big[\exp\Big(-\int_A \Omega\Big)\Big] u(0). \tag{17.32}$$

Proof. Pick a nonvanishing section (hence a frame field) ξ of E over S, assuming S is homeomorphic to a disc. Any section u of E over S is of the form $u = v\xi$ for a complex valued function v on S. Then parallel transport along $\gamma(t) = (x_1(t), \dots, x_n(t))$ is defined by

$$\frac{dv}{dt} = -\Big(\Gamma_k \frac{dx_k}{dt}\Big)\, v. \tag{17.33}$$

The solution to this single first order ODE is

$$v(t) = \Big[\exp\Big(-\int_0^t \Gamma_k(\gamma(s)) \frac{dx_k}{ds}\, ds\Big)\Big]\, v(0). \tag{17.34}$$

Hence

$$v(b) = \Big[\exp\Big(-\int_\gamma \Gamma\Big)\Big] v(0), \tag{17.35}$$

where $\Gamma = \sum \Gamma_k\, dx_k$. The curvature 2-form Ω is given, as a special case of (17.10), by

$$\Omega = d\Gamma, \tag{17.36}$$

and Stokes' theorem gives (17.32), from (17.35), provided A is contractible, so the section ξ can be constructed. The general case follows from cutting A into contractible pieces.

Next, we relate $\int_M \Omega$ to the "index" of a section of a complex line bundle $E \to M$, when M is a compact oriented manifold of dimension 2. Suppose X is a section of E over $M \setminus S$, where S consists of a finite number of points; suppose that X is nowhere vanishing on $M \setminus S$, and that, near each $p_j \in S$, X has the following form. There is a coordinate neighborhood $\mathcal{O}_j$ centered at p_j, with p_j the origin, and a nonvanishing section ξ_j of E near p_j, such that

$$X = v_j \xi_j \quad \text{on} \quad \mathcal{O}_j, \quad v_j : \mathcal{O}_j \setminus p \to \mathbb{C} \setminus 0. \tag{17.37}$$

Taking a small counterclockwise circle γ_j about p_j, $v_j/|v_j| = \omega_j$ maps γ_j to S^1; consider the *degree* ℓ_j of this map, i.e., the winding number of γ_j about S^1. This is the index of X at p_j, and the sum over p_j is the total index of X :

$$\text{Index}\,(X) = \sum_j \ell_j. \tag{17.38}$$

We will establish the following.

Proposition 17.4. *For any connection on a complex line bundle $E \to M$, with curvature form Ω, and X as above, we have*

$$\int_M \Omega = -(2\pi i) \cdot \mathit{Index}(X). \tag{17.39}$$

Proof. You can replace X by a section of $E \setminus 0$ over $M \setminus \{p_j\}$, homotopic to the original, having the form (17.37) with

$$v_j = e^{i\ell_j\theta} + w_j, \tag{17.40}$$

in polar coordinates (r, θ) about p_j, with $w_j \in C^1(\mathcal{O}_j),\ w_j(0) = 0$. Excise small disks $\mathcal{D}_j$ containing p_j; let $\mathcal{D} = \cup \mathcal{D}_j$. Then, by Stokes' theorem,

$$\int_{M \setminus \mathcal{D}} \Omega = -\sum_j \int_{\gamma_j} \Gamma \tag{17.41}$$

where $\gamma_j = \partial \mathcal{D}_j$ and Γ is the connection 1-form with respect to the section X, i.e., with $\nabla_k = \nabla_{\partial_k},\ \partial_k = \partial / \partial x_k$ in local coordinates,

$$\nabla_k X = \Gamma_k X, \quad \Gamma = \sum \Gamma_k \, dx_k. \tag{17.42}$$

Now (17.37) gives (with no summation)

$$\Gamma_k v_j \xi_j = (\partial_k v_j + v_j \widetilde{\Gamma}_{jk}) \xi_j \tag{17.43}$$

on $\overline{\mathcal{D}}_j$, where $\widetilde{\Gamma}_{jk} dx_k$ is the connection 1-form with respect to the section ξ_j. Hence

$$\Gamma_k = v_j^{-1} \partial_k v_j + \widetilde{\Gamma}_{jk} \tag{17.44}$$

with remainder term $\widetilde{\Gamma}_{jk} \in C^1(\mathcal{O}_j)$. By (17.40), we have

$$\int_{\gamma_j} \Gamma = 2\pi i \ell_j + O(r) \tag{17.45}$$

if each $\mathcal{D}_j$ has radius $\le Cr$. Passing to the limit as the disks $\mathcal{D}_j$ shrink to p_j gives (17.39).

Since the left side of (17.39) is independent of the choice of X, it follows that the index of X depends only on E, not on the choice of such X.

In case M is a compact oriented Riemannian 2-manifold, whose tangent bundle can be given the structure of a complex line bundle as noted above, (17.39) is equivalent to

$$\int_M K \, dV = 2\pi \operatorname{Index}(X) \tag{17.46}$$

for any smooth vector field X, nonvanishing, on M minus a finite set of points. This verifies the identity

$$\operatorname{Index}(X) = \chi(M) \tag{17.47}$$

in this case.

As a further comment on the Gauss-Bonnet formula for compact surfaces, let us recall from Exercise 8 of §16 that, if M is a compact oriented surface in $\mathbb{R}^3$, with Gauss map $N : M \to S^2$, then

$$\operatorname{Deg}(N) = \frac{1}{4\pi} \int_M N^* \omega_0 = \frac{1}{4\pi} \int_M K \, dV. \tag{17.48}$$

(Indeed, we can appeal to (15.60) for the second identity here.) Furthermore, Corollary 10.5 yields an independent proof that, in this case,

$$\operatorname{Deg}(N) = \frac{1}{2} \operatorname{Index}(X), \tag{17.49}$$

for any vector field X on M with a finite number of critical points. Hence (17.48)–(17.49) provide another proof of (17.1), at least for a surface in $\mathbb{R}^3$. This line of reasoning will be extended to the higher dimensional case of hypersurfaces of $\mathbb{R}^{n+1}$, in the early part of §20, preparatory to establishing the general Chern-Gauss-Bonnet Theorem.

To end this section, we provide a direct proof of the formula (17.19), using an argument parallel to the proof of Proposition 17.3. Thus, assuming that M is an oriented surface, we give TM the structure of a complex line bundle, and pick a nonvanishing section ξ of TM over a neighborhood of $\overline{\mathcal{O}}$. Let $\gamma = \partial \mathcal{O}$ be parametrized by arc length, $T = \gamma'(s),\ 0 \le s \le b$, with $\gamma(b) = \gamma(0)$. The geodesic curvature κ of γ, appearing in (17.19), is given by

$$\nabla_T T = \kappa N, \quad N = JT. \tag{17.50}$$

If we set $T = u\xi$, where $u : \overline{\mathcal{O}} \to \mathbb{C}$, then, parallel to (17.33), we have (17.50) equivalent to

$$\frac{du}{ds} = -\sum \Gamma_k \frac{dx_k}{ds} u + i\kappa u. \tag{17.51}$$

The solution to this single first order ODE is (parallel to (17.34))

$$u(t) = \Big[\exp\Big(i\int_0^t \kappa(s)\ ds - \int_0^t \Gamma_k\big(\gamma(s)\big)\frac{dx_k}{ds}\ ds\Big)\Big]\ u(0). \tag{17.52}$$

Hence

$$u(b) = \Big[\exp\Big(i\int_\gamma \kappa(s)\ ds - \int_{\mathcal{O}} \Omega\Big)\Big]\ u(0). \tag{17.53}$$

By (17.13), we have

$$\Omega = -iK\ dV, \tag{17.54}$$

and since $u(b) = u(0)$, we have

$$\exp\Big(i\int_\gamma \kappa(s)\ ds + i\int_{\mathcal{O}} K\ dV\Big) = 1, \tag{17.55}$$

or

$$\int_{\mathcal{O}} K\ dV + \int_\gamma \kappa(s)\ ds = 2\pi\nu, \tag{17.56}$$

for some $\nu \in \mathbb{Z}$. Now if $\mathcal{O}$ were a tiny disc in M, it would be clear that $\nu = 1$. Using the contractibility of $\mathcal{O}$ and the fact that the left side of (17.56) cannot jump, we have $\nu = 1$, which proves (17.19).

Exercises

1. Given a triangulation of a compact surface M, within each triangle construct a vector field, vanishing at 7 points as illustrated in Fig. 10.1, with the vertices as attractors, the center as a repeller, and the midpoints of each side as saddle points. Fit these together to produce a smooth vector field X on M. Show directly that

$$\text{Index}\,(X) = V - E + F.$$

2. Let $L \to M$ be a complex line bundle, u and v sections of L with a finite number of zeros. Show directly that u and v have the same index.

Hint. Start with $u = fv$ on $M \setminus Z$, where Z is the union of the zero sets, and $f : M \setminus Z \to \mathbb{C} \setminus 0$.

3. Let M_1 and M_2 be n-dimensional submanifolds of $\mathbb{R}^k$. Suppose a curve γ is contained in the intersection $M_1 \cap M_2$, and assume

$$p = \gamma(s) \Longrightarrow T_p M_1 = T_p M_2.$$

Show that parallel translations along γ in M_1 and in M_2 coincide.
Hint. If $T = \gamma'(s)$ and X is a vector field along γ, tangent to M_1 (hence to M_2), show that $\nabla_T^{M_1} X = \nabla_T^{M_2} X$, using Corollary 16.2.

4. Let $\mathcal{O}$ be the region in $S^2 \subset \mathbb{R}^3$ consisting of points in S^2 of geodesic distance $< r$ from $p = (0,0,1)$, where $r \in (0,\pi)$ is given. Let $\gamma = \partial\mathcal{O}$. Construct a cone, with vertex at $(0,0,\sec r)$, tangent to S^2 along γ. Using this and Exercise 3, show that parallel translation over one circuit of γ is given by

$$\text{counterclockwise rotation by } \theta = 2\pi(1 - \cos r).$$

Hint. Flatten out the cone, as in Fig. 17.3. Notice that γ has length $\ell = 2\pi\ \sin r$. Compare this calculation with the result of (17.32), which in this context implies

$$u(\ell) = \Bigl[\exp i \int\limits_{\mathcal{O}} K \, dV\Bigr] \, u(0).$$

5. Let $\gamma : [a,b] \to \mathbb{R}^3$ be a smooth closed curve, so $\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$. Assume γ is parametrized by arclength, so $\gamma'(t) = T(t)$ and $T : [a,b] \to S^2$; hence T is a smooth closed curve in $\mathbb{S}^2$. Note that the normal space to γ at $p = \gamma(t)$ is naturally identified with the tangent space to S^2 at $T(t) = q$:

$$\nu_p(\gamma) = T_q S^2.$$

(a) Show that parallel translation along γ of a section of the normal bundle $\nu(\gamma)$, with respect to the connection described in Exercise 3 of §16, coincides with parallel translation along the curve T of vectors tangent to S^2.
Hint. Recall Exercise 3 of §13.

(b) Suppose the curvature κ of γ never vanishes, so the torsion τ is well defined, as in (16.62). Show that parallel translation once around γ acts on $\nu_p(\gamma)$ by multiplication by

$$\exp\Bigl(-i \int\limits_{\gamma} \tau(s)\, ds\Bigr).$$

Here we use the complex structure on $\nu_p(\gamma)$ given by $JN = B,\ JB = -N$.
Hint. Use (16.64).
Compare these two results.

6. Let M be a compact surface in $\mathbb{R}^3$, with principal curvatures κ_1, κ_2. Set

$$\begin{aligned}\mathcal{W}(M) &= \frac{1}{2\pi}\int_M \Bigl(\frac{\kappa_1+\kappa_2}{2}\Bigr)^2\,dV,\\ \mathcal{C}(M) &= \frac{1}{2\pi}\int_M \Bigl(\frac{\kappa_1-\kappa_2}{2}\Bigr)^2\,dV.\end{aligned}$$

Show that

$$\mathcal{W}(M) = \mathcal{C}(M) + \chi(M).$$

For further discussion of these functionals, see [Wil2], Chapter 7.

7. Let $x(s)$ be a unit speed curve in $\mathbb{R}^3$, with Frenet apparatus $T(s), N(s), B(s)$, curvature $\kappa(s)$ (which we will assume to be > 0), and torsion $\tau(s)$; cf. (16.62). Fix $a > 0$ (small) and consider the surface S, the boundary of a tube about this curve, given by

$$X(s,t) = x(s) + a(\cos t)N(s) + a(\sin t)B(s).$$

Show that the outward unit normal ν to S is given by

$$\nu(s,t) = (\cos t)N(s) + (\sin t)B(s),$$

and that, in the (s,t)-coordinates,

$$K\,dV = -\kappa(s)(\cos t)\,ds\,dt.$$

8. Let M be a compact surface imbedded in $\mathbb{R}^3$. Let $K^+ = \max{(K,0)}$. Show that

$$\int_M K^+\,dV \ge 4\pi.$$

Hint. Show that the image of $\{x \in M : K(x) \ge 0\}$ under the Gauss map is *all of* S^2.

9. Let $x : [0, L] \to \mathbb{R}^3$ be a smooth, closed, unit speed curve. Show that

$$\int_0^L \kappa(s)\, ds \geq 2\pi.$$

Show that one has equality if and only if it is a convex plane curve. This result is known as Fenchel's theorem. (Assume $\kappa(s) > 0$.)
Hint. Use the results of Exercises 7–8. Note that S has curvature $K \geq 0$ on the set $\pi/2 \leq t \leq 3\pi/2$.

10. Let $\gamma : [0, b] \to M$ be a smooth, closed, unit-speed curve on a two-dimensional, oriented Riemannian manifold. (In particular, assume $\gamma'(b) = \gamma'(0)$.) Let $T = \gamma', N = JT$ along γ. Show the (17.50) extends to

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T.$$

Deduce that, if $U(t)$ is defined along $\gamma(t)$ by parallel translation then

$$U(b) = e^{-J \int_\gamma \kappa(s)\, ds}\, U(0).$$

Show that, if $\gamma = \partial\mathcal{O}$, this result plus Proposition 17.3 again implies the identity (17.55), i.e.,

$$\exp\Bigl(i \int\limits_\gamma \kappa(s)\, ds + i \int\limits_{\mathcal{O}} K\, dV\Bigr) = 1.$$

18. The principal bundle picture

An important tool for understanding vector bundles is the notion of an underlying structure, namely that of a *principal bundle.* If M is a manifold and G a Lie group, then a principal G-bundle $P \xrightarrow{p} M$ is a locally trivial fibration with a G-action on P, such that G acts on each fiber $P_x = p^{-1}(x)$ in a simply transitive fashion. An example is the *frame bundle* of an oriented Riemannian manifold M, $F(M) \rightarrow M$, where $F_x(M)$ consists of the set of ordered oriented orthonormal bases of the tangent space T_x to M at x. If $n = \dim M$, this is a principal $SO(n)$-bundle.

If $P \rightarrow M$ is a principal G-bundle, then associated to each representation π of G on a vector space V is a vector bundle $E \rightarrow M$. The set E is a quotient space of the Cartesian product $P \times V$, under the equivalence relation

$$(y, v) \sim (y \cdot g, \pi(g)^{-1} v), \quad g \in G. \tag{18.1}$$

We have written the G-action on P as a right action. One writes $E = P \times_\pi V$. The space of sections of E is naturally isomorphic to a certain subspace of the space of V-valued functions on P :

$$C^\infty(M, E) \approx \{u \in C^\infty(P, V) : u(y \cdot g) = \pi(g)^{-1} u(y),\ g \in G\}. \tag{18.2}$$

We describe how this construction works for the frame bundle $F(M)$ of an oriented Riemannian manifold, which, as mentioned above, is a principal $SO(n)$-bundle. Thus, a point $y \in F_x(M)$ consists of an n-tuple $(e_1, \dots, e_n)$, forming an ordered oriented orthonormal basis of $T_x M$. If $g = (g_{jk}) \in SO(n)$, the G-action is given by

$$(e_1, \dots, e_n) \cdot g = (f_1, \dots, f_n), \quad f_j = \sum_\ell g_{\ell j} e_\ell. \tag{18.3}$$

One can check that $(f_1, \dots, f_n)$ is also an oriented orthonormal basis of $T_x M$, and that $(y \cdot g) \cdot g' = y \cdot (g g')$ for $g, g' \in SO(n)$. If π is the "standard" representation of $SO(n)$ on $\mathbb{R}^n$, given by matrix multiplication, we claim that there is a natural identification

$$F(M) \times_\pi \mathbb{R}^n \approx TM. \tag{18.4}$$

In fact, if $y = (e_1, \dots, e_n) \in F_x(M)$ and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, the map (18.4) is defined by

$$(y, v) \mapsto \sum_j v_j e_j \in T_x M. \tag{18.5}$$

We need to show that this is constant on equivalence classes, as defined by (18.1), i.e., for any $g \in SO(n)$,

$$z = y \cdot g = (f_1, \dots, f_n),\ w = \pi(g)^{-1} v \Longrightarrow \sum w_k f_k = \sum v_j e_j. \tag{18.6}$$

In fact, setting $g^{-1} = h = (h_{jk})$, we see that

$$\sum_k w_k f_k = \sum_{j,k,\ell} h_{kj}\, v_j\, g_{\ell k}\, e_\ell = \sum_{j,\ell} \delta_{\ell j}\, v_j\, e_\ell,$$

since $\sum_k g_{\ell k} h_{kj} = \delta_{\ell j}$, and this implies (18.6).

Connections are naturally described in terms of a geometrical structure on a principal bundle. This should be expected, since as we saw in §13 a connection on a vector bundle can be described in terms of a "connection 1-form" (13.14), depending on a choice of local frame for the vector bundle.

The geometrical structure giving a connection on a principal bundle $P \to M$ is the following. For each $y \in P$, the tangent space $T_y P$ contains the subspace $V_y P$ of vectors tangent to the fibre $p^{-1}(x),\ x = p(y)$. The space $V_y P$, called the "vertical space," is naturally isomorphic to the Lie algebra $\mathfrak{g}$ of G, via the map

$$\iota_y : \mathfrak{g} \longrightarrow V_y P, \quad \iota_y(X) = \frac{d}{dt}(y \cdot \operatorname{Exp} tX)\big|_{t=0}. \tag{18.7}$$

A connection on P is determined by a choice of complementary subspace, called a "horizontal space:"

$$T_y P = V_y P \oplus H_y P, \tag{18.8}$$

with the G-invariance

$$g_*(H_y P) = H_{y \cdot g} P \tag{18.9}$$

where $g_* : T_y P \to T_{y\cdot g} P$ is the natural derivative map.

Given this structure, a vector field X on M has a uniquely defined "lift" $\widetilde{X}$ to a vector field on P, such that $p_* \widetilde{X}_y = X_x$ $(x = p(y))$ and $\widetilde{X}_y \in H_y P$ for each $y \in P$. Furthermore, if E is a vector bundle determined by a representation of G and $u \in C^\infty(M, V)$ corresponds to a section v of E, the V-valued function $\widetilde{X} \cdot u$ corresponds to a section of E which we denote $\nabla_X v$; ∇ is the covariant derivative on E defined by the connection on P just described. If V has an inner product and π is unitary, E gets a natural metric, and ∇ is a metric connection on E.

If π_j are representations of G on V_j, giving vector bundles $E_j \to M$ associated to a principal bundle $P \to M$ with connection, then $\pi_1 \otimes \pi_2$ is a representation of G on $V_1 \otimes V_2$ and we have a vector bundle $E \to M,\ E = E_1 \otimes E_2$. The prescription above associating a connection to E as well as to E_1 and E_2 agrees with the definition of a connection on a tensor product of two vector bundles given by (13.29). This follows simply from the derivation property of the vector field $\widetilde{X}$, acting as a first order differential operator on functions on P.

The characterization (18.8)–(18.9) of a connection on a principal bundle $P \to M$ is equivalent to the following, in view of the natural isomorphism $V_yP \approx \mathfrak{g}$. The splitting (18.8) corresponds to a projection of T_yP onto V_yP, hence to a linear map $T_yP \to \mathfrak{g}$ which gives the identification $V_yP \approx \mathfrak{g}$ on the linear subspace V_yP of T_yP. This map can be regarded as a $\mathfrak{g}$-valued 1-form ξ on P, called the *connection form.* Explicitly, for $X \in T_yP$,

$$\xi(X) = \iota_y^{-1}(X_v), \quad X = X_v + X_h,\ X_v \in V_yP,\ X_h \in H_yP.$$

Note that the invariance property (18.9) implies $\xi(g_*X) = \xi(g_*X_v)$, or equivalently

$$(g^*\xi)(X) = (g^*\xi)(X_v),$$

where g^* denotes the pull-back of the form ξ induced from the G-action on P. A calculation gives

$$\iota_{y\cdot g}^{-1} \circ g_* \circ \iota_y = Ad_{g^{-1}} \tag{18.10}$$

on $\mathfrak{g}$, and hence

$$g^*\xi = Ad_{g^{-1}}\xi, \quad g \in G. \tag{18.11}$$

The way the Levi-Civita connection on an oriented Riemannian manifold gives rise to a connection on the frame bundle $F(M) \to M$ is the following. Fix $y \in F(M)$, $x = p(y)$. Recall that the point y is an ordered (oriented) orthonormal basis $(e_1, \dots, e_n)$ of the tangent space T_xM. Parallel transport of each e_j along a curve γ through x thus gives a family of orthonormal bases for the tangent space to M at $\gamma(t)$, hence a curve $\gamma^\#$ in $F(M)$ lying over γ. The tangent to $\gamma^\#$ at y belongs to the horizontal space $H_yF(M)$, which in fact consists of all such tangent vectors as the curve γ through x is varied. This construction generalizes to other vector bundles $E \to M$ with connection ∇. One can use the bundle of orthonormal frames for E if ∇ is a metric connection, or the bundle of general frames for a general connection.

Let us re-state how a connection on a principal bundle gives rise to connections on associated vector bundles. Given a principal G-bundle $P \to M$, consider a local section σ of P, over $U \subset M$. If we have a representation π of G on V, the associated vector bundle $E \to M$, and a section u of E, then we have $u \circ \sigma : U \to V$, using the identification (18.2). Given a connection on P, with connection 1-form ξ, we can characterize the covariant derivative induced on sections of E by

$$(\nabla_X u) \circ \sigma = \mathcal{L}_X(u \circ \sigma) + \Gamma(X)u \circ \sigma, \tag{18.12}$$

where $\mathcal{L}_X$ acts componentwise on $u \circ \sigma$, and

$$\Gamma(X) = (d\pi)\big(\xi_y(\widehat{X})\big), \quad y = \sigma(x), \quad \widehat{X} = D\sigma(x)X, \tag{18.13}$$

$d\pi$ denoting the derived representation of $\mathfrak{g}$ on V. To see this, note that the left hand side of (18.2) is, by definition, equal to $(\mathcal{L}_{\widetilde{X}}u)\circ\sigma$. Meanwhile, apply the chain rule to write

$$\begin{aligned}\mathcal{L}_X(u\circ\sigma) &= Du(\sigma)\,D\sigma(X) = Du(\sigma)\widehat{X} \\ &= Du(\sigma)\widetilde{X} + Du(\sigma)V \\ &= (\mathcal{L}_{\widetilde{X}}u)\circ\sigma + (\mathcal{L}_V u)\circ\sigma,\end{aligned}$$

where $V = \widehat{X} - \widetilde{X}$ is *vertical.* It follows from (18.2) that, is u is a section of E and V is vertical, then

$$\mathcal{L}_V u = -d\pi\big(\xi(V)\big)u.$$

Since $\xi(V) = \xi(\widehat{X})$, this establishes (18.12). Note the similarity of (18.12) to (13.7). Note also that Γ depends on σ; cf. Exercise 4 below.

Recall the curvature $R(X,Y)$ of a connection ∇ on a vector bundle $E \to M$, defined by the formula (13.10). In case $E = P\times_\pi V$, and ∇u is defined as above, we have (using the identification (18.2))

$$R(X,Y)u = \mathcal{L}_{[\widetilde{X},\widetilde{Y}]}u - \mathcal{L}_{\widetilde{[X,Y]}}u. \tag{18.14}$$

Alternatively, using (18.12)–(18.13), we see that the curvature of ∇ is given by

$$R(X,Y)u\circ\sigma = \Big\{\mathcal{L}_X\Gamma(Y) - \mathcal{L}_Y\Gamma(X) + \big[\Gamma(X),\Gamma(Y)\big] - \Gamma\big([X,Y]\big)\Big\}u\circ\sigma. \tag{18.15}$$

This is similar to (13.13). Next we want to obtain a formula similar to (but more fundamental than) (13.15).

Fix $y\in P,\ x = p(y)$. It is convenient to calculate (18.15) at x by picking the local section σ to have the property that

$$D\sigma(x) : T_xM \longrightarrow H_yP, \tag{18.16}$$

which is easily arranged. Then $\widehat{X} = \widetilde{X}$ at y, so $\Gamma(X) = 0$ at y. Hence, at x,

$$\begin{aligned}R(X,Y)u\circ\sigma &= \big\{\mathcal{L}_X\Gamma(Y) - \mathcal{L}_Y\Gamma(X)\big\}u\circ\sigma \\ &= (d\pi)\big\{\widehat{X}\cdot\xi(\widehat{Y}) - \widehat{Y}\cdot\xi(\widehat{X})\big\}u\circ\sigma \\ &= (d\pi)\big\{(d\sigma^*\xi)(X,Y) + (\sigma^*\xi)([X,Y])\big\}u\circ\sigma.\end{aligned} \tag{18.17}$$

Of course, $\sigma^*\xi = 0$ at x. Thus we see that

$$R(X,Y)u = (d\pi)\big\{(d\xi)(\widetilde{X},\widetilde{Y})\big\}u, \tag{18.18}$$

at y, and hence everywhere on P. In other words,

$$R(X,Y) = (d\pi)\big(\Omega(\widetilde{X},\widetilde{Y})\big), \tag{18.19}$$

where Ω is the $\mathfrak{g}$-valued 2-form on P defined by

$$\Omega(X^{\#}, Y^{\#}) = (d\xi)(\varkappa X^{\#}, \varkappa Y^{\#}), \tag{18.20}$$

for $X^{\#}, Y^{\#} \in T_yP$. Here, $\varkappa$ is the projection of T_yP onto H_yP, with respect to the splitting (18.9). One calls Ω the curvature 2-form of the connection ξ on P.

If V and W are smooth vector fields on P, then

$$(d\xi)(V, W) = V \cdot \xi(W) - W \cdot \xi(V) - \xi\big([V, W]\big). \tag{18.21}$$

In particular, if $V = \widetilde{X}$, $W = \widetilde{Y}$ are horizontal vector fields on P, then since $\xi(\widetilde{X}) = \xi(\widetilde{Y}) = 0$, we have

$$(d\xi)(\widetilde{X}, \widetilde{Y}) = -\xi\big([\widetilde{X}, \widetilde{Y}]\big). \tag{18.22}$$

Hence, given $X^{\#}, Y^{\#} \in T_yP$, we have

$$\Omega(X^{\#}, Y^{\#}) = -\xi\big([\widetilde{X}, \widetilde{Y}]\big), \tag{18.23}$$

where $\widetilde{X}$ and $\widetilde{Y}$ are any horizontal vector fields on P such that $\widetilde{X} = \varkappa X^{\#}$ and $\widetilde{Y} = \varkappa Y^{\#}$ at $y \in P$. Since ξ annihilates $[\widetilde{X}, \widetilde{Y}]$ if and only if it is horizontal, we see that Ω measures the failure of the bundle of horizontal spaces to be involutive.

It follows from Frobenius' Theorem (see §I) that, if $\Omega = 0$ on P, there is an integral manifold $S \subset P$, such that, for each $y \in S$, $T_yS = H_yP$. Each translate $S \cdot g$ is also an integral manifold. We can use this family of integral manifolds to construct local sections $v_1, \dots v_K$ of E ($K = \dim V$), linearly independent at each point, such that $\nabla v_j = 0$ for all j, given that $\Omega = 0$. Thus we recover Proposition 13.2, in this setting.

The following important result is known as Cartan's formula for the curvature 2-form.

Theorem 18.1. *We have*

$$\Omega = d\xi + \frac{1}{2}[\xi, \xi]. \tag{18.24}$$

The bracket $[\xi, \eta]$ of $\mathfrak{g}$-valued 1-forms is defined as follows. Suppose, in local coordinates,

$$\xi = \sum \xi_j \, dx_j, \quad \eta = \sum \eta_k \, dx_k, \quad \xi_j, \ \eta_k \in \mathfrak{g}. \tag{18.25}$$

Then we set

$$[\xi, \eta] = \sum_{j,k} [\xi_j, \eta_k] \, dx_j \wedge dx_k = \sum_{j<k} \big([\xi_j, \eta_k] + [\eta_j, \xi_k]\big) \, dx_j \wedge dx_k, \tag{18.26}$$

which is a $\mathfrak{g}$-valued 2-form. Equivalently, if U and V are vector fields on P,

$$[\xi,\eta](U,V) = [\xi(U),\eta(V)] + [\eta(U),\xi(V)]. \tag{18.27}$$

In particular,

$$\frac{1}{2}[\xi,\xi](U,V) = [\xi(U),\xi(V)]. \tag{18.28}$$

Note that, if π is a representation of G on a vector space V and $d\pi$ the derived representation of $\mathfrak{g}$ on V, if we set $A_j = d\pi(\xi_j)$, then, for

$$d\pi(\xi) = \alpha = \sum A_j \, dx_j, \tag{18.29}$$

we have

$$\alpha \wedge \alpha = \sum_{j,k} A_j A_k \, dx_j \wedge dx_k = \frac{1}{2} \sum_{j,k} (A_j A_k - A_k A_j) \, dx_j \wedge dx_k. \tag{18.30}$$

Hence

$$\alpha \wedge \alpha = \frac{1}{2}(d\pi)[\xi,\xi]. \tag{18.31}$$

Thus we see the parallel between (18.24) and (13.15).

To prove (18.24), one evaluates each side on $(X^\#, Y^\#)$, for $X^\#, Y^\# \in T_y P$. We write $X^\# = \widetilde{X} + X_v$ with $\widetilde{X} \in H_y P$, $X_v \in V_y P$, and similarly write $Y^\# = \widetilde{Y} + Y_v$. It suffices to check the following four cases:

$$\Omega(\widetilde{X},\widetilde{Y}), \quad \Omega(\widetilde{X},Y_v), \quad \Omega(X_v,\widetilde{Y}), \quad \Omega(X_v,Y_v). \tag{18.32}$$

Without loss of generality, one can assume that $\widetilde{X}$ and $\widetilde{Y}$ are horizontal lifts of vector fields on M, and that $\xi(X_v)$ and $\xi(Y_v)$ are constant $\mathfrak{g}$-valued functions on P. By (18.20) and (18.28), we have

$$\Omega(\widetilde{X},\widetilde{Y}) = (d\xi)(\widetilde{X},\widetilde{Y}), \quad \frac{1}{2}[\xi,\xi](\widetilde{X},\widetilde{Y}) = [\xi(\widetilde{X}),\xi(\widetilde{Y})] = 0, \tag{18.33}$$

so (18.24) holds in this case. Next, clearly

$$\Omega(\widetilde{X},Y_v) = 0, \quad [\xi(\widetilde{X}),\xi(Y_v)] = 0, \tag{18.34}$$

while

$$d\xi(\widetilde{X},Y_v) = \widetilde{X}\cdot\xi(Y_v) - Y_v\cdot\xi(\widetilde{X}) - \xi\big([\widetilde{X},Y_v]\big). \tag{18.35}$$

Now, having arranged that $\xi(Y_v)$ be a constant $\mathfrak{g}$-valued function on P, we have that $\widetilde{X}\cdot\xi(Y_v)=0$. Of course, $Y_v\cdot\xi(\widetilde{X})=0$. Also, $[\widetilde{X},Y_v]=-\mathcal{L}_{Y_v}\widetilde{X}$ is horizontal, by (18.10), so $\xi\big([\widetilde{X},Y_v]\big)=0$. This verifies (18.24) when both sides act on $(\widetilde{X},Y_v)$, and similarly we have (18.24) when both sides act on $(X_v,\widetilde{Y})$. We consider the final case. Clearly

$$\Omega(X_v,Y_v)=0, \tag{18.36}$$

while

$$d\xi(X_v,Y_v)=X_v\cdot\xi(Y_v)-Y_v\cdot\xi(X_v)-\xi\big([X_v,Y_v]\big)=-\xi\big([X_v,Y_v]\big), \tag{18.37}$$

and

$$\frac{1}{2}[\xi,\xi](X_v,Y_v)=[\xi(X_v),\xi(Y_v)]=\xi\big([X_v,Y_v]\big), \tag{18.38}$$

so (18.24) is verified in this last case, and Theorem 18.1 is proved.

We next obtain a form of the Bianchi identity which will play an important role in the next section. Compare (13.42) and (14.13).

Proposition 18.2. *We have*

$$d\Omega=[\Omega,\xi]. \tag{18.39}$$

Here, if $\Omega=\sum\Omega_{jk}\,dx_j\wedge dx_k$ in local coordinates, we set

$$[\Omega,\xi]=\sum_{j,k,\ell}[\Omega_{jk},\xi_\ell]\,dx_j\wedge dx_k\wedge dx_\ell=-\sum_{j,k,\ell}[\xi_\ell,\Omega_{jk}]\,dx_\ell\wedge dx_j\wedge dx_k=-[\xi,\Omega]. \tag{18.40}$$

To get (18.39), apply d to (18.24), obtaining (since $dd\xi=0$)

$$d\Omega=\frac{1}{2}[d\xi,\xi]-\frac{1}{2}[\xi,d\xi]=[d\xi,\xi], \tag{18.41}$$

which differs from $[\Omega,\xi]$ by $(1/2)[[\xi,\xi],\xi]$. We have

$$[[\xi,\xi],\xi]=\sum_{j,k,\ell}[[\xi_j,\xi_k],\xi_\ell]\,dx_j\wedge dx_k\wedge dx_\ell. \tag{18.42}$$

Now cyclic permutations of (j,k,ℓ) leave $dx_j\wedge dx_k\wedge dx_\ell$ invariant, so we can replace $[[\xi_j,\xi_k],\xi_\ell]$ in (18.42) by the average over cyclic permutations of (j,k,ℓ). However, Jacobi's identity for a Lie algebra is

$$[[\xi_j,\xi_k],\xi_\ell]+[[\xi_k,\xi_\ell],\xi_j]+[[\xi_\ell,\xi_j],\xi_k]=0,$$

so $[[\xi,\xi],\xi]=0$, and we have (18.39).

It is worth mentioning that, whenever ξ is a $\mathfrak{g}$-vauled 1-form and Ω is given by (18.24), then (18.39) holds. This observation will be useful in the proof of Lemma 19.3 in the next section.

Exercises

1. Let $P \xrightarrow{p} M$ be a principal G-bundle with connection, where M is a Riemannian manifold. Pick an inner product on $\mathfrak{g}$. For $y \in P$, define an inner product on $T_yP = V_yP \oplus H_yP$ so that, if $Z \in T_yP$ has decomposition $Z = Z_v + Z_h$, then

$$\|Z\|^2 = \|\xi(Z_v)\|^2 + \|Dp(y)Z_h\|^2.$$

Show that this is a G-invariant Riemannian metric on P.

2. Conversely, if $P \xrightarrow{p} M$ is a principal G-bundle, and if P has a G-invariant Riemannian metric, show that this determines a connection on P, by declaring that, for each $y \in P$, H_yP is the *orthogonal complement* of V_yP.

3. A choice of section σ of P over an open set $U \subset M$ produces an isomorphism

$$j_\sigma : C^\infty(U,E) \longrightarrow C^\infty(U,V). \tag{18.43}$$

If $\tilde{\sigma}$ is another section, there is a smooth function $g : U \to G$ such that

$$\tilde{\sigma}(x) = \sigma(x)\cdot g(x), \quad \forall\, x \in U. \tag{18.44}$$

Show that

$$j_{\tilde{\sigma}} \circ j_\sigma^{-1} v(x) = \pi\big(g(x)\big)^{-1} v(x). \tag{18.45}$$

4. According to (18.12), if $u \in C^\infty(U,E)$ and $v = j_\sigma u,\ \tilde{v} = j_{\tilde{\sigma}} u$, we have

$$(\nabla_X u)\circ\sigma = X\cdot v + \Gamma(X)v, \quad (\nabla_X u)\circ\tilde{\sigma} = X\cdot\tilde{v} + \widetilde{\Gamma}(X)\tilde{v}. \tag{18.46}$$

Show that

$$\widetilde{\Gamma}(X) = \pi\big(g(x)\big)^{-1}\Gamma(X)\pi\big(g(x)\big) + d\pi\big(D\lambda_{g(x)}(g(x))\circ Dg(x)X\big), \tag{18.47}$$

where $Dg(x)X \in T_{g(x)}G,\ \lambda_g(h) = g^{-1}h,\ D\lambda_g(g) : T_gG \to T_eG \approx \mathfrak{g}$. Compare (13.42).
Hint. Make use of (18.11), plus the identity $(d\pi)(Ad_{g^{-1}}A) = \pi(g)^{-1}d\pi(A)\pi(g),\ A \in \mathfrak{g}$.

5. Show that, for X, Y vector fields on M, $\Omega(\widetilde{X}, \widetilde{Y})$ satisfies

$$\Omega(\widetilde{X}, \widetilde{Y})(y \cdot g) = Ad(g)^{-1}\Omega(\widetilde{X}, \widetilde{Y}). \tag{18.48}$$

Deduce that setting

$$\Omega^b(X, Y) = \Omega(\widetilde{X}, \widetilde{Y}) \tag{18.49}$$

defines Ω^b as a section of $\Lambda^2T^* \otimes (Ad\ P)$, where $Ad\ P$ is the vector bundle

$$Ad\ P = P \times_{Ad} \mathfrak{g}. \tag{18.50}$$

Hint. Apply (18.10) and (18.23). Use also that $g_*\widetilde{X} = \widetilde{X}$, with a similar result for $[\widetilde{X}, \widetilde{Y}]$, and for its vertical component.

6. If ξ_0 and ξ_1 are connection 1-forms on $P \to M$, show that $t\xi_1 + (1-t)\xi_0$ is also, for any $t \in \mathbb{R}$. Generalize to $\sum t_i\xi_i$, with $\sum t_i = 1$; allow t_i to depend on $x \in M$. Using a partition of unity argument, prove that every principal G-bundle $P \to M$ has a connection.
Hint. If P_0 and P_1 are projections, show that $tP_1 + (1-t)P_0$ is also, *provided* that P_0 and P_1 have the *same range.*

7. Let ξ_0 and ξ_1 be two connection 1-forms for $P \to M$, and let ∇ be an arbitrary third connection on P. Consider

$$\alpha = \xi_1 - \xi_0. \tag{18.51}$$

If X is a vector field on M and $\widetilde{X}$ the horizontal lift determined by ∇, show that

$$\alpha^b(X) = \alpha(\widetilde{X}) \tag{18.52}$$

defines α^b as an element of $C^\infty(M, \Lambda^1T^* \otimes Ad\ P)$. Show that α^b is *independent* of the choice of ∇. Compare (13.37).

8. In the setting of Exercise 7, if Ω_j are the curvatures of the connection 1-forms ξ_j, show that

$$\Omega_1 - \Omega_0 = d\alpha + [\alpha, \xi_0] + \frac{1}{2}[\alpha, \alpha]. \tag{18.53}$$

Compare (14.17) and (15.62). If $d^\nabla \alpha^b$ is the $(Ad\ P)$-valued 2-form defined as in §14, via the connection ξ_0, relate $d^\nabla \alpha^b$ to $d\alpha + [\alpha, \xi_0]$.

9. Let $E \to M$ be a Hermitian vector bundle, with fiber dimension k and with metric connection ∇. Let $F(E) \to M$ be the bundle of ordered orthonormal frames for E. Show that this is a principal $U(k)$-bundle, yielding a bundle isomorphic to E under the "standard" representation of $U(k)$ on $\mathbb{C}^k$. Show that $F(E) \to M$ has a connection, consistent with the connection ∇ on E mentioned above.
Hint. Extend the arguments bearing on the frame bundle $F(M)$, around (18.3).
Extend this to other situations, e.g., considering general complex vector bundles with connection and replacing $U(k)$ by $Gl(k, \mathbb{C})$.

10. Let $P \to M$ be a principal G-bundle, and let $f : N \to M$ be a smooth map. Define a principal G-bundle $f^*P \to N$, with fiber over $x \in N$ equal to the fiber of P over $f(x) \in M$:

$$f^*P = \{(x, y) \in N \times P : y \in P_{f(x)}\}.$$

Show that a connection on P induces a natural connection on f^*P.
Hint. You get a map $\Phi : f^*P \to P$, taking $(f^*P)_x$ to $P_{f(x)}$. Use Φ to pull back the connection 1-form on P.

11. If X is a vector field on M and $\widetilde{X}$ its horizontal lift to P (a principal G-bundle with connection), show that the flow generated by $\widetilde{X}$ *commutes* with the G-action on P.
Hint. Use $g_*\widetilde{X} = \widetilde{X}$.

19. The Chern-Weil construction

Let $P \to M$ be a principal G-bundle, endowed with a connection, as in §18. Let Ω be its curvature form, a $\mathfrak{g}$-valued 2-form on P; equivalently there is the $Ad\ P$-valued 2-form Ω^b on M. The Chern-Weil construction gives closed differential forms on M, whose cohomology classes are independent of the choice of connection on P. These "characteristic classes" are described as follows.

A function $f : \mathfrak{g} \to \mathbb{C}$ is called *invariant* if

$$f(Ad(g)X) = f(X), \quad X \in \mathfrak{g},\ g \in G. \tag{19.1}$$

Denote by $\mathcal{I}_k$ the set of polynomials $p : \mathfrak{g} \to \mathbb{C}$ which are invariant and homogeneous of degree k. If $p \in \mathcal{I}_k$, there is associated a symmetric Ad-invariant k-linear function P on $\mathfrak{g}$, called the polarization of p, given by

$$P(Y_1, \dots, Y_k) = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k} p(t_1 Y_1 + \cdots + t_k Y_k), \tag{19.2}$$

such that $p(X) = P(X, \dots, X)$. Into the entries of P we can plug copies of Ω, or of Ω^b, to get $2k$-forms

$$p(\Omega) = P(\Omega, \dots, \Omega) \in \Lambda^{2k} P, \tag{19.3}$$

and, with Ω^b given by (18.49),

$$p(\Omega^b) = P(\Omega^b, \dots, \Omega^b) \in \Lambda^{2k} M. \tag{19.4}$$

Note that, if $\pi : P \to M$ is the projection, then

$$p(\Omega) = \pi^* p(\Omega^b); \tag{19.5}$$

we say $p(\Omega)$, a form on P, is *basic*, i.e., the pull-back of a form on M. The following two propositions summarize the major basic results about these forms.

Proposition 19.1**.** *For any connection ∇ on $P \to M$, $p \in \mathcal{I}_k$, the forms $p(\Omega)$ and $p(\Omega^b)$ are closed. Hence $p(\Omega^b)$ represents a deRham cohomology class*

$$[p(\Omega^b)] \in \mathcal{H}^{2k}(M, \mathbb{C}). \tag{19.6}$$

If $q \in \mathcal{I}_j$, then $pq \in \mathcal{I}_{j+k}$, and $(pq)(\Omega) = p(\Omega) \wedge q(\Omega)$. Furthermore, if $f : N \to M$ is smooth and ∇_f the connection on f^*P pulled back from ∇ on P, which has curvature $\Omega_f = f^*\Omega$, then

$$p(\Omega_f^b) = f^* p(\Omega^b). \tag{19.7}$$

Proposition 19.2. *The cohomology class (19.6) is independent of the connection on P, so it depends only on the bundle.*

The map $\mathcal{I}_* \to \mathcal{H}^{2*}(M, \mathbb{C})$ is called the Chern-Weil homomorphism. We first prove that $d\,p(\Omega) = 0$ on P, the rest of Proposition 19.1 being fairly straightforward. If we differentiate with respect to t at $t = 0$ the identity

$$P\big(Ad(\text{Exp } tY)X, \dots, Ad(\text{Exp } tY)X\big) = p(X), \tag{19.8}$$

we get

$$\sum P(X, \dots, [Y, X], \dots, X) = 0. \tag{19.9}$$

Into this we can substitute the curvature form Ω for X and the connection form ξ for Y, to get

$$\sum P(\Omega, \dots, [\xi, \Omega], \dots, \Omega) = 0. \tag{19.10}$$

Now the Bianchi identity $d\Omega = -[\xi, \Omega]$ obtained in (18.39) shows that (19.10) is equivalent to $d\,p(\Omega) = 0$ on P. Since $\pi^* : \Lambda^j M \to \Lambda^j P$ is injective and (19.5) holds, we also have $d\,p(\Omega^b) = 0$ on M, and Proposition 19.1 is proved.

The proof of Proposition 19.2 is conveniently established via the following result, which also has further uses.

Lemma 19.3. *Let ξ_0 and ξ_1 be any $\mathfrak{g}$-valued 1-forms on P (or any manifold). Set $\alpha = \xi_1 - \xi_0$, $\xi_t = \xi_0 + t\alpha$, and $\Omega_t = d\xi_t + (1/2)[\xi_t, \xi_t]$. Given $p \in \mathcal{I}_k$, we have*

$$p(\Omega_1) - p(\Omega_0) = k\ d\Big[\int_0^1 P(\alpha, \Omega_t, \dots, \Omega_t)\, dt\Big]. \tag{19.11}$$

Proof. Since $(d/dt)\Omega_t = d\alpha + [\xi_t, \alpha]$, we have

$$\frac{d}{dt} p(\Omega_t) = k\ P(d\alpha + [\xi_t, \alpha], \Omega_t, \dots, \Omega_t). \tag{19.12}$$

It suffices to prove that the right side of (19.12) is equal to $k\ dP(\alpha, \Omega_t, \dots, \Omega_t)$. This follows by the "Bianchi" identity $d\Omega_t = -[\xi_t, \Omega_t]$ and the same sort of arguments used in the proof of Proposition 19.1. Instead of (19.8), one starts with

$$P\big(Ad(\text{Exp } tY)Z, Ad(\text{Exp } tY)X, \dots, Ad(\text{Exp } tY)X\big) = P(Z, X, \dots, X).$$

To apply Lemma 19.3 to Proposition 19.2, let ξ_0 and ξ_1 be the connection forms associated to two connections on $P \to M$, so Ω_0 and Ω_1 are their curvature forms. Note that each ξ_t defines a connection form on P, with curvature form Ω_t. Furthermore, $\alpha = \xi_1 - \xi_0$, acting on $X^\# \in T_yP$, depends only on $\pi_* X^\# \in T_xM$ and gives rise to an $Ad\ P$-valued 1-form α^b on M (cf. Exercise 7 of §18). Thus the right side of (19.11) is the pull back via π^* of the $(2k-1)$-form

$$k\ d\Big[\int_0^1 P(\alpha^b, \Omega_t^b, \dots, \Omega_t^b)\, dt\Big] \tag{19.13}$$

on M, which yields Proposition 19.2.

We can also apply Lemma 19.3 to $\xi_1 = \xi$, a connection 1-form, and $\xi_0 = 0$. Then $\xi_t = t\xi$; denote $d\xi_t + (1/2)[\xi_t, \xi_t]$ by Φ_t. We have the $(2k-1)$-form on P called the *transgressed form*:

$$Tp(\Omega) = k\ \int_0^1 P(\xi, \Phi_t, \dots, \Phi_t)\, dt, \tag{19.14}$$

with

$$\Phi_t = t\ d\xi + \frac{1}{2}t^2[\xi, \xi]. \tag{19.15}$$

Then Lemma 19.3 gives

$$P(\Omega) = d\ Tp(\Omega), \tag{19.16}$$

i.e., $p(\Omega)$ is an *exact* form on P, not merely a closed form. On the other hand, as opposed to $p(\Omega)$ itself, $Tp(\Omega)$ is not necessarily a basic form, i.e., the pull-back of a form on M. In fact, $p(\Omega^b)$ is not necessarily an exact form on M; typically it determines a nontrivial cohomology class on M. Transgressed forms play an important role in Chern-Weil theory.

The Levi-Civita connection on an oriented Riemannian manifold of dimension 2 can be equated with a connection on the associated principal S^1-bundle. If we identify S^1 with the unit circle in $\mathbb{C}$, its Lie algebra is naturally identified with $i\mathbb{R}$, and this identification provides an element of $\mathcal{I}_1$, unique up to a constant multiple. This is of course a constant times the product of the Gauss curvature and the volume form, as shown in (17.13); see also (17.54). The invariance of Proposition 19.2 recovers the independence (17.26) of the integrated curvature from the metric used on a Riemannian manifold of dimension 2. More generally, for any complex line bundle L over M, a manifold of any dimension, L can be associated to a principal S^1-bundle, and the Chern-Weil construction produces the class $[\Omega^b] \in \mathcal{H}^2(M, \mathbb{C})$. The class $c_1(L) = -(1/2\pi i)[\Omega^b] \in \mathcal{H}^2(M, \mathbb{C})$ is called the first Chern class of the line bundle L. In this case, the connection 1-form on P can be identified with an ordinary (complex-valued) 1-form, and it is precisely the transgressed form (19.14).

Note that, if dim $M = 2$, then (17.39) says

$$c_1(L)[M] = \text{ Index } X,$$

for any nonvanishing section X of L over $M \setminus \{p_1, \dots, p_K\}$.

For general G, there may be no nontrivial elements of $\mathcal{I}_1$. In fact, if $p : \mathfrak{g} \to \mathbb{R}$ is a nonzero linear form, $V = \ker\, p$ is a linear subspace of $\mathfrak{g}$ of codimension 1, which is $Ad\; G$ invariant if $p \in \mathcal{I}_1$. This means V is an *ideal*: $[V, \mathfrak{g}] \subset V$. Thus there are no nontrivial elements of $\mathcal{I}_1$ unless $\mathfrak{g}$ has an ideal of codimension 1. In particular, if $\mathfrak{g}$ is semisimple, $\mathcal{I}_1 = 0$.

When G is compact, there are always nontrivial elements of $\mathcal{I}_2$, i.e., Ad invariant quadratic forms on $\mathfrak{g}$. In fact, any bi-invariant metric tensor on G gives a positive definite element of $\mathcal{I}_2$. Applying the Chern-Weil construction in this case then gives cohomology classes in $\mathcal{H}^4(M, \mathbb{C})$.

One way of obtaining elements of $\mathcal{I}_k$ is the following. Let π be a representation of G on a vector space V_π, and set

$$p_{\pi k}(X) = \text{ Tr } \Lambda^k d\pi(X), \quad X \in \mathfrak{g}, \tag{19.17}$$

where $d\pi(X)$ denotes the representation of $\mathfrak{g}$ on V_π. In connection with this, note that

$$\det\big(\lambda I + d\pi(X)\big) = \sum_{j=0}^{M} \lambda^{M-j} \text{ Tr } \Lambda^j d\pi(X), \quad M = \text{ dim } V_\pi. \tag{19.18}$$

If $P \to M$ is a principal $U(n)$-bundle, or a principal $Gl(n, \mathbb{C})$-bundle, π the standard representation on $\mathbb{C}^n$, then consider

$$\det\Big(\lambda - \frac{\Omega}{2\pi i}\Big) = \sum_{k=0}^{n} c_k(\Omega)\lambda^{n-k}. \tag{19.19}$$

The classes $[c_k(\Omega^b)] \in \mathcal{H}^{2k}(M, \mathbb{C})$ are the Chern classes of P. If $E \to M$ is the associated vector bundle, arising via the standard representation π, we also call this the kth Chern class of E :

$$c_k(E) = [c_k(\Omega^b)] \in \mathcal{H}^{2k}(M, \mathbb{C}). \tag{19.20}$$

The object

$$c(E) = \sum c_k(E) \in \bigoplus_{k=0}^{n} \mathcal{H}^{2k}(M, \mathbb{C}) \tag{19.21}$$

is called the total Chern class of such a vector bundle.

If $P \to M$ is a principal $O(n)$-bundle, π the standard representation on $\mathbb{R}^n$, then consider

$$\det\Big(\lambda - \frac{\Omega}{2\pi}\Big) = \sum_{k=0}^{n} d_k(\Omega)\lambda^{n-k}. \tag{19.22}$$

The polynomials $d_k(\Omega)$ vanish for k odd, since $\Omega^t = -\Omega$, and one obtains Pontrjagin classes:

$$p_k(\Omega^b) = d_{2k}(\Omega^b) \in \mathcal{H}^{4k}(M, \mathbb{R}). \tag{19.23}$$

If $F \to M$ is the associated vector bundle, arising from the standard representation π, then $p_k(F)$ is defined to be (19.23).

Exercises

1. Show that $P(Y_1, \dots, Y_k)$, defined by (19.2), is linear in each Y_j. Verify the identity $p(X) = P(X, \dots, X)$.
Hint. Show that, for $a \in \mathbb{R}$, $P(Y_1, \dots, aY_j, \dots, Y_k) = aP(Y_1, \dots, Y_j, \dots, Y_k)$. As for the last part, use $p(t_1 X + \cdots + t_k X) = (t_1 + \cdots + t_k)^k p(X)$.

2. If $X_1, \dots, X_{2k}$ are vector fields on P, show that, for $p \in \mathcal{I}_k$ with associated k-linear form given by (19.2),

$$p(\Omega)(X_1, \dots, X_{2k}) = \frac{1}{2^k} \sum_{\sigma \in S_{2k}} (\operatorname{sgn} \sigma) P\big(\Omega(X_{\sigma(1)}, X_{\sigma(2)}), \dots, \Omega(X_{\sigma(2k-1)}, X_{\sigma(2k)})\big).$$

Hint. If $P(Y_1, \dots, Y_k) = \sum c_{a_1 \cdots a_k} Y_1^{a_1} \cdots Y_k^{a_k}$, where superscripts represent components with respect to some basis chosen for $\mathfrak{g}$, then substitution of Ω for each Y_ν gives

$$\sum c_{a_1 \cdots a_k} \Omega^{a_1} \wedge \cdots \wedge \Omega^{a_k}.$$

To apply this $2k$-form to $(X_1, \dots, X_{2k})$, use a variant of the formula below (6.3).

3. If $X_1, \dots, X_{2k}$ are vector fields on M, with horizontal lifts $\widetilde{X}_1, \dots, \widetilde{X}_{2k}$, and $p \in \mathcal{I}_k$, show that

$$p(\Omega^b)(X_1, \dots, X_{2k}) = \frac{1}{2^k} \sum_{\sigma \in S_{2k}} (\operatorname{sgn} \sigma) P\big(\Omega(\widetilde{X}_{\sigma(1)}, \widetilde{X}_{\sigma(2)}), \dots \Omega(\widetilde{X}_{\sigma(2k-1)}, \widetilde{X}_{\sigma(2k)})\big),$$

regarded as a function on P, is constant on fibers, and hence defines a function on M, so that (19.4)–(19.5) holds.
Hint. Use (18.48).

4. Show that, if X, Y, and Z are vector fields on P, then

$$[\xi, \Omega](X, Y, Z) = [\xi(X), \Omega(Y, Z)] + [\xi(Y), \Omega(Z, X)] + [\xi(Z), \Omega(X, Y)].$$

Verify the identity (19.10).

5. Flesh out the proof of Lemma 19.3. Show that, for $p \in \mathcal{I}_k,\ P$ as in (19.2),

$$P([Y, Z], X, \dots, X) + P(Z, [Y, X], \dots, X) + \cdots + P(Z, X, \dots, [Y, X]) = 0,$$

for $X, Y, Z \in \mathfrak{g}$. Then show that

$$P([\xi_t, \alpha], \Omega_t, \dots, \Omega_t) - P(\alpha, [\xi_t, \Omega_t], \dots, \Omega_t) - \cdots - P(\alpha, \Omega_t, \dots, [\xi_t, \Omega_t]) = 0.$$

Note the *minus* signs. Use this to show that

$$d\,P(\alpha, \Omega_t, \dots, \Omega_t) = P(d\alpha, \Omega_t, \dots, \Omega_t) + P([\xi_t, \alpha], \Omega_t, \dots, \Omega_t),$$

as needed to prove (19.11).
Hint. For example, write

$$\begin{aligned} &P(\alpha, [\xi_t, \Omega_t], \dots, \Omega_t)(X_1, \dots, X_{2k}) \\ &= \frac{1}{3!}\frac{1}{2^{k-2}} \sum_{\sigma \in S_{2k}} (\operatorname{sgn}\sigma) P\big(\alpha(X_{\sigma(1)}), [\xi_t, \Omega_t](X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}), \dots, \Omega_t(X_{\sigma(2k-1)}, X_{\sigma(2k)})\big). \end{aligned}$$

Apply Exercise 4 and show that this equals

$$\frac{1}{2^{k-1}} \sum_{\sigma \in S_{2k}} (\operatorname{sgn}\sigma) P\big(\alpha(X_{\sigma(1)}), [\xi_t(X_{\sigma(2)}), \Omega_t(X_{\sigma(3)}, X_{\sigma(4)})], \dots, \Omega_t(X_{\sigma(2k-1)}, X_{\sigma(2k)})\big).$$

6. Let $E \to M$ be a complex vector bundle over a compact manifold M, with fiber dimension k. It can always be endowed with a Hermitian inner product and a metric connection. Consider the principal $U(k)$-bundle and the principal $Gl(k, \mathbb{C})$-bundle associated with E, as in Exercise 9 of §18. Show that the construction (19.19)–(19.21) applied to these two principal bundles yields the same Chern classes $c_\nu(E) \in \mathcal{H}^{2\nu}(M, \mathbb{C})$.

7. If E and F are complex vector bundles over M, we can form $E \oplus F \to M$. Show that

$$c(E \oplus F) = c(E) \wedge c(F), \tag{19.24}$$

where $c(E)$ is the total Chern class given by (19.21), i.e.,

$$c(E) = \det\Big(I - \frac{\Omega^b}{2\pi i}\Big) \in \mathcal{H}^{\text{even}}(M, \mathbb{C}), \tag{19.25}$$

for a curvature 2-form arising from a connection on E.

8. Define the Chern character of a complex vector bundle $E \to M$ as the cohomology class $ch(E) \in \mathcal{H}^{\text{even}}(M, \mathbb{C})$ of

$$ch(\Omega^b) = \operatorname{Tr} e^{-\Omega^b/2\pi i}, \tag{19.26}$$

writing $\operatorname{Tr} e^{-\Omega^b/2\pi i} \in \bigoplus_{k\ge 0} \Lambda^{2k} M$ via the power series expansion of the exponential function. Show that

$$\begin{aligned} ch(E \oplus F) &= ch(E) + ch(F), \\ ch(E \otimes F) &= ch(E) \wedge ch(F) \end{aligned} \tag{19.27}$$

in $\mathcal{H}^{\text{even}}(M, \mathbb{C})$.

9. If $F \to M$ is a real vector bundle and $E = F \otimes \mathbb{C}$ its complexification, show that

$$p_j(F) = (-1)^j c_{2j}(E). \tag{19.28}$$

10. Using $so(4) \approx so(3) \oplus so(3)$, construct two different characteristic classes, in $\mathcal{H}^4(M, \mathbb{C})$, when M is a compact, oriented 4-dimensional manifold.

11. Let $E \to M$ be a complex vector bundle over a compact manifold M, of fiber dimension k. Show that there exists a trivial bundle $F \approx \mathbb{C}^N \times M \to M$ such that E is isomorphic to a subbundle of F.
Hint. Cover M with open sets $\mathcal{O}_j,\ j = 1, \dots, M$, over which there are local frame fields for E, and use a partition of unity $\{\varphi_j\}$ subordinate to $\{\mathcal{O}_j\}$ to define a bundle map $E \to \mathbb{C}^N \times M$, with $N = kM$.

12. Let $G_{k,N}(\mathbb{C})$ be the Grassmannian of k-dimensional complex linear subspaces of $\mathbb{C}^N$. Let $\tau \to G_{k,N}(\mathbb{C})$ be the "tautological bundle," i.e., if $V \in G_{k,N}(\mathbb{C})$, the fiber over V is $\tau_V = V$. Show that τ has a natural metric connection.
Hint. Recall (13.5).
For more material on these Grassmannians, see Appendix V.

13. When $E \to M$ is a subbundle (of fiber dimension k) of the trivial bundle $\mathbb{C}^N \times M$, you get a natural smooth map $\psi : M \to G_{k,N}(\mathbb{C})$, namely $\psi(x) = E_x \subset \mathbb{C}^N$. Show that, for $\nu \in \mathbb{Z}^+$,

$$c_\nu(E) = \psi^* c_\nu(\tau). \tag{19.29}$$

Hint. Recall (19.7).

20. The Chern-Gauss-Bonnet theorem

Our goal in this section is to generalize the Gauss-Bonnet formula (17.1), producing a characteristic class derived from the curvature tensor Ω of a Riemannian metric on a compact oriented manifold M, say $e(\Omega) \in \Lambda^n(M)$, such that

$$\int_M e(\Omega) = \chi(M), \tag{20.1}$$

the right side being the Euler characteristic of M.

A clue to obtaining $e(\Omega)$ comes from the higher dimensional generalization of the index formula (17.47), i.e.,

$$\text{Index}\,(X) = \chi(M), \tag{20.2}$$

valid for any vector field X on M with isolated critical points. The relation between these two formulas when dim $M = 2$ was noted near the end of §17. It arises from the relation between Index(X) and the degree of the Gauss map.

Indeed, let M be a compact n dimensional submanifold of $\mathbb{R}^{n+k}$, X a (tangent) vector field on M with a finite number of critical points, and $\overline{\mathcal{T}}$ a small tubular neighborhood of M. By Corollary 10.5, we know that, if $N : \partial\mathcal{T} \to S^{n+k-1}$ denotes the Gauss map on $\partial\mathcal{T}$, formed by the outward-pointing normals, then

$$\text{Index}\,(X) = \text{Deg}\,(N). \tag{20.3}$$

As noted at the end of §17, if M is a surface in $\mathbb{R}^3$, with Gauss map N_M, then $\text{Deg}(N_M) = (1/4\pi)\int_M K\,dV$, where K is the Gauss curvature of M, with its induced metric. If $\mathcal{T}$ is a small tubular neighborhood of M in this case, then $\partial\mathcal{T}$ is diffeomorphic to 2 oppositely oriented copies of M, with approximately the same metric tensor. The outer component of $\partial\mathcal{T}$ has Gauss map approximately equal to N_M, and the inner component has Gauss map approximately equal to $-N_M$. From this we see that (20.2)–(20.3) imply (20.1) with $e(\Omega) = (1/2\pi)K\,dV$, in this case.

We make a further comment on the relation between (20.2) and (20.3). Note that the right side of (20.3) is independent of the choice of X. Thus (as noted already in §10) any two vector fields on M, with only isolated critical points, have the same index. Suppose M has a triangulation τ into n-simplices. There is a construction of a vector field X_τ, illustrated in Fig. 10.1 for $n = 2$, with the property that X_τ has a critical point at each vertex, of index $+1$, and a critical point in the middle of each j-simplex in τ, of index $(-1)^j$, so that

$$\text{Index}\,(X_\tau) = \sum_{j=0}^{n} (-1)^j \nu_j(M), \tag{20.4}$$

where $\nu_j(M)$ is the number of j-simplices in the triangulation τ of M. We leave the construction of X_τ in higher dimensions as an exercise.

Now, in view of the invariance of Index(X), it follows that the right side of (20.4) is independent of the triangulation of X. Also, if X has a more general cell decomposition, we can form the sum on the right side of (20.4), where ν_j stands for the number of j-dimensional cells in X. Each cell can be divided into simplices in such a way that a triangulation is obtained, and the sum on the right side of (20.4) is unchanged under such a refinement. This alternating sum is one definition of the Euler characteristic, but there is another definition, namely

$$\chi(M) = \sum_{j=0}^{n} (-1)^j \dim \mathcal{H}^j(M). \tag{20.5}$$

We will temporarily denote the right side of (20.4) by $\chi_c(M)$.

Now we tackle the question of representing (20.3) as an integrated curvature, to produce (20.1). We begin with the case when M is a compact hypersurface in $\mathbb{R}^{n+1}$. In that case we have, by (16.66),

$$\mathrm{Deg}\,(N) = \frac{2}{A_n} \int\limits_M \big(\det A_N\big)\, dV, \ \text{ for } n \text{ even}, \tag{20.6}$$

where A_n is the area of S^n and $A_N : T_pM \to T_pM$ is the Weingarten map. The factor 2 arises because $\partial\mathcal{T}$ consists of two copies of M. We can express $\det A_N$ directly in terms of the Riemann curvature tensor $R_{jk\ell m}$ of M, using Gauss' Theorema Egregium.

In fact, with respect to an oriented orthonormal basis $\{e_j\}$ of T_pM, the matrix of A_N has entries $A_{jk} = \widetilde{II}(e_j, e_k)$, and, by (16.14),

$$R_{jk\ell m} = \langle R(e_\ell, e_m)e_k, e_j\rangle = \det \begin{pmatrix} A_{mk} & A_{mj} \\ A_{\ell k} & A_{\ell j} \end{pmatrix} \tag{20.7}$$

In other words, the curvature tensor captures the action of $\Lambda^2 A_N$ on $\Lambda^2 T_pM$. If $n = 2k$ is even, we can then express $\det A_N$ as a polynomial in the components $R_{jk\ell m}$, using

$$\begin{aligned} (\det A_N)\, e_1 \wedge \cdots \wedge e_n &= \big(\Lambda^n A_N\big)(e_1 \wedge \cdots \wedge e_n) \\ &= (Ae_1 \wedge Ae_2) \wedge \cdots \wedge (Ae_{n-1} \wedge Ae_n). \end{aligned} \tag{20.8}$$

Now, by (20.7),

$$Ae_j \wedge Ae_k = \frac{1}{2} \sum R_{\ell m jk}\, e_\ell \wedge e_m. \tag{20.9}$$

Replacing $(1,\dots,n)$ in (20.8) with all its permutations and summing, we obtain

$$\det A_N = \frac{1}{2^{n/2}n!}\sum_{j,k}(\operatorname{sgn} j)(\operatorname{sgn} k)R_{j_1j_2k_1k_2}\cdots R_{j_{n-1}j_nk_{n-1}k_n}, \tag{20.10}$$

where $j=(j_1,\dots,j_n)$ stands for a permutation of $(1,\dots,n)$. The fact that the quantity (20.10), integrated over M, is equal to $(A_n/2)\chi(M)$, when M is a hypersurface in $\mathbb{R}^{n+1}$, was first established by H. Hopf, as a consequence of his result (20.2). The content of the generalized Gauss-Bonnet formula is that for any compact Riemannian manifold M, of dimension $n=2k$, integrating the right side of (20.10) over M gives $(A_n/2)\chi(M)$.

One key point in establishing the general case is to perceive the right side of (20.10) as arising via the Chern-Weil construction from an invariant polynomial on the Lie algebra $\mathfrak{g}=so(n)$, to produce a characteristic class. Now the curvature 2-form can in this case be considered a section of $\Lambda^2T^*\otimes\Lambda^2T^*$, reflecting the natural linear isomorphism $\mathfrak{g}\approx\Lambda^2T^*$. Furthermore, $\Lambda^*T^*\otimes\Lambda^*T^*$ has a product, satisfying

$$(\alpha_1\otimes\beta_1)\wedge(\alpha_2\otimes\beta_2)=(\alpha_1\wedge\alpha_2)\otimes(\beta_1\wedge\beta_2). \tag{20.11}$$

If we set

$$\Omega=\frac{1}{4}\sum R_{jk\ell m}(e_j\wedge e_k)\otimes(e_\ell\wedge e_m), \tag{20.12}$$

then we form the k-fold product, $k=n/2$, obtaining

$$\Omega\wedge\cdots\wedge\Omega=2^{-n}\sum_{j,k}(\operatorname{sgn} j)(\operatorname{sgn} k)R_{j_1j_2k_1k_2}\cdots R_{j_{n-1}j_nk_{n-1}k_n}(\omega\otimes\omega), \tag{20.13}$$

with $\omega=e_1\wedge\cdots\wedge e_n$. Thus, the right side of (20.10), multiplied by $\omega\otimes\omega$, is equal to $2^{n/2}/n!$ times the right side of (20.13). (Observe the distinction between the product (20.11) and the product on $\operatorname{End}(E)\otimes\Lambda^*T$, used in (19.19) and (19.22), which assigns a different meaning to $\Omega\wedge\cdots\wedge\Omega$.)

Now the Chern-Weil construction produces (20.13), with $\omega\otimes\omega$ replaced by ω, if we use the *Pfaffian*

$$\operatorname{Pf}: so(n)\longrightarrow\mathbb{R},\quad n=2k, \tag{20.14}$$

defined as follows. Let $\xi: so(n)\to\Lambda^2\mathbb{R}^n$ be the isomorphism

$$\xi(X)=\frac{1}{2}\sum X_{jk}\,e_j\wedge e_k,\quad X=(X_{jk})\in so(n). \tag{20.15}$$

Then, if $n=2k$, take a product of k factors of $\xi(X)$, to obtain a multiple of $\omega=e_1\wedge\cdots\wedge e_n$. Then $\operatorname{Pf}(X)$ is uniquely defined by

$$\xi(X)\wedge\cdots\wedge\xi(X)=k!\,\operatorname{Pf}(X)\omega. \tag{20.16}$$

Note that, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then $T^*\xi(X) = \xi(T^t XT)$, so

$$\mathrm{Pf}\,(T^t XT) = (\det T)\mathrm{Pf}\,(X). \tag{20.17}$$

Now any $X \in so(n)$ can be written as $X = T^t YT$, where $T \in SO(n)$, i.e., T is an orthogonal matrix of determinant 1, and Y is a sum of 2×2 skew-symmetric blocks, of the form

$$Y_\nu = \begin{pmatrix} 0 & \lambda_\nu \\ -\lambda_\nu & 0 \end{pmatrix}, \quad \lambda_\nu \in \mathbb{R}. \tag{20.18}$$

Thus $\xi(Y) = \lambda_1 e_1 \wedge e_2 + \cdots + \lambda_k e_{n-1} \wedge e_n$, so

$$\mathrm{Pf}\,(Y) = \lambda_1 \cdots \lambda_k. \tag{20.19}$$

Note that $\det Y = (\lambda_1 \cdots \lambda_k)^2$. Hence, by (20.17), we have

$$\mathrm{Pf}\,(X)^2 = \det X, \tag{20.20}$$

when X is a real skew-symmetric $n \times n$ matrix, $n = 2k$. When (20.17) is specialized to $T \in SO(n)$, it implies that Pf is an invariant polynomial, homogeneous of degree k, i.e., $\mathrm{Pf} \in \mathcal{I}_k,\ k = n/2$.

Now, with Ω in (20.12) regarded as a $\mathfrak{g}$-valued 2-form, we have the left side of (20.13) equal to $(1/k!)\mathrm{Pf}(\Omega)$. Thus we are on the way toward establishing the generalized Gauss-Bonnet theorem, in the following formulation.

Theorem 20.1. *If M is a compact oriented Riemannian manifold of dimension $n = 2k$, then*

$$\chi(M) = (2\pi)^{-k} \int_M Pf(\Omega). \tag{20.21}$$

The factor $(2\pi)^{-k}$ arises as follows. From (20.10) and (20.13), it follows that, when M is a compact hypersurface in $\mathbb{R}^{n+1}$, the right side of (20.6) is equal to $C_k \int_M \mathrm{Pf}\,(\Omega)$, with

$$C_k = \frac{2^{k+1}}{A_n}\ \frac{k!}{n!}. \tag{20.22}$$

Now the area of the unit sphere is given by

$$A_{2k} = \frac{2\pi^{k+\frac{1}{2}}}{\Gamma(k+\frac{1}{2})} = \frac{2\pi^k}{\big(k-\frac{1}{2}\big)\cdots\big(\frac{1}{2}\big)},$$

as is shown in (4.30), and substituting this into (20.22) gives $C_k = (2\pi)^{-k}$.

We give a proof of Theorem 20.1 which extends the proof of (17.24), in which handles are added to a surface. To effect this parallel, we consider how the two sides of (20.21) change when M is altered by a certain type of *surgery*, which we will define in the next paragraph. First, we mention another ingredient in the proof of Theorem 20.1. Namely, the right side of (20.21) is independent of the choice of metric on M. Since different metrics produce different $SO(2k)$ frame bundles, this assertion requires a further argument. We will postpone the proof of this invariance until near the end of this section.

We now describe the "surgeries" alluded to above. To perform surgery on M_0, of dimension n, excise a set H_0 diffeomorphic to $S^{\ell-1} \times B^m$, with $m+\ell-1=n$, where $B^m = \{x \in \mathbb{R}^m : |x| < 1\}$, obtaining a manifold with boundary X, ∂X being diffeomorphic to $S^{\ell-1} \times S^{m-1}$. Then attach to X a copy of $B^\ell \times S^{m-1}$, sewing them together along their boundaries, both diffeomorphic to $S^{\ell-1} \times S^{m-1}$, to obtain M_1. Symbolically, we write

$$M_0 = X \# H_0, \quad M_1 = X \# H_1. \tag{20.23}$$

We say M_1 is obtained from M_0 by a surgery of type (ℓ, m).

We compare the way each side of (20.21) changes when M changes from M_0 to M_1. We also look at how $\chi_c(M)$, defined to be the right side of (20.4), changes. In fact, this definition easily yields

$$\chi(X\#H_1) = \chi(X\#H_0) - \chi(H_0) + \chi(H_1). \tag{20.24}$$

For notational simplicity, we have dropped the "c" subscript. It is more convenient to produce an identity involving only manifolds without boundary, so note that

$$\begin{aligned} \chi(H_0\#H_0) &= 2\chi(H_0) - \chi(\partial H_0) \\ \chi(H_1\#H_1) &= 2\chi(H_1) - \chi(\partial H_1) \end{aligned} \tag{20.25}$$

and, since $\partial H_0 = \partial H_1$, we have

$$\chi(H_1) - \chi(H_0) = \frac{1}{2}\chi(H_1\#H_1) - \frac{1}{2}\chi(H_0\#H_0), \tag{20.26}$$

hence

$$\chi(M_1) = \chi(M_0) + \frac{1}{2}\chi(H_1\#H_1) - \frac{1}{2}\chi(H_0\#H_0). \tag{20.27}$$

Note that $H_0\#H_0 = S^{\ell-1} \times S^m$, $H_1\#H_1 = S^\ell \times S^{m-1}$. To compute the Euler characteristic of these two spaces, we can use multiplicativity of χ. Note that products of cells in Y_1 and Y_2 give cells in $Y_1 \times Y_2$, and

$$\nu_j(Y_1 \times Y_2) = \sum_{i+k=j} \nu_i(Y_1)\nu_k(Y_2); \tag{20.28}$$

then from (20.4) it follows that

$$\chi(Y_1 \times Y_2) = \sum_{j\ge 0}(-1)^j \sum_{i+k=j} \nu_i(Y_1)\nu_k(Y_2) = \chi(Y_1)\chi(Y_2). \tag{20.29}$$

We use the easily established result that χ_c satisfies

$$\chi(S^j) = 2 \text{ if } j \text{ is even}, \quad 0 \text{ if } j \text{ is odd}. \tag{20.30}$$

See (10.13). We have $\chi(H_0\#H_0) - \chi(H_1\#H_1)$ equal to 4 if ℓ is odd and m even, -4 if ℓ is even and m odd, and 0 if ℓ and m have the same parity (which does not arise if dim M is even).

The change in $\chi_c(M)$ just derived in fact coincides with the change in $\chi(M)$, defined by (20.5). This follows from results on deRham cohomology which will be obtained in §§21–24. In fact (24.8) implies (20.24), from which (20.25)–(20.27) follow, (22.45) implies (20.29), when Y_j are smooth compact manifolds, and (22.49)–(22.50) implies (20.30), when χ is defined by (20.5).

Thus, for $e(M) = \int_M e(\Omega)$, to change the same way as $\chi(M)$ under a surgery, we need the following properties, in addition to "functoriality." We need

$$\begin{aligned} e(S^j \times S^k) = 0 \ &\text{ if } \ j \text{ or } k \text{ is odd}, \\ 4 \ &\text{ if } \ j \text{ and } k \text{ are even}. \end{aligned} \tag{20.31}$$

If $e(\Omega)$ is locally defined we have, upon giving X, H_0, and H_1 coherent orientations,

$$\int_{M_1} e(\Omega) = \int_{M_0} e(\Omega) - \int_{H_0} e(\Omega) + \int_{H_1} e(\Omega), \tag{20.32}$$

parallel to (20.24). Place metrics on M_j which are product metrics on $(-\varepsilon,\varepsilon) \times S^{\ell-1} \times S^{m-1}$ on a small neighborhood of ∂X. If we place a metric on $H_j\#H_j$ which is symmetric with respect to the natural involution, we will have

$$\int_{H_j} e(\Omega) = \frac{1}{2} \int_{H_j\#H_j} e(\Omega) \tag{20.33}$$

provided $e(\Omega)$ has the following property. Given an oriented Riemannian manifold Y, let $Y^\#$ be the same manifold with orientation reversed, and let the associated curvature forms be denoted Ω_Y and $\Omega_{Y^\#}$. We require

$$e(\Omega_Y) = -e(\Omega_{Y^\#}). \tag{20.34}$$

Now $e(\Omega) = \mathrm{Pf}\,(\Omega/2\pi)$ certainly satisfies (20.34), in view of the dependence on orientation built into (20.16). To see that (20.31) holds in this case, we need only note that $S^\ell \times S^k$ can be smoothly imbedded as a hypersurface in $\mathbb{R}^{\ell+k+1}$. This can be done via imbedding $S^\ell \times I \times B^k$ into $\mathbb{R}^{\ell+k+1}$, and taking its boundary (and smoothing it out). In that case, since $\mathrm{Pf}\,(\Omega/2\pi)$ is a characteristic class whose integral is independent of the choice of metric, we can use the metric induced from the imbedding. We now have (20.31)–(20.33). Furthermore, for such a hypersurface $M = H_j \# H_j$, we know that the right side of (20.21) is equal to $\chi_c(H_j\#H_j)$, by the argument preceding the statement of Theorem 20.1, and since (20.29) and (20.30) are both valid for both χ and χ_c, we also have this quantity equal to $\chi(H_j\#H_j)$.

It follows that (20.21) holds for any M obtainable from S^n by a finite number of surgeries. With one extra wrinkle we can establish (20.21) for all compact oriented M. The idea for using this technique is one the author learned from J. Cheeger, who used a somewhat more sophisticated variant in work on analytic torsion [Ch].

Assume M is connected. Imbed M in $\mathbb{R}^K$ (for some K), fix $p \in M$, and define $f_0 : M \times \mathbb{R} \to \mathbb{R}$ by $f_0(x,t) = |x-p|^2 + t^2$, where $|x-p|^2$ is the square-norm of $x - p \in \mathbb{R}^K$. For R sufficiently large, $f_0^{-1}(R)$ is diffeomorphic to two copies of M, under $(x,t) \mapsto x$. For $r > 0$ sufficiently small, $f_0^{-1}(r)$ is diffeomorphic to the sphere S^n.

Our argument will use basic results of Morse Theory. A Morse function $f : Z \to \mathbb{R}$ is a smooth function on a manifold Z all of whose critical points are nondegenerate, i.e., if $\nabla f(z) = 0$ then $D^2 f(z)$ is an invertible $\nu \times \nu$ matrix, $\nu =$ dim Z. One also assumes f takes different values at distinct critical points, and that $f^{-1}(K)$ is compact for every compact $K \subset \mathbb{R}$. Now the function f_0 above may not be a Morse function on $Z = M \times \mathbb{R}$, but there will exist a smooth perturbation f of f_0 which is a Morse function. This can be proven using Sard's theorem; see Appendix O. The new f will share with f_0 the property that $f^{-1}(r)$ is diffeomorphic to S^n and $f^{-1}(R)$ is diffeomorphic to two copies of M. Note that an orientation on M induces an orientation on $M \times \mathbb{R}$, and hence an orientation on any level set $f^{-1}(c)$ which is regular, i.e., which contains no critical points. In particular, $f^{-1}(R)$ is a union of two copies of M with *opposite* orientations. One of the fundamental results of Morse Theory is the following.

Theorem 20.2. *If $c_1 < c_2$ are regular values of a Morse function $f : Z \to \mathbb{R}$, and there is exactly one critical point z_0, with $c_1 < f(z_0) < c_2$, then $M_2 = f^{-1}(c_2)$ is obtained from $M_1 = f^{-1}(c_1)$ by a surgery. In fact, if $D^2 f(z_0)$ has signature (ℓ, m), M_2 is obtained from M_1 by a surgery of type (m, ℓ).*

This is a consequence of the following result, known as the Morse Lemma.

Proposition 20.3. *Let f have a nondegenerate critical point at $p \in Z$. Then there is a coordinate system $(x_1, \dots, x_n)$ centered at p in which*

$$f(x) = f(p) + x_1^2 + \cdots + x_\ell^2 - x_{\ell+1}^2 - \cdots - x_{\ell+m}^2, \tag{20.35}$$

near the origin, where $\ell + m = \nu =$ dim Z.

Proof. Suppose that in some coordinate system $D^2 f(p)$ is given by a nondegenerate symmetric $\nu \times \nu$ matrix A. It will suffice to produce a coordinate system in which

$$f(x) = f(p) + \frac{1}{2} Ax \cdot x, \tag{20.36}$$

near the origin, since going from here to (20.35) is a simple exercise in linear algebra. We will arrange (20.36) by an argument due to R. Palais.

Begin with any coordinate system centered at p. Let

$$\omega_1 = df, \quad \omega_0 = dg, \text{ where } g(x) = \frac{1}{2} Ax \cdot x, \tag{20.37}$$

with $A = D^2 f(0)$ in this coordinate system. Set $\omega_t = t\omega_1 + (1-t)\omega_0$, which vanishes at p for each $t \in [0,1]$. The nondegeneracy hypothesis on A implies that the components of each ω_t have linearly independent gradients at p; hence there exists a smooth time-dependent vector field X_t (not unique), such that

$$\omega_t \rfloor X_t = g - f, \quad X_t(p) = 0. \tag{20.38}$$

Let $\mathcal{F}_t$ be the flow generated by X_t, with $\mathcal{F}_0 = Id$. Note that $\mathcal{F}_t$ fixes p. It is then an easy computation using (8.38), plus the identity $\mathcal{L}_X \omega = d(\omega \rfloor X) + (d\omega) \rfloor X$, that

$$\frac{d}{dt}\big(\mathcal{F}_t^* \omega_t\big) = 0. \tag{20.39}$$

Hence $\mathcal{F}_1^* \omega_1 = \omega_0$, so $f \circ \mathcal{F}_1 = g$ and the proof of Proposition 20.3 is complete.

From Theorem 20.2, it follows that, given any compact oriented connected M, of dimension n, a finite number of surgeries on S^n yields two copies of M, with opposite orientations, say M and $M^\#$. Hence (20.21) holds with M replaced by the disjoint union $M \cup M^\#$. But, in view of (20.34), both sides of the resulting identity are equal to twice the corresponding sides of (20.21); for χ_c this follows easily from (20.4), and for χ it follows immediately from (20.5). We hence have the Chern-Gauss-Bonnet formula, and also the identity $\chi(M) = \chi_c(M)$, modulo the task of showing the invariance of the right side of (20.21) under changes of metric on M.

We turn to the task of demonstrating such invariance. Say g_0 and g_1 are two Riemannian metric tensors on M, with associated $SO(n)$-bundles $P_0 \to M$, $P_1 \to M$, having curvature forms Ω_0 and Ω_1. We want to show that $\mathrm{Pf}\,(\Omega_1^\flat) - \mathrm{Pf}\,(\Omega_0^\flat)$ is exact on M. To do this, consider the family of metric tensors $g_t = tg_1 + (1-t)g_0$ on M, with associated $SO(n)$-bundles $P_t \to M$, for $t \in [0,1]$. These bundles fit together to produce a principal $SO(n)$-bundle $\widetilde{P} \to M \times [0,1]$. We know there exists a connection on this principal bundle. Let $T = \partial/\partial t$ on $M \times [0,1]$, and let $\widetilde{T}$ denote its horizontal lift (with respect to a connection chosen on $\widetilde{P}$). The flow generated by $\widetilde{T}$ commutes with the $SO(n)$-action on $\widetilde{P}$. Flowing along one unit of time then yields a diffeomorphism $\Phi : P_0 \to P_1$, commuting with the $SO(n)$-action, hence giving an isomorphism of $SO(n)$-bundles. Now applying Proposition 19.2 to the original connection on P_0 and to that pulled back from P_1 gives the desired invariance.

Before S. S. Chern's work, H. Hopf had established Theorem 20.1 when M is a compact hypersurface in $\mathbb{R}^{2k+1}$. Then C. Allendoerfer [Al] and W. Fenchel [Fen] proved it for the case when M is isometrically imbedded in $\mathbb{R}^{2k+n}$, by relating the integral on the right side of (20.21) to the integral over $\partial\mathcal{T}$ of the Gauss curvature of the boundary of a small tubular neighborhood $\mathcal{T}$ of M, and using the known result that $\chi(\partial\mathcal{T}) = 2\chi(M)$. At that time, it was not known that any compact Riemannian manifold could be isometrically imbedded in Euclidean space. By other means, Allendoerfer and A. Weil [AW] proved Theorem 20.1, at least for real analytic metrics, via a triangulation and local isometric imbedding. Chern then produced an intrinsic proof of Theorem 20.1 and initiated a new understanding of characteristic classes.

In Chern's original paper [Cher], it was established that $\int_M \mathrm{Pf}(\Omega/2\pi)$ is equal to the index of a vector field X on M, by a sophisticated variant of the argument establishing Proposition 17.4, involving a differential form on the unit sphere bundle of M, related to, but more complicated than, the transgressed form (19.14). An exposition of this argument can also be found in [Poo], and in [Wil]. When dim $M = 2$, one can identify the unit sphere bundle and the frame bundle, and in that case the form coincides with the transgressed form, and the argument becomes equivalent to that used to prove Proposition 17.4. An exposition of the proof of Theorem 20.1 using tubes can be found in [Gr].

We mention a further generalization of the Gauss-Bonnet formula. If $E \to X$ is an $SO(2k)$-bundle over a compact manifold X (say of dimension n), with metric connection ∇ and associated curvature Ω, then $\mathrm{Pf}\,(\Omega/2\pi)$ is defined as a $2k$-form on X. This gives a class $\mathrm{Pf}\,(E) \in \mathcal{H}^{2k}(X)$, independent of the choice of connection on E, as long as it is a metric connection. There is an extension of Theorem 20.1, describing the cohomology class of $\mathrm{Pf}\,(\Omega/2\pi)$ in $\mathcal{H}^{2k}(X)$. Treatments of this can be found in [KN] and in [Spi].

Exercises

1. Verify directly that, when Ω is the curvature 2-form arising from the standard metric on S^{2k}, then

$$\int_{S^{2k}} \mathrm{Pf}\,(\Omega/2\pi) = 2.$$

2. Generalize Theorem 20.1 to the nonorientable case.
Hint. If M is not orientable, look at its orientable double cover $\widetilde{M}$. Use (20.4) to show that $\chi(\widetilde{M}) = 2\chi(M)$.

Using (20.16) as a local identity, define a *measure* $\widetilde{\mathrm{Pf}}\,(\Omega)$ in the nonorientable case.

3. If M_j are compact Riemannian manifolds with curvature forms Ω_j and $M_1 \times M_2$ has the product metric, with curvature form Ω, show directly that

$$\pi_1^* \mathrm{Pf}\,(\Omega_1) \wedge \pi_2^* \mathrm{Pf}\,(\Omega_2) = \mathrm{Pf}\,(\Omega),$$

where π_j projects $M_1 \times M_2$ onto M_j. If dim M_j is odd, set $\mathrm{Pf}\,(\Omega_j) = 0$. Use this to reprove (20.31), when $e(\Omega) = \mathrm{Pf}\,(\Omega)$.

4. Show directly that the right sides of (20.2) and (20.3) both vanish when M is a hypersurface of *odd* dimension in $\mathbb{R}^{n+1}$.

5. Work out "more explicitly" the formula (20.21) when dim $M = 4$. Show that (cf. [Av])

$$\chi(M) = \frac{1}{32\pi^2} \int\limits_M \Bigl(|R|^2 - |\mathrm{Ric} - \frac{1}{4} S g|^2 \Bigr)\, dV,$$

where R is the Riemann curvature tensor, Ric the Ricci tensor, and S the scalar curvature.

6. We say a Riemannian manifold is Einstein if its Ricci tensor is a scalar multiple of its metric tensor. Show that, if M is a compact 4-dimensional Einstein manifold, then

$$\chi(M) = \frac{1}{32\pi^2} \int\limits_M |R|^2 \, dV.$$

Draw conclusions. See [Bes] for further material on Einstein manifolds.

7. Let $E \to M$ be a complex vector bundle, with a Hermitian inner product, and let $\widetilde{E} \to M$ denote the real vector bundle obtained by ignoring the complex structure on E (so $\dim_{\mathbb{C}} E_x = k$ and $\dim_{\mathbb{R}} \widetilde{E}_x = 2k$). Show that $\widetilde{E}$ has a natural structure as an $SO(2k)$-bundle, and that

$$\mathrm{Pf}\,(\widetilde{E}) = c_k(E),$$

where the right side is the kth Chern class of E.
Hint. If $\kappa : u(k) \to so(2k)$ is the natural inclusion, show that

$$\mathrm{Pf}\Bigl(\frac{\kappa(X)}{2\pi}\Bigr) = \det\Bigl(-\frac{X}{2\pi i}\Bigr), \quad X \in u(k).$$

8. Let M be a compact complex Hermitian manifold, of complex dimension n (i.e., real dimension $2n$). Denote by $\mathcal{T}$ its tangent bundle, regarded as a complex vector bundle, with fibers $\mathcal{T}_p$ of complex dimension n. Deduce that

$$\int_M c_n(\mathcal{T}) = \chi(M),$$

where $c_n(\mathcal{T})$ is the top Chern class, defined by (19.19)–(19.20).

9. If $E \to M$ is an $SO(2k)$-bundle, show that

$$\mathrm{Pf}\,(E) \wedge \mathrm{Pf}\,(E) = p_k(E),$$

where the right side is the kth Pontrjagin class of E.
Note. This has no content when $E = TM$.

21. The Hodge Laplacian on k-forms

If M is an n-dimensional Riemannian manifold, recall there is the exterior derivative

$$d : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M), \tag{21.1}$$

satisfying

$$d^2 = 0. \tag{21.2}$$

The Riemannian metric on M gives rise to an inner product on T_x^* for each $x \in M$, and then to an inner product on $\Lambda^k T_x^*$, via

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \sum_\pi (\text{sgn } \pi) \langle v_1, w_{\pi(1)} \rangle \cdots \langle v_k, w_{\pi(k)} \rangle, \tag{21.3}$$

where π ranges over the set of permutations of $\{1, \dots, k\}$. Equivalently, if $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_x^* M$, then $\{e_{j_1} \wedge \cdots \wedge e_{j_k} : j_1 < j_2 < \cdots < j_k\}$ is an orthonormal basis of $\Lambda^k T_x^* M$. Consequently, there is an inner product on k-forms, i.e., sections of Λ^k, given by

$$(u, v) = \int_M \langle u, v \rangle \, dV(x). \tag{21.4}$$

Thus there is a first order differential operator

$$\delta : \Lambda^{k+1}(M) \longrightarrow \Lambda^k(M) \tag{21.5}$$

which is the formal adjoint of d, i.e., δ is characterized by

$$(du, v) = (u, \delta v), \quad u \in \Lambda^k(M),\ v \in \Lambda^{k+1}(M), \ \text{compactly supported.} \tag{21.6}$$

We set $\delta = 0$ on 0-forms. Of course, (21.2) implies

$$\delta^2 = 0. \tag{21.7}$$

There is a useful formula for δ, involving d and the "Hodge star operator," which will be derived in §22.

The Hodge Laplacian on k-forms

$$\Delta : \Lambda^k(M) \longrightarrow \Lambda^k(M) \tag{21.8}$$

is defined by

$$-\Delta = (d+\delta)^2 = d\delta + \delta d. \tag{21.9}$$

Consequently,

$$(-\Delta u, v) = (du, dv) + (\delta u, \delta v) \quad \text{for } u, v \in C_0^\infty(M, \Lambda^k). \tag{21.10}$$

Since $\delta = 0$ on $\Lambda^0(M)$, we have $-\Delta = \delta d$ on $\Lambda^0(M)$.

We will obtain an analogue of (21.10) for the case where M is a compact manifold with boundary, so a boundary integral appears. To obtain such a formula, we specialize the general Green-Stokes formula (8B.17) to the cases $P = d$ and $P = \delta$. First, we compute the *symbols* of d and δ. Since, for a k-form u,

$$d(u\, e^{i\lambda\psi}) = i\lambda e^{i\lambda\psi}(d\psi) \wedge u + e^{i\lambda\psi}\ du, \tag{21.11}$$

we see that

$$\frac{1}{i}\sigma_d(x,\xi)u = \xi \wedge u. \tag{21.12}$$

As a special case of (8B.12), we have

$$\sigma_\delta(x,\xi) = \sigma_d(x,\xi)^t. \tag{21.13}$$

The adjoint of the map (21.12) from $\Lambda^k T_x^*$ to $\Lambda^{k+1} T_x^*$ is given by the interior product

$$\iota_\xi u = u \rfloor X, \tag{21.14}$$

where $X \in T_x$ is the vector corresponding to $\xi \in T_x^*$ under the isomorphism $T_x \approx T_x^*$ given by the Riemannian metric. Consequently,

$$\frac{1}{i}\sigma_\delta(x,\xi)u = -\iota_\xi u. \tag{21.15}$$

Now, the Green-Stokes formula (8B.17) implies, for M a compact Riemannian manifold with boundary,

$$\begin{aligned} (du, v) &= (u, \delta v) + \frac{1}{i}\int\limits_{\partial M} \langle \sigma_d(x,\nu)u, v\rangle\, dS \\ &= (u, \delta v) + \int\limits_{\partial M} \langle \nu \wedge u, v\rangle\, dS, \end{aligned} \tag{21.16}$$

and

$$
\begin{aligned}
(\delta u, v) &= (u, dv) + \frac{1}{i} \int\limits_{\partial M} \langle \sigma_\delta(x,\nu) u, v\rangle \, dS \\
&= (u, dv) - \int\limits_{\partial M} \langle \iota_\nu u, v\rangle \, dS.
\end{aligned}
\tag{21.17}
$$

Recall that ν is the outward pointing unit normal to ∂M.

Consequently, our generalization of (21.10) is

$$
\begin{aligned}
-(\Delta u, v) = (du, dv) &+ (\delta u, \delta v) \\
&+ \frac{1}{i} \int\limits_{\partial M} \big[\langle \sigma_d(x,\nu)\delta u, v\rangle + \sigma_\delta(x,\nu) du, v\rangle\big] \, dS,
\end{aligned}
\tag{21.18}
$$

or, equivalently,

$$
\begin{aligned}
-(\Delta u, v) = (du, dv) &+ (\delta u, \delta v) \\
&+ \int\limits_{\partial M} \big[\langle \nu \wedge (\delta u), v\rangle - \langle \iota_\nu (du), v\rangle\big] \, dS.
\end{aligned}
\tag{21.19}
$$

Taking adjoints of the symbol maps, we can also write

$$
\begin{aligned}
-(\Delta u, v) = (du, dv) &+ (\delta u, \delta v) \\
&+ \int\limits_{\partial M} \big[\langle \delta u, \iota_\nu v\rangle - \langle du, \nu \wedge v\rangle\big] \, dS.
\end{aligned}
\tag{21.20}
$$

Let us note what the symbol of Δ is. By (21.12) and (21.15),

$$
-\sigma_\Delta(x,\xi) u = \iota_\xi \xi \wedge u + \xi \wedge \iota_\xi u. \tag{21.21}
$$

If we perform the calculation by picking an orthonormal basis for T_x^* of the for $\{e_1, \dots, e_n\}$ with $\xi = |\xi| e_1$, we see that

$$
\sigma_\Delta(x,\xi) u = -|\xi|^2 u. \tag{21.22}
$$

In other words, in a local coordinate system, we have, for a k-form u,

$$
\Delta u = g^{j\ell}(x)\, \partial_j \partial_\ell u + Y_y u, \tag{21.23}
$$

where Y_k is a first order differential operator.

Generally speaking, a differential operator $P : C^\infty(M, E_0) \to C^\infty(M, E_1)$ is said to be *elliptic* provided $\sigma_P(x,\xi) : E_{0x} \to E_{1x}$ is invertible for each $x \in M$, and each $\xi \neq 0$. By (21.22), the Laplace operator on k-forms is elliptic.

Of course, the definition $-\Delta = \delta d$ for the Laplace operator on 0-forms coincides with the definition given in (8.26). In this regard, it is useful to note explicitly the following result about δ on 1-forms. Let X be a vector field and ξ the 1-form corresponding to X under a given metric:

$$g(Y, X) = \langle Y, \xi \rangle. \tag{21.24}$$

Then

$$\delta\xi = -\operatorname{div} X. \tag{21.25}$$

This identity follows from (8.23), which can be rewritten

$$(\xi, du) = (X, \operatorname{grad} u) = -(\operatorname{div} X, u) + \int_{\partial M} \langle X, \nu \rangle u \, dS,$$

and the definition of δ as the formal adjoint of d.

We end this section with some algebraic implications of the symbol formula (21.21)–(21.22) for the Laplace operator. If we define $\wedge_\xi : \Lambda^* T_x^* \to \Lambda^* T_x^*$ by $\wedge_\xi(\omega) = \xi \wedge \omega$ and define ι_ξ as above, by (10.14), then the content of this calculation is

$$\wedge_\xi \iota_\xi + \iota_\xi \wedge_\xi = |\xi|^2. \tag{21.26}$$

As we have mentioned, this can be established by picking $\xi/|\xi|$ to be the first member of an orthonormal basis of T_x^*. This identity has the following extension:

$$\wedge_\xi \iota_\eta + \iota_\eta \wedge_\xi = \langle \xi, \eta \rangle, \tag{21.27}$$

which follows from formula (6.8). Note the connection with (8.39).

Exercises

1. Show that the adjoint of the exterior product operator $\xi\wedge$ is ι_ξ, as asserted in (21.14).

2. If $\alpha = \sum_k a_{jk}(x)\, dx_j \wedge dx_k$ and $a_j{}^k = g^{k\ell} a_{j\ell}$, relate $\delta\alpha$ to the divergence $a_j{}^k{}_{;k}$, as defined in (12.29).

3. Using (21.20), write down an expression for

$$(\Delta u, v) - (u, \Delta v)$$

as a boundary integral, when u and v are k-forms.

4. Let $\omega \in \Lambda^n(M)$, $n = \dim M$, be the volume form of an oriented Riemannian manifold M. Show that $\delta\omega = 0$.
Hint. Compare (21.6) with the special case of Stokes' formula $\int_M du = 0$ for $u \in \Lambda^{n-1}(M)$, compactly supported.

5. Granted the result of Exercise 4, show that Stokes' formula $\int_M du = \int_{\partial M} u$, for $u \in \Lambda^{n-1}(M)$, follows from (21.16).

6. If $f \in C^\infty(M)$ and $u \in \Lambda^k(M)$, show that

$$\delta(fu) = f\delta u - \iota_{(df)}u.$$

7. For a vector field u on the Riemannian manifold M, let $\widetilde{u}$ denote the associated 1-form. Show that

$$\delta(\widetilde{u} \wedge \widetilde{v}) = (\mathrm{div}\ v)\widetilde{u} - (\mathrm{div}\ u)\widetilde{v} - \widetilde{[u, v]},$$

for $\widetilde{u}, \widetilde{v} \in \Lambda^1(M)$.
Reconsider this problem after reading §22.

22. The Hodge decomposition and harmonic forms

Let M be a compact Riemannian manifold, without boundary. Recall from §21 the Hodge Laplacian on k-forms,

$$\Delta : C^\infty(M, \Lambda^k) \longrightarrow C^\infty(M, \Lambda^k), \tag{22.1}$$

defined by

$$-\Delta = (d + \delta)^2 = d\delta + \delta d, \tag{22.2}$$

where d is the exterior derivative operator and δ its formal adjoint, satisfying

$$(du, v) = (u, \delta v) \tag{22.3}$$

for a smooth k-form u and $(k+1)$-form v; $\delta = 0$ on 0-forms. Note that, for smooth k-forms,

$$-(\Delta u, v) = (du, dv) + (\delta u, \delta v). \tag{22.4}$$

The local coordinate expression

$$\Delta u = g^{j\ell}(x)\partial_j\partial_\ell u + Y_j u, \tag{22.5}$$

where Y_k are first order differential operators, derived in (21.23), implies that the Hodge Laplacian on k-forms is an elliptic differential operator, i.e., its symbol (defined in §8B) $\sigma_\Delta(x, \xi) = -|\xi|^2$ is invertible for all $\xi \neq 0$. We will not present details on the analysis of elliptic operators here, but we will state some of the implications for the Hodge Laplacian, which will be important in the development of the Hodge decomposition. A detailed presentation can be found in Chapter 5 of [T1].

First, for any fixed $C_1 > 0$, $T = (-\Delta + C_1)^{-1}$ is a compact self adjoint operator on $L^2(M, \Lambda^k)$. The identity (22.4) implies

$$0 < (Tu, u) \le C_1^{-1} \|u\|_{L^2}^2, \tag{22.6}$$

for nonzero u. The space $L^2(M, \Lambda^k)$ has an orthonormal basis $u_j^{(k)}$ consisting of eigenfunctions of T :

$$Tu_j^{(k)} = \mu_j^{(k)} u_j^{(k)}; \quad u_j^{(k)} \in H^1(M, \Lambda^k). \tag{22.7}$$

By (22.6), we have

$$0 < \mu_j^{(k)} \le C_1^{-1}. \tag{22.8}$$

For each k, we can order the $u_j^{(k)}$ so that $\mu_j^{(k)} \searrow 0$, as $j \nearrow \infty$. It follows that

$$-\Delta u_j^{(k)} = \lambda_j^{(k)} u_j^{(k)} \tag{22.9}$$

with

$$\lambda_j^{(k)} = \frac{1}{\mu_j^{(k)}} - C_1, \tag{22.10}$$

so

$$\lambda_j^{(k)} \geq 0, \quad \lambda_j^{(k)} \nearrow \infty \text{ as } j \to \infty. \tag{22.11}$$

The standard elliptic regularity results imply that

$$u_j^{(k)} \in C^\infty(M, \Lambda^k). \tag{22.12}$$

In particular, the 0-eigenspace of Δ on k-forms is finite dimensional, and consists of smooth k-forms. These are called *harmonic forms.* We denote this 0-eigenspace by $\mathcal{H}_k$. By (22.4), we see that

$$u \in \mathcal{H}_k \Longleftrightarrow u \in C^\infty(M, \Lambda^k),\ du = 0,\ \text{and } \delta u = 0 \text{ on } M. \tag{22.13}$$

Denote by P_k the orthogonal projection of $L^2(M, \Lambda^k)$ onto $\mathcal{H}_k$. We also define a continuous linear map

$$G : L^2(M, \Lambda^k) \longrightarrow L^2(M, \Lambda^k) \tag{22.14}$$

by

$$\begin{aligned} G u_j^{(k)} = \quad & 0 & \text{if } \lambda_j^{(k)} = 0, \\ & \big(1/\lambda_j^{(k)}\big) u_j^{(k)} & \text{if } \lambda_j^{(k)} > 0. \end{aligned} \tag{22.15}$$

Hence $-\Delta G u_j^{(k)} = (I - P_k) u_j^{(k)}$. It follows that

$$-\Delta G u = (I - P_k) u \ \text{ for } \ u \in C^\infty(M, \Lambda^k). \tag{22.16}$$

Now the elliptic regularity implies

$$G : C^\infty(M, \Lambda^k) \longrightarrow C^\infty(M, \Lambda^k), \tag{22.17}$$

and, if $C^r(M)$ denotes a Hölder space, then, for $j = 0, 1, 2, \dots,\ r \in (0, 1)$,

$$G : C^{j+r}(M, \Lambda^k) \longrightarrow C^{j+2+r}(M, \Lambda^k). \tag{22.18}$$

Using (22.2), we write (22.16) in the following form, known as the *Hodge decomposition.*

Proposition 22.1. *Given* $u \in C^\infty(M, \Lambda^k)$, *we have*

$$u = d\delta G u + \delta d G u + P_k u. \tag{22.19}$$

The three terms on the right are mutually orthogonal in $L^2(M, \Lambda^k)$.

Proof. Only the orthogonality remains to be establishd. But if $u \in C^\infty(M, \Lambda^{k-1})$ and $v \in C^\infty(M, \Lambda^{k+1})$, then

$$(du, \delta v) = (d^2 u, v) = 0, \tag{22.20}$$

and if $w \in \mathcal{H}_k$, so $dw = \delta w = 0$, we have

$$(du, w) = (u, \delta w) = 0, \text{ and } (\delta v, w) = (v, dw) = 0, \tag{22.21}$$

so the orthogonality is established.

A smooth k-form u is said to be *exact* if $u = dv$ for some smooth $(k-1)$-form v, and *closed* if $du = 0$. Since $d^2 = 0$, every exact form is closed:

$$\mathcal{E}^k(M) \subset \mathcal{C}^k(M), \tag{22.22}$$

where $\mathcal{E}^k(M)$ and $\mathcal{C}^k(M)$ denote respectively the spaces of exact and closed k-forms. Similarly, a k-form u is said to be co-exact if $u = \delta v$ for some smooth $(k+1)$-form v, and co-closed if $\delta u = 0$, and since $\delta^2 = 0$ we have

$$\mathcal{CE}^k(M) \subset \mathcal{CC}^k(M), \tag{22.23}$$

with obvious notation. The *deRham cohomology* groups are defined, as in §9, as quotient spaces:

$$\mathcal{H}^k(M) = \mathcal{C}^k(M)/\mathcal{E}^k(M). \tag{22.24}$$

The following is one of the most important consequences of the Hodge decomposition (22.19).

Proposition 22.2. *If* M *is a compact Riemannian manifold, there is a natural isomorphism*

$$\mathcal{H}^k(M) \approx \mathcal{H}_k. \tag{22.25}$$

Proof. Since every harmonic form is closed, there is an injection

$$j : \mathcal{H}_k \hookrightarrow \mathcal{C}^k(M), \tag{22.26}$$

which hence gives rise to a natural map

$$J : \mathcal{H}_k \longrightarrow \mathcal{H}^k(M), \tag{22.27}$$

by passing to the quotient (22.24). It remains to show that J is bijective. The orthogonality (22.21) shows that

$$(\text{Image } j) \cap \mathcal{E}^k(M) = 0,$$

so J is injective. Also (22.21) shows that, if $u \in \mathcal{C}^k(M)$, then $\delta dGu = 0$ in (22.19), so $u = d\delta Gu + P_k u$, or $u = P_k u \bmod \mathcal{E}^k(M)$. Hence J is surjective, and the proof is complete.

Clearly the space $\mathcal{H}^k(M)$ is independent of the Riemannian metric chosen for M. Thus the *dimension* of the space $\mathcal{H}_k$ of harmonic k-forms is independent of the metric. Indeed, since the isomorphism (22.25) is natural, we can say the following. Given two Riemannian metrics g and g' for M, with associated spaces $\mathcal{H}_k$ and $\mathcal{H}'_k$ of harmonic k-forms, there is a natural isomorphism $\mathcal{H}_k \approx \mathcal{H}'_k$. Otherwise said, each $u \in \mathcal{H}_k$ is cohomologous to a unique $u' \in \mathcal{H}'_k$.

An important theorem of deRham states that $\mathcal{H}^k(M)$, defined by (22.24), is isomorphic to a certain singular cohomology group. A variant is an isomorphism of $\mathcal{H}^k(M, \mathbb{R})$ with a certain Cech cohomology group. We refer to [SiT], [BoT] for material on this.

We now introduce the *Hodge star operator*

$$* : C^\infty(M, \Lambda^k) \longrightarrow C^\infty(M, \Lambda^{m-k}) \quad (m = \text{ dim } M) \tag{22.28}$$

in fact, a bundle map

$$* : \Lambda^k T_x^* \longrightarrow \Lambda^{m-k} T_x^*,$$

which will be seen to relate δ to d. For (22.28) to be defined, we need to assume M is an *oriented* Riemannian manifold, so there is a distinguished volume form

$$\omega \in C^\infty(M, \Lambda^m). \tag{22.29}$$

Then the star operator (22.28) is uniquely specified by the relation

$$u \wedge *v = \langle u, v \rangle \omega, \tag{22.30}$$

where $\langle u, v \rangle$ is the inner product on $\Lambda^k T_x^*$, which was defined by (21.3). In particular, it follows that $*1 = \omega$. Furthermore, if $\{e_1, \dots, e_m\}$ is an oriented orthonormal basis of $T_x^* M$, we have

$$*(e_{j_1} \wedge \cdots \wedge e_{j_k}) = (\text{sgn } \pi)\ e_{\ell_1} \wedge \cdots \wedge e_{\ell_{m-k}} \tag{22.31}$$

where $\{j_1, \dots, j_k, \ell_1, \dots, \ell_{m-k}\} = \{1, \dots, m\}$, and π is the permutation mapping the one ordered set to the other. It follows that

$$** = (-1)^{k(m-k)} \text{ on } \Lambda^k(M), \tag{22.32}$$

where, for short, we are denoting $C^\infty(M, \Lambda^k)$ by $\Lambda^k(M)$. We denote (22.32) by $\overline{w}$, and also set

$$w = (-1)^k \text{ on } \Lambda^k(M), \tag{22.33}$$

so

$$d(u \wedge v) = du \wedge v + w(u) \wedge dv. \tag{22.34}$$

It follows that, if $u \in \Lambda^{k-1}(M), v \in \Lambda^k(M)$, then $w(u) \wedge d * v = -u \wedge d * w(v)$, so

$$\begin{aligned} d(u \wedge *v) &= du \wedge *v - u \wedge d * w(v) \\ &= du \wedge *v - u \wedge *\overline{w} * d * w(v), \end{aligned} \tag{22.35}$$

since $*\overline{w}* = \text{id.}$, by (22.32). Integrating over M, since $\partial M = \emptyset$, we have, by Stokes' formula, $\int_M d(u \wedge *v) = 0$, and hence

$$\begin{aligned} (du, v) = \int_M du \wedge *v &= \int_M u \wedge *\overline{w} * d * w(v) \\ &= (u, \overline{w} * d * w(v)). \end{aligned} \tag{22.36}$$

In other words,

$$\begin{aligned} \delta &= \overline{w} * d * w \\ &= (-1)^{k(m-k)-m+k-1} * d * \text{ on } \Lambda^k(M). \end{aligned} \tag{22.37}$$

Thus, by the characterization (22.13) of harmonic k-forms, we have

$$* : \mathcal{H}_k \longrightarrow \mathcal{H}_{m-k}, \tag{22.38}$$

and, by (22.32), this map is an isomorphism. In view of Proposition 22.2, we have the following special case of *Poincaré duality.*

Corollary 22.3. *If M is a compact oriented Riemannian manifold, there is an isomorphism of deRham cohomology groups*

$$\mathcal{H}^k(M) \approx \mathcal{H}^{m-k}(M). \tag{22.39}$$

As a further application of the Hodge decomposition, we prove the following result on the deRham cohomology groups of a Cartesian product $M \times N$ of two compact manifolds, a special case of the Kunneth formula.

Proposition 22.4. *If M and N are compact manifolds, of dimension m and n respectively, then, for $0 \le k \le m+n$,*

$$\mathcal{H}^k(M \times N) \approx \bigoplus_{i+j=k} \Big[\mathcal{H}^i(M) \otimes \mathcal{H}^j(N)\Big]. \tag{22.40}$$

Proof. Endow M and N with Riemannian metrics, and give $M \times N$ the product metric. If $\{u_\mu^{(i)}\}$ is an orthonormal basis of $L^2(M, \Lambda^i)$ and $\{v_\nu^{(j)}\}$ is an orthonormal basis of $L^2(N, \Lambda^j)$, each consisting of eigenfunctions of the Hodge Laplace operator, then $\{u_\mu^{(i)} \wedge v_\nu^{(j)} : i+j=k\}$ is an orthonormal basis of $L^2(M \times N, \Lambda^k)$, consisting of eigenfunctions of the Hodge Laplacian, and since all these Laplace operators are negative semi-definite, we have the isomorphism

$$\mathcal{H}_k(M \times N) \approx \bigoplus_{i+j=k} \Big[\mathcal{H}_i(M) \otimes \mathcal{H}_j(N)\Big], \tag{22.41}$$

where $\mathcal{H}_i(M)$ denotes the space of harmonic i-forms on M, etc., and by (22.25) this proves the proposition.

We define the ith *Betti number* of M to be

$$b_i(M) = \dim \mathcal{H}^i(M). \tag{22.42}$$

Thus, (22.40) implies the identity

$$b_k(M \times N) = \sum_{i+j=k} b_i(M) b_j(N). \tag{22.43}$$

This identity has an application to the *Euler characteristic* of a product. The Euler characteristic of M is defined by

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \, b_i(M), \tag{22.44}$$

where $m = \dim M$. From (22.43) follows directly the product formula

$$\chi(M \times N) = \chi(M)\chi(N). \tag{22.45}$$

Exercises

1. Let $\alpha \in \Lambda^1(M^n), \beta \in \Lambda^k(M^n)$. Show that

$$*(\iota_\alpha \beta) = \pm \alpha \wedge *\beta. \tag{22.46}$$

Find the sign.
Hint. Start with the identity $\sigma \wedge \alpha \wedge *\beta = \langle \sigma \wedge \alpha, \beta \rangle$, given $\sigma \in \Lambda^{k-1}(M)$.
Alternative. Show $*\delta = \pm d*$, which implies (22.46) by passing to symbols.

2. Show that, if X is a smooth vector field on M, and $\beta \in \Lambda^k(M^n)$, then

$$\nabla_X(*\beta) = *(\nabla_X \beta).$$

3. Show that, if $F : M \to M$ is an isometry which preserves orientation, then $F^*(*\beta) = *(F^*\beta)$.

4. If $f : M \to N$ is a smooth map between compact manifolds, show that the pull-back $f^* : \Lambda^k(N) \to \Lambda^k(M)$ induces a homomorphism $f^* : \mathcal{H}^k(N) \to \mathcal{H}^k(M)$. If $f_t,\ 0 \le t \le 1$ is a smooth family of such maps, show that $f_0^* = f_1^*$ on $\mathcal{H}^k(N)$.
Hint. For the latter, recall formulas (6.31)–(6.35), or Exercise 6 of §6.

5. If M is compact, connected, and oriented, and dim $M = n$, show that

$$\mathcal{H}^0(M) \approx \mathcal{H}^n(M) \approx \mathbb{R}.$$

Relate this to Proposition 9.5.

In Exercises 6–8, let G be a compact connected Lie group, endowed with a bi-invariant Riemannian metric. For each $g \in G$, there are left and right translations $L_g(h) = gh,\ R_g(h) = hg$. Let $\mathcal{B}_k$ denote the space of bi-invariant k-forms on G,

$$\mathcal{B}_k = \{\beta \in \Lambda^k(G) : R_g^*\beta = \beta = L_g^*\beta \text{ for all } g \in G\}. \tag{22.47}$$

6. Show that every harmonic k-form on G belongs to $\mathcal{B}_k$.
Hint. If $\beta \in \mathcal{B}_k$, show $R_g^*\beta$ and $L_g^*\beta$ are both harmonic and cohomologous to β.

7. Show that every $\beta \in \mathcal{B}_k$ is closed, i.e., $d\beta = 0$. Also, show that $* : \mathcal{B}_k \to \mathcal{B}_{n-k}$ ($n = \dim G$). Hence conclude

$$\mathcal{B}_k = \mathcal{H}_k.$$

Hint. To show that $d\beta = 0$, note that, if $\iota : G \to G$ is $\iota(g) = g^{-1}$, then $\iota^*\beta \in \mathcal{B}_k$ and $\iota^*\beta(e) = (-1)^k\beta(e)$. Since also $d\beta \in \mathcal{B}_{k+1}$ deduce that $\iota^* d\beta$ equals both $(-1)^k d\beta$ and $(-1)^{k+1}d\beta$.

8. With G as above, show that $\mathcal{B}_1$ is linearly isomorphic to the *center* $\mathcal{Z}$ of the Lie algebra $\mathfrak{g}$ of G. Conclude that, if $\mathfrak{g}$ has trivial center, then $\mathcal{H}^1(G) = 0$.

Exercises 9–10 look at $\mathcal{H}^k(S^n)$.

9. Let β be any harmonic k-form on S^n. Show that $g^*\beta = \beta$, where g is any element of $SO(n+1)$, the group of rotations on $\mathbb{R}^{n+1}$, acting as a group of isometries of S^n.
Hint. Compare the argument used in Exercise 6.

10. Consider the point $p = (0, \dots, 0, 1) \in S^n$. The group $SO(n)$, acting on $\mathbb{R}^n \subset \mathbb{R}^{n+1}$, fixes p. Show that $\mathcal{H}^k(S^n)$ is isomorphic to (a linear subspace of)

$$V_k = \{\beta \in \Lambda^k\mathbb{R}^n : g^*\beta = \beta \text{ for all } g \in SO(n)\}. \tag{22.48}$$

Show that $V_k = 0$ if $0 < k < n$. Deduce that

$$\mathcal{H}^k(S^n) = 0 \ \text{ if } \ 0 < k < n. \tag{22.49}$$

Hint. Given $\beta \in \Lambda^k\mathbb{R}^n$, $1 \le j, \ell \le n$, average $g^*\beta$ over g in the group of rotations in the $x_j - x_\ell$ plane.
Note. By Exercise 5, if $n \ge 1$,

$$\mathcal{H}^k(S^n) = \mathbb{R} \ \text{ if } \ k = 0 \text{ or } n. \tag{22.50}$$

Recall the elementary proof of this, for $k = n$, in Proposition 9.5.

11. Suppose M is compact, connected, but not orientable, $\dim M = n$. Show that $\mathcal{H}^n(M) = 0$.
Hint. Let $\tilde{M}$ be an orientable double cover, with natural involution ι. A harmonic n-form on M would lift to a harmonic form on $\tilde{M}$, invariant under ι^*; but ι reverses orientation.

Auxiliary exercises on the Hodge star operator

In most of the exercises to follow, adopt the following notational convention. For a vector field u on an oriented Riemannian manifold, let $\widetilde{u}$ denote the associated 1-form.

1. Show that

$$f = \text{ div } u \Longleftrightarrow f = *d * \tilde{u}.$$

If $M = \mathbb{R}^3$, show that

$$v = \text{ curl } u \Longleftrightarrow \tilde{v} = *d\tilde{u}.$$

2. If u and v are vector fields on $\mathbb{R}^3$, show that

$$w = u \times v \Longleftrightarrow \tilde{w} = *(\tilde{u} \wedge \tilde{v}).$$

Show that, for $\tilde{u}, \tilde{v} \in \Lambda^1(M^n),\ *(\tilde{u} \wedge \tilde{v}) = (*\tilde{u}) \rfloor v$.
If $u \times v$ is defined by this formula for vector fields on an oriented Riemannian 3-fold, show that $u \times v$ is orthogonal to u and v.

3. Show that the identity

$$\text{div } (u \times v) = v \cdot \text{curl } u - u \cdot \text{curl } v, \tag{22.51}$$

for u and v vector fields on $\mathbb{R}^3$, is a special case of

$$*d(\tilde{u} \wedge \tilde{v}) = \langle *d\tilde{u}, \tilde{v} \rangle - \langle \tilde{u}, *d\tilde{v} \rangle, \quad \tilde{u}, \tilde{v} \in \Lambda^1(M^3).$$

Deduce this from $d(\tilde{u} \wedge \tilde{v}) = (d\tilde{u}) \wedge \tilde{v} - \tilde{u} \wedge d\tilde{v}$.

In Exercises 4–6, we produce a generalization of the identity

$$\begin{aligned} \text{curl } (u \times v) &= v \cdot \nabla u - u \cdot \nabla v + (\text{div } v)u - (\text{div } u)v \\ &= [v, u] + (\text{div } v)u - (\text{div u})v, \end{aligned} \tag{22.52}$$

valid for u and v vector fields on $\mathbb{R}^3$.

4. For $\tilde{u}, \tilde{v} \in \Lambda^1(M^n)$, use Exercise 2 to show that

$$d * (\tilde{u} \wedge \tilde{v}) = -(d * \tilde{u}) \rfloor v + \mathcal{L}_v(*\tilde{u}).$$

5. If $\omega \in \Lambda^n(M^n)$ is the volume form, show that $*(\omega \rfloor v) = \tilde{v}$. Deduce that

$$*\big[d(*\tilde{u})\rfloor v\big] = (\text{div } u)\tilde{v}.$$

6. Applying $\mathcal{L}_v$ to $(*\tilde{u}) \wedge \tilde{w} = \langle u, \tilde{w}\rangle\omega$, show that

$$*\mathcal{L}_v(*\tilde{u}) = \widetilde{[v, u]} + (\text{div } v)\tilde{u},$$

and hence

$$*d * (\tilde{u} \wedge \tilde{v}) = \widetilde{[v, u]} + (\text{div } v)\tilde{u} - (\text{div } u)\tilde{v},$$

generalizing (22.52).

In Exercises 7–10, we produce a generalization of the identity

$$\text{grad}\,(u \cdot v) = u \cdot \nabla v + v \cdot \nabla u + u \times \text{curl } v + v \times \text{curl } u, \tag{22.53}$$

valid for u and v vector fields on $\mathbb{R}^3$. Only Exercise 10 makes contact with the Hodge star operator.

7. Noting that, for $\tilde{u}, \tilde{v} \in \Lambda^1(M^n)$, $d(\tilde{u}\rfloor v) = \mathcal{L}_v\tilde{u} - (d\tilde{u})\rfloor v$, show that

$$2d(\tilde{u}\rfloor v) = \mathcal{L}_v\tilde{u} + \mathcal{L}_u\tilde{v} - (d\tilde{u})\rfloor v - (d\tilde{v})\rfloor u.$$

8. Show that

$$\mathcal{L}_v\tilde{u} = \widetilde{[v, u]} + (\mathcal{L}_v g)(\cdot, u),$$

where g is the metric tensor, and where $h(\cdot, u) = w$ means $h(X, u) = g(X, w) = \langle X, w\rangle$. Hence

$$\mathcal{L}_v\tilde{u} + \mathcal{L}_u\tilde{v} = (\mathcal{L}_v g)(\cdot, u) + (\mathcal{L}_u g)(\cdot, v).$$

9. Show that

$$(\mathcal{L}_v g)(\cdot, u) + (\mathcal{L}_u g)(\cdot, v) = d(\tilde{u}\rfloor v) + \nabla_u\tilde{v} + \nabla_v\tilde{u}.$$

10. Deduce that

$$d\langle u, v\rangle = \nabla_u \tilde{v} + \nabla_v \tilde{u} - (d\tilde{u})\rfloor v - (d\tilde{v})\rfloor u.$$

To relate this to (22.53), show using Exercises 1–2 that, for vector fields on $\mathbb{R}^3$,

$$w = v \times \text{curl } u \Longleftrightarrow \tilde{w} = -(d\tilde{u})\rfloor v.$$

11. If $u, v \in \Lambda^k(M^n)$ and $w \in \Lambda^{n-k}(M^n)$, show that

$$(w, *v) = (-1)^{k(n-k)}(*w, v),$$

and

$$(*u, *v) = (u, v).$$

12. Show that $*d = (-1)^{k+1}\delta *$ on $\Lambda^k(M)$.

13. Verify carefully that $\Delta * = *\Delta$. In particular, on $\Lambda^k(M^n)$,

$$*\Delta = \Delta * = (\pm 1)\big[(\pm 1) d * d + (\pm 1) * d * d *\big].$$

Find the signs.

23. The Hodge decomposition on manifolds with boundary

Let $\overline{M}$ be a compact Riemannian manifold with boundary, dim $M = m$. We have the Hodge Laplace operator

$$\Delta : C^\infty(\overline{M}, \Lambda^k) \longrightarrow C^\infty(\overline{M}, \Lambda^k).$$

As shown in §21, we have a generalization of Green's formula, expressing $-(\Delta u, v)$ as $(du, dv) + (\delta u, \delta v)$ plus a boundary integral. Two forms of this, equivalent to formula (21.18), are

$$(23.1)\quad -(\Delta u, v) = (du, dv) + (\delta u, \delta v) + \frac{1}{i} \int\limits_{\partial M} \big[\langle \sigma_d(x,\nu)\delta u, v\rangle + \langle du, \sigma_d(x,\nu)v\rangle\big]\, dS$$

and

$$(23.2)\quad -(\Delta u, v) = (du, dv) + (\delta u, \delta v) + \frac{1}{i} \int\limits_{\partial M} \big[\langle \delta u, \sigma_\delta(x,\nu)v\rangle + \langle \sigma_\delta(x,\nu)du, v\rangle\big]\, dS.$$

Recall from (21.12)–(21.14) that

$$(23.3)\qquad \frac{1}{i}\sigma_d(x,\nu)u = \nu \wedge u, \quad \frac{1}{i}\sigma_\delta(x,\nu)u = -\iota_\nu u.$$

We have studied the Dirichlet and Neumann boundary problems for Δ on 0-forms in previous sections. Here we will see that for each $k \in \{0, \dots, m\}$ there is a pair of boundary conditions generalizing these. For starters, suppose M is half of a compact Riemannian manifold without boundary N, having an *isometric* involution $\tau : N \to N$, fixing ∂M and switching M and $N \setminus M$. For short, will say N is the *isometric double* of M. Note that elements of $C^\infty(N)$ which are *odd* with respect to τ vanish on ∂M, i.e., satisfy the Dirichlet boundary condition, while elements *even* with respect to τ have vanishing normal derivatives on ∂M, i.e., satisfy the Neumann boundary condition. Now, if $u \in \Lambda^k(N)$, then the hypothesis $\tau^* u = -u$ (which implies $\tau^* du = -du$ and $\tau^*\delta u = -\delta u$) implies

$$(23.4)\qquad \sigma_d(x,\nu)u = 0 \text{ and } \sigma_d(x,\nu)\delta u = 0 \text{ on } \partial M,$$

while the hypothesis $\tau^* u = u$ (hence $\tau^* du = du$ and $\tau^*\delta u = \delta u$) implies

$$(23.5)\qquad \sigma_\delta(x,\nu)u = 0 \text{ and } \sigma_\delta(x,\nu)du = 0 \text{ on } \partial M.$$

We call the boundary conditions (23.4) and (23.5) *relative* boundary conditions and *absolute* boundary conditions, respectively. Thus, specialized to 0-forms, relative boundary condiitons are Dirichlet boundary conditions, and absolute boundary conditions are Neumann boundary conditions.

It is easy to see that

$$\nu \wedge u\big|_{\partial M} = 0 \Longleftrightarrow j^* u = 0, \text{ where } j : \partial M \hookrightarrow \overline{M}. \tag{23.6}$$

Thus the relative boundary conditions (23.4) can be rewritten

$$j^* u = 0, \quad j^*(\delta u) = 0. \tag{23.7}$$

Using (23.3), we can rewrite the absolute boundary conditions (23.5) as

$$u \rfloor \nu = 0 \text{ and } (du) \rfloor \nu = 0 \text{ on } \partial M. \tag{23.8}$$

Also, from Exercise 1 of §22, it follows that

$$\begin{aligned} \sigma_d(x,\nu)(*u) &= \pm * \sigma_\delta(x,\nu)u \\ \sigma_d(x,\nu)\delta * u &= \pm * \sigma_\delta(x,\nu)du. \end{aligned} \tag{23.9}$$

Thus the Hodge star operator interchanges absolute and relative boundary conditions. In particular, the absolute boundary conditions are also equivalent to

$$j^*(*u) = 0, \quad j^*(\delta * u) = 0. \tag{23.10}$$

Note that, if u and v satisfy relative boundary conditions, then the boundary integral in (23.1) vanishes. Similarly, if u and v satisfy absolute boundary conditions, then the boundary integral in (23.2) vanishes.

There are elliptic regularity results for both the boundary conditions (23.4) and (23.5). A detailed analysis is given in Chapter 5 of [T1], but we will just summarize some relevant results here. The boundary conditions define self adjoint extensions of $-\Delta$, which we will denote $\mathcal{L}_R$ and $\mathcal{L}_A$, respectively. We also use the notation $\mathcal{L}_b$, where b stands for R or A.

The maps

$$T_b = (\mathcal{L}_b + 1)^{-1} \tag{23.11}$$

are compact self adjoint operators on $L^2(M, \Lambda^k)$, so we have orthonormal bases $\{u_j^{(k)}\}$ and $\{v_j^{(k)}\}$ of $L^2(M, \Lambda^k)$ satisfying

$$T_R u_j^{(k)} = \mu_j^{(k)} u_j^{(k)}, \quad u_j^{(k)} \in C_R^\infty(\overline{M}, \Lambda^k), \tag{23.12}$$

and

$$T_A v_j^{(k)} = \nu_j^{(k)} v_j^{(k)}, \quad v_j^{(k)} \in C_A^\infty(\overline{M}, \Lambda^k), \tag{23.13}$$

where $C_b^\infty(\overline{M}, \Lambda^k)$ is the space of smooth k-forms on $\overline{M}$ satisfying the boundary condition (23.4) if $b = R$, (23.5) if $b = A$. The eigenvalues of T_R and T_A all have magnitude ≤ 1, and we can order them so that, for each $k, \mu_j^{(k)}$ and $\nu_j^{(k)} \searrow 0$ as $j \to \infty$. It follows that, for each k,

$$\mathcal{L}_R u_j^{(k)} = \rho_j^{(k)} u_j^{(k)}, \quad \rho_j^{(k)} = \frac{1}{\mu_j^{(k)}} - 1 \nearrow \infty, \tag{23.14}$$

and

$$\mathcal{L}_A v_j^{(k)} = \alpha_j^{(k)} v_j^{(k)}, \quad \alpha_j^{(k)} = \frac{1}{\nu_j^{(k)}} - 1 \nearrow \infty. \tag{23.15}$$

Here, $\rho_j^{(k)} \geq 0$ and $\alpha_j^{(k)} \geq 0$, and only finitely many of these quantities are equal to zero.

The 0-eigenspaces of $\mathcal{L}_R$ and $\mathcal{L}_A$ are finite dimensional spaces in $C^\infty(\overline{M}, \Lambda^k)$; denote them by $\mathcal{H}_k^R$ and $\mathcal{H}_k^A$, respectively. We see that, for $b = R$ or A,

$$u \in \mathcal{H}_k^b \Longleftrightarrow u \in C^\infty(\overline{M}, \Lambda^k),\ B_b^{(0)} u = 0 \text{ on } \partial M, \text{ and } du = \delta u = 0 \text{ on } M, \tag{23.16}$$

where

$$B_R^{(0)} u = \nu \wedge u, \quad B_A^{(0)} u = u \rfloor \nu. \tag{23.17}$$

Also recall that we can replace $\nu \wedge u$ by $j^* u$. We call $\mathcal{H}_k^R$ and $\mathcal{H}_k^A$ the spaces of harmonic k-forms, satisfying relative and absolute boundary conditions, respectively.

Denote by P_h^R and P_h^A the orthogonal projections of $L^2(M, \Lambda^k)$ onto $\mathcal{H}_k^R$ and $\mathcal{H}_k^A$. Parallel to (22.15)–(22.16), we have continuous linear maps

$$G^b : C^\infty(\overline{M}, \Lambda^k) \longrightarrow C_b^\infty(\overline{M}, \Lambda^k), \quad b = R \text{ or } A, \tag{23.18}$$

such that G^b annihlates $\mathcal{H}_k^b$ and inverts $-\Delta$ on the orthogonal complement of $\mathcal{H}_k^b$:

$$-\Delta G^b u = (I - P_h^b) u \text{ for } u \in C^\infty(\overline{M}, \Lambda^k). \tag{23.19}$$

Furthermore, for $j \geq 0, r \in (0,1)$,

$$G^b : C^{j+r}(\overline{M}, \Lambda^k) \longrightarrow C^{j+2+r}(\overline{M}, \Lambda^k). \tag{23.20}$$

The identity (23.19) then produces the following two Hodge decompositions for a compact Riemannian manifold with boundary.

Proposition 23.1. *Given* $u \in C^\infty(\overline{M}, \Lambda^k)$, $j \geq 0$, *we have*

$$u = d\delta G^R u + \delta d G^R u + P_h^R u = P_d^R u + P_\delta^R u + P_h^R u, \tag{23.21}$$

and

$$u = d\delta G^A u + \delta d G^A u + P_h^A u = P_d^A u + P_\delta^A u + P_h^A u. \tag{23.22}$$

In both cases, the three terms on the right side are mutually orthogonal in $L^2(M, \Lambda^k)$.

Proof. It remains only to check orthogonality, which requires a slightly longer argument than used in Proposition 22.1. We will use the identity

$$(du, v) = (u, \delta v) + \gamma(u, v), \tag{23.23}$$

for $u \in \Lambda^{j-1}(\overline{M})$ and $v \in \Lambda^j(\overline{M})$, with

$$\gamma(u, v) = \frac{1}{i} \int\limits_{\partial M} \langle \sigma_d(x, \nu) u, v \rangle \, dS = \frac{1}{i} \int\limits_{\partial M} \langle u, \sigma_\delta(x, \nu) v \rangle \, dS. \tag{23.24}$$

Note that $\gamma(u, v) = 0$ if either

$$\begin{aligned} & u \in C_R^1(\overline{M}, \Lambda^{j-1}) = \big\{ u \in C^1(\overline{M}, \Lambda^{j-1}) : j^* u\big|_{\partial M} = 0 \big\} \\ \text{or} \quad & \\ & v \in C_A^1(\overline{M}, \Lambda^j) = \big\{ u \in C^1(\overline{M}, \Lambda^j) : u \rfloor \nu = 0 \text{ on } \partial M \big\}. \end{aligned} \tag{23.25}$$

In particular, we see that

$$\begin{aligned} u \in C_R^1(\overline{M}, \Lambda^{k-1}) &\Longrightarrow du \perp \ \ker \delta \cap C^1(\overline{M}, \Lambda^k), \\ v \in C_A^1(\overline{M}, \Lambda^k) &\Longrightarrow \delta v \perp \ \ker d \cap C^1(\overline{M}, \Lambda^{k-1}). \end{aligned} \tag{23.26}$$

From the definitions, we have

$$\delta : C_R^2(\overline{M}, \Lambda^j) \longrightarrow C_R^1(\overline{M}, \Lambda^{j-1}), \quad d : C_A^2(\overline{M}, \Lambda^j) \longrightarrow C_A^1(\overline{M}, \Lambda^{j+1}), \tag{23.27}$$

where $C_b^2(\overline{M}, \Lambda^j)$ consists of $u \in C^2(\overline{M}, \Lambda^j)$ satisfying the boundary condition b. Thus

$$\begin{aligned} d\delta C_R^2(\overline{M}, \Lambda^k) &\perp \ \ker \delta \cap C^1(\overline{M}, \Lambda^k), \\ \delta d C_A^2(\overline{M}, \Lambda^k) &\perp \ \ker d \cap C^1(\overline{M}, \Lambda^k). \end{aligned} \tag{23.28}$$

Now (23.28) implies for the ranges:

$$\mathcal{R}(P_d^R) \perp \mathcal{R}(P_\delta^R) + \mathcal{R}(P_h^R) \text{ and } \mathcal{R}(P_\delta^A) \perp \mathcal{R}(P_d^A) + \mathcal{R}(P_h^A). \tag{23.29}$$

Furthermore, if $u \in \mathcal{H}_k^R$ and $v = dG^R w$, then $\gamma(u,v) = 0$, so $(u, \delta v) = (du, v) = 0$. Similarly, if $v \in \mathcal{H}_k^A$ and $u = \delta G^A w$, then $\gamma(u,v) = 0$, so $(du, v) = (u, \delta v) = 0$. Thus

$$\mathcal{R}(P_\delta^R) \perp \mathcal{R}(P_h^R) \text{ and } \mathcal{R}(P_d^A) \perp \mathcal{R}(P_h^A). \tag{23.30}$$

The proposition is proved.

We can produce an analogue of Proposition 22.2, relating the spaces $\mathcal{H}_k^b$ to cohomology groups. We first look at the case $b = R$. Set

$$C_r^\infty(\overline{M}, \Lambda^k) = \{u \in C^\infty(\overline{M}, \Lambda^k) : j^* u = 0\}. \tag{23.31}$$

Since $d \circ j^* = j^* \circ d$, it is clear that

$$d : C_r^\infty(\overline{M}, \Lambda^k) \longrightarrow C_r^\infty(\overline{M}, \Lambda^{k+1}). \tag{23.32}$$

Our spaces of "closed" and "exact" forms are

$$\mathcal{C}_R^k(\overline{M}) = \{u \in C_r^\infty(\overline{M}, \Lambda^k) : du = 0\}, \quad \mathcal{E}_R^k(\overline{M}) = d\, C_r^\infty(\overline{M}, \Lambda^{k-1}). \tag{23.33}$$

We set

$$\mathcal{H}^k(\overline{M}, \partial M) = \mathcal{C}_R^k(\overline{M}) / \mathcal{E}_R^k(\overline{M}). \tag{23.34}$$

Proposition 23.2. *If $\overline{M}$ is a compact Riemannian manifold with boundary, there is a natural isomorphism*

$$\mathcal{H}^k(\overline{M}, \partial M) \approx \mathcal{H}_k^R. \tag{23.35}$$

Proof. By (23.16) we have an injection

$$j : \mathcal{H}_k^R \longrightarrow \mathcal{C}_R^k(\overline{M}),$$

which yields a map

$$J : \mathcal{H}_k^R \longrightarrow \mathcal{H}^k(\overline{M}, \partial M),$$

by composing with (23.34). The orthogonality of the terms in (23.21) implies (Image $j) \cap \mathcal{E}_R^k(\overline{M}) = 0$, so J is injective. Furthermore, if $u \in \mathcal{C}_R^k(\overline{M})$, then u is orthogonal to δv for any $v \in C^\infty(\overline{M}, \Lambda^{k+1})$, so the term $\delta(dG^R u)$ in (23.21) vanishes, and hence J is surjective. This proves the proposition.

As in §22, it is clear that $\mathcal{H}^k(\overline{M}, \partial M)$ is independent of a metric on M. Thus the dimension of $\mathcal{H}_k^R$ is independent of such a metric.

Associated to absolute boundary conditions is the family of spaces

$$C_a^\infty(\overline{M}, \Lambda^k) = \{u \in C^\infty(\overline{M}, \Lambda^k) : \iota_\nu u = \iota_\nu(du) = 0\}, \tag{23.36}$$

replacing (23.31). Note that $C_a^\infty(\overline{M}, \Lambda^k) = C_A^\infty(\overline{M}, \Lambda^k)$, while $C_r^\infty(\overline{M}, \Lambda^k) \neq C_R^\infty(\overline{M}, \Lambda^k)$. We have

$$d : C_a^\infty(\overline{M}, \Lambda^k) \longrightarrow C_a^\infty(\overline{M}, \Lambda^{k+1}), \tag{23.37}$$

and, with $\mathcal{C}_A^k(\overline{M})$ the kernel of d in (23.37) and $\mathcal{E}_A^{k+1}(\overline{M})$ its image, we can form quotients. The following result is parallel to Proposition 23.2.

Proposition 23.3. *There is a natural isomorphism*

$$\mathcal{H}_k^A \approx \mathcal{C}_A^k(\overline{M})/\mathcal{E}_A^k(\overline{M}). \tag{23.38}$$

Proof. Exactly parallel to the last proof.

We have refrained from denoting the right side of (23.38) by $\mathcal{H}^k(\overline{M})$, since the deRham cohomology of $\overline{M}$ has the standard definition

$$\mathcal{H}^k(\overline{M}) = \mathcal{C}^k(\overline{M})/\mathcal{E}^k(\overline{M}), \tag{23.39}$$

where $\mathcal{C}^k(\overline{M})$ is the kernel and $\mathcal{E}^{k+1}(\overline{M})$ the image of d in

$$d : C^\infty(\overline{M}, \Lambda^k) \longrightarrow C^\infty(\overline{M}, \Lambda^{k+1}). \tag{23.40}$$

Note that *no* boundary conditions are imposed here. We now establish that (23.38) is isomorphic to $\mathcal{H}^k(\overline{M})$.

Proposition 23.4. *The quotient spaces $\mathcal{C}_A^k(\overline{M})/\mathcal{E}_A^k(\overline{M})$ and $\mathcal{H}^k(\overline{M})$ are naturally isomorphic. Hence*

$$\mathcal{H}_k^A \approx \mathcal{H}^k(\overline{M}). \tag{23.41}$$

Proof. It is clear that there is a natural map

$$\kappa : \mathcal{C}_A^k(\overline{M})/\mathcal{E}_A^k(\overline{M}) \longrightarrow \mathcal{H}^k(\overline{M}),$$

since $\mathcal{C}_A^k(\overline{M}) \subset \mathcal{C}^k(\overline{M})$ and $\mathcal{E}_A^k(\overline{E}) \subset \mathcal{E}^k(\overline{M})$. To show that κ is surjective, let $\alpha \in C^\infty(\overline{M}, \Lambda^k)$ be closed; we want $\tilde{\alpha} \in \mathcal{C}_a^k(\overline{M})$ such that $\alpha - \tilde{\alpha} = d\beta$ for some $\beta \in C^\infty(\overline{M}, \Lambda^{k-1})$.

To arrange this, we use a 1-parameter family of maps

$$\varphi_t : \overline{M} \longrightarrow \overline{M}, \quad 0 \le t \le 1, \tag{23.42}$$

such that φ_0 is the identity map, and as $t \to 1$, φ_t retracts a collar neighborhood $\mathcal{O}$ of ∂M onto ∂M, along geodesics normal to ∂M. Set $\tilde{\alpha} = \varphi_1^* \alpha$. It is easy to see that $\tilde{\alpha} \in \mathcal{C}_a^k(\overline{M})$. Furthermore, $\alpha - \tilde{\alpha} = d\beta$ with

$$\beta = -\int_0^1 \varphi_t^*\big(\alpha \rfloor X(t)\big)\, dt \in C^\infty(\overline{M}, \Lambda^{k-1}), \tag{23.43}$$

where $X(t) = (d/dt)\varphi_t$. Compare the proof of the Poincaré Lemma, Theorem 6.2. It follows that κ is surjective.

Consequently we have a natural surjective homomorphism

$$\tilde{\kappa} : \mathcal{H}_k^A \longrightarrow \mathcal{H}^k(\overline{M}). \tag{23.44}$$

It remains to prove that $\tilde{\kappa}$ is injective. But if $\alpha \in \mathcal{H}_k^A$ and $\alpha = d\beta$, $\beta \in C^\infty(\overline{M}, \Lambda^{k-1})$, then the identity (23.23) with $du = d\beta$, $v = \alpha$ implies $(\alpha, \alpha) = 0$, hence $\alpha = 0$. This completes the proof.

One can give a proof of (23.41) without using such a homotopy argument, in fact without using $\mathcal{C}_A^k(\overline{M})/\mathcal{E}_A^k(\overline{M})$ at all. See Exercise 5 in the exercises on cohomology after this section. On the other hand, homotopy arguments similar to that used above are also useful, and will arise in a number of problems in this set of exercises.

We can now establish the following Poincaré duality theorem, whose proof is immediate, since by (23.9) the Hodge star operator interchanges absolute and relative boundary conditions.

Proposition 23.5. *If $\overline{M}$ is an oriented compact Riemannian manifold with boundary, then*

$$* : \mathcal{H}_k^R \longrightarrow \mathcal{H}_{m-k}^A \tag{23.45}$$

is an isomorphism, where $m =$ dim M. Consequently,

$$\mathcal{H}^k(\overline{M}, \partial M) \approx \mathcal{H}^{m-k}(\overline{M}). \tag{23.46}$$

We end this section with a brief description of a sequence of maps on cohomology, associated to a compact manifold $\overline{M}$ with boundary. The sequence takes the form

$$\cdots \to \mathcal{H}^{k-1}(\partial M) \xrightarrow{\delta} \mathcal{H}^k(\overline{M}, \partial M) \xrightarrow{\pi} \mathcal{H}^k(\overline{M}) \xrightarrow{\iota} \mathcal{H}^k(\partial M) \xrightarrow{\delta} \mathcal{H}^{k+1}(\overline{M}, \partial M) \to \cdots \tag{23.47}$$

These maps are defined as follows. The inclusion

$$C_r^\infty(\overline{M}, \Lambda^k) \hookrightarrow C^\infty(\overline{M}, \Lambda^k),$$

yielding $\mathcal{C}_R^k(\overline{M}) \subset \mathcal{C}^k(\overline{M})$ and $\mathcal{E}_R^k(\overline{M}) \subset \mathcal{E}^k(\overline{M})$, gives rise to π in a natural fashion. The map ι comes from the pull-back

$$j^* : C^\infty(\overline{M}, \Lambda^k) \longrightarrow C^\infty(\partial M, \Lambda^k),$$

which induces a map on cohomology since $j^*d = dj^*$. Note that j^* annihilates $C_r^\infty(\overline{M}, \Lambda^k)$, so $\iota \circ \pi = 0$.

The "coboundary map" δ is defined on the class $[\alpha] \in \mathcal{H}^{k-1}(\partial M, \mathbb{R})$ of a closed form $\alpha \in \Lambda^{k-1}(\partial M)$ by choosing a form $\beta \in C^\infty(\overline{M}, \Lambda^{k-1})$ such that $j^*\beta = \alpha$ and taking the class $[d\beta]$ of $d\beta \in \mathcal{C}_R^k(\overline{M})$. Note that $d\beta$ might not belong to $\mathcal{E}_R^k(\overline{M})$ if

$j^*\beta$ is not exact. If another $\tilde{\beta}$ is picked such that $j^*\tilde{\beta} = \alpha + d\gamma$, then $d(\beta - \tilde{\beta})$ does belong to $\mathcal{E}_R^k(\overline{M})$, so δ is well defined:

$$\delta[\alpha] = [d\beta] \text{ with } j^*\beta = \alpha.$$

Note that, if $[\alpha] = \iota[\tilde{\beta}]$, via $\alpha = j^*\tilde{\beta}$ with $\tilde{\beta} \in \mathcal{C}^k(\overline{M})$, then $d\tilde{\beta} = 0$, so $\delta \circ \iota = 0$. Also, since $d\beta \in \mathcal{E}^k(\overline{M})$, $\pi \circ \delta = 0$.

In fact, the sequence (23.47) is *exact*, i.e., the image of each map is equal to the kernel of the map which follows. This "long exact sequence" in cohomology is a useful computational tool. Exactness will be sketched in some of the following exercises on cohomology.

Another important exact sequence, the Mayer-Vietoris sequence, is discussed in §24.

Exercises

1. Let u be a 1-form on M with associated vector field U. Show that the relative boundary conditions (23.4) are equivalent to

$$U \perp \partial M \text{ and div } U = 0 \text{ on } \partial M.$$

If dim $M = 3$, show that the absolute boundary conditions (23.5) are equivalent to

$$U \parallel \partial M \text{ and curl } U \perp \partial M.$$

Treat the case dim $M = 2$.

2. Form the orthogonal projections $P_d^b = d\delta G^b$, $P_\delta^b = \delta d G^b$. With $b = R$ or A, show that the four operators

$$G^b, P_h^b, P_d^b, \text{ and } P_\delta^b$$

all commute. Deduce that one can arrange the eigenfunctions $u_j^{(k)}$, forming an orthonormal basis of $L^2(M, \Lambda^k)$, such that each one appears in exactly one term in the Hodge decomposition (23.21), and that the same can be done with the eigenfunctions $v_j^{(k)}$, relative to the decomposition (23.22).

3. If $\overline{M}$ is oriented, and $*$ the Hodge star operator, show that

$$T_A * = * T_R,$$

where T_A and T_R are as in (23.26). Show that

$$P_h^A * = * P_h^R \text{ and } G^A * = * G^R.$$

Also, with P_d^b and P_δ^b the projections defined above, show that

$$P_d^A * = * P_\delta^R \text{ and } P_\delta^A * = * P_d^R.$$

Exercises on cohomology

1. Let $\overline{M}$ be a compact, connected manifold with nonempty boundary, and double N. Endow N with a Riemannian metric invariant under the involution τ. Show that

$$\mathcal{H}^k(\overline{M}, \partial M) \approx \{u \in \mathcal{H}_k(N) : \tau^* u = -u\}. \tag{23.48}$$

Deduce that, if M is also orientable,

$$\mathcal{H}^n(\overline{M}, \partial M) = \mathbb{R}, \quad n = \text{ dim } M. \tag{23.49}$$

2. If $\overline{M}$ is connected, show directly that

$$\mathcal{H}^0(\overline{M}) = \mathbb{R}.$$

By Poincaré duality, this again implies (23.49), when M is orientable.

3. Show that, if M is connected and $\partial M \neq \emptyset$,

$$\mathcal{H}^0(\overline{M}, \partial M) = 0.$$

Deduce that, if M is also orientable, $n = \text{dim } M$, then

$$\mathcal{H}^n(\overline{M}) = 0.$$

Give a proof of this that also works in the non-orientable case.

4. Show directly, using the proof of the Poincaré Lemma, Theorem 6.2 that

$$\mathcal{H}^k(\overline{B^n}) = 0, \quad 1 \le k \le n, \tag{23.50}$$

where $\overline{B^n}$ is the closed unit ball in $\mathbb{R}^n$, with boundary S^{n-1}. Deduce that

$$\begin{aligned} \mathcal{H}^k(\overline{B^n}, S^{n-1}) = 0, \quad & 0 \le k < n, \\ \mathbb{R}, \quad & k = n. \end{aligned} \tag{23.51}$$

5. Use (23.28) to show directly from Proposition 23.1 (not using Proposition 23.4) that, if $\alpha \in C^\infty(\overline{M}, \Lambda^k)$ is closed, then $\alpha = d\beta + P_h^A \alpha$ for some $\beta \in C^\infty(\overline{M}, \Lambda^{k-1})$, in fact, for $\beta = \delta G^A \alpha$. Hence conclude that

$$\mathcal{H}_k^A \approx \mathcal{H}^k(\overline{M})$$

without using the homotopy argument of Proposition 23.4.

Let M be a smooth manifold without boundary. The cohomology with *compact supports* $\mathcal{H}_c^k(M)$ is defined via

$$d : C_0^\infty(M, \Lambda^k) \longrightarrow C_0^\infty(M, \Lambda^{k+1}), \tag{23.52}$$

as

$$\mathcal{H}_c^k(M) = \mathcal{C}_c^k(M)/\mathcal{E}_c^k(M),$$

where the kernel of d in (23.52) is $\mathcal{C}_c^k(M)$ and its image is $\mathcal{E}_c^{k+1}(M)$.

In Exercises 6–7, we assume M is the interior of a compact manifold with boundary $\overline{M}$.

6. Via $C_0^\infty(M, \Lambda^k) \hookrightarrow C_r^\infty(\overline{M}, \Lambda^k)$, we have a well defined homomorphism

$$\rho : \mathcal{H}_c^k(M) \longrightarrow \mathcal{H}^k(\overline{M}, \partial M).$$

Show that ρ is injective.
Hint. Let $\varphi_t : \overline{M} \to \overline{M}$ be as in (23.42); also, given $K \subset\subset M$, arrange that each φ_t is the identity on K. If $\alpha \in \mathcal{C}_c^k(M)$ has support in K and $\alpha = d\beta$, with $\beta \in C_r^\infty(\overline{M}, \Lambda^{k-1})$, show that $\tilde{\beta} = \varphi_1^* \beta$ has compact support and $d\tilde{\beta} = \alpha$.

7. Show that ρ is *surjective*, and hence

$$\mathcal{H}_c^k(M) \approx \mathcal{H}^k(\overline{M}, \partial M). \tag{23.53}$$

Hint. If $\alpha \in \mathcal{C}_R^k(\overline{M})$, set $\tilde{\alpha} = \varphi_1^* \alpha$ and parallel the argument using (23.43), in the proof of Proposition 23.4.

8. If M is connected and oriented, and dim $M = n$, show that

$$\mathcal{H}_c^n(M) = \mathbb{R},$$

even if M cannot be compactified to a manifold with smooth boundary.
Hint. If $\alpha \in C_0^\infty(M, \Lambda^n)$ and $\int_M \alpha = 0$, fit the support of α in the interior Y of a compact smooth manifold with boundary $\overline{Y} \subset M$. Then apply arguments outlined above.

9. Let X be a compact connected manifold; given $p \in X$, let $M = X \setminus \{p\}$. Then $C_0^\infty(M, \Lambda^k) \hookrightarrow C^\infty(X, \Lambda^k)$ induces a homomorphism

$$\gamma : \mathcal{H}_c^k(M) \longrightarrow \mathcal{H}^k(X).$$

Show that γ is an *isomorphism*, for $0 < k \le$ dim X.
Hint. Construct a family of maps $\psi_t : X \to X$, with properties like φ_t used in Exercises 6–7, this time collapsing a neighborhood $\mathcal{O}$ of p onto p as $t \to 1$. Establish the injectivity and surjectivity of γ by arguments similar to those used in Exercises 6 and 7, noting that the analogue of the argument in Exercise 7 fails in this case when $k = 0$.

10. Using Exercise 9, deduce that

$$\mathcal{H}^k(S^n) \approx \mathcal{H}_c^k(\mathbb{R}^n), \quad 0 < k \le n. \tag{23.54}$$

In light of Exercises 4 and 7, show that this leads to

$$\begin{aligned} \mathcal{H}^k(S^n) = 0 & \quad \text{if } 0 < k < n, \\ \mathbb{R} & \quad \text{if } k = 0 \text{ or } n, \end{aligned} \tag{23.55}$$

provided $n \ge 1$, giving therefore a demonstration of (22.49)–(22.50) different from that suggested in Exercise 9 of §22.

Exercises 11–13 establish the exactness of the sequence (23.47).

11. Show that $\ker \iota \subset \operatorname{im} \pi$.
Hint. Given $u \in \mathcal{C}^k(\overline{M}),\ j^*u = dv$, pick $w \in \Lambda^{k-1}(\overline{M})$ such that $j^*w = v$, to get $u - dw \in C_r^\infty(\overline{M}, \Lambda^k)$, closed.

12. Show that $\ker \delta \subset \operatorname{im} \iota$.
Hint. Given $\alpha \in \mathcal{C}^k(\partial M)$, if $\alpha = j^*\beta$ with $[d\beta] = 0$ in $\mathcal{H}^{k+1}(M, \partial M)$, i.e., $d\beta = d\tilde{\beta},\ \tilde{\beta} \in C_r^\infty(\overline{M}, \Lambda^k)$, show that $[\alpha] = \iota[\beta - \tilde{\beta}]$.

13. Show that $\ker \pi \subset \operatorname{im} \delta$.
Hint. Given $u \in \mathcal{C}_R^k(\overline{M})$, if $u = dv,\ v \in \Lambda^{k-1}(\overline{M})$, show that $[u] = \delta[v]$.

14. Applying (23.47) to $\overline{M} = \overline{B^{n+1}}$, the closed unit ball in $\mathbb{R}^{n+1}$, yields

$$\mathcal{H}^k(\overline{B^{n+1}}) \xrightarrow{\iota} \mathcal{H}^k(S^n) \xrightarrow{\delta} \mathcal{H}^{k+1}(\overline{B^{n+1}}, S^n) \xrightarrow{\pi} \mathcal{H}^{k+1}(\overline{B^{n+1}}). \tag{23.56}$$

Deduce that

$$\mathcal{H}^k(S^n) \approx \mathcal{H}^{k+1}(\overline{B^{n+1}}, S^n), \ \text{ for } k \geq 1,$$

since by (23.50) the endpoints of (23.56) vanish for $k \geq 1$. Then, by (23.51), there follows a third demonstration of the computation (23.55) of $\mathcal{H}^k(S^n)$.

15. Using Exercise 3, show that, if M is connected and $\partial M \neq \emptyset$, the long exact sequence (23.47) begins with

$$0 \to \mathcal{H}^0(\overline{M}) \xrightarrow{\iota} \mathcal{H}^0(\partial M) \xrightarrow{\delta} \mathcal{H}^1(\overline{M}, \partial M) \to \cdots$$

and ends with

$$\cdots \to \mathcal{H}^{n-1}(\overline{M}) \xrightarrow{\iota} \mathcal{H}^{n-1}(\partial M) \xrightarrow{\delta} \mathcal{H}^n(\overline{M}, \partial M) \to 0.$$

16. Define the relative Euler characteristic

$$\chi(\overline{M}, \partial M) = \sum_{k \geq 0} (-1)^k \dim \mathcal{H}^k(\overline{M}, \partial M). \tag{23.57}$$

Define $\chi(\overline{M})$ and $\chi(\partial M)$ as in (22.44). Show that

$$\chi(\overline{M}) = \chi(\overline{M}, \partial M) + \chi(\partial M). \tag{23.58}$$

Hint. For any exact sequence of the form

$$0 \to V_1 \to \cdots \to V_N \to 0,$$

with V_k finite dimensional vector spaces over $\mathbb{R}$, show that $\sum\limits_{k\ge 0} (-1)^k \dim V_k = 0$.

17. Using Poincaré duality show that, if $\overline{M}$ is orientable, $n = \dim M$,

$$\chi(\overline{M}) = (-1)^n \chi(\overline{M}, \partial M). \tag{23.59}$$

Deduce that, if n is odd, and $\overline{M}$ orientable,

$$\chi(\partial M) = 2\chi(\overline{M}). \tag{23.60}$$

18. If N is the double of $\overline{M}$, show that

$$\dim \mathcal{H}^k(N) = \dim \mathcal{H}^k(\overline{M}) + \dim \mathcal{H}^k(\overline{M}, \partial M). \tag{23.61}$$

Deduce that, if M is orientable and $\dim M$ is even, then

$$\chi(N) = 2\chi(\overline{M}). \tag{23.62}$$

In Exercises 19–20, let $\overline{M}$ be a compact, oriented, n-dimensional manifold with boundary ∂M, and with double N. Let X be a vector field, all of whose critical points are isolated and in the interior M, and assume that, on ∂M, X points *out of* M. Our goal is to demonstrate that

$$\operatorname{Index} X = \chi(\overline{M}). \tag{23.63}$$

Recall that, in case $\overline{M} = M$ has no boundary, this arose in (10.11), as a definition of $\chi(M)$. Its identity with $\chi(M)$ as defined by (22.44) was established in §20. We also mention that (23.63) extends the identity (10.18).

19. Show that X and $-X$ can be fitted together on N to produce a vector field $\widetilde{X}$ on N such that

$$\operatorname{Index} \widetilde{X} = \big(1 + (-1)^n\big) \operatorname{Index} X.$$

Use this result and (23.62) to prove (23.63) when n is even.

20. Let Y be a (tangent) vector field on ∂M with isolated critical points (so we know $\text{Index}\, Y = \chi(\partial M)$). Use a construction like that in the proof of Proposition 10.4 to construct a vector field Z on a collar neighborhood $\mathcal{C}$ of ∂M in N such that Z points away from $\mathcal{C}$ on $\partial\mathcal{C}$ (diffeomorphic to two copies of ∂M) and $\text{Index}\, Z = \text{Index}\, Y$. Fit together X, $-Z$ and X to produce a vector field on N, and deduce that

$$2\ \text{Index}\, X + (-1)^n \chi(\partial M) = \chi(N). \tag{23.64}$$

Deduce (23.63) from this, using (23.62) if n is even and using (23.60) if n is odd.

In Exercises 21–23, let $\overline{\Omega}_j$ be compact oriented manifolds of dimension n, with boundary. Assume that $\partial\Omega_j \neq \emptyset$ and that Ω_2 is connected. Let $F : \overline{\Omega}_1 \to \overline{\Omega}_2$ be a smooth map with the property that $f = F\big|_{\partial\Omega_1} : \partial\Omega_1 \to \partial\Omega_2$. Recall that we have defined $\text{Deg}\ f$ in §9, when $\partial\Omega_2$ is connected.

21. Let $\sigma \in \Lambda^n(\overline{\Omega}_2)$ satisfy $\int_{\Omega_2} \sigma = 1$. Show that $\int_{\Omega_1} F^*\sigma$ is independent of the choice of such σ, using $\mathcal{H}^n(\overline{\Omega}_j, \partial\Omega_j) = \mathbb{R}$. Compare Lemma 9.6. Define

$$\text{Deg}\ F = \int_{\Omega_1} F^*\sigma.$$

22. Produce a formula for $\text{Deg}\ F$, similar to (9.16), using $F^{-1}(y_0),\ y_0 \in \Omega_2$.

23. Prove that $\text{Deg}\ F = \text{Deg}\ f$, assuming $\partial\Omega_2$ is connected.
Hint. Pick $\omega \in \Lambda^{n-1}(\partial\Omega_2)$ such that $\int_{\partial\Omega_2} \omega = 1$, pick $\tilde{\omega} \in \Lambda^{n-1}(\overline{\Omega}_2)$ such that $j^*\tilde{\omega} = \omega$, and let $\sigma = d\tilde{\omega}$. Formulate an extension of this result to cases where $\partial\Omega_2$ has several connected components.

24. Using the results of Exercises 21–23, establish the "argument principle," in complex analysis.
Hint. A holomorphic map is always orientation preserving.

In Exercise 25, we assume that $\overline{M}$ is a compact manifold with boundary, with interior M. Define $\mathcal{H}^k(M)$ via the deRham complex, $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$. It is desired to establish the isomorphism of this with $\mathcal{H}^k(\overline{M})$.

25. Let $\mathcal{C}$ be a small collar neighborhood of ∂M, so $\overline{M}_1 = \overline{M} \setminus \mathcal{C}$ is diffeomorphic to $\overline{M}$. With $j : \overline{M}_1 \hookrightarrow M$, show that the pull-back $j^* : \Lambda^k(M) \to \Lambda^k(\overline{M}_1)$ induces an isomorphism of cohomology:

$$\mathcal{H}^k(M) \approx \mathcal{H}^k(\overline{M}_1).$$

Hint. For part of the argument, it is useful to consider a smooth family

$$\varphi_t : \overline{M} \longrightarrow \overline{M}_t, \quad 0 \le t \le 1$$

of diffeomorphisms of $\overline{M}$ onto manifolds $\overline{M}_t$, with $\overline{M}_0 = \overline{M}$ and $\varphi_0 = id$. If $\beta \in \Lambda^k(M)$ and $d\beta = 0$, and if $\beta_1 = \varphi_1^* j^* \beta$, then

$$\beta = \beta_1 - d\Bigl(\int_0^1 \varphi_t^* \beta \rfloor X(t)\, dt\Bigr),$$

where $X(t)(x) = (d/dt)\varphi_t(x)$. Contrast this with the proof of Proposition 23.4.

24. The Mayer-Vietoris sequence in DeRham cohomology

Here we establish a useful complement to the long exact sequence (23.47), and illustrate some of its implications. Let X be a smooth manifold, and suppose X is the union of two open sets, M_1 and M_2. Let $U = M_1 \cap M_2$. The Mayer-Vietoris sequence has the form

(24.1)

$$\cdots \to \mathcal{H}^{k-1}(U) \xrightarrow{\delta} \mathcal{H}^k(X) \xrightarrow{\rho} \mathcal{H}^k(M_1) \oplus \mathcal{H}^k(M_2) \xrightarrow{\gamma} \mathcal{H}^k(U) \xrightarrow{\delta} \mathcal{H}^{k+1}(X) \to \cdots$$

These maps are defined as follows. A closed form $\alpha \in \Lambda^k(X)$ restricts to a pair of closed forms on M_1 and M_2, yielding ρ in a natural fashion. The map γ also comes from restriction; if $\iota_\nu : U \hookrightarrow M_\nu$, a pair of closed forms $\alpha_\nu \in \Lambda^k(M_\nu)$ goes to $\iota_1^*\alpha_1 - \iota_2^*\alpha_2$, defining γ. Clearly $\iota_1^*(\alpha|_{M_1}) = \iota_2^*(\alpha|_{M_2})$ if $\alpha \in \Lambda^k(X)$, so $\gamma \circ \rho = 0$.

To define the "coboundary map" δ on a class $[\alpha]$, with $\alpha \in \Lambda^k(U)$ closed, pick $\beta_\nu \in \Lambda^k(M_\nu)$ such that $\alpha = \beta_1 - \beta_2$. Thus $d\beta_1 = d\beta_2$ on U. Set

$$\delta[\alpha] = [\sigma] \text{ with } \sigma = d\beta_\nu \text{ on } M_\nu. \tag{24.2}$$

To show that (24.2) is well defined, suppose $\beta_\nu \in \Lambda^k(M_\nu)$ and $\beta_1 - \beta_2 = d\omega$ on U. Let $\{\varphi_\nu\}$ be a smooth partition of unity supported on $\{M_\nu\}$, and consider $\psi = \varphi_1\beta_1 + \varphi_2\beta_2$, where $\varphi_\nu\beta_\nu$ is extended by 0 off M_ν. We have $d\psi = \varphi_1\, d\beta_1 + \varphi_2\, d\beta_2 + d\varphi_1 \wedge (\beta_1 - \beta_2) = \sigma + d\varphi_1 \wedge (\beta_1 - \beta_2)$. Since $d\varphi_1$ is supported on U, we can write

$$\sigma = d\psi - d(d\varphi_1 \wedge \omega),$$

an exact form on X, so (24.2) makes δ well defined. Obviously the restriction of σ to each M_ν is always exact, so $\rho \circ \delta = 0$. Also, if $\alpha = \iota_1^*\alpha_1 - \iota_2^*\alpha_2$ on U, we can pick $\beta_\nu = \alpha_\nu$ to define $\delta[\alpha]$. Then $d\beta_\nu = d\alpha_\nu = 0$, so $\delta \circ \gamma = 0$.

In fact, the sequence (24.1) is exact, i.e.,

$$\text{im } \delta = \text{ ker } \rho, \quad \text{im } \rho = \text{ ker } \gamma, \quad \text{im } \gamma = \text{ ker } \delta. \tag{24.3}$$

We leave the verification of this as an exercise, which can be done with arguments similar to those sketched in Exercises 11–13 in the set of exercises on cohomology after §23.

If M_ν are the interiors of compact manifolds with smooth boundary, and $\overline{U} = \overline{M_1} \cap \overline{M_2}$ has smooth boundary, the argument above extends directly to produce an exact sequence

(24.4)

$$\cdots \to \mathcal{H}^{k-1}(\overline{U}) \xrightarrow{\delta} \mathcal{H}^k(X) \xrightarrow{\rho} \mathcal{H}^k(\overline{M}_1) \oplus \mathcal{H}^k(\overline{M}_2) \xrightarrow{\gamma} \mathcal{H}^k(\overline{U}) \xrightarrow{\delta} \mathcal{H}^{k+1}(X) \to \cdots$$

Furthermore, suppose that instead $X = \overline{M}_1 \cup \overline{M}_2$ and $\overline{M}_1 \cap \overline{M}_2 = Y$ is a smooth hypersurface in X. One also has an exact sequence
(24.5)
$$\cdots \to \mathcal{H}^{k-1}(Y) \xrightarrow{\delta} \mathcal{H}^k(X) \xrightarrow{\rho} \mathcal{H}^k(\overline{M}_1) \oplus \mathcal{H}^k(\overline{M}_2) \xrightarrow{\gamma} \mathcal{H}^k(Y) \xrightarrow{\delta} \mathcal{H}^{k+1}(X) \to \cdots$$

To relate (24.4) and (24.5), let U be a collar neighborhood of Y, and form (24.4) with $\overline{M}_\nu$ replaced by $\overline{M}_\nu \cup \overline{U}$. There is a map $\pi : \overline{U} \to Y$, collapsing orbits of a vector field transversal to Y, and π^* induces an isomorphism of cohomology groups, $\pi^* : \mathcal{H}^k(\overline{U}) \approx \mathcal{H}^k(Y)$.

To illustrate the use of (24.5), suppose $X = S^n, Y = S^{n-1}$ is the equator, and $\overline{M}_\nu$ are the upper and lower hemispheres, each diffeomorphic to the ball $\overline{B^n}$. Then we have an exact sequence
(24.6)
$$\cdots \to \mathcal{H}^{k-1}(\overline{B^n}) \oplus \mathcal{H}^{k-1}(\overline{B^n}) \xrightarrow{\gamma} \mathcal{H}^{k-1}(S^{n-1}) \xrightarrow{\delta} \mathcal{H}^k(S^n) \xrightarrow{\rho} \mathcal{H}^k(\overline{B^n}) \oplus \mathcal{H}^k(\overline{B^n}) \to \cdots$$

As in (23.50), $\mathcal{H}^k(\overline{B^n}) = 0$ except for $k = 0$, when you get $\mathbb{R}$. Thus

$$\delta : \mathcal{H}^{k-1}(S^{n-1}) \xrightarrow{\approx} \mathcal{H}^k(S^n) \text{ for } k > 1. \tag{24.7}$$

Granted that the computation $\mathcal{H}^1(S^1) \approx \mathbb{R}$ is elementary, (or see Exercise 1 below) this implies $\mathcal{H}^n(S^n) \approx \mathbb{R}$, for $n \geq 1$. Looking at the segment

$$0 \to \mathcal{H}^0(S^n) \xrightarrow{\rho} \mathcal{H}^0(\overline{B^n}) \oplus \mathcal{H}^0(\overline{B^n}) \xrightarrow{\gamma} \mathcal{H}^0(S^{n-1}) \xrightarrow{\delta} \mathcal{H}^1(S^n) \to 0,$$

we see that, if $n \geq 2$, then $\ker \gamma \approx \mathbb{R}$, so γ is surjective, hence $\delta = 0$, so $\mathcal{H}^1(S^n) = 0$ for $n \geq 2$. Also, if $0 < k < n$, we see by iterating (24.7) that $\mathcal{H}^k(S^n) \approx \mathcal{H}^1(S^{n-k+1})$, so $\mathcal{H}^k(S^n) = 0$ for $0 < k < n$. Since obviously $\mathcal{H}^0(S^n) = \mathbb{R}$ for $n \geq 1$, we have a fourth computation of $\mathcal{H}^k(S^n)$, distinct from those sketched in Exercise 10 of §22 and in Exercises 10 and 14 of the set of exercises on cohomology after §23.

We note an application of (24.5) to the computation of Euler characteristics, namely

$$\chi(\overline{M}_1) + \chi(\overline{M}_2) = \chi(X) + \chi(Y). \tag{24.8}$$

Note that this result contains some of the implications of Exercises 17 and 18 in the set of exercises on cohomology, in §23.

Using this, it is an exercise to show that if one two dimensional surface X_1 is obtained from another X_0 by adding a handle, then $\chi(X_1) = \chi(X_0) - 2$. (Compare Exercise 13 of §10.) In particular, if M^g is obtained from S^2 by adding g handles, then $\chi(M^g) = 2 - 2g$. Thus, if M^g is orientable, since $\mathcal{H}^0(M^g) \approx \mathcal{H}^2(M^g) \approx \mathbb{R}$, we have

$$\mathcal{H}^1(M^g) \approx \mathbb{R}^{2g}. \tag{24.9}$$

It is useful to examine the beginning of the sequence (24.5):

$$0 \to \mathcal{H}^0(X) \xrightarrow{\rho} \mathcal{H}^0(\overline{M}_1) \oplus \mathcal{H}^0(\overline{M}_2) \xrightarrow{\gamma} \mathcal{H}^0(Y) \xrightarrow{\delta} \mathcal{H}^1(X) \to \cdots \tag{24.10}$$

Suppose C is a smooth closed curve in S^2. Apply (24.10) with $M_1 = \mathcal{C}$, a collar neighborhood of C, and $\overline{M}_2 = \overline{\Omega}$, the complement of $\mathcal{C}$. Since $\partial\mathcal{C}$ is diffeomorphic to two copies of C, and since $\mathcal{H}^1(S^2) = 0$, (24.10) becomes

$$0 \to \mathbb{R} \xrightarrow{\rho} \mathbb{R} \oplus \mathcal{H}^0(\overline{\Omega}) \xrightarrow{\gamma} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} 0 \tag{24.11}$$

Thus γ is surjective while ker $\gamma =$ im $\rho \approx \mathbb{R}$. This forces

$$\mathcal{H}^0(\overline{\Omega}) \approx \mathbb{R} \oplus \mathbb{R}. \tag{24.12}$$

In other words, $\overline{\Omega}$ has exactly two connected components. This is the smooth case of the Jordan curve theorem. Jordan's theorem holds when C is a homeomorphic image of S^1, but the trick of putting a collar about C does not extend to this case.

More generally, if X is a compact connected smooth oriented manifold such that $\mathcal{H}^1(X) = 0$, and if Y is a smooth compact connected oriented hypersurface, then letting $\mathcal{C}$ be a collar neighborhood of Y and $\overline{\Omega} = X \setminus \mathcal{C}$, we again obtain the sequence (24.11) and hence the conclusion (24.12). The orientability insures that $\partial\mathcal{C}$ is diffeomorphic to two copies of Y. This result is (the smooth case of) the following generalized Jordan-Brouwer separation theorem, which we state formally.

Theorem 24.1. *If X is a smooth manifold, Y a smooth submanifold of codimension 1, and both are*

compact, connected, and oriented,

and if

$$\mathcal{H}^1(X) = 0,$$

then $X \setminus Y$ has precisely two connected components.

If all these conditions hold, *except* that Y is *not* orientable, then we replace $\mathbb{R} \oplus \mathbb{R}$ by $\mathbb{R}$ in (24.11), and conclude that $X \setminus Y$ is connected, in that case. As an example, real projective space $\mathbb{RP}^2$ sits in $\mathbb{RP}^3$ in such a fashion.

Recall from §9 the elementary proof of Theorem 24.1 when $X = \mathbb{R}^{n+1}$, in particular the argument using degree theory that, if Y is a compact oriented surface in $\mathbb{R}^{n+1}$ (hence, in S^{n+1}), then its complement has at least 2 connected components. One can extend the degree theory argument to the nonorientable case, as follows.

There is a notion of degree mod 2 of a map $F : Y \to S^n$, which is well defined whether or not Y is orientable. For one approach, see [Mil2]. This is also invariant under homotopy. Now, if in the proof of Theorem 9.11, one drops the hypothesis that the hypersurface Y (denoted X there) is orientable, it still follows that the mod 2 degree of F_p must jump by ± 1 when p crosses Y, so $\mathbb{R}^{n+1} \setminus Y$ still must have at least two connected components. In view of the result noted after Theorem 24.1, this situation cannot arise. This establishes the following.

Proposition 24.2. *If Y is a compact hypersurface of $\mathbb{R}^{n+1}$ (or S^{n+1}), then Y is orientable.*

Exercises

1. Show that the case $n = 1$ of (24.6) yields

$$\mathcal{H}^0(S^1) \xrightarrow{\rho} \mathcal{H}^0(I) \oplus \mathcal{H}^0(I) \xrightarrow{\gamma} \mathcal{H}^0(S^0) \xrightarrow{\delta} \mathcal{H}^1(S^1) \xrightarrow{\rho} \mathcal{H}^1(I) \oplus \mathcal{H}^1(I)$$

where I is a closed interval, so $\mathcal{H}^1(I) = 0$.
(a) Note that, since S^0 consists of two points, $\mathcal{H}^0(S^0) \approx \mathbb{R}^2$, so γ above has the form $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$.
(b) Show that the image $\rho\big(\mathcal{H}^0(S^1)\big)$ in $\mathcal{H}^0(I) \oplus \mathcal{H}^0(I)$ has dimension 1, so $\ker \gamma$ has dimension 1.
(c) Deduce that $\ker \delta$ has dimension 1, and hence that

$$\mathcal{H}^1(S^1) \approx \mathbb{R}, \tag{24.13}$$

thus showing that this fact follows from the Mayer-Vietoris sequence.
(d) On the other hand, show that the specialization of the proof of Proposition 9.5 to the case $M = S^1$, implying (24.13), is trivial.

25. Operators of Dirac type

Let M be a Riemannian manifold, $E_j \to M$ vector bundles with Hermitian metrics. A first order elliptic differential operator

$$D : C^\infty(M, E_0) \longrightarrow C^\infty(M, E_1) \tag{25.1}$$

is said to be of *Dirac type* provided D^*D has scalar principal symbol. This implies

$$\sigma_{D^*D}(x, \xi) = g(x, \xi)I : E_{0x} \longrightarrow E_{0x}, \tag{25.2}$$

where $g(x, \xi)$ is a positive quadratic form on $T^*_x M$. Thus g itself arises from a Riemannian metric on M. Now the calculation of (25.2) is independent of the choice of Riemannian metric on M. We will suppose M is endowed with the Riemannian metric inducing the form $g(x, \xi)$ on T^*M.

If $E_0 = E_1$ and $D = D^*$, we say D is a symmetric Dirac-type operator. Given a general operator D of Dirac type, if we set $E = E_0 \oplus E_1$ and define $\tilde{D}$ on $C^\infty(M, E)$ as

$$\tilde{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \tag{25.3}$$

then D is a symmetric Dirac-type operator.

Let $\vartheta(x, \xi)$ denote the principal symbol of a symmetric Dirac-type operator. With $x \in M$ fixed, set $\vartheta(\xi) = \vartheta(x, \xi)$. Thus ϑ is a linear map from $T^*_x M = \{\xi\}$ into $\mathrm{End}(E_x)$, satisfying

$$\vartheta(\xi) = \vartheta(\xi)^* \tag{25.4}$$

and

$$\vartheta(\xi)^2 = \langle \xi, \xi \rangle I. \tag{25.5}$$

Here, $\langle\ ,\ \rangle$ is the inner product on $T^*_x M$; let us denote this vector space by V. We will show how ϑ extends from V to an algebra homomorphism, defined on a *Clifford algebra* $Cl(V, g)$, which we now proceed to define.

Let V be a finite dimensional real vector space, g a quadratic form on V. We allow g to be definite or indefinite if nondegenerate; we even allow g to be degenerate. The Clifford algebra $Cl(V, g)$ is the quotient algebra of the tensor algebra

$$\bigotimes V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots \tag{25.6}$$

by the ideal $\mathcal{I} \subset \bigotimes V$ generated by

$$\{v \otimes w + w \otimes v - 2\langle v, w\rangle \cdot 1 : v, w \in V\}, \tag{25.7}$$

where $\langle \ , \ \rangle$ is the symmetric bilinear form on V arising from g. Thus, in $Cl(V, g)$, V occurs naturally as a linear subspace, and there is the anticommutation relation

$$vw + wv = 2\langle v, w\rangle \cdot 1 \quad \text{in} \quad Cl(V, g), \quad v, w \in V. \tag{25.8}$$

We will look more closely at the structure of Clifford algebras in the next section.

Now if $\vartheta : V \to \operatorname{End}(E)$ is a linear map of the V into the space of endomorphisms of a vector space E, satisfying (25.5), i.e.,

$$\vartheta(v)^2 = \langle v, v\rangle I, \quad v \in V, \tag{25.9}$$

it follows from expanding $\vartheta(v + w)^2 = [\vartheta(v) + \vartheta(w)]^2$ that

$$\vartheta(v)\vartheta(w) + \vartheta(w)\vartheta(v) = 2\langle v, w\rangle I, \quad v, w \in V. \tag{25.10}$$

Then, from the construction of $Cl(V, g)$, it follows that ϑ extends uniquely to an algebra homomorphism

$$\vartheta : Cl(V, g) \longrightarrow \operatorname{End}(E), \quad \vartheta(1) = I. \tag{25.11}$$

This gives E the structure of a module over $Cl(V, g)$, i.e., of a Clifford module. If E has a Hermitian metric and (25.4) also holds, i.e.,

$$\vartheta(v) = \vartheta(v)^*, \quad v \in V, \tag{25.12}$$

we call E a Hermitian Clifford module. For this notion to be useful, we need g to be positive definite.

In the case where $E = E_0 \oplus E_1$ is a direct sum of Hermitian vector spaces, we say a homomorphism $\vartheta : Cl(V, g) \to \operatorname{End}(E)$ gives E the structure of a *graded* Clifford module provided $\vartheta(v)$ interchanges E_0 and E_1, for $v \in V$, in addition to the hypotheses above. The principal symbol of (25.3) has this property, if D is of Dirac type.

Let us give some examples of operators of Dirac type. If M is a Riemannian manifold, the exterior derivative operator

$$d : \Lambda^j M \longrightarrow \Lambda^{j+1} M \tag{25.13}$$

has a formal adjoint

$$\delta = d^* : \Lambda^{j+1} M \longrightarrow \Lambda^j M, \tag{25.14}$$

discussed in §§21–23. Thus we have

$$d + \delta : \Lambda^* M \longrightarrow \Lambda^* M, \tag{25.15}$$

where, with $n = \dim M$,

$$\Lambda^* M = \bigoplus_{j=0}^{n} \Lambda^j M.$$

As was shown in §21, $(d+\delta)^*(d+\delta) = d^*d + dd^*$ is the negative of the Hodge Laplacian on each $\Lambda^j M$, so (25.15) is a symmetric Dirac-type operator. There is more structure. Indeed, we have

$$d + \delta : \Lambda^{\text{even}} M \longrightarrow \Lambda^{\text{odd}} M. \tag{25.16}$$

If D is this operator, then $D^* = d + \delta : \Lambda^{\text{odd}} M \to \Lambda^{\text{even}} M$, and an operator of type (25.3) arises.

Computations implying that (25.15) is of Dirac type were done in §21, leading to (21.22) there. If we define

$$\wedge_v : \Lambda^j V \longrightarrow \Lambda^{j+1} V, \quad \wedge_v(v_1 \wedge \cdots \wedge v_j) = v \wedge v_1 \wedge \cdots \wedge v_j, \tag{25.17}$$

on a vector space V with a positive definite inner product, and then define

$$\iota_v : \Lambda^{j+1} V \longrightarrow \Lambda^j V \tag{25.18}$$

to be its adjoint, then the principal symbol of $d + \delta$ on $V = T_x^* M$ is $1/i$ times $\wedge_\xi - \iota_\xi$. That is to say,

$$iM(v) = \wedge_v - \iota_v \tag{25.19}$$

defines a linear map from V into $\text{End}(\Lambda^* V)$ which extends to an algebra homomorphism

$$M : Cl(V, g) \longrightarrow \text{End}(\Lambda^* V).$$

Granted $\wedge_v \wedge_w = - \wedge_w \wedge_v$ and its analogue for ι, the anticommutation relation

$$M(v)M(w) + M(w)M(v) = 2\langle v, w \rangle I \tag{25.20}$$

also follows from the identity

$$\wedge_v \iota_w + \iota_w \wedge_v = \langle v, w \rangle I. \tag{25.21}$$

In this context we note the role that (25.21) played as the algebraic identity behind Cartan's formula for the Lie derivative of a differential form:

$$\mathcal{L}_X \alpha = d(\alpha \rfloor X) + (d\alpha) \rfloor X; \tag{25.22}$$

cf. (6.20).

Another Dirac-type operator arises from (25.15) as follows. Suppose dim $M = n = 2k$ is even. Recall from §22 that $d^* = \delta$ is given in terms of the Hodge star operator on $\Lambda^j M$ by

$$d^* = (-1)^{j(n-j)+j} * d* \\ = *d* \qquad \text{if } \; n = 2k. \tag{25.23}$$

Also recall that, on $\Lambda^j M$,

$$*^2 = (-1)^{j(n-j)} = (-1)^j \quad \text{ if } \; n = 2k. \tag{25.24}$$

Now define on the complexification $\Lambda^*_{\mathbb{C}} M$ of the real vector bundle $\Lambda^* M$

$$\alpha : \Lambda^j_{\mathbb{C}} M \longrightarrow \Lambda^{n-j}_{\mathbb{C}} M \tag{25.25}$$

by

$$\alpha = i^{j(j-1)+k} * \quad \text{ on } \; \Lambda^j_{\mathbb{C}} M. \tag{25.26}$$

It follows that

$$\alpha^2 = 1 \tag{25.27}$$

and

$$\alpha(d+\delta) = -(d+\delta)\alpha. \tag{25.28}$$

Thus we can write

$$\Lambda^*_{\mathbb{C}} M = \Lambda^+ M \oplus \Lambda^- M \; \text{ with } \; \alpha = \pm I \; \text{ on } \; \Lambda^\pm M, \tag{25.29}$$

and we have

$$D_H^\pm = d + \delta : C^\infty(M, \Lambda^\pm) \longrightarrow C^\infty(M, \Lambda^\mp). \tag{25.30}$$

Thus D_H^+ is an operator of Dirac type, with adjoint D_H^-. This operator is called the Hirzebruch signature operator.

Both of the examples discussed above give rise to Hermitian Clifford modules. We now show conversely that generally such modules produce operators of Dirac type. More precisely, if M is a Riemannian manifold, $T_x^* M$ has an induced inner product, giving rise to a bundle $Cl(M) \to M$ of Clifford algebras. We suppose $E \to M$ is a Hermitian vector bundle such that each fiber is a Hermitian $Cl_x(M)$-module (in a smooth fashion). Let $E \to M$ have a connection ∇, so

$$\nabla : C^\infty(M, E) \longrightarrow C^\infty(M, T^* \otimes E). \tag{25.31}$$

Now if E_x is a $Cl_x(M)$-module, the inclusion $T_x^* \hookrightarrow Cl_x$ gives rise to a linear map

$$m : C^\infty(M, T^* \otimes E) \longrightarrow C^\infty(M, E), \tag{25.32}$$

called "Clifford multiplication." We compose these two operators; set

$$D = i\ m \circ \nabla : C^\infty(M, E) \longrightarrow C^\infty(M, E). \tag{25.33}$$

We see that, for $v \in E_x$,

$$\sigma_D(x, \xi)v = m(\xi \otimes v) = \xi \cdot v, \tag{25.34}$$

so $\sigma_D(x, \xi)$ is $|\xi|_x$ times an isometry on E_x. Hence D is of Dirac type.

If U is an open subset of M, on which we have an orthonormal frame $\{e_j\}$ of smooth vector fields, with dual orthonormal frame $\{v_j\}$ of 1-forms, then, for a section φ of E,

$$D\varphi = i \sum v_j \cdot \nabla_{e_j} \varphi \text{ on } U. \tag{25.35}$$

Note that $\sigma_D(x, \xi)^* = \sigma_D(x, \xi)$, so D can be made symmetric, by at most altering it by a zero order term. Given a little more structure, we have more. We say ∇ is a *Clifford connection* on E if ∇ is a metric connection which is also compatible with Clifford multiplication, in that

$$\nabla_X(v \cdot \varphi) = (\nabla_X v) \cdot \varphi + v \cdot \nabla_X \varphi, \tag{25.36}$$

for a vector field X, a 1-form v, and a section φ of E. Here, of course, $\nabla_X v$ arises from the Levi-Civita connection on M.

Proposition 25.1. *If ∇ is a Clifford connection on E, then D is symmetric.*

Proof. Let $\varphi, \psi \in C_0^\infty(M, E)$. We want to show that

$$\int_M \Big[\langle D\varphi, \psi\rangle - \langle \varphi, D\psi\rangle\Big] dV = 0. \tag{25.37}$$

We can suppose φ, ψ have compact support in a set U on which local orthonormal frames e_j, v_j as above are given. Define a vector field X on U by

$$\langle X, v\rangle = \langle \varphi, v \cdot \psi\rangle, \quad v \in \Lambda^1 U.$$

If we show that, pointwise in U,

$$i \text{ div } X = \langle D\varphi, \psi\rangle - \langle \varphi, D\psi\rangle, \tag{25.38}$$

then (25.37) will follow from the Divergence Theorem. Indeed, starting with

$$\text{div } X = \sum \langle \nabla_{e_j} X, v_j \rangle, \tag{25.39}$$

and using the metric and derivation properties of ∇, we have

$$\begin{aligned} \text{div } X &= \sum \Big[e_j \cdot \langle X, v_j \rangle - \langle X, \nabla_{e_j} v_j \rangle \Big] \\ &= \sum \Big[e_j \langle \varphi, v_j \cdot \psi \rangle - \langle \varphi, (\nabla_{e_j} v_j) \cdot \psi \rangle \Big]. \end{aligned}$$

Looking at the last quantity, we expand the first part into a sum of three terms, one of which cancels the last part, and obtain

$$\text{div } X = \sum \Big[\langle \nabla_{e_j} \varphi, v_j \cdot \psi \rangle + \langle \varphi, v_j \cdot \nabla_{e_j} \psi \rangle \Big], \tag{25.40}$$

which gives (25.38) and completes the proof.

If $E = E_0 \oplus E_1$ is a graded Hermitian $Cl(M)$-module, and if E_0 and E_1 are each provided with metric connections, and (25.36) holds, then the construction above gives an operator of Dirac type, of the form (25.3).

The examples (25.15) and (25.30) described above can be obtained from Hermitian Clifford modules via Clifford connections. The Clifford module is $\Lambda^* M \to M$, with natural inner product on each factor $\Lambda^k M$ and $Cl(M)$-module structure given by (25.19). The connection is the natural connection on $\Lambda^* M$, extending that on $T^* M$, so that the derivation identity

$$\nabla_X (\varphi \wedge \psi) = (\nabla_X \varphi) \wedge \psi + \varphi \wedge (\nabla_X \psi) \tag{25.41}$$

holds for a j-form φ and a k-form ψ. In this case it is routine to verify the compatibility condition (25.36) and to see that the construction (25.33) gives rise to the operator $d + d^*$ on differential forms.

We remark that it is common to use Clifford algebras associated to negative definite forms rather than positive definite ones. The two types of algebras are simply related. If a linear map $\vartheta : V \to \text{End}(E)$ extends to an algebra homomorphism $Cl(V, g) \to \text{End}(E)$, then $i\vartheta$ extends to an algebra homomorphism $Cl(V, -g) \to \text{End}(E)$. If one uses a negative form, the condition (25.12) that E be a Hermitian Clifford module should be changed to $\vartheta(v) = -\vartheta(v)^*,\ v \in V$. In such a case, we should drop the factor of i in (25.33) to associate the Dirac-type operator D to a $Cl(M)$-module E. In fact, getting rid of this factor of i in (25.33) and (25.35) is perhaps the principal reason some people use the negative quadratic form to construct Clifford algebras.

Exercises

1. Let E be a $Cl(M)$-module with connection ∇. If φ is a section of E and f a scalar function, show that

$$D(f\varphi) = f\ D\varphi + i(df) \cdot \varphi,$$

where the last term involves a Clifford multiplication.

2. If ∇ is a Clifford connection on E, and u is a 1-form, show that

$$D(u \cdot \varphi) = -u \cdot D\varphi + 2i\nabla_U \varphi + i(\mathcal{D}u) \cdot \varphi$$

where U is the vector field corresponding to u via the metric tensor on M and

$$\mathcal{D} : C^\infty(M, \Lambda^1) \longrightarrow C^\infty(M, Cl)$$

is given by

$$\mathcal{D}u = i \sum v_j \cdot \nabla_{e_j} u,$$

with respect to local dual orthonormal frames e_j, v_j, and ∇ arising from the Levi-Civita connection.

3. Show that $\mathcal{D}(df) = i\Delta f$.
Note. Compare Exercise 6 of §26.

4. If D arises from a Clifford connection on E, show that

$$D^2(f\varphi) = f\ D^2\varphi - 2\nabla_{\mathrm{grad}\ f}\varphi - (\Delta f)\varphi.$$

26. Clifford algebras

In this section we discuss some further results about the structure of Clifford algebras, which were defined in §25.

First we note that by construction $Cl(V,g)$ has the following universal property. Let A_0 be any associative algebra over $\mathbb{R}$, with unit, containing V as a linear subset, generated by V, such that the anticommutation relation (25.8) holds in A_0, for all $v, w \in V$, i.e., $vw + wv = 2\langle v, w\rangle \cdot 1$ in A_0. Then there is a natural surjective homomorphism

$$\alpha : Cl(V,g) \longrightarrow A_0. \tag{26.1}$$

If $\{e_1, \dots, e_n\}$ is a basis of V, any element of $Cl(V,g)$ can be written as a polynomial in the e_j. Since $e_j e_k = -e_k e_j + 2\langle e_j, e_k\rangle \cdot 1$ and in particular $e_j^2 = \langle e_j, e_j\rangle \cdot 1$, we can, starting with terms of highest order, rearrange each monomial in such a polynomial so the e_j appear with j in ascending order, and no exponent greater than one occurs on any e_j. In other words, each element $w \in Cl(V,g)$ can be written in the form

$$w = \sum_{i_\nu = 0 \text{ or } 1} a_{i_1 \cdots i_n}\, e_1^{i_1} \cdots e_n^{i_n}, \tag{26.2}$$

with real coefficients $a_{i_1 \cdots i_n}$.

Denote by A the set of formal expressions of the form (26.2), a real vector space of dimension 2^n; we have a natural inclusion $V \subset A$. We can define a "product" $A \otimes A \to A$ in which a product of monomials $(e_1^{i_1} \cdots e_n^{i_n}) \cdot (e_1^{j_1} \cdots e_n^{j_n})$ with each i_ν and each j_μ equal to either 0 or 1, is a linear combination of monomials of such a form, by pushing each $e_\mu^{j_\mu}$ past the $e_\nu^{i_\nu}$ for $\nu > \mu$, invoking the anticommutation relations. It is routine to verify that this gives A the structure of an associative algebra, generated by V. The universal property mentioned above implies that A is isomorphic to $Cl(V,g)$. Thus each $w \in Cl(V,g)$ has a unique representation in the form (26.2), and $\dim Cl(V,g) = 2^n$ if $\dim V = n$.

Recall from §25 the algebra homomorphism $M : Cl(V,g) \to \operatorname{End}(\Lambda^* V)$, defined there provided g is positive definite (which can be extended to include general g). Then, we can define a linear map

$$\tilde{M} : Cl(V,g) \longrightarrow \Lambda^* V; \quad \tilde{M}(w) = M(w)(1), \tag{26.3}$$

for $w \in Cl(V,g)$. Note that, if $v \in V \subset Cl(V,g)$, then $\tilde{M}(v) = v$. Comparing the anticommutation relations of $Cl(V,g)$ with those of $\Lambda^* V$, we see that, if $w \in Cl(V,g)$ is one of the monomials in (26.2), say $w = e_1^{j_1} \cdots e_n^{j_n}$, all j_ν either 0 or 1, $k = j_1 + \cdots + j_n$, then

$$\tilde{M}(e_1^{j_1} \cdots e_n^{j_n}) - e_1^{j_1} \wedge \cdots \wedge e_n^{j_n} \in \Lambda^{k-1} V. \tag{26.4}$$

It follows easily that (26.3) is an isomorphism of vector spaces. This observation also shows that the representation of an element of $Cl(V, g)$ in the form (26.2) is unique. If g is positive definite and e_j is an orthonormal basis of V, the difference in (26.4) vanishes.

In the case $g = 0$, the anticommutation relation (25.8) becomes $vw = -wv$, for $v, w \in V$, and we have the exterior algebra:

$$Cl(V, 0) = \Lambda^* V.$$

Through the remainder of this section we will restrict attention to the case where g is positive definite. We denote $\langle v, v\rangle$ by $|v|^2$. For $V = \mathbb{R}^n$ with g its standard Euclidean inner product, we denote $Cl(V, g)$ by $Cl(n)$.

It is useful to consider the complexified Clifford algebra

$$\mathbb{C}l(n) = \mathbb{C} \otimes Cl(n),$$

as it has a relatively simple structure, specified as follows.

Proposition 26.1. *There are isomorphisms of complex algebras*

$$\mathbb{C}l(1) \approx \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C}l(2) \approx End(\mathbb{C}^2), \tag{26.5}$$

and

$$\mathbb{C}l(n+2) \approx \mathbb{C}l(n) \otimes \mathbb{C}l(2); \tag{26.6}$$

hence, with $\kappa = 2^k$,

$$\mathbb{C}l(2k) \approx End(\mathbb{C}^\kappa), \quad \mathbb{C}l(2k+1) \approx End(\mathbb{C}^\kappa) \oplus End(\mathbb{C}^\kappa). \tag{26.7}$$

Proof. The isomorphisms (26.5) are simple exercises. To prove (26.6), embed $\mathbb{R}^{n+2}$ into $\mathbb{C}l(n) \otimes \mathbb{C}l(2)$ by picking an orthonormal basis $\{e_1, \dots, e_{n+2}\}$ and taking

$$\begin{aligned} &e_j \mapsto i\; e_j \otimes e_{n+1}e_{n+2} && \text{for } \ 1 \le j \le n, \\ &e_j \mapsto 1 \otimes e_j && \text{for } \ j = n+1 \text{ or } n+2. \end{aligned} \tag{26.8}$$

Then the universal property of $\mathbb{C}l(n+2)$ leads to the isomorphism (26.6). Given (26.5)–(26.6), then (26.7) follows by induction.

While, parallel to (26.5), one has $Cl(1) = \mathbb{R} \oplus \mathbb{R}$ and $Cl(2) = \mathrm{End}(\mathbb{R}^2)$, other algebras $Cl(n)$ are more complicated than their complex analogues; in place of (26.6) one has a form of periodicity with period 8. We refer to [LM] for more on this.

It follows from Proposition 26.1 that $\mathbb{C}^{2^k}$ has the structure of an irreducible $Cl(2k)$ module, though making the identification (26.7) explicit involves some untangling, in a way that depends strongly on a choice of basis. It is worthwhile to note the following explicit, invariant construction, for V, a vector space of real dimension $2k$, with a positive inner product $\langle\ ,\ \rangle$, endowed with one other piece of structure, namely a complex structure J. Assume J is an isometry for $\langle\ ,\ \rangle$. Denote the complex vector space (V,J) by $\mathcal{V}$, which has complex dimension k. On $\mathcal{V}$ we have a positive Hermitian form

$$(u,v)=\langle u,v\rangle+i\langle u,Jv\rangle. \tag{26.9}$$

Form the complex exterior algebra

$$\Lambda^*_{\mathbb{C}}\mathcal{V}=\bigoplus_{j=0}^{k}\Lambda^j_{\mathbb{C}}\mathcal{V}, \tag{26.10}$$

with its natural Hermitian form. For $v\in\mathcal{V}$, one has the exterior product $v\wedge:\Lambda^j_{\mathbb{C}}\mathcal{V}\to\Lambda^{j+1}_{\mathbb{C}}\mathcal{V}$; denote its adjoint, the interior product, by $j_v:\Lambda^{j+1}_{\mathbb{C}}\mathcal{V}\to\Lambda^j_{\mathbb{C}}\mathcal{V}$. Set

$$i\ \mu(v)\varphi=v\wedge\varphi-j_v\varphi,\quad v\in\mathcal{V},\ \varphi\in\Lambda^*_{\mathbb{C}}\mathcal{V}. \tag{26.11}$$

Note that $v\wedge\varphi$ is $\mathbb{C}$-linear in v and $j_v\varphi$ is conjugate linear in v, so $\mu(v)$ is only $\mathbb{R}$-linear in v. As in (25.20), we obtain

$$\mu(u)\mu(v)+\mu(v)\mu(u)=2\langle u,v\rangle\cdot I, \tag{26.12}$$

so $\mu:V\to\mathrm{End}(\Lambda^*_{\mathbb{C}}\mathcal{V})$ extends to a homomorphism of algebras

$$\mu:Cl(V,g)\longrightarrow\mathrm{End}(\Lambda^*_{\mathbb{C}}\mathcal{V}), \tag{26.13}$$

hence to a homomorphism of $\mathbb{C}$-algebras

$$\mu:\mathbb{C}l(V,g)\longrightarrow\mathrm{End}(\Lambda^*_{\mathbb{C}}\mathcal{V}), \tag{26.14}$$

where $\mathbb{C}l(V,g)$ denotes $\mathbb{C}\otimes Cl(V,g)$.

Proposition 26.2. *The homomorphism (26.14) is an isomorphism, when V is a real vector space of dimension $2k$, with complex structure J, $\mathcal{V}$ the associated complex vector space.*

Proof. We already know that both $\mathbb{C}l(V,g)$ and $\mathrm{End}(\Lambda^*_{\mathbb{C}}\mathcal{V})$ are isomorphic to $\mathrm{End}(\mathbb{C}^\kappa)$, $\kappa=2^k$. We will make use of the algebraic fact that this is a complex algebra which has no proper 2-sided ideals. Now the kernel of μ in (26.14) would have to be a 2-sided ideal, so either $\mu=0$ or μ is an isomorphism. But for $v\in V,\ \mu(v)\cdot 1=v$, so $\mu\neq 0$; thus μ is an isomorphism.

We next mention that a grading can be put on $Cl(V,g)$. Namely, let $Cl^0(V,g)$ denote the set of sums of the form (26.2) with $i_1+\cdots+i_n$ even, and let $Cl^1(V,g)$

denote the set of sums of that form with $i_1 + \cdots + i_n$ odd. It is easy to see that this specification is independent of the choice of basis $\{e_j\}$. Also we clearly have

$$u \in Cl^j(V,g),\ w \in Cl^k(V,g) \Longrightarrow uw \in Cl^{j+k}(V,g), \tag{26.15}$$

where j and k are each 0 or 1, and we compute $j+k$ mod 2. If (V,g) is $\mathbb{R}^n$ with its standard Euclidean metric, we denote $Cl^j(V,g)$ by $Cl^j(n)$, $j = 0$ or 1.

We note that there is an isomorphism

$$j : Cl(2k-1) \longrightarrow Cl^0(2k) \tag{26.16}$$

uniquely specified by the property that, for $v \in \mathbb{R}^{2k-1}$, $j(v) = ve_{2k}$, where $\{e_1, \dots, e_{2k-1}\}$ denotes the standard basis of $\mathbb{R}^{2k-1}$, with e_{2k} added to form a basis of $\mathbb{R}^{2k}$. This will be useful in the next section for constructing spinors on odd dimensional spaces.

We can construct a finer grading on $Cl(V,g)$. Namely set

$$Cl^{[k]}(V,g) = \text{ set of sums of the form (26.2) with } i_1 + \cdots + i_n = k. \tag{26.17}$$

Thus $Cl^{[0]}(V,g)$ is the set of scalars and $Cl^{[1]}(V,g)$ is V. If we insist that $\{e_j\}$ be an orthonormal basis of V, then $Cl^{[k]}(V,g)$ is invariantly defined, for all k. In fact, using the isomorphism (26.3), we have

$$Cl^{[k]}(V,g) = \tilde{M}^{-1}\big(\Lambda^k V\big). \tag{26.18}$$

Note that

$$Cl^0(V,g) = \bigoplus_{k \text{ even}} Cl^{[k]}(V,g), \quad Cl^1(V,g) = \bigoplus_{k \text{ odd}} Cl^{[k]}(V,g).$$

Let us also note that $Cl^{[2]}(V,g)$ has a natural Lie algebra structure. In fact, if $\{e_j\}$ is orthonormal,

$$\begin{aligned}[e_ie_j, e_ke_\ell] &= e_ie_je_ke_\ell - e_ke_\ell e_ie_j \\ &= 2(\delta_{jk}e_ie_\ell - \delta_{\ell j}e_ie_k + \delta_{ik}e_\ell e_j - \delta_{\ell i}e_ke_j).\end{aligned} \tag{26.19}$$

The construction (26.17) makes $Cl(V,g)$ a graded vector space, but not a graded algebra, since typically $Cl^{[j]}(V,g) \cdot Cl^{[k]}(V,g)$ is not contained in $Cl^{[j+k]}(V,g)$, as (26.19) illustrates. We can set

$$Cl^{(k)}(V,g) = \bigoplus \big\{Cl^{[j]}(V,g) : j \le k,\ j = k \text{ mod } 2\big\}, \tag{26.20}$$

and then $Cl^{(j)}(V,g) \cdot Cl^{(k)}(V,g) \subset Cl^{(j+k)}(V,g)$. As k ranges over the even or the odd integers, the spaces (26.20) provide *filtrations* of $Cl^0(V,g)$ and $Cl^1(V,g)$.

Exercises

1. Let V have an oriented orthonormal basis $\{e_1, \cdots, e_n\}$. Set

$$\nu = e_1 \cdots e_n \in Cl(V, g).$$

Show that ν is independent of the choice of such a basis.
Note. $\tilde{M}(\nu) = e_1 \wedge \cdots \wedge e_n \in \Lambda^n V$, with $\tilde{M}$ as in (26.3).

2. Show that $\nu^2 = (-1)^{n(n-1)/2}$.

3. Show that, for all $u \in V$, $\nu u = (-1)^{n-1} u \nu$.

4. With μ as in (26.11)–(26.14), show that

$$\mu(\nu)^* = (-1)^{n(n-1)/2} \mu(\nu), \text{ and } \mu(\nu)^* \mu(\nu) = I.$$

5. Show that

$$\tilde{M}(\nu w) = c_{nk} * \tilde{M}(w),$$

for $w \in Cl^{[k]}(V, g)$, where $* : \Lambda^k V \to \Lambda^{n-k} V$ is the Hodge star operator. Find the constants c_{nk}.

6. Let $\mathcal{D} : C^\infty(M, T^*) \to C^\infty(M, Cl)$ be as in Exercise 2 of §25, i.e.,

$$\mathcal{D}u = i \sum v_j \cdot \nabla_{e_j} u,$$

where $\{e_j\}$ is a local orthonormal frame of vector fields, $\{v_j\}$ the dual frame. Show that

$$\tilde{M}(\mathcal{D}v) = -i(d + d^*)v.$$

7. Show that $\mathrm{End}(\mathbb{C}^m)$ has no proper 2-sided ideals.
Hint. Suppose $M_0 \neq 0$ belongs to such an ideal $\mathcal{I}$, and $v_0 \neq 0$ belongs to the range of M_0. Show that every $v \in \mathbb{C}^m$ belongs to the range of some $M \in \mathcal{I}$, and hence that every one-dimensional projection belongs to $\mathcal{I}$.

27. Spinors

We define the spinor groups $\mathrm{Pin}(V,g)$ and $\mathrm{Spin}(V,g)$, for a vector space V with a positive quadratic form g; set $|v|^2 = g(v,v) = \langle v,v\rangle$. We set

$$\mathrm{Pin}(V,g) = \{v_1\cdots v_k \in Cl(V,g) : v_j \in V,\ |v_j| = 1\}, \tag{27.1}$$

with the induced multiplication. Since $(v_1\cdots v_k)(v_k\cdots v_1) = 1$, it follows that $\mathrm{Pin}(V,g)$ is a group. We can define an action of $\mathrm{Pin}(V,g)$ on V as follows. If $u \in V$ and $x \in V$, then $ux + xu = 2\langle x,u\rangle \cdot 1$ implies

$$uxu = -xuu + 2\langle x,u\rangle u = -|u|^2 x + 2\langle x,u\rangle u. \tag{27.2}$$

If also $y \in V$,

$$\langle uxu, uyu\rangle = |u|^2\langle x,y\rangle = \langle x,y\rangle \quad \text{if} \quad |u| = 1. \tag{27.3}$$

Thus if $u = v_1\cdots v_k \in \mathrm{Pin}(V,g)$ and if we define a conjugation on $Cl(V,g)$ by

$$u^* = v_k\cdots v_1, \quad v_j \in V, \tag{27.4}$$

it follows that

$$x \mapsto uxu^*, \quad x \in V \tag{27.5}$$

is an isometry on V for each $u \in \mathrm{Pin}(V,g)$. It will be more convenient to use

$$u^\# = (-1)^k u^*, \quad u = v_1\cdots v_k. \tag{27.6}$$

Then we have a group homomorphism

$$\tau : \mathrm{Pin}(V,g) \longrightarrow O(V,g), \tag{27.7}$$

defined by

$$\tau(u)x = uxu^\#, \quad x \in V,\ u \in \mathrm{Pin}(V,g). \tag{27.8}$$

Note that, if $v \in V,\ |v| = 1$, then, by (27.2),

$$\tau(v)x = x - 2\langle x,v\rangle v \tag{27.9}$$

is reflection across the hyperplane in V orthogonal to v. It is easy to show that any orthogonal transformation $T \in O(V,g)$ is a product of a finite number of such reflections, so the group homomorphism (27.7) is surjective.

Note that each isometry (27.9) is orientation reversing. Thus, if we define

$$\begin{aligned} \mathrm{Spin}(V,g) &= \{v_1\cdots v_k \in Cl(V,g) : v_j \in V,\ |v_j| = 1,\ k \text{ even}\} \\ &= \mathrm{Pin}(V,g) \cap Cl^0(V,g), \end{aligned} \tag{27.10}$$

then

$$\tau : \mathrm{Spin}(V,g) \longrightarrow SO(V,g) \tag{27.11}$$

and in fact $\mathrm{Spin}(V,g)$ is the inverse image of $SO(V,g)$ under τ in (27.7). We now show that τ is a 2-fold covering map.

Proposition 27.1. *τ is a 2-fold covering map. In fact, ker $\tau = \{\pm 1\}$.*

Proof. Note that $\pm 1 \in \mathrm{Spin}(V, g) \subset Cl(V, g)$ and ± 1 acts trivially on V, via (27.8). Now, if $u = v_1 \cdots v_k \in$ ker τ, k must be even, since $\tau(u)$ must preserve orientation, so $u^\# = u^*$. Since $uxu^* = x$ for all $x \in V$, we have $ux = xu$, so $uxu = |x|^2 u$, $x \in V$. If we pick an orthonormal basis $\{e_1, \dots, e_n\}$ of V and write $u \in$ ker τ in the form (26.2), each $i_1 + \cdots + i_n$ even, since $e_j u e_j = u$ for each j, we deduce that, for each j,

$$u = \sum (-1)^{i_j} \ a_{i_1 \cdots i_n} \ e^{i_1 \cdots i_n} \text{ if } u \in \text{ ker } \tau.$$

Hence $i_j = 0$ for all j, so u is a scalar; hence $u = \pm 1$.

We next consider the connectivity properties of $\mathrm{Spin}(V, g)$.

Proposition 27.2. $\mathrm{Spin}(V, g)$ *is the connected* 2*-fold cover of* $SO(V, g)$, *provided g is positive definite and dim $V \geq 2$.*

Proof. It suffices to connect $-1 \in \mathrm{Spin}(V, g)$ to the identity element $1 \in \mathrm{Spin}(V, g)$ via a continuous curve in $\mathrm{Spin}(V, g)$. In fact, pick orthogonal $e_1,\ e_2$, and set

$$\gamma(t) = e_1 \cdot \big[-(\cos t) e_1 + (\sin t) e_2\big], \quad 0 \leq t \leq \pi.$$

If $V = \mathbb{R}^n$ with its standard Euclidean inner product g, denote $\mathrm{Spin}(V, g)$ by $\mathrm{Spin}(n)$. It is a known topological fact that $SO(n)$ has fundamental group $\mathbb{Z}_2$, and $\mathrm{Spin}(n)$ is simply connected, for $n \geq 3$. Though we make no use of this result, we mention that one route to it is via the "homotopy exact sequence" (see [BTu]) for $S^n = SO(n+1)/SO(n)$. This leads to $\pi_1\big(SO(n+1)\big) \approx \pi_1\big(SO(n)\big)$ for $n \geq 3$. Meanwhile, one sees directly that $SU(2)$ is a double cover of $SO(3)$, and it is homeomorphic to S^3.

We next produce representations of $\mathrm{Pin}(V, g)$ and $\mathrm{Spin}(V, g)$, arising from the homomorphism (26.13). First assume V has real dimension $2k$, with complex structure J; let $\mathcal{V} = (V, J)$ be the associated complex vector space, of complex dimension k, and set

$$S(V, g, J) = \Lambda^*_{\mathbb{C}} \mathcal{V}, \tag{27.12}$$

with its induced Hermitian metric, arising from the metric (26.9) on $\mathcal{V}$. The inclusion $\mathrm{Pin}(V, g) \subset Cl(V, g) \subset \mathbb{C}l(V, g)$ followed by (26.14) gives the representation

$$\rho : \mathrm{Pin}(V, g) \longrightarrow Aut\big(S(V, g, J)\big). \tag{27.13}$$

Proposition 27.3. *The representation ρ of* $\mathrm{Pin}(V, g)$ *is irreducible and unitary.*

Proof. Since the $\mathbb{C}$-subalgebra of $\mathbb{C}l(V, g)$ generated by $\mathrm{Pin}(V, g)$ is all of $\mathbb{C}l(V, g)$, the irreducibility follows from the fact that μ in (26.14) is an isomorphism. For unitarity, it follows from (26.11) that $\mu(v)$ is self adjoint for $v \in V$; by (26.12),

$\mu(v)^2 = |v|^2 I$, so $v \in V,\ |v| = 1$ implies that $\rho(v)$ is unitary, and unitarity of ρ on $\mathrm{Pin}(V, g)$ follows.

The restriction of ρ to $\mathrm{Spin}(V, g)$ is not irreducible. In fact, set

$$S_+(V, g, J) = \Lambda_{\mathbb{C}}^{\mathrm{even}} \mathcal{V}, \quad S_-(V, g, J) = \Lambda_{\mathbb{C}}^{\mathrm{odd}} \mathcal{V}. \tag{27.14}$$

Under ρ, the action of $\mathrm{Spin}(V, g)$ preserves both S_+ and S_-. In fact, we have (26.14) restricting to

$$\mu : \mathbb{C}l^0(V, g) \longrightarrow \mathrm{End}_{\mathbb{C}}\big(S_+(V, g, J)\big) \oplus \mathrm{End}_{\mathbb{C}}\big(S_-(V, g, J)\big), \tag{27.15}$$

this map being an isomorphism. On the other hand,

$$z \in \mathbb{C}l^1(V, g) \Longrightarrow \mu(z) : S_\pm \longrightarrow S_\mp. \tag{27.16}$$

From (27.15) we get representations

$$D^\pm_{1/2} : \mathrm{Spin}(V, g) \longrightarrow Aut\big(S_\pm(V, g, J)\big) \tag{27.17}$$

which are irreducible and unitary.

If $V = \mathbb{R}^{2k}$ with its standard Euclidean metric, standard orthonormal basis $e_1, \dots, e_{2k}$, we impose the complex structure $Je_i = e_{i+k},\ Je_{i+k} = -e_i,\ 1 \le i \le k$, and set

$$S(2k) = S(\mathbb{R}^{2k}, |\ |^2, J), \quad S_\pm(2k) = S_\pm(\mathbb{R}^{2k}, |\ |^2, J). \tag{27.18}$$

Then (27.17) defines representations

$$D^\pm_{1/2} : \mathrm{Spin}(2k) \longrightarrow Aut\big(S_\pm(2k)\big). \tag{27.19}$$

We now consider the odd dimensional case. If $V = \mathbb{R}^{2k-1}$, we use the isomorphism

$$Cl(2k-1) \longrightarrow Cl^0(2k) \tag{27.20}$$

produced by the map

$$v \mapsto ve_{2k}, \quad v \in \mathbb{R}^{2k-1}. \tag{27.21}$$

Then the inclusion $\mathrm{Spin}(2k-1) \subset Cl(2k-1)$ composed with (27.20) gives an inclusion

$$\mathrm{Spin}(2k) \hookrightarrow \mathrm{Spin}(2k). \tag{27.22}$$

Composing with $D^+_{1/2}$ from (27.19) gives a representation

$$D^+_{1/2} : \mathrm{Spin}(2k-1) \longrightarrow Aut\ S_+(2k). \tag{27.23}$$

We also have a representation $D^-_{1/2}$ of $\mathrm{Spin}(2k-1)$ on $S_-(2k)$, but these two representations are equivalent. They are intertwined by the map

$$\mu(e_{2k}) : S_+(2k) \to S_-(2k). \tag{27.24}$$

We now study spinor bundles on an oriented Riemannian manifold M, with metric tensor g. Over M lies the bundle of oriented orthonormal frames,

$$P \longrightarrow M, \tag{27.25}$$

a principal $SO(n)$-bundle, $n = \dim M$. A spin structure on M is a "lift,"

$$\tilde{P} \longrightarrow M, \tag{27.26}$$

a principal $\mathrm{Spin}(n)$-bundle, such that $\tilde{P}$ is a double covering of P in such a way that the action of $\mathrm{Spin}(n)$ on the fibers of $\tilde{P}$ is compatible with the action of $SO(n)$ on the fibers of P, via the covering homomorphism $\tau : \mathrm{Spin}(n) \to SO(n)$. There are topological obstructions to the existence of a spin structure, which we will not discuss here. See [LM]. It turns out that there is a naturally defined element of $\mathcal{H}^2(M, \mathbb{Z}_2)$ whose vanishing guarantees the existence of a lift, and when such lifts exist, equivalence classes of such lifts are parametrized by elements of $\mathcal{H}^1(M, \mathbb{Z}_2)$.

Given a spin structure (27.26), spinor bundles are constructed via the representations of $\mathrm{Spin}(n)$ described above. Two cases arise, depending on whether $n = \dim M$ is even or odd. If $n = 2k$, we form the bundle of spinors

$$S(\tilde{P}) = \tilde{P} \times_\rho S(2k), \tag{27.27}$$

where $\rho = D^+_{1/2} \oplus D^-_{1/2}$ is the sum of the representations (27.19); this is a sum of the two vector bundles

$$S_\pm(\tilde{P}) = \tilde{P} \times_{D^\pm_{1/2}} S_\pm(2k). \tag{27.28}$$

Recall that, as in §18, the sections of $S(\tilde{P})$ are in natural correspondence with the functions f on $\tilde{P}$, taking values in the vector space $S(2k)$, which satisfy the compatibility conditions

$$f(p \cdot g) = \rho(g)^{-1} f(p), \quad p \in \tilde{P},\ g \in \mathrm{Spin}(2k), \tag{27.29}$$

where we write the $\mathrm{Spin}(n)$-action on $\tilde{P}$ as a right action.

Recall that $S(2k)$ is a $Cl(2k)$-module, via (26.13). This result extends to the bundle level.

Proposition 27.4. *The spinor bundle $S(\tilde{P})$ is a natural $Cl(M)$-module.*

Proof. Given a section u of $Cl(M)$ and a section φ of $S(\tilde{P})$, we need to define $u \cdot \varphi$ as a section of $S(\tilde{P})$. We regard u as a function on $\tilde{P}$ with values in $Cl(n)$ and φ as a function on $\tilde{P}$ with values in $S(n)$. Then u is a function on $\tilde{P}$ with values in $S(n)$; we need to verify the compatibility condition (27.29). Indeed, for $p \in \tilde{P},\ g \in \mathrm{Spin}(2k)$,

$$\begin{aligned} u \cdot \varphi(p \cdot g^{-1}) &= \tau(g)u(p) \cdot \rho(g)\varphi(p) \\ &= gu(p)g^{\#} g\varphi(p) \\ &= gu(p) \cdot \varphi(p), \end{aligned} \tag{27.30}$$

since $gg^{\#} = 1$ for $g \in \mathrm{Spin}(n)$. This completes the proof.

Whenever (M, g) is an oriented Riemannian manifold, the Levi-Civita connection provides a connection on the principal $SO(n)$-bundle of frames P. If M has a spin structure, this choice of horizontal space for P lifts in a unique natural fashion to provide a connection on $\tilde{P}$. Thus the spinor bundle constructed above has a natural connection, which we might call the Dirac-Levi-Civita connection.

Proposition 27.5. *The Dirac-Levi-Civita connection ∇ on $S(\tilde{P})$ is a Clifford connection.*

Proof. Clearly ∇ is a metric connection, since the representation ρ of $\mathrm{Spin}(2k)$ on $S(2k)$ is unitary. It remains to verify the compatibility condition (25.36), i.e.,

$$\nabla_X(v \cdot \varphi) = (\nabla_X v) \cdot \varphi + v \cdot \nabla_X \varphi, \tag{27.31}$$

for a vector field X, a 1-form v, and a section φ of $S(\tilde{P})$. To see this, we first note that, as stated in (27.30), the bundle $Cl(M)$ can be obtained from $\tilde{P} \to M$ as $\tilde{P} \times_\kappa Cl(2k)$, where κ is the representation of $\mathrm{Spin}(2k)$ on $Cl(2k)$ given by $\kappa(g)w = gwg^{\#}$. Furthermore, T^*M can be regarded as a subbundle of $Cl(M)$, obtained from $\tilde{P} \times_\kappa \mathbb{R}^{2k}$ with the same formula for κ. The connection on T^*M obtained from that on $\tilde{P}$ is identical to the usual connection on T^*M defined via the Levi-Civita formula. Given this, (27.31) is a straightforward derivation identity.

Using the prescription (25.31)–(25.33) we can define the *Dirac operator* on a Riemannian manifold of dimension $2k$, with a spin structure:

$$D : C^\infty(M, S(\tilde{P})) \longrightarrow C^\infty(M, S(\tilde{P})). \tag{27.32}$$

We see that Proposition 25.1 applies; D is symmetric. Note also the grading:

$$D : C^\infty(M, S_\pm(\tilde{P})) \longrightarrow C^\infty(M, S_\mp(\tilde{P})). \tag{27.33}$$

In other words, this Dirac operator is of the form (25.3).

On a Riemannian manifold of dimension $2k$ with a spin structure $\tilde{P} \to M$, let $F \to M$ be another vector bundle. Then the tensor product $E = S(\tilde{P}) \otimes F$ is a $Cl(M)$-module in a natural fashion. If F has a connection, then E gets a natural product connection. Then the construction (25.31)–(25.33) yields an operator D_F of Dirac type on sections of E; in fact

$$D_F : C^\infty(M, E_\pm) \longrightarrow C^\infty(M, E_\mp), \quad E_\pm = S_\pm(\tilde{P}) \otimes F. \tag{27.34}$$

If F has a metric connection, then E gets a Clifford connection. The operator D_F is called a *twisted Dirac operator*. Sometimes it will be convenient to distinguish notationally the two pieces of D_F; we write

$$D_F^+ : C^\infty(M, E_+) \longrightarrow C^\infty(M, E_-), \quad D_F^- : C^\infty(M, E_-) \longrightarrow C^\infty(M, E_+). \tag{27.35}$$

When dim $M = 2k - 1$ is odd, we use the representation (27.24) to form the bundle of spinors $S_+(\tilde{P}) = \tilde{P} \times_{D^+_{1/2}} S_+(2k)$. The inclusion $Cl(2k-1) \hookrightarrow Cl^0(2k)$ defined by (27.20)–(27.21) makes $S_+(2k)$ a $Cl(2k-1)$-module, and analogues of Propositions 27.4 and 27.5 hold. Consequently there arises a Dirac operator, $D : C^\infty(M, S_+(\tilde{P})) \to C^\infty(M, S_+(\tilde{P}))$, and twisted Dirac operators also arise; in place of (27.34) we have $D_F : C^\infty(M, E_+) \to C^\infty(M, E_+)$, with $E_+ = S_+(\tilde{P}) \otimes F$.

Exercises

1. Verify that the map (27.15) is an isomorphism, and that the representations (27.17) of $\mathrm{Spin}(V, g)$ are irreducible, when dim $V = 2k$.

2. Let ν be as in Exercises 1–4 of §26, with $n = 2k$. Show that:
   a) The center of $\mathrm{Spin}(V, g)$ consists of $\{1, -1, \nu, -\nu\}$.
   b) $\mu(\nu)$ leaves S_+ and S_- invariant.
   c) $\mu(\nu)$ commutes with the action of $Cl^0(V, g)$ under μ, hence with the representations $D^\pm_{1/2}$ of $\mathrm{Spin}(V, g)$.
   d) $\mu(\nu)$ acts as a pair of scalars on S_+ and S_- respectively. These scalars are the two square roots of $(-1)^k$.

3. Calculate $\mu(\nu) \cdot 1$ directly, making use of the definition (26.11). Hence match the scalars in exercise 2d) to S_+ and S_-.
Hint. $\mu(e_{k+1} \cdots e_{2k}) \cdot 1 = (-i)^k e_{k+1} \wedge \cdots \wedge e_{2k}$ in $\Lambda^k_{\mathbb{C}} \mathcal{V}$.
Using $e_{j+k} = i\, e_j$ in $\mathcal{V}$, for $1 \le j \le k$, we have

$\mu(\nu)\cdot 1 = \mu(e_1 \cdots e_k)(e_1 \wedge \cdots \wedge e_k),$
and there are k interior products to compute.

4. Show that $Cl^{[2]}(V,g)$, with the Lie algebra structure (26.19), is naturally isomorphic to the Lie algebra of $\mathrm{Spin}(V,g)$. In fact, if (a_{jk}) is a real antisymmetric matrix, in the Lie algebra of $SO(n)$, which is the same as that of $\mathrm{Spin}(n)$, show that there is the correspondence

$$A = (a_{jk}) \mapsto \frac{1}{4}\sum a_{jk}\ e_j\ e_k = \kappa(A).$$

In particular, show that $\kappa(A_1A_2 - A_2A_1) = \kappa(A_1)\kappa(A_2) - \kappa(A_2)\kappa(A_1)$.

5. If X is a spin manifold and $M \subset X$ is an oriented submanifold of codimension 1, show that M has a spin structure. Deduce that an oriented hypersurface in $\mathbb{R}^n$ has a spin structure.

6. As noted in §18, a section σ of $P \to M$ (over an open set $U \subset M$) defines an isomorphism

$$C^\infty(U,E) \approx C^\infty(U,V),$$

for a vector bundle $E = P \times_\pi V$, where π is a representation of $G = SO(n)$ on V. Given such σ, there are two possible lifts to sections $\tilde{\sigma}_j$ of $\widetilde{P} \to M$ (over U, which we assume is diffeomorphic to a ball), defining two isomorphisms

$$\tau_j : C^\infty(U,F) \longrightarrow C^\infty(U,W),$$

for a vector bundle $F = P \times_\lambda W$, where λ is a representation of $\widetilde{G} =$ Spin(n) on W. Show that $\tau_1 = -\tau_2$.

7. Recall the formula (18.12) for the covariant derivative of a section of E. Suppose $E = TM$. Then, with respect to σ, we get connection coefficients $\Gamma^\alpha{}_{\beta j}$, defined over U. Show that, for each j,

$$\Gamma_j = (\Gamma^\alpha{}_{\beta j}) \in \ \mathrm{Skew}(n) = so(n).$$

Let $F_\pm = S_\pm(\widetilde{P})$. Then, with respect to $\tilde{\sigma}_j$, we have connection coefficients $\widetilde{\Gamma}^A{}_{Bj}$, and, for each j,

$$\widetilde{\Gamma}_j \in \ \mathrm{End}\big(S_\pm(2k)\big), \quad 2k = n \ \text{ or } \ n+1.$$

Show that

$$\widetilde{\Gamma}_j = (dD^\pm_{1/2})(\Gamma_j),$$

where $dD^\pm_{1/2}$ is the derived representation of $so(n)$ on $S_\pm(2k)$, associated to the representation $D^\pm_{1/2}$ of $\mathrm{Spin}(n)$ on $S_\pm(2k)$.

28. Weitzenbock formulas

Let $E \to M$ be a Hermitian vector bundle with a metric connection ∇. Suppose E is also a $Cl(M)$-module, and that ∇ is a Clifford connection. If we consider the Dirac operator $D : C^\infty(M,E) \to C^\infty(M,E)$ and the covariant derivative $\nabla : C^\infty(M,E) \to C^\infty(M,T^* \otimes E)$, then D^2 and $\nabla^*\nabla$ are operators on $C^\infty(M,E)$ with the *same* principal symbol. It is of interest to examine their difference, clearly a differential operator of order ≤ 1. In fact, the difference has order 0. This can be seen in principle from the following considerations. From Exercise 4 of §25, we have

$$D^2(f\varphi) = f\ D^2\varphi - 2\nabla_{\text{grad } f}\varphi - (\Delta f)\varphi \tag{28.1}$$

when $\varphi \in C^\infty(M,E)$, f a scalar function. Similarly we compute $\nabla^*\nabla(f\varphi)$. The derivation property of ∇ implies

$$\nabla(f\varphi) = f\nabla\varphi + df \otimes \varphi. \tag{28.2}$$

To apply ∇^* to this, first a short calculation gives

$$\nabla^* f(u \otimes \varphi) = f\nabla^*(u \otimes \varphi) - \langle df, u\rangle\varphi \tag{28.3}$$

for $u \in C^\infty(M,T^*)$, $\varphi \in C^\infty(M,E)$, and hence

$$\nabla^*(f\nabla\varphi) = f\nabla^*\nabla\varphi - \nabla_{\text{grad } f}\varphi. \tag{28.4}$$

This gives ∇^* applied to the first term on the right side of (28.2). To apply ∇^* to the other term, we can use the identity

$$\nabla^*(u \otimes \varphi) = -\nabla_U\varphi - (\text{div } U)\varphi, \tag{28.5}$$

where U is the vector field corresponding to u via the metric on M. Compare (13.35). Hence

$$\nabla^*(df \otimes \varphi) = -\nabla_{\text{grad } f}\varphi - (\Delta f)\varphi. \tag{28.6}$$

Then (28.6) and (28.4) applied to (28.2) give

$$\nabla^*\nabla(f\varphi) = f\nabla^*\nabla\varphi - 2\nabla_{\text{grad } f}\varphi - (\Delta f)\varphi. \tag{28.7}$$

Comparing (28.1) and (28.7), we have

$$(D^2 - \nabla^*\nabla)(f\varphi) = f(D^2 - \nabla^*\nabla)\varphi, \tag{28.8}$$

which implies $D^2 - \nabla^*\nabla$ has order zero, i.e., is given by a bundle map on E. We now derive the Weitzenbock formula for what this difference is.

Proposition 28.1. *If $E \to M$ is a $Cl(M)$-module with Clifford connection, and associated Dirac-type operator D, then, for $\varphi \in C^\infty(M, E)$,*

$$D^2\varphi = \nabla^*\nabla\varphi - \sum_{j>k} v_k v_j K(e_k, e_j)\varphi, \tag{28.9}$$

where $\{e_j\}$ is a local orthonormal frame of vector fields, with dual frame field $\{v_j\}$, and K is the curvature tensor of (E, ∇).

Proof. Starting with $D\varphi = i\sum v_j \nabla_{e_j}\varphi$, we obtain

$$\begin{aligned} D^2\varphi &= -\sum_{j,k} v_k \nabla_{e_k}\big(v_j \nabla_{e_j}\varphi\big) \\ &= -\sum_{j,k} v_k \Big[v_j \nabla_{e_k}\nabla_{e_j}\varphi + \big(\nabla_{e_k} v_j\big)\nabla_{e_j}\varphi\Big], \end{aligned} \tag{28.10}$$

using the compatibility condition (25.36). We replace $\nabla_{e_k}\nabla_{e_j}$ by the Hessian, using the identity

$$\nabla^2_{e_k,e_j}\varphi = \nabla_{e_k}\nabla_{e_j}\varphi - \nabla_{\nabla_{e_k}e_j}\varphi; \tag{28.11}$$

cf. (14.4). We obtain

$$\begin{aligned} D^2\varphi = -&\sum_{j,k} v_k v_j \nabla^2_{e_k,e_j}\varphi \\ &- \sum_{j,k} v_k \Big[v_j \nabla_{\nabla_{e_k}e_j}\varphi + \big(\nabla_{e_k} v_j\big)\nabla_{e_j}\varphi\Big]. \end{aligned} \tag{28.12}$$

Let us look at each of the two double sums on the right. Using $v_j^2 = 1$ and the anticommutator property $v_k v_j = -v_j v_k$ for $k \neq j$, we see that the first double sum becomes

$$-\sum_j \nabla^2_{e_j,e_j}\varphi - \sum_{j>k} v_k v_j K(e_k, e_j)\varphi, \tag{28.13}$$

since the antisymmetric part of the Hessian is the curvature. This is equal to the right side of (28.9), in light of the formula for $\nabla^*\nabla$ established in Proposition 14.1. As for the remaining double sum in (28.12), for any $p \in M$, we can choose a local orthonormal frame field $\{e_j\}$ such that $\nabla_{e_j} e_k = 0$ at p, and then this term vanishes at p. This proves (28.9).

We denote the difference $D^2 - \nabla^*\nabla$ by $\mathcal{K}$, so

$$(D^2 - \nabla^*\nabla)\varphi = \mathcal{K}\varphi, \quad \mathcal{K} \in C^\infty(M, \mathrm{End}\ E). \tag{28.14}$$

The formula for $\mathcal{K}$ in (28.9) can also be written

$$\mathcal{K}\varphi = -\sum_{j,k} v_k v_j K(e_k, e_j)\varphi. \tag{28.15}$$

Since a number of formulas that follow will involve multiple summation, we will use the summation convention.

This general formula for $\mathcal{K}$ simplifies further in some important special cases. The first simple example of this will be useful for further calculations.

Proposition 28.2. *Let $E = \Lambda^* M$, with $Cl(M)$-module structure and connection described in §1, so $\mathcal{K} \in C^\infty(M, End\, \Lambda^*)$. In this case,*

$$u \in \Lambda^1 M \Longrightarrow \mathcal{K} u = Ric\,(u). \tag{28.16}$$

Proof. The curvature of $\Lambda^* M$ is a sum of curvatures of each factor $\Lambda^k M$. In particular, if $\{e_j, v_j\}$ is a local dual pair of frame fields,

$$K(e_i, e_j) v_k = -R^k{}_{\ell ij} v_\ell, \tag{28.17}$$

where $R^k{}_{\ell ij}$ are the components of the Riemann tensor, with respect to these frame fields, and we use the summation convention. In light of (28.15), the desired identity (28.16) will hold provided

$$\frac{1}{2} v_i v_j v_\ell R^k{}_{\ell ij} = \mathrm{Ric}\,(v_k), \tag{28.18}$$

so it remains to establish this identity. Since, if (i, j, ℓ) are distinct, $v_i v_j v_\ell = v_\ell v_i v_j = v_j v_\ell v_i$ and since by Bianchi's first identity

$$R^k{}_{\ell ij} + R^k{}_{j\ell i} + R^k{}_{ij\ell} = 0,$$

it follows that in summing the left side of (4.18), the sum over (i, j, ℓ) distinct vanishes. By antisymmetry of $R^k{}_{\ell ij}$, the terms with $i = j$ vanish. Thus the only contributions arise from $i = \ell \neq j$ and $i \neq \ell = j$. Therefore the left side of (28.18) is equal to

$$\frac{1}{2}\big(-v_j R^k{}_{iij} + v_i R^k{}_{jij}\big) = v_i R^k{}_{jij} = \mathrm{Ric}\,(v_k), \tag{28.19}$$

which completes the proof.

We next derive Lichnerowicz's calculation of $\mathcal{K}$ when $E = S(\tilde{P})$, the spinor bundle of a manifold M with spin structure. First we need an expression for the curvature of $S(\tilde{P})$.

Lemma 28.3. *The curvature tensor of the spinor bundle $S(\tilde{P})$ is given by*

$$K(e_i, e_j)\varphi = \frac{1}{4} R^k{}_{\ell ij} v_k v_\ell \varphi. \tag{28.20}$$

Proof. This follows from the relation between curvatures on vector bundles and on principal bundles established in §18, together with the identification of the Lie algebra of $\mathrm{Spin}(n)$ with $Cl^{[2]}(n)$ given in Exercise 4 of §27.

Proposition 28.4. *For the spin bundle $S(\tilde{P})$, $\mathcal{K} \in C^\infty(M, End\, S(\tilde{P}))$ is given by*

$$\mathcal{K}\varphi = \frac{1}{4} S\varphi, \tag{28.21}$$

where S is the scalar curvature of M.

Proof. Using (28.20), the general formula (28.15) yields

$$\mathcal{K}\varphi = -\frac{1}{8} R^k{}_{\ell ij}\ v_i v_j v_k v_\ell \varphi = \frac{1}{8} v_i v_j v_\ell R^k{}_{\ell ij}\ v_k \varphi, \tag{28.22}$$

the last identity holding by the anticommutation relations; note that only the sum over $k \neq \ell$ counts. Now, by (28.18), this becomes

$$\begin{aligned} \mathcal{K}\varphi &= \frac{1}{4} v_i v_k R^k{}_{jij} \varphi \\ &= \frac{1}{4}\ \mathrm{Ric}_{ii} \quad \text{(by symmetry)} \\ &= \frac{1}{4} S\varphi, \end{aligned} \tag{28.23}$$

completing the proof.

We record the generalization of Proposition 28.4 to the case of twisted Dirac operators.

Proposition 28.5. *Let $E \to M$ have a metric connection ∇, with curvature R^E. For the twisted Dirac operator on sections of $F = S(\tilde{P}) \otimes E$, the section $\mathcal{K}$ of End F has the form*

$$\mathcal{K}\varphi = \frac{1}{4} S\varphi + \frac{1}{2} \sum_{i,j} v_i v_j R^E(e_i, e_j)\varphi. \tag{28.24}$$

Proof. Here $R^E(e_i, e_j)$ is shorthand for $I \otimes R^E(e_i, e_j)$ acting on $S(\tilde{P}) \otimes E$. This formula is a consequence of the general formula (28.15) and the argument proving Proposition 4.4, since the curvature of $S(\tilde{P}) \otimes E$ is $K \otimes I + I \otimes R^E$, K being the curvature of $S(\tilde{P})$, given by (28.20).

We draw some interesting conclusions from some of these Weitzenbock formulas, due to S.Bochner and A.Lichnerowitz.

Proposition 28.6. *If M is compact and connected, and the section $\mathcal{K}$ in (28.14)–(28.15) has the property that $\mathcal{K} \geq 0$ on M and $\mathcal{K} > 0$ at some point, then ker $D = 0$.*

Proof. This is immediate from

$$(D^2\varphi, \varphi) = (\mathcal{K}\varphi, \varphi) + \|\nabla\varphi\|^2_{L^2}.$$

Proposition 28.7. *If M is a compact Riemannian manifold with positive Ricci tensor, then $b_1(M) = 0$, i.e., the deRham cohomology group $\mathcal{H}^1(M, \mathbb{R}) = 0$.*

Proof. Via Hodge theory, we want to show that if $u \in \Lambda^1(M)$ and $du = d^*u = 0$, then $u = 0$. This hypothesis implies $Du = 0$, where D is the Dirac-type operator dealt with in Proposition 28.2. Consequently we have, for a 1-form u on M,

$$\|Du\|_{L^2}^2 = \big(\mathrm{Ric}\,(u), u\big) + \|\nabla u\|_{L^2}^2, \tag{28.25}$$

so the result follows.

Proposition 28.8. *If M is a compact connected Riemannian manifold with a spin structure whose scalar curvature is ≥ 0 on M and > 0 at some point, then M has no nonzero harmonic spinors, i.e., $ker D = 0$ in $C^\infty(M, S(\tilde{P}))$.*

Proof. In light of (28.21), this is a special case of Proposition 28.6.

Exercises

1. Let Δ be the Laplace operator on functions (0-forms) on a compact Riemannian manifold M, Δ_k the Hodge Laplacian on k-forms. If $\mathrm{Spec}\,(-\Delta)$ consists of $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, show that $\lambda_1 \in \mathrm{Spec}\,(-\Delta_1)$.

2. If $\mathrm{Ric} \geq c_0 I$ on M, show that $\lambda_1 \geq c_0$.

3. Recall the deformation tensor of a vector field u :

$$\mathrm{Def}\; u = \frac{1}{2}\mathcal{L}_u g = \frac{1}{2}\big(\nabla u + \nabla u^t\big), \quad \mathrm{Def} : C^\infty(M, T) \to C^\infty(M, S^2).$$

Show that

$$\mathrm{Def}^*\, v = -\mathrm{div}\, v,$$

where $(\mathrm{div}\, v)^j = v^{jk}{}_{;k}$. Establish the Weitzenbock formula

$$2\ \mathrm{div}\ \mathrm{Def}\ u = -\nabla^*\nabla u + \ \mathrm{grad}\ \mathrm{div}\ u + \ \mathrm{Ric}(u). \tag{28.26}$$

The operator div on the right is the usual divergence operator on vector fields.

4. Suppose M is a compact connected Riemannian manifold, whose Ricci tensor satisfies

$$\text{(28.27)} \qquad \operatorname{Ric}(x) \le 0 \text{ on } M, \quad \operatorname{Ric}(x_0) < 0 \text{ for some } x_0 \in M.$$

Show that the operator Def is injective, i.e., there are no nontrivial Killing fields on M, hence no nontrivial one-parameter groups of isometries.
Hint. From (28.26), we have

$$\text{(28.28)} \qquad 2\|\text{Def } u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 + \|\text{div } u\|^2 - \big(\operatorname{Ric}(u), u\big)_{L^2}.$$

5. As shown in (12.39), the equation of a conformal Killing field on an n-dimensional Riemannian manifold M is

$$\text{(28.29)} \qquad \text{Def } X - \frac{1}{n}\big(\text{div } X\big)g = 0.$$

Note that the left side is the *trace free part* of Def $X \in C^\infty(M, S^2T^*)$. Denote it by $\mathcal{D}_{TF}X$. Show that

$$\text{(28.30)} \qquad \mathcal{D}_{TF}^* = -\text{div}\big|_{S_0^2T^*}, \quad \mathcal{D}_{TF}^*\mathcal{D}_{TF}X = -\text{div Def } X + \frac{1}{n}\big(\text{grad div } X\big),$$

where $S_0^2T^*$ is the trace free part of S^2T^*. Show that

$$\text{(28.31)} \qquad \|\mathcal{D}_{TF}X\|_{L^2}^2 = \frac{1}{2}\|\nabla X\|_{L^2}^2 + \big(\frac{1}{2} - \frac{1}{n}\big)\|\text{div } X\|_{L^2}^2 - \frac{1}{2}\big(\operatorname{Ric}(X), X\big)_{L^2}.$$

Deduce that, if M is compact and satisfies (28.27), then M has no nontrivial one-parameter group of *conformal* diffeomorphisms.

6. Show that, if M is a compact Riemannian manifold which is Ricci flat, i.e., Ric $= 0$, then every conformal Killing field is a Killing field, and the dimension of the space of Killing fields is given by

$$\text{(28.32)} \qquad \dim_{\mathbb{R}} \ker \text{Def} \ = \ \dim \mathcal{H}^1(M, \mathbb{R}).$$

Hint. Combine (28.25) and (28.28).

7. Suppose dim $M = 2$ and M is compact and connected. Show that, for $u \in C^\infty(M, S_0^2T^*)$,

$$\|\mathcal{D}_{TF}^*u\|_{L^2}^2 = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \int_M K|u|^2\, dV,$$

where K is the Gauss curvature. Deduce that, if $K \ge 0$ on M, and $K(x_0) > 0$ for some $x_0 \in M$, then Ker $\mathcal{D}_{TF}^* = 0$.

8. If u and v are vector fields on a Riemannian manifold M, show that

$$\text{(28.33)} \qquad \text{div } \nabla_u v = \nabla_u(\text{div } v) + \operatorname{Tr}\big((\nabla u)(\nabla v)\big) - \operatorname{Ric}(u, v).$$

Relate this identity to the Weitzenbock formula for Δ on one-forms (a special case of Proposition 28.2).

29. Minimal surfaces

A minimal surface is one which is critical for the area functional. To begin, we consider a k-dimensional manifold M (generally with boundary), in $\mathbb{R}^n$. Let ξ be a compactly supported normal field to M, and consider the 1-parameter family of manifolds $M_s \subset \mathbb{R}^n$, images of M under the maps

$$\varphi_s(x) = x + s\xi(x), \quad x \in M. \tag{29.1}$$

We want a formula for the derivative of the k-dimensional area of M_s, at $s = 0$. Let us suppose ξ is supported on a single coordinate chart, and write

$$A(s) = \int\limits_{\Omega} \bigl\|\partial_1 X \wedge \cdots \wedge \partial_k X\bigr\| \, du_1 \cdots du_k, \tag{29.2}$$

where $\Omega \subset \mathbb{R}^k$ parametrizes M_s by $X(s,u) = X_0(u) + s\xi(u)$. We can also suppose this chart is chosen so that $\|\partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\| = 1$. Then we have

$$A'(0) = \sum_{j=1}^{k} \int \bigl\langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \bigr\rangle \, du_1 \cdots du_k. \tag{29.3}$$

By the Weingarten formula (16.9) we can replace $\partial_j \xi$ by $-A_\xi E_j$, where $E_j = \partial_j X_0$. Without loss of generality, for any fixed $x \in M$, we can assume that $E_1, \dots, E_k$ is an orthonormal basis of $T_x M$. Then

$$\bigl\langle E_1 \wedge \cdots \wedge A_\xi E_j \wedge \cdots \wedge E_k, E_1 \wedge \cdots \wedge E_k \bigr\rangle = \bigl\langle A_\xi E_j, E_j \bigr\rangle, \tag{29.4}$$

at x. Summing over j yields $\operatorname{Tr} A_\xi(x)$, which is invariantly defined, so we have

$$A'(0) = -\int\limits_{M} \operatorname{Tr} A_\xi(x) \, dA(x), \tag{29.5}$$

where $A_\xi(x) \in \operatorname{End}(T_x M)$ is the Weingarten map of M and $dA(x)$ the Riemannian k-dimensional area element. We say M is a minimal submanifold of $\mathbb{R}^n$ provided $A'(0) = 0$ for all variations of the form (29.1), for which the normal field ξ vanishes on ∂M.

If we specialize to the case where $k = n-1$ and M is an oriented hypersurface of $\mathbb{R}^n$, letting N be the "outward" unit normal to M, for a variation M_s of M given by

$$\varphi_s(x) = x + sf(x)N(x), \quad x \in M, \tag{29.6}$$

we hence have

$$A'(0) = -\int_M \mathrm{Tr}\ A_N(x)\ f(x)\ dA(x). \tag{29.7}$$

The criterion for a hypersurface M of $\mathbb{R}^n$ to be minimal is hence that Tr $A_N = 0$ on M.

Recall from §16 that $A_N(x)$ is a symmetric operator on T_xM. Its eigenvalues, which are all real, are called the principal curvatures of M at x. Various symmetric polynomials in these principal curvatures furnish quantities of interest. The *mean* curvature $H(x)$ of M at x is defined to be the mean value of these principal curvatures, i.e.,

$$H(x) = \frac{1}{k}\ \mathrm{Tr}\ A_N(x). \tag{29.8}$$

Thus a hypersurface $M \subset \mathbb{R}^n$ is a minimal submanifold of $\mathbb{R}^n$ precisely when $H = 0$ on M.

Note that changing the sign of N changes the sign of A_N, hence of H. Under such a sign change, the *mean curvature vector*

$$\mathfrak{H}(x) = H(x)N(x) \tag{29.9}$$

is invariant. In particular, this is well defined whether or not M is orientable, and its vanishing is the condition for M to be a minimal submanifold. There is the following useful formula for the mean curvature of a hypersurface $M \subset \mathbb{R}^n$. Let $X : M \hookrightarrow \mathbb{R}^n$ be the isometric imbedding. We claim that

$$\mathfrak{H}(x) = \frac{1}{k}\Delta X, \tag{29.10}$$

with $k = n - 1$, where Δ is the Laplace operator on the Riemannian manifold M, acting componentwise on X. This is easy to see at a point $p \in M$ if we translate and rotate $\mathbb{R}^n$ to make $p = 0$ and represent M as the image of $\mathbb{R}^k = \mathbb{R}^{n-1}$ under

$$Y(x') = \bigl(x', f(x')\bigr), \quad x' = (x_1, \dots, x_k),\ \nabla f(0) = 0. \tag{29.11}$$

Then one verifies that $\Delta X(p) = \partial_1^2 Y(0) + \cdots + \partial_k^2 Y(0) = \bigl(0, \dots, 0, \partial_1^2 f(0) + \cdots + \partial_k^2 f(0)\bigr)$, and (29.10) follows from the formula

$$\langle A_N(0)X, Y\rangle = \sum_{i,j=1}^{k} \partial_i\partial_j f(0)\ X_iY_j \tag{29.12}$$

for the second fundamental form of M at p, derived in (16.19).

More generally, if $M \subset \mathbb{R}^n$ has dimension $k \le n-1$, we can define the mean curvature vector $\mathfrak{H}(x)$ by

$$\langle \mathfrak{H}(x), \xi \rangle = \frac{1}{k} \operatorname{Tr} A_\xi(x), \quad \mathfrak{H}(x) \perp T_x M, \tag{29.13}$$

so the criterion for M to be a minimal submanifold is that $\mathfrak{H} = 0$. Furthermore, (29.10) continues to hold. This can be seen by the same type of argument used above; represent M as the image of $\mathbb{R}^k$ under (6.11), where now $f(x') = (x_{k+1}, \dots, x_n)$. Then (29.12) generalizes to

$$\langle A_\xi(0)X, Y \rangle = \sum_{i,j=1}^{k} \langle \xi, \partial_i \partial_j f(0) \rangle \; X_i Y_j, \tag{29.14}$$

which yields (29.10). We record this observation.

Proposition 29.1. *Let $X : M \to \mathbb{R}^n$ be an isometric immersion of a Riemannian manifold into $\mathbb{R}^n$. Then M is a minimal submanifold of $\mathbb{R}^n$ if and only if the coordinate functions $x_1, \dots, x_n$ are harmonic functions on M.*

A two-dimensional minimal submanifold of $\mathbb{R}^n$ is called a minimal surface. The theory is most developed in this case, and we will concentrate on the two-dimensional case in the material below.

When dim $M = 2$, we can extend Proposition 29.1 to cases where $X : M \to \mathbb{R}^n$ is not an isometric map. This occurs because, in such a case, the class of harmonic functions on M is invariant under conformal changes of metric. In fact, if Δ is the Laplace operator for a Riemannian metric g_{ij} on M and Δ_1 that for $g_{1ij} = e^{2u} g_{ij}$, then, since $\Delta f = g^{-1/2} \partial_i (g^{ij} g^{1/2} \partial_j f)$, and $g_1^{ij} = e^{-2u} g^{ij}$, while $g_1^{1/2} = e^{ku} g^{1/2}$ (if dim $M = k$), we have

$$\Delta_1 f = e^{-2u} \; \Delta f + e^{-ku} \langle df, de^{(k-2)u} \rangle = e^{-2u} \Delta f, \text{ if } k = 2. \tag{29.15}$$

Hence ker $\Delta =$ ker Δ_1, if $k = 2$. We hence have the following.

Proposition 29.2. *If Ω is a Riemannian manifold of dimension 2 and $X : \Omega \to \mathbb{R}^n$ a smooth immersion, with image M, then M is a minimal surface provided X is harmonic and $X : \Omega \to M$ is conformal.*

In fact, granted that $X : \Omega \to M$ is conformal, M is minimal if and only if X is harmonic on Ω.

We can use this result to produce lots of examples of minimal surfaces, by the following classical device. Take Ω to be an open set in $\mathbb{R}^2 = \mathbb{C}$, with coordinates (u_1, u_2). Given a map $X : \Omega \to \mathbb{R}^n$, with components $x_j : \Omega \to \mathbb{R}$, form the complex valued functions

$$\psi_j(\zeta) = \frac{\partial x_j}{\partial u_1} - i \frac{\partial x_j}{\partial u_2} = 2 \frac{\partial}{\partial \zeta} x_j, \quad \zeta = u_1 + i u_2. \tag{29.16}$$

Clearly ψ_j is holomorphic if and only if x_j is harmonic (for the standard flat metric on Ω), since $\Delta = 4(\partial/\partial\overline{\zeta})(\partial/\partial\zeta)$. Furthermore, a short calculation gives

$$\sum_{j=1}^{n} \psi_j(\zeta)^2 = \left|\partial_1 X\right|^2 - \left|\partial_2 X\right|^2 - 2i\ \partial_1 X \cdot \partial_2 X. \tag{29.17}$$

Granted that $X : \Omega \to \mathbb{R}^n$ is an immersion, the criterion that it be conformal is precisely that this quantity vanish. We have the following result.

Proposition 29.3. *If $\psi_1, \dots, \psi_n$ are holomorphic functions on $\Omega \subset \mathbb{C}$ such that*

$$\sum_{j=1}^{n} \psi_j(\zeta)^2 = 0 \quad on \quad \Omega, \tag{29.18}$$

while $\sum |\psi_j(\zeta)|^2 \neq 0$ on Ω, then setting

$$x_j(u) = \ Re \int \psi_j(\zeta)\ d\zeta \tag{29.19}$$

defines an immersion $X : \Omega \to \mathbb{R}^n$ whose image is a minimal surface.

If Ω is not simply connected, the domain of X is actually the universal covering surface of Ω.

We mention some particularly famous minimal surfaces in $\mathbb{R}^3$ that arise in such a fashion. Surely the premier candidate for (29.18) is

$$\sin^2\zeta + \cos^2\zeta - 1 = 0. \tag{29.20}$$

Here, take $\psi_1(\zeta) = \sin\zeta,\ \psi_2(\zeta) = -\cos\zeta,\ \psi_3(\zeta) = -i$. Then (29.19) yields

$$x_1 = (\cos u_1)(\cosh u_2), \quad x_2 = (\sin u_1)(\cosh u_2), \quad x_3 = u_2. \tag{29.21}$$

The surface obtained in $\mathbb{R}^3$ is called the *catenoid.* It is the surface of revolution about the x_3-axis of the curve $x_1 = \cosh x_3$ in the x_1-x_3 plane. Whenever $\psi_j(\zeta)$ are holomorphic functions satisfying (29.18), so are $e^{i\theta}\psi_j(\zeta)$, for any $\theta \in \mathbb{R}$. The resulting immersions $X_\theta : \Omega \to \mathbb{R}^n$ give rise to a family of minimal surfaces $M_\theta \subset \mathbb{R}^n$ which are said to be *associated.* In particular, $M_{\pi/2}$ is said to be *conjugate* to $M = M_0$. When M_0 is the catenoid, defined by (29.21), the conjugate minimal surface arises from $\psi_1(\zeta) = i\ \sin\zeta$, $\psi_2(\zeta) = -i\ \cos\zeta$, $\psi_3(\zeta) = 1$, and is given by

$$x_1 = (\sin u_1)(\sinh u_2), \quad x_2 = (\cos u_1)(\sinh u_2), \quad x_3 = u_1. \tag{29.22}$$

This surface is called the *helicoid.* We mention that associated minimal surfaces are locally isometric, but generally not congruent, i.e., the isometry between the surfaces does not extend to a rigid motion of the ambient Euclidean space.

The catenoid and helicoid were given as examples of minimal surfaces by Meusnier, in 1776.

One systematic way to produce triples of holomorphic functions $\psi_j(\zeta)$ satisfying (29.18) is to take

$$\psi_1 = \frac{1}{2} f(1-g^2), \quad \psi_2 = \frac{i}{2} f(1+g^2), \quad \psi_3 = fg, \tag{29.23}$$

for arbitrary holomorphic functions f and g on Ω. More generally, g can be meromorphic on Ω, as long as f has a zero of order $2m$ at each point where g has a pole of order m. The resulting map $X : \Omega \to M \subset \mathbb{R}^3$ is called the Weierstrass-Enneper representation of the minimal surface M. It has an interesting connection with the Gauss map of M, which will be sketched in the exercises. The example arising from $f = 1,\ g = \zeta$ produces "Enneper's surface." This surface is immersed in $\mathbb{R}^3$ but not imbedded.

For a long time the only known examples of complete imbedded minimal surfaces in $\mathbb{R}^3$ of finite topological type were the plane, catenoid, and helicoid, but in the 1980s it was proved by [HM1] that the surface obtained by taking $g = \zeta$ and $f(\zeta) = \wp(\zeta)$ (the Weierstrass $\wp$-function), is another example. Further examples have been found; computer graphics have been a valuable aid in this search; see [HM2].

A natural question is how general is the class of minimal surfaces arising from the construction in Proposition 29.3. In fact, it is easy to see that every minimal $M \subset \mathbb{R}^n$ is at least *locally* representable in such a fashion, using the existence of local isothermal coordinates, established in §N. Thus any $p \in M$ has a neighborhood $\mathcal{O}$ such that there is a conformal diffeomorphism $X : \Omega \to \mathcal{O}$, for some open set $\Omega \subset \mathbb{R}^2$. By Proposition 29.2 and the remark following it, if M is minimal, then X must be harmonic, so (29.16) furnishes the functions $\psi_j(\zeta)$ used in Proposition 29.3. Incidentally, this shows that any minimal surface in $\mathbb{R}^n$ is real analytic.

As for the question of whether the construction of Proposition 29.3 globally represents every minimal surface, the answer here is also "yes." A proof uses the fact that every noncompact Riemann surface (without boundary) is covered by either $\mathbb{C}$ or the unit disc in $\mathbb{C}$. A proof of this "uniformization theorem" can be found in [FaKr]. A positive answer, for simply connected compact minimal surfaces, with smooth boundary, is implied by the following result, which will also be useful for an attack on the Plateau problem.

Proposition 29.4. *If $\overline{M}$ is a compact simply connected Riemannian manifold of dimension 2, with smooth boundary, then there exists a conformal diffeomorphism*

$$\Phi : \overline{M} \longrightarrow \overline{D}, \tag{29.24}$$

where $\overline{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

This is a slight generalization of the Riemann Mapping Theorem. The following is a sketch of the proof.

Fix $p \in M$, and let $G \in \mathcal{D}'(M) \cap C^\infty(\overline{M} \setminus p)$ be the unique solution to

$$\Delta G = 2\pi\delta, \quad G = 0 \text{ on } \partial M. \tag{29.25}$$

Since M is simply connected, it is orientable, so we can pick a Hodge star operator, and $*dG = \beta$ is a smooth closed 1-form on $\overline{M} \setminus p$. If γ is a curve in M of degree 1 about p, then $\int_\gamma \beta$ can be calculated by deforming γ to be a small curve about p. Basic theory of elliptic PDE gives $G(x) \sim \log\ \mathrm{dist}(x,p)$, and one establishes that $\int_\gamma \beta = 2\pi$. Thus we can write $\beta = dH$, where H is a smooth function on $\overline{M} \setminus p$, well defined mod $2\pi\mathbb{Z}$. Hence $\Phi(x) = e^{G+iH}$ is a single valued function, tending to 0 as $x \to p$, which one verifies to be the desired conformal diffeomorphism (29.24). More details can be found in Chapter 14 of [T1].

An immediate corollary is that the argument given above for local representation of a minimal surface in the form (29.19) extends to a global representation of a compact simply connected minimal surface, with smooth boundary

So far we have dealt with smooth surfaces, at least immersed in $\mathbb{R}^n$. The theorem of Douglas and Rado which we now tackle deals with "generalized" surfaces, which we will simply define to be the images of 2-dimensional manifolds under smooth maps into $\mathbb{R}^n$ (or some other manifold). The theorem, a partial answer to the "Plateau problem," asserts the existence of an area minimizing generalized surface whose boundary is a given simple closed curve in $\mathbb{R}^n$.

To be precise, let γ be a smooth simple closed curve in $\mathbb{R}^n$, i.e., a diffeomorphic image of S^1. Let

$$\mathfrak{X}_\gamma = \{\varphi \in C(\overline{D}, \mathbb{R}^n) \cap C^\infty(D, \mathbb{R}^n) : \varphi : S^1 \to \gamma \text{ monotone, and } \alpha(\varphi) < \infty\}, \tag{29.26}$$

where α is the area functional:

$$\alpha(\varphi) = \int_D |\partial_1 \varphi \wedge \partial_2 \varphi|\ dx_1 dx_2. \tag{29.27}$$

Then let

$$\mathcal{A}_\gamma = \inf\ \{\alpha(\varphi) : \varphi \in \mathfrak{X}_\gamma\}. \tag{29.28}$$

The existence theorem of Douglas and Rado is:

Theorem 29.5. *There is a map $\varphi \in \mathfrak{X}_\gamma$ such that $\alpha(\varphi) = \mathcal{A}_\gamma$.*

We can choose $\varphi_\nu \in \mathfrak{X}_\gamma$ such that $\alpha(\varphi_\nu) \searrow \mathcal{A}_\gamma$, but $\{\varphi_\nu\}$ could hardly be expected to have a convergent subsequence unless some structure is imposed on the maps φ_ν. The reason is that $\alpha(\varphi) = \alpha(\varphi \circ \psi)$ for any C^∞ diffeomorphism $\psi : \overline{D} \to \overline{D}$. We say $\varphi \circ \psi$ is a *reparametrization* of φ. The key to success is to take

φ_ν which approximately minimize not only the area functional $\alpha(\varphi)$ but also the energy functional

$$\vartheta(\varphi) = \int_D |\nabla\varphi(x)|^2 \, dx_1 dx_2, \tag{29.29}$$

so that we will also have $\vartheta(\varphi_\nu) \searrow d_\gamma$, where

$$d_\gamma = \inf\, \{\vartheta(\varphi) : \varphi \in \mathfrak{X}_\gamma\}. \tag{29.30}$$

To relate these, we compare (29.29) and the area functional (29.27).

To compare integrands, we have

$$|\nabla\varphi|^2 = |\partial_1\varphi|^2 + |\partial_2\varphi|^2, \tag{29.31}$$

while the square of the integrand in (29.27) is equal to

(29.32)
$$|\partial_1\varphi \wedge \partial_2\varphi|^2 = |\partial_1\varphi|^2|\partial_2\varphi|^2 - \langle\partial_1\varphi, \partial_2\varphi\rangle \le |\partial_1\varphi|^2|\partial_2\varphi|^2 \le \frac{1}{4}\bigl(|\partial_1\varphi|^2 + |\partial_2\varphi|^2\bigr)^2,$$

where equality holds if and only if

$$|\partial_1\varphi| = |\partial_2\varphi|, \text{ and } \langle\partial_1\varphi, \partial_2\varphi\rangle = 0. \tag{29.33}$$

Whenever $\nabla\varphi \neq 0$, this is the condition that φ be *conformal.* More generally, if (29.33) holds, but we allow $\nabla\varphi(x) = 0$, we say that φ is *essentially conformal.* Thus, we have seen that, for each $\varphi \in \mathfrak{X}_\gamma$,

$$\alpha(\varphi) \le \frac{1}{2}\vartheta(\varphi), \tag{29.34}$$

with equality if and only if φ is essentially conformal. The following result allows us to transform the problem of minimizing $\alpha(\varphi)$ over $\mathfrak{X}_\gamma$ into that of minimizing $\vartheta(\varphi)$ over $\mathfrak{X}_\gamma$, which will be an important tool in the proof of Theorem 29.5. Set

$$\mathfrak{X}_\gamma^\infty = \{\varphi \in C^\infty(\overline{D}, \mathbb{R}^n) : \varphi : S^1 \to \gamma \text{ diffeo}\}. \tag{29.35}$$

Proposition 29.6. *Given $\varepsilon > 0$, any $\varphi \in \mathfrak{X}_\gamma^\infty$ has a reparametrization $\varphi \circ \psi$ such that*

$$\frac{1}{2}\vartheta(\varphi \circ \psi) \le \alpha(\varphi) + \varepsilon. \tag{29.36}$$

Proof. We will obtain this from Proposition 29.4, but that result may not apply to $\varphi(\overline{D})$, so we do the following. Take $\delta > 0$ and define $\Phi_\delta : \overline{D} \to \mathbb{R}^{n+2}$ by $\Phi_\delta(x) = \bigl(\varphi(x), \delta x\bigr)$. For any $\delta > 0$, Φ_δ is a diffeomorphism of $\overline{D}$ onto its image, and if δ is very small, area $\Phi_\delta(\overline{D})$ is only a little larger than area $\varphi(D)$. Now, by Proposition 29.4, there is a conformal diffeomorphism $\Psi : \Phi_\delta(\overline{D}) \to \overline{D}$. Set $\psi = \psi_\delta = \big(\Psi \circ \Phi_\delta\big)^{-1} : \overline{D} \to \overline{D}$. Then $\Phi_\delta \circ \psi = \Psi^{-1}$ and, as established above, $(1/2)\vartheta(\Psi^{-1}) = \text{area}(\Psi^{-1}(\overline{D}))$, i.e.,

$$\frac{1}{2}\vartheta(\Phi_\delta \circ \psi) = \text{area}\big(\Phi_\delta(\overline{D})\big). \tag{29.37}$$

Since $\vartheta(\varphi \circ \psi) \le \vartheta(\Phi_\delta \circ \psi)$, the result (29.34) follows, if δ is taken small enough.

One can show that

$$\mathcal{A}_\gamma = \inf\,\{\alpha(\varphi) : \varphi \in \mathfrak{X}_\gamma^\infty\}, \quad d_\gamma = \inf\,\{\vartheta(\varphi) : \varphi \in \mathfrak{X}_\gamma^\infty\}. \tag{29.38}$$

It then follows from Proposition 29.6 that $\mathcal{A}_\gamma = (1/2)d_\gamma$ and, if $\varphi_\nu \in \mathfrak{X}_\gamma^\infty$ is chosen so that $\vartheta(\varphi_\nu) \to d_\gamma$, then a fortiori $\alpha(\varphi_\nu) \to \mathcal{A}_\gamma$.

There is still an obstacle to obtaining a convergent subsequence of such $\{\varphi_\nu\}$. Namely, the energy integral (29.29) is invariant under reparametrizations $\varphi \mapsto \varphi \circ \psi$ for which $\psi : \overline{D} \to \overline{D}$ is a *conformal* diffeomorphism. We can put a clamp on this by noting that, given any two triples of (distinct) points $\{p_1, p_2, p_2\}$ and $\{q_1, q_2, q_3\}$ in $S^1 = \partial D$, there is a unique conformal diffeomorphism $\psi : \overline{D} \to \overline{D}$ such that $\psi(p_j) = q_j, 1 \le j \le 3$. Let us now make one choice of $\{p_j\}$ on S^1, e.g., the 3 cube roots of 1, and make one choice of a triple $\{q_j\}$ of distinct points in γ. The following key compactness result will enable us to prove Theorem 29.5.

Proposition 29.7. *For any $d \in (d_\gamma, \infty)$, the set*

$$\Sigma_d = \big\{\varphi \in \mathfrak{X}_\gamma^\infty : \varphi \textit{ harmonic}, \varphi(p_j) = q_j, \textit{ and } \vartheta(\varphi) \le d\big\} \tag{29.39}$$

is relatively compact in $C(\overline{D}, \mathbb{R}^n)$.

In view of known estimates for harmonic functions, including mapping properties of the Poisson integral, which can be found in Chapter 5 of [T1], this result is equivalent to the relative compactness in $C(\partial D, S^1)$ of

$$\mathcal{S}_K = \{u \in C^\infty(S^1, S^1) \text{ diffeo} : u(p_j) = q_j, \text{ and } \|\nabla\, \mathrm{PI}\, u\|_{L^2(D)} \le K\}, \tag{29.40}$$

for any given $K < \infty$. Here PI u denotes the harmonic $\mathbb{R}^n$-valued function on D, equal to u on $\partial D = S^1$.

We will show that the oscillation of u over any arc $I \subset S^1$ of length 2δ is $\le CK/\sqrt{\log \frac{1}{\delta}}$. This modulus of continuity will imply the compactness, by Ascoli's theorem.

Pick a point $z \in S^1$. Let C_r denote the portion of the circle of radius r and center z which lies in $\overline{D}$. Thus C_r is an arc, of length $\le \pi r$. Let $\delta \in (0, 1)$. As r

varies from δ to $\sqrt{\delta}$, C_r sweeps out part of an annulus, as illustrated in Fig. 29.1. We claim there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\int_{C_\rho} |\nabla\varphi| \; ds \le K \sqrt{\frac{2\pi}{\log \frac{1}{\delta}}}, \tag{29.41}$$

if $K = \|\nabla\varphi\|_{L^2(D)}$, $\varphi = \mathrm{PI}\; u$. To establish this, let

$$\omega(r) = r \int_{C_r} |\nabla\varphi|^2 \; ds.$$

Then

$$\int_\delta^{\sqrt{\delta}} \omega(r) \; \frac{dr}{r} = \int_\delta^{\sqrt{\delta}} \int_{C_r} |\nabla\varphi|^2 \; ds \; dr = I \le K^2.$$

By the mean value theorem, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$I = \omega(\rho) \int_\delta^{\sqrt{\delta}} \frac{dr}{r} = \frac{\omega(\rho)}{2} \; \log \frac{1}{\delta}.$$

For this value of ρ, we have

$$\rho \int_{C_\rho} |\nabla\varphi|^2 \; ds = \frac{2I}{\log \frac{1}{\delta}} \le \frac{2K^2}{\log \frac{1}{\delta}}. \tag{29.42}$$

Then Cauchy's inequality yields (29.41), since $\mathrm{length}(C_\rho) \le \pi\rho$.

This almost gives the desired modulus of continuity. The arc C_ρ is mapped by φ into a curve of length $\le K\sqrt{2\pi/\log \frac{1}{\delta}}$, whose endpoints divide γ into 2 segments, one rather short (if δ is small), one not so short. There are two possibilities; $\varphi(z)$ is contained in either the short segment (as in Fig. 29.2) or the long segment (as in Fig. 29.3). However, as long as $\varphi(p_j) = p_j$ for three points p_j, this latter possibility cannot occur. We see that $|u(a) - u(b)| \le K\sqrt{2\pi/\log \frac{1}{\delta}}$, if a and b are the points where C_ρ intersects S^1. Now the monotonicity of u along S^1 guarantees that the total variation of u on the (small) arc from a to b in S^1 is $\le K\sqrt{2\pi/\log \frac{1}{\delta}}$. This establishes the modulus of continuity and concludes the proof.

Now that we have Proposition 29.7, we proceed as follows. Pick a sequence φ_ν in $\mathfrak{X}_\gamma^\infty$ such that $\vartheta(\varphi_\nu) \to d_\gamma$, so also $\alpha(\varphi_\nu) \to \mathcal{A}_\gamma$. Now we do not increase $\vartheta(\varphi_\nu)$ if we replace φ_ν by the Poisson integral of $\varphi_\nu\big|_{\partial D}$, and we no not alter this energy integral if we reparametrize via a conformal diffeomorphism to take $\{p_j\}$ to $\{q_j\}$.

Thus we may as well suppose that $\varphi_\nu \in \Sigma_d$. Using Proposition 29.7 and passing to a subsequence, we can assume

$$\varphi_\nu \longrightarrow \varphi \ \text{ in } \ C(\overline{D}, \mathbb{R}^n), \tag{29.43}$$

and we can furthermore arrange

$$\varphi_\nu \longrightarrow \varphi \ \text{ weakly in } \ H^1(D, \mathbb{R}^n). \tag{29.44}$$

By interior estimates for harmonic functions, we have

$$\varphi_\nu \longrightarrow \varphi \ \text{ in } \ C^\infty(D, \mathbb{R}^n). \tag{29.45}$$

The limit function φ is certainly harmonic on D. By (29.44), we have

$$\vartheta(\varphi) \le \lim_{\nu\to\infty} \ \vartheta(\varphi_\nu) = d_\gamma. \tag{29.46}$$

Now (29.34) applies to φ, so we have

$$\alpha(\varphi) \le \frac{1}{2}\vartheta(\varphi) \le \frac{1}{2}d_\gamma = \mathcal{A}_\gamma. \tag{29.47}$$

On the other hand, (29.43) implies that $\varphi : \partial D \to \gamma$ is monotone. Thus φ belongs to $\mathfrak{X}_\gamma$. Hence we have

$$\alpha(\varphi) = \mathcal{A}_\gamma. \tag{29.48}$$

This proves Theorem 29.5, and most of the following more precise result.

Theorem 29.8**.** *If γ is a smooth simple closed curve in $\mathbb{R}^n$, there exists a continuous map $\varphi : \overline{D} \to \mathbb{R}^n$ such that*

$$\vartheta(\varphi) = d_\gamma, \quad \alpha(\varphi) = \mathcal{A}_\gamma, \tag{29.49}$$

$$\varphi : D \longrightarrow \mathbb{R}^n \ \textit{is harmonic and essentially conformal}, \tag{29.50}$$

$$\varphi : S^1 \longrightarrow \gamma, \ \textit{homeomorphically}. \tag{29.51}$$

Proof. We have (29.49) from (29.46)–(29.48). By the argument involving (29.31)–(29.32), this forces φ to be essentially conformal. It remains to demonstrate (29.51).

We know that $\varphi : S^1 \to \gamma$, monotonically. If it fails to be a homeomorphism, there must be an interval $I \subset S^1$ on which φ is constant. Using a linear fractional transformation to map D conformally onto the upper half plane $\Omega^+ \subset \mathbb{C}$, we can regard φ as a harmonic and essentially conformal map of $\Omega^+ \to \mathbb{R}^n$, constant on an interval I on the real axis $\mathbb{R}$. Via the Schwartz reflection principle, we can extend φ to a harmonic function

$$\varphi : \mathbb{C} \setminus (\mathbb{R} \setminus I) \longrightarrow \mathbb{R}^n.$$

Now consider the holomorphic function $\psi : \mathbb{C}\setminus(\mathbb{R}\setminus I) \to \mathbb{C}^n$, given by $\psi(\zeta) = \partial\varphi/\partial\zeta$. As in the calculations leading to Proposition 29.3, the identities

$$|\partial_1\varphi|^2 - |\partial_2\varphi|^2 = 0, \quad \partial_1\varphi \cdot \partial_2\varphi = 0, \tag{29.52}$$

which hold on Ω^+, imply $\sum_{j=1}^n \psi_j(\zeta)^2 = 0$ on Ω^+; hence this holds on $\mathbb{C} \setminus (\mathbb{R} \setminus I)$, and so does (29.52). But since $\partial_1\varphi = 0$ on I, we deduce that $\partial_2\varphi = 0$ on I, hence $\psi = 0$ on I, hence $\psi \equiv 0$. This implies that φ, being both $\mathbb{R}^n$-valued and anti-holomorphic, must be constant, which is impossible. This contradiction establishes (29.51).

Theorem 29.8 furnishes a generalized minimal surface whose boundary is a given smooth closed curve in $\mathbb{R}^n$. We know that φ is smooth on D. It has been shown by S.Hildebrandt that φ is C^∞ on $\overline{D}$ when the curve γ is C^∞, as we have assumed here. It should be mentioned that Douglas and others treated the Plateau problem for simple closed curves γ which were *not* smooth. We have restricted attention to smooth γ for simplicity. A treatment of the general case can be found in [Nit].

There remains the question of the smoothness of the image surface $M = \varphi(D)$. The map $\varphi : D \to \mathbb{R}^n$ would fail to be an immersion at a point $z \in D$ where $\nabla\varphi(z) = 0$. At such a point, the $\mathbb{C}^n$-valued holomorphic function $\psi = \partial\varphi/\partial\zeta$ must vanish, i.e., each of its components must vanish. Since a holomorphic function on $D \subset \mathbb{C}$ which is not identically zero can only vanish on a discrete set, we have:

Proposition 29.9. *The map $\varphi : D \to \mathbb{R}^n$ parametrizing the generalized minimal surface in Theorem 29.8 has injective derivative except at a discrete set of points in D.*

If $\nabla\varphi(z) = 0$, then $\varphi(z) \in M = \varphi(D)$ is said to be a *branch point* of the generalized minimal surface M; we say M is a branched surface. If $n \geq 4$, there are indeed generalized minimal surfaces with branch points which arise via Theorem 29.8. Results of R. Osserman, complemented by work of R. Gulliver, show that, if $n = 3$, the construction of Theorem 29.8 yields a smooth minimal surface, immersed in $\mathbb{R}^3$. Such a minimal surface need not be imbedded; for example, if γ is a knot in $\mathbb{R}^3$, such a surface with boundary equal to γ is certainly not imbedded. If γ is analytic, it is known that there cannot be branch points on the boundary, though this is open for merely smooth γ. An extensive discussion of boundary regularity is given in Vol. 2 of [DHKW].

The following result of Rado yields one simple criterion for a generalized minimal surface to have no branch points.

Proposition 29.10. *Let γ be a smooth closed curve in $\mathbb{R}^n$. If a minimal surface with boundary γ produced by Theorem 29.8 has any branch points, then γ has the*

property that
(29.53)
for some $p \in \mathbb{R}^n$, *every hyperplane through* p *intersects* γ *in at least 4 points.*

Proof. Suppose $z_0 \in D$ and $\nabla\varphi(z_0) = 0$, so $\psi = \partial\varphi/\partial\zeta$ vanishes at z_0. Let $L(x) = \alpha \cdot x + c = 0$ be the equation of an arbitrary hyperplane through $p = \varphi(z_0)$. Then $h(x) = L\big(\varphi(x)\big)$ is a (real valued) harmonic function on D, satisfying

$$\Delta h = 0 \text{ on } D, \quad \nabla h(z_0) = 0. \tag{29.54}$$

The proposition is then proved, by the following:

Lemma 29.11. *Any real valued* $h \in C^\infty(D) \cap C(\overline{D})$ *having the property (29.54) must assume the value* $h(z_0)$ *on at least 4 points on* ∂D.

Proof. First, composing h with a linear fractional transformation preserving D and taking 0 to z_0, we reduce the problem to proving the lemma when $z_0 = 0$. Also, without loss of generality we can assume that $h(0) = 0$. In such a case, one can deduce from the Poisson integral formula that the hypotheses in (29.54) imply

$$\int_{\partial D} f\, ds = \int_{\partial D} xf\, ds = \int_{\partial D} yf\, ds = 0,$$

where $f = h|_{\partial D} \in C(\partial D)$. Our task is to deduce from this that f vanishes at four points on ∂D.

Assume that f is not identically zero. Then $\int_{\partial D} f\, ds = 0$ implies $f > 0$ on some arc I_0 in $\partial D = S^1$, and vanishes at the endpoints. Let I be the largest arc containing I_0 on which $f \geq 0$. Rotating S^1, we can assume $I = \{\theta : -\alpha \leq \theta \leq \alpha\}$ for some $\alpha \in (0, \pi)$. We also have

$$\int_{S^1} (x - a) f\, ds = 0, \quad a = \cos \alpha.$$

Note that $\int_I (x - a) f\, ds > 0$, while $x - a < 0$ on $J = \{\theta : \alpha < \theta < 2\pi - \alpha\}$, the arc in S^1 complementary to I. Now $\int_J (x - a) f\, ds < 0$, so we deduce that $f > 0$ on some arc J_1 inside J, so f must vanish at 4 points, as asserted.

The following result gives a condition under which a minimal surface constructed by Theorem 29.8 is the graph of a function.

Proposition 29.12. *Let* $\mathcal{O}$ *be a bounded convex domain in* $\mathbb{R}^2$ *with smooth boundary. Let* $g : \partial\mathcal{O} \to \mathbb{R}^{n-2}$ *be smooth. Then there exists a function*

$$f \in C^\infty(\mathcal{O}, \mathbb{R}^{n-2}) \cap C(\overline{\mathcal{O}}, \mathbb{R}^{n-2}), \tag{29.55}$$

whose graph is a minimal surface, whose boundary is the curve $\gamma \subset \mathbb{R}^n$ which is the graph of g, so

$$f = g \quad on \quad \partial\mathcal{O}. \tag{29.56}$$

Proof. Let $\varphi : \overline{D} \to \mathbb{R}^n$ be the function constructed in Theorem 29.8. Set $F(x) = \big(\varphi_1(x), \varphi_2(x)\big)$. Then $F : \overline{D} \to \mathbb{R}^2$ is harmonic on D and F maps $S^1 = \partial D$ homeomorphically onto $\partial\mathcal{O}$. It follows from convexity of $\mathcal{O}$ and the maximum principle for harmonic functions that $F : \overline{D} \to \overline{\mathcal{O}}$.

We claim that $DF(x)$ is invertible for each $x \in D$. Indeed, if $x_0 \in D$ and $DF(x_0)$ is singular, we can choose nonzero $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that, at $x = x_0$,

$$\alpha_1 \frac{\partial\varphi_1}{\partial x_j} + \alpha_2 \frac{\partial\varphi_2}{\partial x_j} = 0, \quad j = 1, 2.$$

Then the function $h(x) = \alpha_1\varphi_1(x) + \alpha_2\varphi_2(x)$ has the property (29.54), so $h(x)$ must take the value $h(x_0)$ at 4 distinct points of ∂D. Since $F : \partial D \to \partial\mathcal{O}$ is a homeomorphism, this forces the *linear* function $\alpha_1 x_1 + \alpha_2 x_2$ to take the same value at 4 distinct points of $\partial\mathcal{O}$, which contradicts convexity of $\mathcal{O}$.

Thus $F : D \to \mathcal{O}$ is a local diffeomorphism. Since F gives a homeomorphism of the boundaries of these regions, degree theory implies that F is a diffeomorphism of D onto $\mathcal{O}$ and a homeomorphism of $\overline{D}$ onto $\overline{\mathcal{O}}$. Consequently, the desired function in (29.55) is $f = \widetilde{\varphi} \circ F^{-1}$, where $\widetilde{\varphi}(x) = \big(\varphi_2(x), \dots, \varphi_n(x)\big)$.

Functions whose graphs are minimal surfaces satisfy a certain nonlinear PDE, called the minimal surface equation, which we will derive and study in §31.

Let us mention that, while one ingredient in the solution to the Plateau problem presented above was a version of the Riemann mapping theorem, Proposition 29.4, there are presentations for which the Riemann mapping theorem is a consequence of the argument, rather than an ingredient; see e.g., [Nit].

It is also of interest to consider the analogue of the Plateau problem when, instead of immersing the disc in $\mathbb{R}^n$ as a minimal surface with given boundary, one takes a surface of higher genus, and perhaps several boundary components. An extra complication is that Proposition 6.4 must be replaced by something more elaborate, since two compact surfaces with boundary which are diffeomorphic to each other but not to the disc may not be conformally equivalent. One needs to consider spaces of “moduli” of such surfaces. This problem was tackled by Douglas [Dou2] and by Courant [Cou], but their work has been criticised by [ToT] and [Jos], who present alternative solutions. The paper [Jos] also treats the Plateau problem for surfaces in Riemannian manifolds, extending results of [Mor1].

There have been successful attacks on problems in the theory of minimal submanifolds, particularly in higher dimension, using very different techniques, involving geometric measure theory, currents, and varifolds. Material on these important developments can be found in [Alm], [Fed], and [Morg].

So far in this section, we have devoted all our attention to minimal submanifolds of Euclidean space. It is also interesting to consider minimal submanifolds of other Riemannian manifolds. We make a few brief comments on this topic. A great deal more can be found in [Law], [Law2], and [Mor1], and in survey articles in [Bom].

Let Y be a smooth compact Riemannian manifold. Assume Y is isometrically imbedded in $\mathbb{R}^n$, which can always be arranged, by Nash's theorem (cf. Appendix Y). Let M be a compact k-dimensional submanifold of Y. We say M is a minimal submanifold of Y if its k-dimensional volume is a critical point with respect to small variations of M, within Y. The computations (29.1)–(29.13) extend to this case. We need to take $X = X(s,u)$ with $\partial_s X(s,u) = \xi(s,u)$, tangent to Y, rather than $X(s,u) = X_0(u) + s\xi(u)$. Then these computations show that M is a minimal submanifold of Y if and only if, for each $x \in M$,

$$\mathfrak{H}(x) \perp T_x Y, \tag{29.57}$$

where $\mathfrak{H}(x)$ is the mean curvature vector of M (as a submanifold of $\mathbb{R}^n$), defined by (29.13).

There is also a well defined mean curvature vector $\mathfrak{H}_Y(x) \in T_x Y$, orthogonal to $T_x M$, obtained from the second fundamental form of M as a submanifold of Y. One sees that $\mathfrak{H}_Y(x)$ is the orthogonal projection of $\mathfrak{H}(x)$ onto $T_x Y$, so the condition that M be a minimal submanifold of Y is that $\mathfrak{H}_Y = 0$ on M.

The formula (29.10) continues to hold, for the isometric imbedding $X : M \to \mathbb{R}^n$. Thus M is a minimal submanifold of Y if and only if, for each $x \in M$,

$$\Delta X(x) \perp T_x Y. \tag{29.58}$$

If dim $M = 2$, the formula (29.15) holds, so if M is given a new metric, conformally scaled by a factor e^{2u}, the new Laplace operator Δ_1 has the property that $\Delta_1 X = e^{-2u} \Delta X$, hence is parallel to ΔX. Thus the property (29.58) is unaffected by such a conformal change of metric; we have the following extension of Proposition 29.2.

Proposition 29.13. *If M is a Riemannian manifold of dimension 2 and $X : M \to \mathbb{R}^n$ is a smooth imbedding, with image $M_1 \subset Y$, then M_1 is a minimal submanifold of Y provided $X : M \to M_1$ is conformal and, for each $x \in M$,*

$$\Delta X(x) \perp T_{X(x)} Y. \tag{29.59}$$

We note that (29.59) alone specifies that X is a *harmonic* map from M into Y.

Exercises

1. Consider the Gauss map $N : M \to S^2$, for a smooth oriented surface $M \subset \mathbb{R}^3$. Show that N is *antiholomorphic* if and only if M is a minimal surface.

Hint. If $N(p) = q,\ DN(p) : T_pM \to T_qS^2 \approx T_pM$ is identified with $-A_N$. (Compare (16.67).) Check when $A_N J = -JA_N$, where J is counterclockwise rotation by 90°, on T_pM.
Thus, if we define the antipodal Gauss map $\widetilde{N} : M \to S^2$ by $\widetilde{N}(p) = -N(p)$, this map is holomorphic precisely when M is a minimal surface.

2. If $x \in S^2 \subset \mathbb{R}^3$, pick $v \in T_xS^2 \subset \mathbb{R}^3$, set $w = Jv \in T_xS^2 \subset \mathbb{R}^3$, and take $\xi = v + iw \in \mathbb{C}^3$. Show that the 1-dimensional complex span of ξ is independent of the choice of v, and that we hence have a holomorphic map

$$\Xi : S^2 \longrightarrow \mathbb{CP}^3.$$

Show that the image $\Xi(S^2) \subset \mathbb{CP}^3$ is contained in the image of $\{\zeta \in \mathbb{C}^3 \setminus 0 : \zeta_1^2 + \zeta_2^2 + \zeta_3^3 = 0\}$ under the natural map $\mathbb{C}^3 \setminus 0 \to \mathbb{CP}^3$.

3. Suppose that $M \subset \mathbb{R}^3$ is a minimal surface constructed by the method of Proposition 29.3, via $X : \Omega \to M \subset \mathbb{R}^3$. Define $\Psi : \Omega \to \mathbb{C}^3 \setminus 0$ by $\Psi = (\psi_1, \psi_2, \psi_3)$, and define $\mathfrak{X} : \Omega \to \mathbb{CP}^3$ by composing Ψ with the natural map $\mathbb{C}^3 \setminus 0 \to \mathbb{CP}^3$. Show that, for $u \in \Omega$,

$$\mathfrak{X}(u) = \Xi \circ \widetilde{N}\big(X(u)\big).$$

For the relation between ψ_j and the Gauss map for minimal surfaces in $\mathbb{R}^n,\ n > 3$, see [Law].

4. Give a detailed demonstration of (29.38).

5. If $\widetilde{II}$ is the second fundamental form of a minimal hypersurface $M \subset \mathbb{R}^n$, show that $\widetilde{II}$ has divergence zero. As in §12, we define the divergence of a second order tensor field T by $T^{jk}{}_{;k}$.
Hint. Use the Codazzi equation (cf. (16.18)) plus the zero trace condition.

6. Similarly, if $\widetilde{II}$ is the second fundamental form of a minimal submanifold M of codimension 1 in S^n (with its standard metric), show that $\widetilde{II}$ has divergence zero.
Hint. The Codazzi equation (16.16) is

$$(\nabla_Y \widetilde{II})(X, Z) - (\nabla_Y \widetilde{II})(Y, Z) = \langle R(X, Y)Z, N\rangle,$$

where ∇ is the Levi-Civita connection on M, X, Y, Z are tangent to M, Z is normal to M (but tangent to S^n), and R is the curvature tensor of S^n. In such a case, the right side *vanishes.* (See Exercise 6 in §16.) Thus the argument needed for Exercise 5 above extends.

7. Extend the result of Exercises 5–6 to the case where M is a codimension 1 minimal submanifold in any Riemannian manifold Ω with constant sectional curvature.

8. Let M be a two-dimensional minimal submanifold of S^3, with its standard metric. Assume M is diffeomorphic to S^2. Show that M must be a "great sphere" in S^3.
Hint. By Exercise 6, $\widetilde{II}$ is a symmetric trace-free tensor of divergence zero, i.e. $\widetilde{II}$ belongs to

$$\mathcal{V} = \{u \in C^\infty(M, S_0^2 T^*) : \text{div } u = 0\} = \text{ ker } \mathcal{D}_{TF}^*,$$

where $\mathcal{D}_{TF}^*$ is defined by (28.30). See Exercise 7 of §28 for a proof that $\mathcal{V} = 0$ if M has positive curvature everywhere. Next, guided by Exercises 5–7 of §28, show that $\mathcal{V} = 0$ provided the analogous space vanishes when M is given some conformally equivalent metric.

Now it is known that, if M is any Riemannian manifold diffeomorphic to S^2, then it has a conformally equivalent metric *isometric* to S^2, with its standard metric. A proof of this (using the Riemann-Roch theorem) can be found in Chapter 10 of [T1].

30. Second variation of area

We take up a computation of the second variation of the area integral, and some implications, for a family of manifolds of dimension k, immersed in a Riemannian manifold Y. First, we take $Y = \mathbb{R}^n$ and suppose the family is given by $X(s,u) = X_0(u) + s\xi(u)$, as in (29.1)–(29.5).

Suppose as in the computation (29.2)–(29.5) that $\|\partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\| = 1$ on M, while $E_j = \partial_j X_0$ form an orthonormal basis of T_xM, for a given point $x \in M$. Then, extending (29.3), we have

$$(30.1)\quad A'(s) = \sum_{j=1}^{k} \int \frac{\langle \partial_1 X \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X, \partial_1 X \wedge \cdots \wedge \partial_k X \rangle}{\|\partial_1 X \wedge \cdots \wedge \partial_k X\|}\, du_1 \cdots du_k.$$

Consequently, $A''(0)$ will be the integral with respect to $du_1 \cdots du_k$ of a sum of three terms:

$$(30.2)\quad \begin{aligned} &-\sum_{i,j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \\ &\qquad\qquad \times \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \\ &+ 2\sum_{i<j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \\ &+ \sum_{i,j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_k X_0 \rangle. \end{aligned}$$

Let us write

$$(30.3)\qquad A_\xi E_i = \sum_\ell a_\xi^{i\ell} E_\ell,$$

with $E_j = \partial_j X_0$ as before. Then, as in (29.4), the first sum in (30.2) is equal to

$$(30.4)\qquad -\sum_{i,j} a_\xi^{ii} a_\xi^{jj}.$$

Let us move to the last sum in (30.2). We use the Weingarten formula $\partial_j \xi = \nabla_j^1 \xi - A_\xi E_j$, to write this sum as

$$(30.5)\qquad \sum_{i,j} a_\xi^{jj} a_\xi^{ii} + \sum_{i,j} \langle \nabla_j^1 \xi, \nabla_i^1 \xi \rangle,$$

at x. Note that the first sum in (30.5) *cancels* (30.4), while the last sum in (30.5) can be written $\|\nabla^1\xi\|^2$. Here, ∇^1 is the connection induced on the normal bundle of M.

Now we look at the middle term in (30.2), i.e.,

$$2\sum_{i<j}\sum_{\ell,m} a_\xi^{i\ell} a_\xi^{jm} \big\langle E_1\wedge\cdots\wedge E_\ell\wedge\cdots\wedge E_m\wedge\cdots\wedge E_k, E_1\wedge\cdots\wedge E_k\big\rangle, \tag{30.6}$$

at x, where E_ℓ appears in the ith slot and E_m appears in the jth slot, in the k-fold wedge product. This is equal to

$$2\sum_{i<j}\big(a_\xi^{ii}a_\xi^{jj} - a_\xi^{ij}a_\xi^{ji}\big) = 2\ \mathrm{Tr}\ \Lambda^2 A_\xi, \tag{30.7}$$

at x. Thus we have

$$A''(0) = \int_M \Big[\|\nabla^1\xi\|^2 + 2\ \mathrm{Tr}\ \Lambda^2 A_\xi\Big]\ dA(x). \tag{30.8}$$

If M is a hypersurface of $\mathbb{R}^n$, and we take $\xi = fN$, where N is a unit normal field, then $\|\nabla^1\xi\|^2 = \|\nabla f\|^2$ and (30.7) is equal to

$$2\sum_{i<j}\big\langle R(E_j,E_i)E_i,E_j\big\rangle f^2 = Sf^2, \tag{30.9}$$

by the Theorema Egregium, where S is the scalar curvature of M. Consequently, if $M\subset\mathbb{R}^n$ is a hypersurface (with boundary), and M_s are given by (29.6), with area integral (29.2), then

$$A''(0) = \int_M \Big[\|\nabla f\|^2 + S(x)f^2\Big]\ dA(x). \tag{30.10}$$

Recall that, when dim $M = 2$, so $M\subset\mathbb{R}^3$,

$$S = 2K, \tag{30.11}$$

where K is the Gauss curvature, which is ≤ 0 whenever M is a minimal surface in $\mathbb{R}^3$.

If M has general codimension in $\mathbb{R}^n$, we can rewrite (30.8) using the identity

$$2\ \mathrm{Tr}\ \Lambda^2 A_\xi = (\mathrm{Tr}\ A_\xi)^2 - \|A_\xi\|^2, \tag{30.12}$$

where $\|A_\xi\|$ denotes the Hilbert-Schmidt norm of A_ξ, i.e., $\|A_\xi\|^2 = \mathrm{Tr}\,(A_\xi^* A_\xi)$. Recalling (29.13), if $k = \dim M$, we get

$$A''(0) = \int_M \Big[\|\nabla^1\xi\|^2 - \|A_\xi\|^2 + k^2\langle\mathfrak{H}(x),\xi\rangle^2\Big]\, dA(x). \tag{30.13}$$

Of course, the last term in the integrand vanishes for all compactly supported fields ξ normal to M, when M is a minimal submanifold of $\mathbb{R}^n$.

We next suppose the family of manifolds M_s is contained in a manifold $Y \subset \mathbb{R}^n$. Hence, as before, instead of $X(s,u) = X_0(u) + s\xi(u)$, we require $\partial_s X(s,u) = \xi(s,u)$ to be tangent to Y. We take $X(0,u) = X_0(u)$. Then (30.1) holds, and we need to add to (30.2) the following term, in order to compute $A''(0)$:

$$(30.14)\quad \Phi = \sum_{j=1}^{k} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \kappa \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle, \quad \kappa = \partial_s \xi = \partial_s^2 X.$$

If, as before, $\partial_j X_0 = E_j$ form an orthonormal basis of $T_x M$, for a given $x \in M$, then

$$(30.15)\qquad \Phi = \sum_{j=1}^{k} \langle \partial_j \kappa, E_j \rangle, \text{ at } x.$$

Now, given the compactly supported field $\xi(0,u)$, tangent to Y and normal to M, let us suppose that, for each u, $\gamma_u(s) = X(s,u)$ is a constant speed *geodesic* in Y, such that $\gamma_u'(0) = \xi(0,u)$. Thus $\kappa = \gamma_u''(0)$ is normal to Y, and, by the Weingarten formula for $M \subset \mathbb{R}^n$,

$$(30.16)\qquad \partial_j \kappa = \nabla^1_{E_j} \kappa - A_\kappa E_j$$

at x, where ∇^1 is the connection on the normal bundle to $M \subset \mathbb{R}^n$ and A is as before the Weingarten map for $M \subset \mathbb{R}^n$. Thus

$$(30.17)\qquad \Phi = -\sum_j \langle A_\kappa E_j, E_j \rangle = -\ \mathrm{Tr}\ A_\kappa = -k \langle \mathfrak{H}(x), \kappa \rangle,$$

where $k = \dim M$.

If we suppose M is a minimal submanifold of Y, then $\mathfrak{H}(x)$ is normal to Y, so, for any compactly supported field ξ, normal to M and tangent to Y, the computation (30.13) supplemented by (30.14)–(30.17) gives

$$(30.18)\qquad A''(0) = \int_M \Big[\|\nabla^1 \xi\|^2 - \|A_\xi\|^2 - k\langle \mathfrak{H}(x), \kappa \rangle \Big]\, dA(x).$$

Recall that A_ξ is the Weingarten map of $M \subset \mathbb{R}^n$.

We prefer to use B_ξ, the Weingarten map of $M \subset Y$. It is readily verified that

$$(30.19)\qquad A_\xi = B_\xi \in \mathrm{End}\ T_x M,$$

if $\xi \in T_x Y$ and $\xi \perp T_x M$; see problem 13 in §16. Thus in (30.18) we can simply replace $\|A_\xi\|^2$ by $\|B_\xi\|^2$. Also recall that ∇^1 in (30.18) is the connection on the normal bundle to $M \subset \mathbb{R}^n$. We prefer to use the connection on the normal bundle to $M \subset Y$, which we denote $\nabla^{\#}$. To relate these two objects, we use the identities

$$\partial_j \xi = \nabla^1_j \xi - A_\xi E_j, \quad \partial_j \xi = \widetilde{\nabla}_j \xi + II^Y(E_j, \xi), \quad \widetilde{\nabla}_j \xi = \nabla^{\#}_j \xi - B_\xi E_j, \tag{30.20}$$

where $\widetilde{\nabla}$ denotes the covariant derivative on Y and II^Y the fundamental form of $Y \subset \mathbb{R}^n$. In view of (30.19), we obtain

$$\nabla^1_j \xi = \nabla^{\#}_j \xi + II^Y(E_j, \xi), \tag{30.21}$$

a sum of terms tangent to Y and normal to Y, respectively. Hence

$$\|\nabla^1 \xi\|^2 = \|\nabla^{\#} \xi\|^2 + \sum_j \|II^Y(E_j, \xi)\|^2. \tag{30.22}$$

Thus we can rewrite (30.18) as

$$A''(0) = \int_M \Big[\|\nabla^{\#} \xi\|^2 - \|B_\xi\|^2 + \sum_j \|II^Y(E_j, \xi)\|^2 - \ \mathrm{Tr}\ A_\kappa \Big]\, dA(x). \tag{30.23}$$

We want to replace the last two terms in this integrand by a quantity defined intrinsically by $M_s \subset Y$, not by the way Y is imbedded in $\mathbb{R}^n$. Now $\mathrm{Tr}\ A_\kappa = \sum \langle II^M(E_j, E_j), \kappa \rangle$, where II^M is the second fundamental form of $M \subset \mathbb{R}^n$. On the other hand, it is easily verified that

$$\kappa = \gamma_u''(0) = II^Y(\xi, \xi). \tag{30.24}$$

Thus the last two terms in the integrand sum to

$$\Psi = \sum_j \Big[\|II^Y(E_j, \xi)\|^2 - \langle II^Y(\xi, \xi), II^M(E_j, E_j) \rangle \Big]. \tag{30.25}$$

We can replace $II^M(E_j, E_j)$ by $II^Y(E_j, E_j)$ here, since these two objects have the same component normal to Y. Then Gauss' formula implies

$$\Psi = \sum_j \langle R^Y(\xi, E_j)\xi, E_j \rangle, \tag{30.26}$$

where R^Y is the Riemann curvature tensor of Y. We define $\overline{\mathfrak{R}} \in\ \mathrm{End}\ N_x M$, where $N(M)$ is the normal bundle of $N \subset Y$, by

$$\langle \overline{\mathfrak{R}}(\xi), \eta \rangle = \sum_j \langle R^Y(\xi, E_j)\eta, E_j \rangle, \tag{30.27}$$

at x, where $\{E_j\}$ is an orthonormal basis of T_xM. It follows easily that this is independent of the choice of such an orthonormal basis.

Our calculation of $A''(0)$ becomes

$$A''(0) = \int_M \Big[\|\nabla^\# \xi\|^2 - \|B_\xi\|^2 + \langle \overline{\mathfrak{R}}(\xi), \xi\rangle\Big]\, dA(x), \tag{30.28}$$

when M is a minimal submanifold of Y, where $\nabla^\#$ is the connection on the normal bundle to $M \subset Y$, B is the Weingarten map for $M \subset Y$, and $\overline{\mathfrak{R}}$ is defined by (30.27). If we define a second order differential operator $\mathfrak{L}_0$ and a zero order operator $\mathfrak{B}$ on $C_0^\infty\big(M, N(M)\big)$ by

$$\mathfrak{L}_0 \xi = (\nabla^\#)^* \nabla^\# \xi, \quad \langle \mathfrak{B}(\xi), \eta\rangle = \operatorname{Tr}\,(B_\eta^* B_\xi), \tag{30.29}$$

we can write this as

$$A''(0) = (\mathfrak{L}\xi, \xi)_{L^2(M)}, \quad \mathfrak{L}\xi = \mathfrak{L}_0 \xi - \mathfrak{B}(\xi) + \overline{\mathfrak{R}}(\xi). \tag{30.30}$$

We emphasize that these formulas, and the ones below, for $A''(0)$ are valid for immersed minimal submanifolds of Y as well as for imbedded submanifolds.

Suppose M has codimension 1 in Y, and that Y and M are orientable. Complete the basis $\{E_j\}$ of T_xM to an orthonormal basis $\{E_j : 1 \le j \le k+1\}$ of T_xY. In this case, $E_{k+1}(x)$ and $\xi(x)$ are parallel, so $\langle R^Y(\xi, E_{k+1})\eta, E_{k+1}\rangle = 0$. Thus (30.27) becomes

$$\overline{\mathfrak{R}}(\xi) = -\operatorname{Ric}^Y \xi, \ \text{ if } \dim Y = \dim M + 1, \tag{30.31}$$

where Ric^Y denotes the Ricci tensor of Y. In such a case, taking $\xi = fE_{k+1} = f\nu$, where ν is a unit normal field to M, tangent to Y, we obtain

$$A''(0) = \int_M \Big[\|\nabla f\|^2 - \big(\|B_\nu\|^2 + \langle \operatorname{Ric}^Y \nu, \nu\rangle\big)|f|^2\Big]\, dA(x) = (Lf, f)_{L^2(M)}, \tag{30.32}$$

where

$$Lf = -\Delta f + \varphi f, \quad \varphi = -\|B_\nu\|^2 - \langle \operatorname{Ric}^Y \nu, \nu\rangle. \tag{30.33}$$

We can express φ in a different form, noting that

$$\langle \operatorname{Ric}^Y \nu, \nu\rangle = S^Y - \sum_{j=1}^k \langle \operatorname{Ric}^Y E_j, E_j\rangle, \tag{30.34}$$

where S^Y is the scalar curvature of Y. From Gauss' formula we readily obtain, for general $M \subset Y$, of any codimension,

$$\begin{aligned}\langle \mathrm{Ric}^Y E_j, E_j\rangle &= \langle R^Y(E_j,\nu)\nu, E_j\rangle + \langle \mathrm{Ric}^M E_j, E_j\rangle \\ &\qquad + \sum_\ell \|II(E_j,E_\ell)\|^2 - k\big\langle \mathfrak{H}_Y, II(E_j,E_j)\big\rangle,\end{aligned} \tag{30.35}$$

where II denotes the second fundamental form of $M \subset Y$. Summing over $1 \le j \le k$, when M has codimension 1 in Y, and ν is a unit normal to M, we get

$$2\langle \mathrm{Ric}^Y \nu, \nu\rangle = S^Y - S^M - \|B_\nu\|^2 + \|\mathfrak{H}_Y\|^2. \tag{30.36}$$

If M is a minimal submanifold of Y, of codimension 1, this implies that

$$\begin{aligned}\varphi &= \frac{1}{2}(S^M - S^Y) - \frac{1}{2}\|B_\nu\|^2 \\ &= \frac{1}{2}(S^M - S^Y) + \ \mathrm{Tr}\ \Lambda^2 B_\nu.\end{aligned} \tag{30.37}$$

We also note that, when dim $M = 2$ and dim $Y = 3$, then, for $x \in M$,

$$\mathrm{Tr}\ \Lambda^2 B_\nu(x) = K^M(x) - K^Y(T_xM), \tag{30.38}$$

where $K^M = (1/2)S^M$ is the Gauss curvature of M and $K^Y(T_xM)$ is the sectional curvature of Y, along the plane T_xM.

We consider another special case, where dim $M = 1$. We have

$$\langle \overline{\mathfrak{R}}(\xi), \xi\rangle = -|\xi|^2 K^Y(\Pi_{M\xi}),$$

where $K^Y(\Pi_{M\xi})$ is the sectional curvature of Y along the plane in T_xY spanned by T_xM and ξ. In this case, to say M is minimal is to say it is a geodesic; hence $B_\xi = 0$ and $\nabla^\# \xi = \widetilde{\nabla}_T \xi$, where $\widetilde{\nabla}$ is the covariant derivative on Y, and T is a unit tangent vector to M. Thus (30.28) becomes the familiar formula for the second variation of arc-length for a geodesic:

$$\ell''(0) = \int_\gamma \Big[\|\widetilde{\nabla}_T \xi\|^2 - |\xi|^2 K^Y(\Pi_{\gamma\xi})\Big]\, ds, \tag{30.39}$$

where we have used γ instead of M to denote the curve, and also ℓ instead of A and ds instead of dA, to denote arc-length. Compare (15.61).

The operators $\mathfrak{L}$ and L are second order elliptic operators, which are self-adjoint, with domain $H^2(M)$, if M is compact, without boundary, and domain $H^2(M) \cap H_0^1(M)$, if $\overline{M}$ is compact with boundary. In such cases, the spectra of these operators consist of eigenvalues $\lambda_j \nearrow +\infty$. If M is not compact, but B and

$\overline{\mathfrak{R}}$ are bounded, we can use the Friedrichs method to define self adjoint extensions $\mathfrak{L}$ and L, which might have continuous spectrum.

We say a minimal submanifold $M \subset Y$ is *stable* if $A''(0) \geq 0$ for all smooth compactly supported variations ξ, normal to M (and vanishing on ∂M). Thus the condition that M be stable is that the spectrum of $\mathfrak{L}$ (equivalently, of L, if codim $M = 1$) be contained in $[0, \infty)$. In particular, if M is actually *area minimizing* with respect to small perturbations, leaving ∂M fixed (which we will just call "area minimizing"), then it must be stable, so

$$M \text{ area minimizing} \implies \text{spec}\, \mathfrak{L} \subset [0, \infty). \tag{30.40}$$

The second variational formulas above provide necessary conditions for a minimal immersed submanifold to be stable. For example, suppose M is a boundaryless, codimension 1 minimal submanifold of Y, and both are orientable. Then we can take $f = 1$ in (30.32), to get

$$M \text{ stable} \implies \int_M \Big(\|B_\nu\|^2 + \langle \mathrm{Ric}^Y \nu, \nu \rangle \Big)\, dA \leq 0. \tag{30.41}$$

If dim $M = 2$ and dim $Y = 3$, then, by (30.37), we have

$$M \text{ stable} \implies \int_M \Big(\|B_\nu\|^2 + S^Y - 2K^M \Big)\, dA \leq 0. \tag{30.42}$$

In this case, if M has genus g, the Gauss-Bonnet theorem implies that $\int K^M \, dA = 4\pi(1-g)$, so

$$M \text{ stable} \implies \int_M \Big(\|B_\nu\|^2 + S^Y \Big)\, dA \leq 8\pi(1-g). \tag{30.43}$$

This implies some nonexistence results.

Proposition 30.1. *Assume Y is a compact, oriented Riemannian manifold, and that Y and M have no boundary.*

If the Ricci tensor Ric^Y is positive definite, then Y cannot contain any compact, oriented area-minimizing immersed hypersurface M. If Ric^Y is positive semidefinite, then any such M would have to be totally geodesic in Y.

Now assume dim $Y = 3$. If Y has scalar curvature $S^Y > 0$ everywhere, then Y cannot contain any compact, oriented area-minimizing immersed surface M of genus $g \geq 1$.

More generally, if $S^Y \geq 0$ everywhere, and if M is a compact, oriented immersed hypersurface of genus $g \geq 1$, then for M to be area-minimizing it is necessary that $g = 1$ and that M be totally geodesic in Y.

Schoen and Yau [SY] obtained topological consequences, for a compact, oriented 3-manifold Y, from this together with the following existence theorem. Suppose M is a compact, oriented surface of genus $g \geq 1$, and suppose the fundamental group $\pi_1(Y)$ contains a subgroup isomorphic to $\pi_1(M)$. Then, given any Riemannian metric on Y, there is a smooth immersion of M into Y which is area-minimizing with respect to small perturbations, as shown in [SY]. It follows that if Y is a compact, oriented Riemannian 3-manifold, whose scalar curvature S^Y is everywhere positive, then $\pi_1(Y)$ cannot have a subgroup isomorphic to $\pi_1(M)$, for any compact Riemann surface M of genus $g \geq 1$.

We will not prove the result of [SY] on the existence of such minimal immersions. Instead, we demonstrate a topological result, due to Synge, of a similar flavor but simpler to prove. It makes use of the second variational formula (30.39) for arc-length.

Proposition 30.2. *If Y is a compact, oriented Riemannian manifold of even dimension, with positive sectional curvature everywhere, then Y is simply connected.*

Proof. It is a simple consequence of Ascoli's theorem that there is a length-minimizing closed geodesic in each homotopy class of maps from S^1 to Y. Thus, if $\pi_1(Y) \neq 0$, there is a nontrivial stable geodesic, γ. Pick $p \in \gamma$, ξ_p normal to γ at p, (i.e., $\xi_p \in N_p(\gamma)$), and parallel translate ξ about γ, obtaining $\overline{\xi}_p \in N_p(\gamma)$ after one circuit. This defines an orientation-preserving, orthogonal linear transformation $\tau : N_p\gamma \to N_p\gamma$. If Y has dimension $2k$, then $N_p\gamma$ has dimension $2k-1$, so $\tau \in SO(2k-1)$. It follows that τ must have an eigenvector in $N_p\gamma$, with eigenvalue *one*. Thus we get a nontrivial smooth section ξ of $N(\gamma)$ which is parallel over γ, so (30.39) implies

$$\int_\gamma K^Y(\Pi_{\gamma\xi})\, ds \leq 0. \tag{30.44}$$

If $K^Y(\Pi) > 0$ everywhere, this is impossible.

One might compare these results with Proposition 28.7, which states that, if Y is a compact Riemannian manifold and $\mathrm{Ric}^Y > 0$, then the first cohomology group $\mathcal{H}^1(Y) = 0$.

Regarding the hypotheses of Proposition 30.2, note that $Y = P(\mathbb{R}^n)$, the real projective space, with double cover S^n, provides examples where the conclusion fails, for non-orientable even-dimensional manifolds and for orientable odd-dimensional manifolds.

Exercises

1. If $M \subset Y$ is a minimal submanifold, and $p \in M$, show that there is a neighborhood U of p in Y such that $\widetilde{M} = M \cap U$ is stable.

31. The minimal surface equation

We now study a nonlinear PDE for functions whose graphs are minimal surfaces. We begin with a formula for the mean curvature of a hypersurface $M \subset \mathbb{R}^{n+1}$ defined by $u(x) = c$, where $\nabla u \neq 0$ on M. If $N = \nabla u / |\nabla u|$, we have the formula

$$\langle A_N X, Y\rangle = -|\nabla u|^{-1} (D^2 u)(X, Y), \tag{31.1}$$

for $X, Y \in T_x M$, as shown in (16.26). To take the trace of the restriction of $D^2 u$ to $T_x M$, we merely take $\operatorname{Tr}(D^2 u) - D^2 u(N, N)$. Of course, $\operatorname{Tr}(D^2 u) = \Delta u$. Thus, for $x \in M$,

$$\operatorname{Tr} A_N(x) = -|\nabla u(x)|^{-1} \Big[\Delta u - |\nabla u|^{-2} D^2 u(\nabla u, \nabla u)\Big]. \tag{31.2}$$

Suppose now that M is given by the equation $x_{n+1} = f(x')$, $x' = (x_1, \dots, x_n)$. Thus we take $u(x) = x_{n+1} - f(x')$, with $\nabla u = (-\nabla f, 1)$. We obtain for the mean curvature the formula

$$nH(x) = -\frac{1}{\langle \nabla f\rangle^3} \Big[\langle \nabla f\rangle^2 \Delta f - D^2 f(\nabla f, \nabla f)\Big] = \mathcal{M}(f), \tag{31.3}$$

where $\langle \nabla f\rangle^2 = 1 + |\nabla f(x')|^2$. Written out more fully, the quantity in brackets above is

$$\big(1 + |\nabla f|^2\big)\Delta f - \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\, \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = \widetilde{\mathcal{M}}(f). \tag{31.4}$$

Thus the equation stating that a hypersurface $x_{n+1} = f(x')$ be a minimal submanifold of $\mathbb{R}^{n+1}$ is

$$\widetilde{\mathcal{M}}(f) = 0. \tag{31.5}$$

In case $n = 2$, we have the *minimal surface equation*, which can also be written as

$$\big(1 + |\partial_2 f|^2\big)\partial_1^2 f - 2\big(\partial_1 f \cdot \partial_2 f\big)\partial_1 \partial_2 f + \big(1 + |\partial_1 f|^2\big)\partial_2^2 f = 0. \tag{31.6}$$

It can be verified that this PDE also holds for a minimal surface in $\mathbb{R}^n$ described by $x'' = f(x')$, where $x'' = (x_3, \dots, x_n)$, if (31.6) is regarded as a system of k equations in k unknowns, $k = n - 2$, and $(\partial_1 f \cdot \partial_2 f)$ is the dot product of $\mathbb{R}^k$-valued functions. We continue to denote the left side of (31.6) by $\widetilde{\mathcal{M}}(f)$.

Proposition 29.12 can be translated immediately into the following existence theorem for the minimal surface equation.

Proposition 31.1. *Let $\mathcal{O}$ be a bounded convex domain in $\mathbb{R}^2$ with smooth boundary. Let $g \in C^\infty(\partial\mathcal{O}, \mathbb{R}^k)$ be given. Then there is a solution*

$$u \in C^\infty(\mathcal{O}, \mathbb{R}^k) \cap C(\overline{\mathcal{O}}, \mathbb{R}^k) \tag{31.7}$$

to the boundary problem

$$\widetilde{\mathcal{M}}(u) = 0, \quad u\big|_{\partial\mathcal{O}} = g. \tag{31.8}$$

When $k = 1$, we also have uniqueness, as a consequence of the following.

Proposition 31.2. *Let $\mathcal{O}$ be any bounded domain in $\mathbb{R}^n$. Let $u_j \in C^\infty(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ be real-valued solutions to*

$$\widetilde{\mathcal{M}}(u_j) = 0, \quad u_j = g_j \text{ on } \partial\mathcal{O}, \tag{31.9}$$

for $j = 1, 2$. Then

$$g_1 \le g_2 \text{ on } \partial\mathcal{O} \Longrightarrow u_1 \le u_2 \text{ on } \overline{\mathcal{O}}. \tag{31.10}$$

Proof. We prove this by deriving a linear PDE for the difference $v = u_2 - u_1$ and applying the maximum principle. In general,

$$\Phi(u_2) - \Phi(u_1) = Lv, \quad L = \int_0^1 D\Phi\big(\tau u_2 + (1-\tau)u_1\big)\, d\tau. \tag{31.11}$$

Suppose Φ is a second order differential operator:

$$\Phi(u) = F(u, \partial u, \partial^2 u), \quad F = F(u, p, \zeta). \tag{31.12}$$

Then

$$D\Phi(u) = F_\zeta(u, \partial u, \partial^2 u)\partial^2 v + F_p(u, \partial u, \partial^2 u)\partial v + F_u(u, \partial u, \partial^2 u)v. \tag{31.13}$$

When $\Phi(u) = \widetilde{\mathcal{M}}(u)$ is given by (31.4), $F_u(u, \xi, \zeta) = 0$, and we have

$$D\widetilde{\mathcal{M}}(u)v = A(u)v + B(u)v, \tag{31.14}$$

where

$$A(u)v = \big(1 + |\nabla u|^2\big)\Delta v - \sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\ \frac{\partial^2 v}{\partial x_i \partial x_j} \tag{31.15}$$

is strongly elliptic, and $B(u)$ is a first order differential operator. Consequently, we have

$$\widetilde{\mathcal{M}}(u_2) - \widetilde{\mathcal{M}}(u_1) = Av + Bv, \tag{31.16}$$

where $A = \int_0^1 A\big(\tau u_2 + (1-\tau)u_2\big)\, d\tau$ is strongly elliptic of order 2 at each point of $\mathcal{O}$, and B is a first order differential operator, which annihilates constants. If (31.9) holds, then $Av + Bv = 0$. Now (31.10) follows from the maximum principle for elliptic differential equations (see Chapter 5 of [T1]).

While Proposition 31.2 is a sort of result that holds for a large class of second order scalar elliptic PDE, the next result is much more special, and has interesting consequences. It implies that the size of a solution to the minimal surface equation (31.8) can sometimes be controlled by the behavior of g on *part* of the boundary.

Proposition 31.3. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a domain contained in the annulus $r_1 < |x| < r_2$, and let $u \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ solve $\widetilde{\mathcal{M}}(u) = 0$. Set*

$$G(x; r) = r \ \cosh^{-1}\left(\frac{|x|}{r}\right), \quad \text{for} \ \ |x| > r, \quad G(x; r) \le 0. \tag{31.17}$$

If

$$u(x) \le G(x; r_1) + M \quad \text{on} \ \ \{x \in \partial\mathcal{O} : |x| > r_1\}, \tag{31.18}$$

for some $M \in \mathbb{R}$, then

$$u(x) \le G(x; r_1) + M \quad \text{on} \ \ \mathcal{O}. \tag{31.19}$$

Here, $z = G(x; r_1)$ defines the lower half of a catenoid, over $\{x \in \mathbb{R}^2 : |x| \ge r_1\}$. This function solves the minimal surface equation on $|x| > r_1$, and vanishes on $|x| = r_1$.

Proof. Given $s \in (r_1, r_2)$, let

$$\varepsilon(s) = \max_{s \le |x| \le r_2} \left|G(x; r_1) - G(x; s)\right|. \tag{31.20}$$

The hypothesis (31.18) implies that

$$u(x) \le G(x; s) + M + \varepsilon(s) \tag{31.21}$$

on $\{x \in \partial\mathcal{O} : |x| \ge s\}$. We claim that (31.21) holds for x in

$$\mathcal{O}(s) = \mathcal{O} \cap \{x : s < |x| < r_2\}. \tag{31.22}$$

Once this is established, (31.19) follows by taking $s \searrow r_1$. To prove this, it suffices by Proposition 31.2 to show that (31.21) holds on $\partial\mathcal{O}(s)$. Since it holds on $\partial\mathcal{O}$, it remains to show that (31.21) holds for x in

$$\mathcal{C}(s) = \mathcal{O} \cap \{x : |x| = s\}, \tag{31.23}$$

illustrated by a broken arc in Fig. 31.1. If not, then $u(x) - G(x; s)$ would have a maximum $M_1 > M + \varepsilon(s)$ at some point $p \in \mathcal{C}(s)$. By Proposition 31.1, we have $u(x) - G(x; s) \le M_1$ on $\mathcal{O}(s)$. However, $\nabla u(x)$ is bounded on a neighborhood of p, while

$$\frac{\partial}{\partial r} G(x; s) = -\infty \ \ \text{on} \ \ |x| = s. \tag{31.24}$$

This implies that $u(x) - G(x; s) > M_1$ for all points in $\mathcal{O}(s)$ sufficiently near p. This contradiction shows that (31.21) must hold on $\mathcal{C}(s)$, and the proposition is proved.

One implication is that, if $\mathcal{O} \subset \mathbb{R}^2$ is as illustrated in Fig. 31.1, it is not possible to solve the boundary problem (31.8) with g prescribed arbitrarily on all of $\partial\mathcal{O}$. A more precise statement about domains $\mathcal{O} \subset \mathbb{R}^2$ for which (31.8) is always solvable is the following.

Proposition 31.4. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded connected domain with smooth boundary. Then (31.8) has a solution for all $g \in C^\infty(\partial\mathcal{O})$ if and only if $\mathcal{O}$ is convex.*

Proof. The positive result is given in Proposition 31.1. Now, if $\mathcal{O}$ is not convex, let $p \in \partial\mathcal{O}$ be a point where $\mathcal{O}$ is concave, as illustrated in Fig. 31.2. Pick a disc $\mathcal{D}$ whose boundary C is tangent to $\partial\mathcal{O}$ at p and such that, near p, C intersects the complement $\mathcal{O}^c$ only at p. Then apply Proposition 31.3 to the domain $\widetilde{\mathcal{O}} = \mathcal{O} \setminus \overline{\mathcal{D}}$, taking the origin to be the center of $\mathcal{D}$ and r_1 to be the radius of $\mathcal{D}$. We deduce that, if u solves $\widetilde{\mathcal{M}}(u) = 0$ on $\mathcal{O}$, then

$$u(x) \le M + G(x; r_1) \ \text{ on } \ \partial\mathcal{O} \setminus \mathcal{D} \Longrightarrow u(p) \le M, \tag{31.25}$$

which certainly restricts the class of functions g for which (31.8) can be solved.

Note that the function $v(x) = G(x; r)$ defined by (31.17) also provides an example of a solution to the minimal surface equation (31.8) on an annular region

$$\mathcal{O} = \{x \in \mathbb{R}^2 : r < |x| < s\},$$

with smooth (in fact, locally constant) boundary values

$$v = 0 \ \text{ on } \ |x| = r, \quad v = -r \ \cosh^{-1}\left(\frac{s}{r}\right) \ \text{ on } \ |x| = s,$$

which is not a smooth function, or even a Lipschitz function, on $\overline{\mathcal{O}}$. This is another phenomenon that is different when $\mathcal{O}$ is convex. We will establish the following.

Proposition 31.5. *If $\mathcal{O} \subset \mathbb{R}^2$ is a bounded region with smooth boundary which is strictly convex (i.e., $\partial\mathcal{O}$ has positive curvature), and $g \in C^\infty(\partial\mathcal{O})$ is real valued, then the solution to (31.8) is Lipschitz at each point $x_0 \in \partial\mathcal{O}$.*

Proof. Given $x_0 \in \partial\mathcal{O}$, we have $z_0 = \big(x_0, g(x_0)\big) \in \gamma \subset \mathbb{R}^3$, where γ is the boundary of the minimal surface M which is the graph of $z = u(x)$. The strict convexity hypothesis on $\mathcal{O}$ implies that there are two planes Π_j in $\mathbb{R}^3$ through z_0, such that Π_1 lies below γ and Π_2 above γ, and Π_j are given by $z = \alpha_j \cdot (x - x_0) + g(x_0) = w_{jx_0}(x)$, $\alpha_j = \alpha_j(x_0) \in \mathbb{R}^3$. There is an estimate of the form

$$|\alpha_j(x_0)| \le K(x_0) \|g \circ \rho_{x_0}\|_{C^2}, \tag{31.26}$$

where ρ_{x_0} is radial projection (from the center of $\mathcal{O}$) of $\partial\mathcal{O}$ onto a circle $\mathcal{C}(x_0)$ containing $\mathcal{O}$ and tangent to $\partial\mathcal{O}$ at x_0, and $K(x_0)$ depends on the curvature of $\mathcal{C}(x_0)$. Now Proposition 31.2 applies to give

$$w_{1x_0}(x) \le u(x) \le w_{2x_0}(x), \quad x \in \overline{\mathcal{O}}, \tag{31.27}$$

since linear functions solve the minimal surface equation. This establishes the Lipschitz continuity, with the quantitative estimate

$$|u(x_0) - u(x)| \le A|x - x_0|, \quad x_0 \in \partial\mathcal{O}, \ x \in \overline{\mathcal{O}}, \tag{31.28}$$

where

$$A = \sup_{x_0 \in \partial\mathcal{O}} |\alpha_1(x_0)| + |\alpha_2(x_0)|. \tag{31.29}$$

This result points toward an estimate on $|\nabla u(x)|$, $x \in \overline{\mathcal{O}}$, for a solution to (31.8). We begin the line of reasoning which leads to such an estimate, a line which applies to other situations. First, let's re-derive the minimal surface equation, as the stationary condition for

$$I(u) = \int_{\mathcal{O}} F\big(\nabla u(x)\big)\, dx, \tag{31.30}$$

where

$$F(p) = \Big(1 + |p|^2\Big)^{1/2}, \tag{31.31}$$

so (31.30) gives the area of the graph of $z = u(x)$. The method of the calculus of variations yields the PDE

$$\sum A^{ij}(\nabla u)\partial_i\partial_j u = 0, \tag{31.32}$$

where

$$A^{ij}(p) = \frac{\partial^2 F}{\partial p_i \partial p_j}. \tag{31.33}$$

When $F(p)$ is given by (31.31), we have

$$A^{ij}(p) = \langle p\rangle^{-3/2}\Big(\delta_{ij}\langle p\rangle^2 - p_i p_j\Big), \tag{31.34}$$

so in this case (31.32) is equal to $-\mathcal{M}(u)$, defined by (31.3). Now, when u is a sufficiently smooth solution to (31.32), we can apply $\partial_\ell = \partial/\partial x_\ell$ to this equation, and obtain the PDE

$$\sum \partial_i A^{ij}(\nabla u)\partial_j w_\ell = 0, \tag{31.35}$$

for $w_\ell = \partial_\ell u$, not for all PDE of the form (31.32), but whenever $A^{ij}(p)$ is symmetric in (i, j) and satisfies

$$\frac{\partial A^{ij}}{\partial p_m} = \frac{\partial A^{im}}{\partial p_j}, \tag{31.36}$$

which happens when $A^{ij}(p)$ has the form (31.33). If (31.35) satisfies the ellipticity condition

$$\sum A^{ij}\big(\nabla u(x)\big)\ \xi_i\xi_j \geq C(x)|\xi|^2, \quad C(x) > 0, \tag{31.37}$$

for $x \in \mathcal{O}$, then we can apply the maximum principle, to obtain the following.

Proposition 31.6. *Assume $u \in C^1(\overline{\mathcal{O}})$ is real valued and satisfies the PDE (31.32), with coefficients given by (31.33). If the ellipticity condition (31.37) holds, then $\partial_\ell u(x)$ assumes its maximum and minimum values on $\partial\mathcal{O}$; hence*

$$\sup_{x\in\overline{\mathcal{O}}} |\nabla u(x)| = \sup_{x\in\partial\mathcal{O}} |\nabla u(x)|. \tag{31.38}$$

Combining this result with Proposition 31.5, we have the following.

Proposition 31.7. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded region with smooth boundary which is strictly convex, $g \in C^\infty(\partial\Omega)$ real valued. If $u \in C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}})$ is a solution to (31.8), then there is an estimate*

$$\|u\|_{C^1(\overline{\mathcal{O}})} \le C(\mathcal{O})\ \|g\|_{C^2(\partial\mathcal{O})}. \tag{31.39}$$

Note that the existence result of Proposition 31.1 does not provide us with the knowledge that u belongs to $C^1(\overline{\mathcal{O}})$, and thus it takes further work to demonstrate that the estimate (31.39) actually holds for an arbitrary real valued solution to (31.8), when $\mathcal{O} \subset \mathbb{R}^2$ is strictly convex and g is smooth. This analysis can be found in Chapter 14 of [T1].

We next look at the Gauss curvature of a minimal surface M, given by $z = u(x),\ x \in \mathcal{O} \subset \mathbb{R}^2$. For a general u, the curvature is given by

$$K = \big(1 + |\nabla u|^2\big)^{-2}\ \det\Big(\frac{\partial^2 u}{\partial x_j \partial x_k}\Big). \tag{31.40}$$

See (16.29). When u satisfies the minimal surface equation, there are some other formulas for K, in terms of operations on

$$\Phi(x) = F(\nabla u)^{-1} = \big(1 + |\nabla u|^2\big)^{-1/2}, \tag{31.41}$$

which we will list, leaving their verification as an exercise:

$$K = -\frac{|\nabla\Phi|^2}{1-\Phi^2}, \tag{31.42}$$

$$K = \frac{1}{2\Phi}\ \Delta\Phi, \tag{31.43}$$

$$K = \Delta\ \log(1+\Phi). \tag{31.44}$$

Now if we alter the metric g induced on M via its imbedding in $\mathbb{R}^3$, by a conformal factor:

$$g' = (1+\Phi)^2 g = e^{2v} g, \quad v = \log(1+\Phi), \tag{31.45}$$

then we see that the Gauss curvature k of M in the new metric is

$$k = (-\Delta v + K)e^{-2v} = 0, \tag{31.46}$$

i.e., the metric $g' = (1+\Phi)^2 g$ is flat! Using this observation, we can establish the following remarkable theorem of S. Bernstein:

Theorem 31.8. *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an everywhere defined C^2 solution to the minimal surface equation, then u is a linear function.*

Proof. Consider the minimal surface M given by $z = u(x),\ x \in \mathbb{R}^2$, in the metric $g' = (1+\Phi)^2 g$, which, as we have seen, is flat. Now $g' \geq g$, so this is a *complete* metric on M. Thus (M, g') is isometrically equivalent to $\mathbb{R}^2$. Hence (M, g) is *conformally* equivalent to $\mathbb{C}$.

On the other hand, the antipodal Gauss map

$$\widetilde{N} : M \longrightarrow S^2, \quad \widetilde{N} = (\nabla u, -1)/\langle \nabla u \rangle \tag{31.47}$$

is holomorphic; see Exercise 1 of §29. But the range of $\widetilde{N}$ is contained in the lower hemisphere of S^2, so if we take $S^2 = \mathbb{C} \cup \{\infty\}$ with the point at infinity identified with the "north pole" $(0,0,1)$, we see that $\widetilde{N}$ yields a bounded holomorphic function on $M \approx \mathbb{C}$. By Liouville's theorem, $\widetilde{N}$ must be *constant.* Thus M is a flat plane in $\mathbb{R}^3$.

It turns out that Bernstein's Theorem extends to $u : \mathbb{R}^n \to \mathbb{R}$, for $n \leq 7$, by work of E. deGiorgi, F. Almgren, and J. Simons, but not to $n \geq 8$.

Exercises

1. If $D\widetilde{\mathcal{M}}(u)$ is the differential operator given by (31.14)–(31.15), show that its principal symbol satisfies

$$-\sigma_{D\widetilde{\mathcal{M}}(u)}(x,\xi) = \bigl(1 + |p|^2\bigr)|\xi|^2 - (p \cdot \xi)^2 \geq |\xi|^2, \tag{31.48}$$

where $p = \nabla u(x)$.

2. Show that the formula (31.3) for $\mathcal{M}(f)$ is equivalent to

$$\mathcal{M}(f) = \sum_j \partial_j \Bigl(\langle \nabla f \rangle^{-1}\ \partial_j f \Bigr). \tag{31.49}$$

3. Give a detailed demonstration of the estimate (31.26) on the slope of planes which can lie above and below the graph of g over $\partial\mathcal{O}$ (assumed to have positive curvature), needed for the proof of Proposition 31.5.
Hint. In case $\partial\mathcal{O}$ is the unit circle S^1, consider the cases $g(\theta) = \cos^k \theta$.

4. Establish the formulas (31.42)–(31.44), for the Gauss curvature of a minimal surface.

A. Metric spaces, compactness, and all that

A metric space is a set X, together with a distance function $d : X \times X \to [0, \infty)$, having the properties that

$$
\begin{aligned}
&d(x, y) = 0 \Longleftrightarrow x = y,\\
&d(x, y) = d(y, x),\\
&d(x, y) \le d(x, z) + d(y, z).
\end{aligned}
\tag{A.1}
$$

The third of these properties is called the triangle inequality. An example of a metric space is the set of rational numbers $\mathbb{Q}$, with $d(x, y) = |x - y|$. Another example is $X = \mathbb{R}^n$, with $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

If (x_ν) is a sequence in X, indexed by $\nu = 1, 2, 3, \ldots$, i.e., by $\nu \in \mathbb{Z}^+$, one says $x_\nu \to y$ if $d(x_\nu, y) \to 0$, as $\nu \to \infty$. One says (x_ν) is a Cauchy sequence if $d(x_\nu, x_\mu) \to 0$ as $\mu, \nu \to \infty$. One says X is a *complete* metric space if every Cauchy sequence converges to a limit in X. Some metric spaces are not complete; for example,$\mathbb{Q}$ is not complete. You can take a sequence (x_ν) of rational numbers such that $x_\nu \to \sqrt{2}$, which is not rational. Then (x_ν) is Cauchy in $\mathbb{Q}$, but it has no limit in $\mathbb{Q}$.

If a metric space X is not complete, one can construct its completion $\widehat{X}$ as follows. Let an element ξ of $\widehat{X}$ consist of an *equivalence class* of Cauchy sequences in X, where we say $(x_\nu) \sim (y_\nu)$ provided $d(x_\nu, y_\nu) \to 0$. We write the equivalence class containing (x_ν) as $[x_\nu]$. If $\xi = [x_\nu]$ and $\eta = [y_\nu]$, we can set $d(\xi, \eta) = \lim_{\nu\to\infty} d(x_\nu, y_\nu)$, and verify that this is well defined, and makes $\widehat{X}$ a complete metric space.

If the completion of $\mathbb{Q}$ is constructed by this process, you get $\mathbb{R}$, the set of real numbers.

A metric space X is said to be *compact* provided any sequence (x_ν) in X has a convergent subsequence. Clearly every compact metric space is complete. There are two useful conditions, each equivalent to the characterization of compactness just stated, on a metric space. The reader can establish the equivalence, as an exercise.

(i) If $S \subset X$ is a set with infinitely many elements, then there is an accumulation point, i.e., a point $p \in X$ such that every neighborhood U of p contains infinitely many points in S.

Here, a *neighborhood* of $p \in X$ is a set containing the ball

$$
B_\varepsilon(p) = \{x \in X : d(x, p) < \varepsilon\},
\tag{A.2}
$$

for some $\varepsilon > 0$.

(ii) Every open cover $\{U_\alpha\}$ of X has a finite subcover.

Here, a set $U \subset X$ is called open if it contains a neighborhood of each of its points. The complement of an open set is said to be *closed.* Equivalently, $K \subset X$ is closed provided that

$$x_\nu \in K,\ x_\nu \to p \in X \Longrightarrow p \in K. \tag{A.3}$$

It is clear that any closed subset of a compact metric space is also compact.

If $X_j,\ 1 \leq j \leq m$, is a finite collection of metric spaces, with metrics d_j, we can define a product metric space

$$X = \prod_{j=1}^{m} X_j, \quad d(x,y) = d_1(x_1,y_1) + \cdots + d_m(x_m,y_m). \tag{A.4}$$

Another choice of metric is $\delta(x,y) = \sqrt{d_1(x_1,y_1)^2 + \cdots + d_m(x_m,y_m)^2}$. The metrics d and δ are *equivalent*, i.e., there exist constants $C_0, C_1 \in (0,\infty)$ such that

$$C_0\delta(x,y) \leq d(x,y) \leq C_1\delta(x,y), \quad \forall\ x,y \in X. \tag{A.5}$$

We describe some useful classes of compact spaces.

Proposition A.1. *If X_j are compact metric spaces, $1 \leq j \leq m$, so is $X = \prod_{j=1}^{m} X_j$.*

Proof. If (x_ν) is an infinite sequence of points in X, say $x_\nu = (x_{1\nu}, \dots, x_{m\nu})$, pick a convergent subsequence $(x_{1\nu})$ in X_1, and consider the corresponding subsequence of (x_ν), which we relabel (x_ν). Using this, pick a convergent subsequence $(x_{2\nu})$ in X_2. Continue. Having a subsequence such that $x_{j\nu} \to y_j$ in X_j for each $j = 1, \dots, m$, we then have a convergent subsequence in X.

The following result is useful for calculus on $\mathbb{R}^n$.

Proposition A.2. *If K is a closed bounded subset of $\mathbb{R}^n$, then K is compact.*

Proof. The discussion above reduces the problem to showing that any closed interval $I = [a,b]$ in $\mathbb{R}$ is compact. Suppose S is a subset of I with infinitely many elements. Divide I into 2 equal subintervals, $I_1 = [a,b_1],\ I_2 = [b_1,b],\ b_1 = (a+b)/2$. Then either I_1 or I_2 must contain infinitely many elements of S. Say I_j does. Let x_1 be any element of S lying in I_j. Now divide I_j in two equal pieces, $I_j = I_{j1} \cup I_{j2}$. One of these intervals (say I_{jk}) contains infinitely many points of S. Pick $x_2 \in I_{jk}$ to be one such point (different from x_1). Then subdivide I_{jk} into two equal subintervals, and continue. We get an infinite sequence of distinct points $x_\nu \in S$, and $|x_\nu - x_{\nu+k}| \leq 2^{-\nu}(b-a)$, for $k \geq 1$. Since $\mathbb{R}$ is complete, (x_ν) converges, say to $y \in I$. Any neighborhood of y contains infinitely many points in S, so we are done.

If X and Y are metric spaces, a function $f : X \to Y$ is said to be continuous provided $x_\nu \to x$ in X implies $f(x_\nu) \to f(x)$ in Y.

Proposition A.3. *If X and Y are metric spaces, $f : X \to Y$ continuous, and $K \subset X$ compact, then $f(K)$ is a compact subset of Y.*

Proof. If (y_ν) is an infinite sequence of points in $f(K)$, pick $x_\nu \in K$ such that $f(x_\nu) = y_\nu$. If K is compact, we have a subsequence $x_{\nu_j} \to p$ in X, and then $y_{\nu_j} \to f(p)$ in Y.

If $F : X \to \mathbb{R}$ is continuous, we say $f \in C(X)$. A useful corollary of Proposition A.3 is:

Proposition A.4. *If X is a compact metric space and $f \in C(X)$, then f assumes a maximum and a minimum value on X.*

Proof. We know from Proposition A.3 that $f(X)$ is a compact subset of $\mathbb{R}$. Hence $f(X)$ is bounded, say $f(X) \subset I = [a, b]$. Repeatedly subdividing I into equal halves, as in the proof of Proposition A.2, at each stage throwing out intervals that do not intersect $f(X)$, and keeping only the leftmost and rightmost interval amongst those remaining, we obtain points $\alpha \in f(X)$ and $\beta \in f(X)$ such that $f(X) \subset [\alpha, \beta]$. Then $\alpha = f(x_0)$ for some $x_0 \in X$ is the minimum and $\beta = f(x_1)$ for some $x_1 \in X$ is the maximum.

A function $f \in C(X)$ is said to be *uniformly continuous* provided that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X,\ d(x, y) \le \delta \Longrightarrow |f(x) - f(y)| \le \varepsilon. \tag{A.6}$$

An equivalent condition is that f have a *modulus of continuity*, i.e., a monotonic function $\omega : [0, 1) \to [0, \infty)$ such that $\delta \searrow 0 \Rightarrow \omega(\delta) \searrow 0$, and such that

$$x, y \in X,\ d(x, y) \le \delta \le 1 \Longrightarrow |f(x) - f(y)| \le \omega(\delta). \tag{A.7}$$

Not all continuous functions are uniformly continuous. For example, if $X = (0, 1) \subset \mathbb{R}$, then $f(x) = \sin 1/x$ is continuous, but not uniformly continuous, on X. The following result is useful, for example, in the development of the Riemann integral in §11.

Proposition A.5. *If X is a compact space and $f \in C(X)$, then f is uniformly continuous.*

Proof. If not, there exist $x_\nu, y_\nu \in X$ and $\varepsilon > 0$ such that $d(x_\nu, y_\nu) \le 2^{-\nu}$ but

$$|f(x_\nu) - f(y_\nu)| \ge \varepsilon. \tag{A.8}$$

Taking a convergent subsequence $x_{\nu_j} \to p$, we also have $y_{\nu_j} \to p$. Now continuity of f at p implies $f(x_{\nu_j}) \to f(p)$ and $f(y_{\nu_j}) \to f(p)$, contradicting (A.8).

If X and Y are metric spaces, the space $C(X, Y)$ of continuous maps $f : X \to Y$ has a natural metric structure, under some additional hypotheses. We use

$$D(f, g) = \sup_{x \in X} d\big(f(x), g(x)\big). \tag{A.9}$$

This sup exists provided $f(X)$ and $g(X)$ are *bounded* subsets of Y, where to say $B \subset Y$ is bounded is to say $d : B \times B \to [0, \infty)$ has bounded image. In particular, this supremum exists if X is compact. The following result is useful in the proof of the fundamental local existence theorem for ODE, in §G.

Proposition A.6. *If X is a compact metric space and Y is a complete metric space, then $C(X, Y)$, with the metric (A.9), is complete.*

We leave the proof as an exercise.

We end this appendix with a couple of slightly more sophisticated results on compactness. The following extension of Proposition A.1 is a special case of Tychonov's Theorem.

Proposition A.7. *If $\{X_j : j \in \mathbb{Z}^+\}$ are compact metric spaces, so is $X = \prod_{j=1}^{\infty} X_j$.*

Here, we can make X a metric space by setting

$$d(x, y) = \sum_{j=1}^{\infty} \frac{2^{-j} \ d_j(x, y)}{1 + d_j(x, y)}. \tag{A.10}$$

It is easy to verify that, if $x_\nu \in X$, then $x_\nu \to y$ in X, as $\nu \to \infty$, if and only if, for each j, $p_j(x_\nu) \to p_j(y)$ in X_j, where $p_j : X \to X_j$ is the projection onto the jth factor.

Proof. Following the argument in Proposition A.1, if (x_ν) is an infinite sequence of points in X, we obtain a nested family of subsequences

$$(x_\nu) \supset (x^1{}_\nu) \supset (x^2{}_\nu) \supset \cdots \supset (x^j{}_\nu) \supset \cdots \tag{A.11}$$

such that $p_\ell(x^j{}_\nu)$ converges in X_ℓ, for $1 \le \ell \le j$. The next step is a *diagonal construction.* We set

$$\xi_\nu = x^\nu{}_\nu \in X. \tag{A.12}$$

Then, for each j, after throwing away a finite number $N(j)$ of elements, one obtains from (ξ_ν) a subsequence of the sequence $(x^j{}_\nu)$ in (A.11), so $p_\ell(\xi_\nu)$ converges in X_ℓ for all ℓ. Hence (ξ_ν) is a convergent subsequence of (x_ν).

Before stating the next result, we establish a simple lemma.

Lemma A.8. *If X is a compact metric space, then there is a countable dense subset Σ of X. (One says X is separable.)*

Proof. For each $\nu \in \mathbb{Z}^+$, the collection of balls $B_{1/\nu}(x)$ covers X, so there is a finite subcover, $\{B_{1/\nu}(x_{\nu j}) : 1 \le j \le N(\nu)\}$. It follows that $\Sigma = \{x_{\nu j} : j \le N(\nu), \nu \in \mathbb{Z}^+\}$ is a countable dense subset of X.

The next result is a special case of Ascoli's Theorem.

Proposition A.9. *Let X and Y be compact metric spaces, and fix a modulus of continuity $\omega(\delta)$. Then*

$$\mathcal{C}_\omega = \big\{ f \in C(X,Y) : d\big(f(x), f(y)\big) \le \omega\big(d(x,y)\big) \big\} \tag{A.13}$$

is a compact subset of $C(X,Y)$.

Proof. Let (f_ν) be a sequence in $\mathcal{C}_\omega$. Let Σ be a countable dense subset of X. For each $x \in \Sigma$, $(f_\nu(x))$ is a sequence in Y, which hence has a convergent subsequence. Using a diagonal construction similar to that in the proof of Proposition A.7, we obtain a subsequence (φ_ν) of (f_ν) with the property that $\varphi_\nu(x)$ converges in Y, for *each* $x \in \Sigma$, say

$$x \in \Sigma \Longrightarrow \varphi_\nu(x) \to \psi(x), \tag{A.14}$$

where $\psi : \Sigma \to Y$.

So far, we have not used (A.13), but this hypothesis readily yields

$$d\big(\psi(x), \psi(y)\big) \le \omega\big(d(x,y)\big), \tag{A.15}$$

for all $x, y \in \Sigma$. Using the denseness of $\Sigma \subset X$, we can extend ψ uniquely to a continuous map of $X \to Y$, which we continue to denote ψ. Also, (A.15) holds for all $x, y \in X$, i.e., $\psi \in \mathcal{C}_\omega$.

It remains to show that $\varphi_\nu \to \psi$ uniformly on X. Pick $\varepsilon > 0$. Then pick $\delta > 0$ such that $\omega(\delta) < \varepsilon/3$. Since X is compact, we can cover X by finitely many balls $B_\delta(x_j)$, $1 \le j \le N$, $x_j \in \Sigma$. Pick M so large that $\varphi_\nu(x_j)$ is within $\varepsilon/3$ of its limit for all $\nu \ge M$ (when $1 \le j \le N$). Now, for any $x \in X$, picking $\ell \in \{1, \dots, N\}$ such that $d(x, x_\ell) \le \delta$, we have, for $k \ge 0$, $\nu \ge M$,

$$\begin{aligned} d\big(\varphi_{\nu+k}(x), \varphi_\nu(x)\big) &\le d\big(\varphi_{\nu+k}(x), \varphi_{\nu+k}(x_\ell)\big) + d\big(\varphi_{\nu+k}(x_\ell), \varphi_\nu(x_\ell)\big) \\ &\qquad + d\big(\varphi_\nu(x_\ell), \varphi_\nu(x)\big) \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3, \end{aligned} \tag{A.16}$$

proving the proposition.

B. Topological spaces

As stated in §A, a metric space is a set X together with a "distance function" $d : X \times X \to [0, \infty)$, satisfying

$$\begin{aligned} &d(x, y) = d(y, x), \\ &d(x, y) = 0 \text{ if and only if } x = y, \\ &d(x, y) \le d(x, z) + d(y, z), \end{aligned} \tag{B.1}$$

the last condition known as the "triangle inequality." For a metric space, there is the following notion of convergence of a sequence (x_n); $x_n \to y$ if and only if $d(x_n, y) \to 0$ as $n \to \infty$.

A ball $B_\delta(p)$ of radius δ centered at a point p in a metric space X is $\{x \in X : d(x, p) < \delta\}$. A set $U \subset X$ is said to be *open* provided that for each $p \in X$, there is a ball $B_\delta(p) \subset U$, $\delta > 0$. A set $S \subset X$ is *closed* provided $X \setminus S$ is open. An equivalent statement is that S is closed provided $x_n \in S$, $x_n \to y$ in X imply $y \in S$.

A more general notion is that of a topological space. This is a set X, together with a family $\mathcal{O}$ of subsets, called *open*, satisfying the following conditions:

$$\begin{aligned} &X, \emptyset \text{ open}, \\ &U_j \text{ open}, 1 \le j \le N \Rightarrow \bigcap_{j=1}^{N} U_j \text{ open}, \\ &U_\alpha \text{ open}, \alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha \text{ open}, \end{aligned} \tag{B.2}$$

where A is *any* index set. It is obvious that the collection of open subsets of a metric space, defined above, satisfies these conditions. As before, a set $S \subset X$ is closed provided $X \setminus S$ is open. Also, we say a subset $N \subset X$ containing p is a *neighborhood* of p provided N contains an open set U which in turn contains p.

If X is a topological space and S is a subset, S gets a topology as follows. For each U open in X, $U \cap S$ is declared to be open in S. This is called the induced topology.

A topological space X is said to be *Hausdorff* provided that any distinct $p, q \in X$ have disjoint neighborhoods. Clearly any metric space is Hausdorff. Most important topological spaces are Hausdorff.

A Hausdorff topological space is said to be compact provided the following condition holds. If $\{U_\alpha : \alpha \in A\}$ is any family of open subsets of X, covering X, i.e., $X = \bigcup_{\alpha \in A} U_\alpha$, then there is a finite subcover, i.e., a finite subset $\{U_{\alpha_1}, \dots, U_{\alpha_N} : \alpha_j \in A\}$ such that $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_N}$. An equivalent formulation is the following, known as the finite intersection property. Let $\{S_\alpha : \alpha \in A\}$ be any collection of closed subsets of X. If each finite collection of these closed sets has nonempty intersection, then the complete intersection $\bigcap_{\alpha \in A} S_\alpha$ is nonempty. It is not hard to show that any compact metric space, defined in §A, satisfies this condition.

Any closed subset of a compact space is compact. Furthermore, any compact subset of a Hausdorff space is necessarily closed.

A Hausdorff space X is said to be *locally compact* provided every $p \in X$ has a neighborhood N which is compact (with the induced topology).

A Hausdorff space is said to be *paracompact* provided every open cover $\{U_\alpha : \alpha \in A\}$ has a locally finite refinement, i.e., an open cover $\{V_\beta : \beta \in B\}$ such that each V_β is contained in some U_α and each $p \in X$ has a neighborhood N_p such that $N_p \cap V_\beta$ is nonempty for only finitely many $\beta \in B$. A typical example of a paracompact space is a locally compact Hausdorff space X which is also σ-compact, i.e., $X = \bigcup_{n=1}^\infty X_n$ with X_n compact. Paracompactness is a natural condition under which to construct partitions of unity, as will be illustrated in §§E–F.

A map $F : X \to Y$ between two topological spaces is said to be *continuous* provided $F^{-1}(U)$ is open in X whenever U is open in Y. If $F : X \to Y$ is one to one and onto, and both F and F^{-1} are continuous, F is said to be a *homeomorphism.* For a bijective map $F : X \to Y$, the continuity of F^{-1} is equivalent to the statement that $F(V)$ is open in Y whenever V is open in X; another equivalent statement is that $F(S)$ is closed in Y whenever S is closed in X.

If X and Y are Hausdorff, $F : X \to Y$ continuous, then $F(K)$ is compact in Y whenever K is compact in X. In view of the discussion above, there arises the following useful sufficient condition for a continuous map $F : X \to Y$ to be a homeomorphism. Namely, if X is compact, Y Hausdorff, and F one to one and onto, then F is a homeomorphism.

C. The derivative

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and $F : \mathcal{O} \to \mathbb{R}^m$ a continuous function. We say F is differentiable at a point $x \in \mathcal{O}$, with derivative L, if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that, for $y \in \mathbb{R}^n$, small,

$$F(x+y) = F(x) + Ly + R(x,y) \tag{C.1}$$

with

$$\frac{\|R(x,y)\|}{\|y\|} \to 0 \text{ as } y \to 0. \tag{C.2}$$

We denote the derivative at x by $DF(x) = L$. With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, $DF(x)$ is simply the matrix of partial derivatives,

$$DF(x) = \frac{\partial F_j}{\partial x_k}, \tag{C.3}$$

so that, if $v = (v_1, \dots, v_n)$, (regarded as a column vector) then

$$DF(x)v = \Bigl(\sum_k \frac{\partial F_1}{\partial x_k} v_k, \dots, \sum_k \frac{\partial F_m}{\partial x_k} v_k\Bigr). \tag{C.4}$$

It will be shown below that F is differentiable whenever all the partial derivatives exist and are *continuous* on $\mathcal{O}$. In such a case we say F is a C^1 function on $\mathcal{O}$. More generally, F is said to be C^k if all its partial derivatives of order $\le k$ exist and are continuous.

In (C.2), we can use the *Euclidean* norm on $\mathbb{R}^n$ and $\mathbb{R}^m$. This norm is defined by

$$\|x\| = \bigl(x_1^2 + \cdots + x_n^2\bigr)^{1/2} \tag{C.5}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Any other norm would do equally well.

We now derive the *chain rule* for the derivative. Let $F : \mathcal{O} \to \mathbb{R}^m$ be differentiable at $x \in \mathcal{O}$, as above, let U be a neighborhood of $z = F(x)$ in $\mathbb{R}^m$, and let $G : U \to \mathbb{R}^k$ be differentiable at z. Consider $H = G \circ F$. We have

$$\begin{aligned} H(x+y) &= G(F(x+y)) \\ &= G\bigl(F(x) + DF(x)y + R(x,y)\bigr) \\ &= G(z) + DG(z)\bigl(DF(x)y + R(x,y)\bigr) + R_1(x,y) \\ &= G(z) + DG(z)DF(x)y + R_2(x,y) \end{aligned} \tag{C.6}$$

with

$$\frac{\|R_2(x,y)\|}{\|y\|} \to 0 \text{ as } y \to 0.$$

Thus $G \circ F$ is differentiable at x, and

$$D(G\circ F)(x) = DG(F(x)) \cdot DF(x). \tag{C.7}$$

Another useful remark is that, by the fundamental theorem of calculus, applied to $\varphi(t) = F(x+ty)$,

$$F(x+y) = F(x) + \int_0^1 DF(x+ty)y\, dt, \tag{C.8}$$

provided F is C^1.

A closely related application of the fundamental theorem of calculus is that, if we assume $F : \mathcal{O} \to \mathbb{R}^m$ is differentiable in each variable separately, and that each $\partial F/\partial x_j$ is continuous on $\mathcal{O}$, then

$$\begin{gathered} F(x+y) = F(x) + \sum_{j=1}^n \big[F(x+z_j) - F(x+z_{j-1})\big] = F(x) + \sum_{j=1}^n A_j(x,y)y_j, \\ A_j(x,y) = \int_0^1 \frac{\partial F}{\partial x_j}\big(x+z_{j-1}+ty_je_j\big)\, dt, \end{gathered} \tag{C.9}$$

where $z_0 = 0$, $z_j = (y_1, \dots, y_j, 0, \dots, 0)$, and $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. Now (C.9) implies F is differentiable on $\mathcal{O}$, as we stated below (C.4). As is shown in many calculus texts, one can use the mean value theorem instead of the fundamental theorem of calculus, and obtain a slightly sharper result. We leave the reconstruction of this argument to the reader.

We now describe two convenient notations to express higher order derivatives of a C^k function $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open. In one, let J be a k-tuple of integers between 1 and n; $J = (j_1, \dots, j_k)$. We set

$$f^{(J)}(x) = \partial_{j_k} \cdots \partial_{j_1} f(x), \quad \partial_j = \partial/\partial x_j. \tag{C.10}$$

We set $|J| = k$, the total order of differentiation. A basic result of calculus is that $\partial_i\partial_j f = \partial_j\partial_i f$ provided $f \in C^2(\Omega)$. It follows that, if $f \in C^k(\Omega)$, then $\partial_{j_k} \cdots \partial_{j_1} f = \partial_{\ell_k} \cdots \partial_{\ell_1} f$ whenever $\{\ell_1, \dots, \ell_k\}$ is a permutation of $\{j_1, \dots, j_k\}$. Thus, another convenient notation to use is the following. Let α be an n-tuple of non-negative integers, $\alpha = (\alpha_1, \dots, \alpha_n)$. Then we set

$$f^{(\alpha)}(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f(x), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{C.11}$$

Note that, if $|J| = |\alpha| = k$ and $f \in C^k(\Omega)$,

$$f^{(J)}(x) = f^{(\alpha)}(x), \text{ with } \alpha_i = \#\{\ell : j_\ell = i\}. \tag{C.12}$$

Correspondingly, there are two expressions for monomials in $x = (x_1, \dots, x_n)$:

$$x^J = x_{j_1} \cdots x_{j_k}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{C.13}$$

and $x^J = x^\alpha$ provided J and α are related as in (C.12). Both these notations are called "multi-index" notations.

We now derive Taylor's formula with remainder for a smooth function $F : \Omega \to \mathbb{R}$, making use of these multi-index notations. We will apply the one variable formula, i.e.,

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{1}{2}\varphi''(0)t^2 + \cdots + \frac{1}{k!}\varphi^{(k)}(0)t^k + r_k(t), \tag{C.14}$$

with

$$r_k(t) = \frac{1}{k!}\int_0^t (t-s)^k \varphi^{(k+1)}(s)\, ds, \tag{C.15}$$

given $\varphi \in C^{k+1}(I),\ I = (-a, a)$. Let us assume $0 \in \Omega$, and that the line segment from 0 to x is contianed in Ω. We set $\varphi(t) = F(tx)$, and apply (C.14)–(C.15) with $t = 1$. Applying the chain rule, we have

$$\varphi'(t) = \sum_{j=1}^n \partial_j F(tx) x_j = \sum_{|J|=1} F^{(J)}(tx) x^J. \tag{C.16}$$

Differentiating again, we have

$$\varphi''(t) = \sum_{|J|=1,|K|=1} F^{(J+K)}(tx) x^{J+K} = \sum_{|J|=2} F^{(J)}(tx) x^J, \tag{C.17}$$

where, if $|J| = k,\ |K| = \ell$, we take $J + K = (j_1, \dots, j_k, k_1, \dots, k_\ell)$. Inductively, we have

$$\varphi^{(k)}(t) = \sum_{|J|=k} F^{(J)}(tx) x^J. \tag{C.18}$$

Hence, from (C.14) with $t = 1$,

$$F(x) = F(0) + \sum_{|J|=1} F^{(J)}(0) x^J + \cdots + \frac{1}{k!}\sum_{|J|=k} F^{(J)}(0) x^J + R_k(x),$$

or, more briefly,

$$F(x) = \sum_{|J|\le k} \frac{1}{|J|!} F^{(J)}(0) x^J + R_k(x), \tag{C.19}$$

where

$$R_k(x) = \frac{1}{k!} \sum_{|J|=k+1} \Bigl(\int_0^1 (1-s)^k F^{(J)}(sx)\, ds \Bigr) x^J. \tag{C.20}$$

This gives Taylor's formula with remainder for $F \in C^{k+1}(\Omega)$, in the J-multi-index notation.

As we stated above, for any $j, k \in \{1, \dots, n\}$,

$$\partial_j \partial_k f = \partial_k \partial_j f \tag{C.21}$$

on $\Omega \subset \mathbb{R}^n$, if $f \in C^2(\Omega)$. We will sketch a short proof of this, under the stronger hypothesis that $f \in C^5(\Omega)$. It suffices to prove (C.21) at 0, assumed to belong to Ω. Now (C.19) gives

$$f(x) = \sum_{|J| \le 2} c_J x^J + R_2(x),$$

where c_J are constants and

$$R_2(x) = \frac{1}{2} \sum_{|J|=3} \Bigl(\int_0^1 (1-s)^2 f^{(J)}(sx)\, ds \Bigr) x^J.$$

Now one can easily verify that $\partial_j \partial_k R_2(0) = 0$, provided $f \in C^5(\Omega)$, so $\partial_j \partial_k f(0) = \partial_j \partial_k P_2(0)$, where $P_2(x)$ is the polynomial $\sum_{|J| \le 2} c_J x^J$. It is also straightforward to check that $\partial_j \partial_k P_2 = \partial_k \partial_j P_2$, so we have (C.21).

We also want to write the Taylor formula in the α-multi-index notation. We have

$$\sum_{|J|=k} F^{(J)}(tx) x^J = \sum_{|\alpha|=k} \nu(\alpha) F^{(\alpha)}(tx) x^\alpha, \tag{C.22}$$

where

$$\nu(\alpha) = \#\{J : \alpha = \alpha(J)\},$$

and we define the relation $\alpha = \alpha(J)$ to hold provided the condition (C.12) holds, or equivalently provided $x^J = x^\alpha$. Thus $\nu(\alpha)$ is uniquely defined by

$$\sum_{|\alpha|=k} \nu(\alpha) x^\alpha = \sum_{|J|=k} x^J = (x_1 + \cdots + x_n)^k. \tag{C.23}$$

One sees that, if $|\alpha| = k$, then $\nu(\alpha)$ is equal to the product of the number of combinations of k objects, taken α_1 at a time, times the number of combinations of $k - \alpha_1$ objects, taken α_2 at a time, $\cdots$ times the number of combinations of $k - (\alpha_1 + \cdots + \alpha_{n-1})$ objects, taken α_n at a time. Thus

$$\nu(\alpha) = \binom{k}{\alpha_1} \binom{k-\alpha_1}{\alpha_2} \cdots \binom{k - \alpha_1 - \cdots - \alpha_{n-1}}{\alpha_n} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!} \tag{C.24}$$

In other words, for $|\alpha| = k$,

$$\nu(\alpha) = \frac{k!}{\alpha!}, \quad \text{where} \quad \alpha! = \alpha_1! \cdots \alpha_n! \tag{C.25}$$

Thus the Taylor formula (C.19) can be rewritten

$$F(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_k(x), \tag{C.26}$$

where

$$R_k(x) = \sum_{|\alpha| = k+1} \frac{k+1}{\alpha!} \Bigl(\int_0^1 (1-s)^k F^{(\alpha)}(sx)\, ds \Bigr) x^\alpha. \tag{C.27}$$

D. Inverse function and implicit function theorem

The Inverse Function Theorem, together with its corollary the Implicit Function Theorem is a fundamental result in several variable calculus. First we state the Inverse Function Theorem.

Theorem D.1. *Let F be a C^k map from an open neighborhood Ω of $p_0 \in \mathbb{R}^n$ to $\mathbb{R}^n$, with $q_0 = F(p_0)$. Suppose the derivative $DF(p_0)$ is invertible. Then there is a neighborhood U of p_0 and a neighborhood V of q_0 such that $F : U \to V$ is one to one and onto, and $F^{-1} : V \to U$ is a C^k map. (One says $F : U \to V$ is a diffeomorphism.)*

Using the chain rule, it is easy to reduce to the case $p_0 = q_0 = 0$ and $DF(p_0) = I$, the identity matrix, so we suppose this has been done. Thus

$$F(u) = u + R(u), \quad R(0) = 0, \ DR(0) = 0. \tag{D.1}$$

For v small, we want to solve

$$F(u) = v. \tag{D.2}$$

This is equivalent to $u + R(u) = v$, so let

$$T_v(u) = v - R(u). \tag{D.3}$$

Thus solving (D.2) is equivalent to solving

$$T_v(u) = u. \tag{D.4}$$

We look for a *fixed point* $u = K(v) = F^{-1}(v)$. Also, we want to prove that $DK(0) = I$, i.e., that $K(v) = v + r(v)$ with $r(v) = o(\|v\|)$. If we succeed in doing this, it follows easily that, for general x close to 0,

$$DK(x) = \Big(DF\big(K(x)\big)\Big)^{-1},$$

and a simple inductive argument shows that K is C^k if F is C^k.

A tool we will use to solve (D.4) is the following general result, known as the Contraction Mapping Principle.

Theorem D.2. *Let X be a complete metric space, and let $T : X \to X$ satisfy*

$$\mathit{dist}\,(Tx, Ty) \le r \ \mathit{dist}\,(x, y), \tag{D.5}$$

for some $r < 1$. (We say T is a contraction.) Then T has a unique fixed point x. For any $y_0 \in X$, $T^k y_0 \to x$ as $k \to \infty$.

Proof. Pick $y_0 \in X$ and let $y_k = T^k y_0$. Then $\operatorname{dist}(y_{k+1}, y_k) \le r^k \operatorname{dist}(y_1, y_0)$, so

$$\begin{aligned} \operatorname{dist}(y_{k+m}, y_k) &\le \operatorname{dist}(y_{k+m}, y_{k+m-1}) + \cdots + \operatorname{dist}(y_{k+1}, y_k) \\ &\le \big(r^k + \cdots + r^{k+m-1}\big) \operatorname{dist}(y_1, y_0) \\ &\le r^k \big(1 - r\big)^{-1} \operatorname{dist}(y_1, y_0). \end{aligned} \tag{D.6}$$

It follows that (y_k) is a Cauchy sequence, so it converges; $y_k \to x$. Since $Ty_k = y_{k+1}$ and T is continuous, it follows that $Tx = x$, i.e., x is a fixed point. Uniqueness of the fixed point is clear from the estimate $\operatorname{dist}(Tx, Tx') \le r \operatorname{dist}(x, x')$, which implies $\operatorname{dist}(x, x') = 0$ if x and x' are fixed points. This proves Theorem D.2.

Returning to the problem of solving (D.4), we consider

$$T_v : X_v \longrightarrow X_v \tag{D.7}$$

with

$$X_v = \{u \in \Omega : \|u - v\| \le A_v\} \tag{D.8}$$

where we set

$$A_v = \sup_{\|w\| \le 2\|v\|} \|R(w)\|. \tag{D.9}$$

We claim that (D.7) holds if $\|v\|$ is sufficiently small. To prove this, note that $T_v(u) - v = -R(u)$, so we need to show that, provided $\|v\|$ is small, $u \in X_v$ implies $\|R(u)\| \le A_v$. But indeed, if $u \in X_v$, then $\|u\| \le \|v\| + A_v$, which is $\le 2\|v\|$ if $\|v\|$ is small, so then

$$\|R(u)\| \le \sup_{\|w\| \le 2\|v\|} \|R(w)\| = A_v;$$

this establishes (D.7).

Note that if $\|v\|$ is small enough, the map (D.7) is a contraction map, so there exists a unique fixed point $u = K(v) \in X_v$. Note that, since $u \in X_v$,

$$\|K(v) - v\| \le A_v = o(\|v\|), \tag{D.10}$$

so the inverse function theorem is proved.

Thus if DF is invertible on the domain of F, F is a local diffeomorphism, though stronger hypotheses are needed to guarantee that F is a global diffeomorphism onto its range. Here is one result along these lines.

Proposition D.3. *If $\Omega \subset \mathbb{R}^n$ is open and convex, $F : \Omega \to \mathbb{R}^n$ is C^1, and the symmetric part of $DF(u)$ is positive definite for each $u \in \Omega$, then F is one to one on Ω.*

Proof. Suppose $F(u_1) = F(u_2)$, $u_2 = u_1 + w$. Consider $\varphi : [0,1] \to \mathbb{R}$ given by

$$\varphi(t) = w \cdot F(u_1 + tw).$$

Thus $\varphi(0) = \varphi(1)$, so $\varphi'(t_0)$ must vanish for some $t_0 \in (0,1)$, by the mean value theorem. But $\varphi'(t) = w \cdot DF(u_1 + tw)w > 0$, if $w \neq 0$, by the hypothesis on DF. This shows F is one to one.

We can obtain the following Implicit Function Theorem as a consequence of the Inverse Function Theorem.

Theorem D.4. *Suppose U is a neighborhood of $x_0 \in \mathbb{R}^k$, V a neighborhood of $z_0 \in \mathbb{R}^\ell$, and*

$$F : U \times V \longrightarrow \mathbb{R}^\ell \tag{D.11}$$

is a C^k map. Assume $D_z F(x_0, z_0)$ is invertible; say $F(x_0, z_0) = u_0$. Then the equation $F(x,z) = u_0$ defines $z = f(x, u_0)$ for x near x_0, with f a C^k map.

To prove this, consider $H : U \times V \to \mathbb{R}^k \times \mathbb{R}^\ell$ defined by

$$H(x,z) = \big(x, F(x,z)\big). \tag{D.12}$$

We have

$$DH = \begin{pmatrix} I & D_x F \\ 0 & D_z F \end{pmatrix} \tag{D.13}$$

Thus $DH(x_0, z_0)$ is invertible, so $J = H^{-1}$ exists and is C^k, by the Inverse Function Theorem. It is clear that $J(x, u_0)$ has the form

$$J(x, u_0) = \big(x, f(x, u_0)\big), \tag{D.14}$$

and f is the desired map.

E. Manifolds

A manifold is a Hausdorff topological space with an *atlas*, i.e., a covering by open sets U_j together with homeomorphisms $\varphi_j : U_j \to V_j,\ V_j$ open in $\mathbb{R}^n$. The number n is called the dimension of M. We say that M is a *smooth* manifold provided the atlas has the following property. If $U_{jk} = U_j \cap U_k \neq \emptyset$, then the map

$$\psi_{jk} : \varphi_j(U_{jk}) \to \varphi_k(U_{jk})$$

given by $\varphi_k \circ \varphi_j^{-1}$, is a smooth *diffeomorphism* from the open set $\varphi_j(U_{jk})$ to the open set $\varphi_k(U_{jk})$ in $\mathbb{R}^n$. By this, we mean that ψ_{jk} is C^∞, with a C^∞ inverse. If the ψ_{jk} are all C^ℓ smooth, M is said to be C^ℓ smooth. The pairs (U_j, φ_j) are called local coordinate charts.

A continuous map from M to another smooth manifold N is said to be smooth if it is smooth in local coordinates. Two different atlasses on M, giving a priori two structures of M as a smooth manifold, are said to be equivalent if the identity map on M is smooth from each one of these two manifolds to the other. Really a smooth manifold is considered to be defined by equivalence classes of such atlasses, under this equivalence relation.

One way manifolds arise is the following. Let $f_1, \dots, f_k$ be smooth functions on an open set $U \subset \mathbb{R}^n$. Let $M = \{x \in U : f_j(x) = c_j\}$ for a given $(c_1, \dots, c_k) \in \mathbb{R}^k$. Suppose that $M \neq \emptyset$ and, for each $x \in M$, the gradients ∇f_j are linearly independent at x. It follows easily from the implicit function theorem that M has a natural structure of a smooth manifold of dimension $n - k$. We say M is a submanifold of U. More generally, let $F : X \to Y$ be a smooth map between smooth manifolds, $c \in Y,\ M = F^{-1}(c)$, and assume that $M \neq \emptyset$ and that, at each point $x \in M$, there is a coordinate neighborhood U of x and V of c such that the derivative DF at x has rank k. More pedantically, (U, φ) and (V, ψ) are the coordinate charts, and we assume the derivative of $\psi \circ F \circ \varphi^{-1}$ has rank k at $\varphi(x)$; there is a natural notion of $DF(x) : T_xX \to T_cY$, which will be defined in the next section. In such a case, again the implicit function theorem gives M the structure of a smooth manifold.

We mention a couple of other methods for producing manifolds. For one, given any connected smooth manifold M, its universal covering space $\tilde{M}$ has the natural structure of a smooth manifold. $\tilde{M}$ can be described as follows. Pick a base point $p \in M$. For $x \in M$, consider smooth paths from p to $x,\ \gamma : [0,1] \to M$. We say two such paths γ_0 and γ_1 are equivalent if they are homotopic, i.e., if there is a smooth map $\sigma : I \times I \to M(I = [0,1])$, such that $\sigma(0,t) = \gamma_0(t),\ \sigma(1,t) = \gamma_1(t)$, and $\sigma(s,0) = p,\ \sigma(s,1) = x$. Points in $\tilde{M}$ lying over any given $x \in M$ consist of such equivalence classes.

Another construction produces *quotient* manifolds. In this situation, we have a smooth manifold M, and a discrete group Γ of diffeomorphisms on M. The quotient space $\Gamma \setminus M$ consists of equivalence classes of points of M, where we set $x \sim \gamma(x)$ for for each $x \in M,\ \gamma \in \Gamma$. If we assume that each $x \in M$ has a neighborhood U containing no $\gamma(x)$, for $\gamma \neq e$, the identity element of Γ, then $\Gamma \setminus M$ has a natural smooth manifold structure.

We next discuss *partitions of unity.* Supose M is paracompact. In this case, using a locally finite covering of M by coordinate neighborhoods, we can construct $\psi_j \in C_0^\infty(M)$ such that for any compact $K \subset M$, only finitely many ψ_j are nonzero on K, (we say the sequence ψ_j is locally finite), and such that for any $p \in M$, some $\psi_j(p) \neq 0$. Then

$$\varphi_j(x) = \frac{\psi_j(x)^2}{\sum\limits_k \psi_k(x)^2}$$

is a locally finite sequence of functions in $C_0^\infty(M)$, satisfying $\sum_j \varphi_j(x) = 1$. Such a sequence is called a partition of unity. It has many uses.

A more general notion than manifold is that of a smooth manifold *with boundary.* In this case, $\overline{M}$ is again a Hausdorff topological space, and there are two types of coordinate charts (U_j, φ_j). Either φ_j takes U_j to an open subset V_j of $\mathbb{R}^n$ as before, or φ_j maps U_j homeomorphically onto an open subset of $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Again appropriate transition maps are required to be smooth. In case $\overline{M}$ is paracompact, there is again the construction of partitions of unity. For one simple but effective application of this construction, see the proof of the Stokes formula in §7.

F. Vector bundles

We begin with an intrinsic definition of a tangent vector to a smooth manifold M, at a point $p \in M$. It is an equivalence class of smooth curves through p, i.e., of smooth maps $\gamma : I \to M$, I an interval containing 0, such that $\gamma(0) = p$. The equivalence relation is $\gamma \sim \gamma_1$ provided that, for some coordinate chart (U, φ) about p, $\varphi : U \to V \subset \mathbb{R}^n$, we have

$$\frac{d}{dt}(\varphi \circ \gamma)(0) = \frac{d}{dt}(\varphi \circ \gamma_1)(0). \tag{F.1}$$

This equivalence is independent of choice of coordinate chart about p.

If $V \subset \mathbb{R}^n$ is open, we have a natural identification of the set of tangent vectors to V at $p \in V$ with $\mathbb{R}^n$. In general, the set of tangent vectors to M at p is denoted T_pM. A coordinate cover of M induces a coordinate cover of TM, the disjoint union of T_pM as p runs over M, making TM a smooth manifold. TM is called the tangent bundle of M. Note that each T_pM has the natural structure of a vector space of dimension n, if n is the dimension of M. If $F : X \to M$ is a smooth map between manifolds, $x \in X$, there is a natural linear map $DF(x) : T_xX \to T_pM$, $p = F(x)$, which agrees with the derivative as defined in §C, in local coordinates. $DF(x)$ takes the equivalence class of a smooth curve γ through x to that of the curve $F \circ \gamma$ through p.

The tangent bundle TM of a smooth manifold M is a special case of a vector bundle. Generally, a smooth vector bundle $E \to M$ is a smooth manifold E, together with a smooth map $\pi : E \to M$ with the following properties. For each $p \in M$, the "fibre" $E_p = \pi^{-1}(p)$ has the structure of a vector space, of dimension k, independent of p. Furthermore, there exists a cover of M by open sets U_j, and diffeomorphisms $\Phi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$ with the property that, for each $p \in U_j$, $\Phi_j : E_p \to \{p\} \times \mathbb{R}^k \to \mathbb{R}^k$ is a linear isomorphism, and if $U_{j\ell} = U_j \cap U_\ell \neq \emptyset$, we have smooth "transition functions"

$$\Phi_\ell \circ \Phi_j^{-1} = \Psi_{j\ell} : U_{j\ell} \times \mathbb{R}^k \to U_{j\ell} \times \mathbb{R}^k \tag{F.2}$$

which are the identity on the first factor and such that for each $p \in U_{j\ell}$, $\Psi_{j\ell}(p)$ is a linear isomorphism on $\mathbb{R}^k$. In the case of complex vector bundles, we systematically replace $\mathbb{R}^k$ by $\mathbb{C}^k$ in the discussion above.

The structure above arises for the tangent bundle as follows. Let (U_j, φ_j) be a coordinate cover of M, $\varphi_j : U_j \to V_j \subset \mathbb{R}^n$. Then $\Phi_j : TU_j \to U_j \times \mathbb{R}^n$ takes the equivalence class of smooth curves through $p \in U_j$ containing an element γ to the pair $\big(p, (\varphi_j \circ \gamma)'(0)\big) \in U_j \times \mathbb{R}^n$.

A *section* of a vector bundle $E \to M$ is a smooth map $\beta : M \to E$ such that $\pi(\beta(p)) = p$ for all $p \in M$. For example, a section of the tangent bundle $TM \to M$

is a vector field on M. If X is a vector field on M, generating a flow $\mathcal{F}^t$, then $X(p) \in T_pM$ coincides with the equivalence class of $\gamma(t) = \mathcal{F}^t p$.

Any smooth vector bundle $E \to M$ has associated a vector bundle $E^* \to M$, the "dual bundle" with the property that there is a natural duality of E_p and E_p^* for each $p \in M$. In case E is the tangent bundle TM, this dual bundle is called the cotangent bundle, and denoted T^*M.

More generally, given a vector bundle $E \to M$, other natural constructions involving vector spaces yield other vector bundles over M, such as tensor bundles $\otimes^j E \to M$ with fibre $\otimes^j E_p$, mixed tensor bundles with fibre $\big(\otimes^j E_p\big) \otimes \big(\otimes^k E_p^*\big)$, exterior algebra bundles with fibre ΛE_p, etc. Note that a k-form, as defined in §5, is a section of $\Lambda^k T^*M$. A section of $\big(\otimes^j T\big) \otimes \big(\otimes^k T^*\big)M$ is called a tensor field of type (j,k).

A Riemannian metric tensor on a smooth manifold M is a smooth symmetric section g of $\otimes^2 T^*M$ which is positive definite at each point $p \in M$, i.e., $g_p(X,X) > 0$ for each nonzero $X \in T_pM$. For any fixed $p \in M$, using a local coordinate patch (U,φ) containing p one can construct a positive symmetric section of $\otimes^2 T^*U$. Using a partition of unity, we can hence construct a Riemannian metric tensor on any smooth paracompact manifold M. If we define the length of a path $\gamma : [0,1] \to M$ to be $L(\gamma) = \int_0^1 g\big(\gamma'(t), \gamma'(0)\big)^{1/2} dt$, then

$$d(p,q) = \inf\{L(\gamma) : \gamma(0) = p, \gamma(1) = q\} \tag{F.3}$$

is a distance function making M a metric space, provided M is connected.

G. Fundamental local existence theorem for ODE

The goal of this section is to establish existence of solutions to an ODE

$$\frac{dy}{dt} = F(t,y), \quad y(t_0) = y_0. \tag{G.1}$$

We will prove the following fundamental result.

Theorem G.1. *Let $y_0 \in \mathcal{O}$, an open subset of $\mathbb{R}^n$, $I \subset \mathbb{R}$ an interval containing t_0. Suppose F is continuous on $I \times \mathcal{O}$ and satisfies the following Lipschitz estimate in y :*

$$\|F(t,y_1) - F(t,y_2)\| \le L\|y_1 - y_2\| \tag{G.2}$$

for $t \in I,\ y_j \in \mathcal{O}$. Then the equation (G.1) has a unique solution on some t-interval containing t_0.

To begin the proof, we note that the equation (G.1) is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(s,y(s))\, ds. \tag{G.3}$$

Existence will be established via the Picard iteration method, which is the following. Guess $y_0(t)$, e.g., $y_0(t) = y_0$. Then set

$$y_k(t) = y_0 + \int_{t_0}^{t} F\big(s, y_{k-1}(s)\big)\, ds. \tag{G.4}$$

We aim to show that, as $k \to \infty,\ y_k(t)$ converges to a (unique) solution of (G.3), at least for t close enough to t_0.

To do this, we use the *Contraction Mapping Principle*, established in §D. We look for a fixed point of T, defined by

$$(Ty)(t) = y_0 + \int_{t_0}^{t} F(s,y(s))\, ds. \tag{G.5}$$

Let

$$X = \big\{u \in C(J,\mathbb{R}^n) : u(t_0) = y_0,\ \sup_{t\in J} \|u(t) - y_0\| \le K\big\}. \tag{G.6}$$

Here $J = [t_0 - \varepsilon, t_0 + \varepsilon]$, where ε will be chosen, sufficiently small, below. K is picked so $\{y : \|y - y_0\| \le K\}$ is contained in $\mathcal{O}$, and we also suppose $J \subset I$. Then there exists M such that

$$\sup_{s \in J, \|y - y_0\| \le K} \|F(s, y)\| \le M. \tag{G.7}$$

Then, provided

$$\varepsilon \le \frac{K}{M}, \tag{G.8}$$

we have

$$T : X \to X. \tag{G.9}$$

Now, using the Lipschitz hypothesis (3.2), we have, for $t \in J$,

$$\begin{aligned} \|(Ty)(t) - (Tz)(t)\| &\le \int_{t_0}^{t} L\|y(s) - z(s)\|\,ds \\ &\le \varepsilon\ L\ \sup_{s \in J} \|y(s) - z(s)\| \end{aligned} \tag{G.10}$$

assuming y and z belong to X. It follows that T is a contraction on X provided one has

$$\varepsilon < \frac{1}{L}, \tag{G.11}$$

in addition to the hypotheses above. This proves Theorem G.1.

In view of the lower bound on the length of the interval J on which the existence theorem works, it is easy to show that the only way a solution can fail to be globally defined, i.e., to exist for all $t \in I$, is for $y(t)$ to "explode to infinity" by leaving every compact set $K \subset \mathcal{O}$, as $t \to t_1$, for some $t_1 \in I$.

Often one wants to deal with a higher order ODE. There is a standard method of reducing an nth order ODE

$$y^{(n)}(t) = f(t, y, y', \dots, y^{(n-1)}) \tag{G.12}$$

to a first order system. One sets $u = (u_0, \dots, u_{n-1})$ with

$$u_0 = y, \quad u_j = y^{(j)}, \tag{G.13}$$

and then

$$\frac{du}{dt} = \bigl(u_1, \dots, u_{n-1}, f(t, u_0, \dots, u_{n-1})\bigr) = g(t, u). \tag{G.14}$$

If y takes values in $\mathbb{R}^k$, then u takes values in $\mathbb{R}^{kn}$.

If the system (G.1) is non-autonomous, i.e., if F explicitly depends on t, it can be converted to an autonomous system (one with no explicit t-dependence) as follows. Set $z = (t, y)$. We then have

$$\frac{dz}{dt} = \Big(1, \frac{dy}{dt}\Big) = \big(1, F(z)\big) = G(z). \tag{G.15}$$

Sometimes this process destroys important features of the original system (G.1). For example, if (G.1) is linear, (G.15) might be nonlinear. Nevertheless, the trick of converting (G.1) to (G.15) has some uses.

Many systems of ODE are difficult to solve explicitly. There is one very basic class of ODE which can be solved explicitly, in terms of integrals, namely the single first order linear ODE:

$$\frac{dy}{dt} = a(t)y + b(t), \quad y(0) = y_0, \tag{G.16}$$

where $a(t)$ and $b(t)$ are continuous real or complex valued functions. Set

$$A(t) = \int_0^t a(s)\, ds. \tag{G.17}$$

Then (G.16) can be written as

$$e^{A(t)} \frac{d}{dt}\big(e^{-A(t)}y\big) = b(t). \tag{G.18}$$

From (G.18) we get

$$y(t) = e^{A(t)}y_0 + e^{A(t)} \int_0^t e^{-A(s)} b(s)\, ds. \tag{G.19}$$

The solution to (G.1) is a function of t and also of the initial data:

$$y = y(t, y_0). \tag{G.20}$$

If F is a C^∞ function of its arguments, then y is also a C^∞ function of its arguments. This result is important; it implies that a smooth vector field generates a smooth flow. A proof can be found in Chapter 1 of [T1].

H. Lie groups

A Lie group G is a group which is also a smooth manifold, such that the group operations $G \times G \to G$ and $G \to G$ given by $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth maps. Let e denote the identity element of G. For each $g \in G$, we have left and right translations, L_g and R_g, diffeomorphisms on G, defined by

$$L_g(h) = gh, \quad R_g(h) = hg. \tag{H.1}$$

The set of left invariant vector fields X on G, i.e., vector fields satisfying

$$(DL_g)X(h) = X(gh), \tag{H.2}$$

is called the Lie algebra of G, and denoted $\mathfrak{g}$. If $X, Y \in \mathcal{G}$, then the Lie bracket $[X,Y]$ belongs to $\mathfrak{g}$. Evaluation of $X \in \mathfrak{g}$ at e provides a linear isomorphism of $\mathfrak{g}$ with T_eG.

A vector field X on G belongs to $\mathfrak{g}$ if and only if the flow $\mathcal{F}_X^t$ it generates commutes with L_g for all $g \in G$, i.e., $g(\mathcal{F}_X^t h) = \mathcal{F}_X^t(gh)$ for all $g, h \in G$. If we set

$$\gamma_X(t) = \mathcal{F}_X^t e, \tag{H.3}$$

we obtain $\gamma_X(t+s) = \mathcal{F}_X^s(\mathcal{F}_X^t e) \cdot e = (\mathcal{F}_X^t e)(\mathcal{F}_X^s e)$, and hence

$$\gamma_X(s+t) = \gamma_X(s)\gamma_X(t) \tag{H.4}$$

for $s, t \in \mathbb{R}$; we say γ_X is a smooth one parameter subgroup of G. Clearly

$$\gamma_X'(0) = X(e). \tag{H.5}$$

Conversely, if γ is any smooth one parameter group satisfying $\gamma'(0) = X(e)$, then $\mathcal{F}^t g = g \cdot \gamma(t)$ defines a flow generated by the vector field $X \in \mathfrak{g}$ coinciding with $X(e)$ at e.

The exponential map

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow G \tag{H.6}$$

is defined by

$$\mathrm{Exp}(X) = \gamma_X(1). \tag{H.7}$$

Note that $\gamma_{sX}(t) = \gamma_X(st)$, so $\mathrm{Exp}(tX) = \gamma_X(t)$. In particular, under the identification $\mathfrak{g} \approx T_eG$,

$$D\,\mathrm{Exp}(0) : T_eG \longrightarrow T_eG \text{ is the identity map.} \tag{H.8}$$

The fact that each element $X \in \mathfrak{g}$ generates a one parameter group has the following generalization, to a fundamental result of S. Lie. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra, i.e., $\mathfrak{h}$ is a linear subspace and $X_j \in \mathfrak{h} \Rightarrow [X_1, X_2] \in \mathfrak{h}$. By Frobenius' Theorem (which we will establish in §I), through each point p of G there is a smooth manifold M_p of dimension $k = \dim \mathfrak{h}$, which is an integral manifold for $\mathfrak{h}$, i.e., $\mathfrak{h}$ spans the tangent space of M_p at each $q \in M_p$. We can take M_p to be the maximal such (connected) manifold, and then it is unique. Let H be the maximal integral manifold of $\mathfrak{h}$ containing the identity element e.

Proposition H.1. *H is a subgroup of G.*

Proof. Take $h_0 \in H$ and consider $H_0 = h_0^{-1} H$; clearly $e \in H_0$. By left invariance, H_0 is also an integral manifold of $\mathcal{H}$, so $H_0 = H$. This shows that $h_0, h_1 \in H \Rightarrow h_0^{-1} h_1 \in H$, so H is a group.

Given any $\alpha_0 \in \Lambda^k T_e^* G$, there is a unique k-form α on G, invariant under L_g, i.e., satisfying $L_g^* \alpha = \alpha$ for all $g \in G$, equal to α_0 at e. In case $k = n = \dim G$, if ω_0 is a nonzero element of $\Lambda^n T_e^* G$, the corresponding left invariant n-form ω on G defines also an orientation on G, and hence a left invariant volume form on G, called (left) *Haar measure.* It is uniquely defined up to a constant multiple. Similarly one has right Haar measure. In many but not all cases left Haar measure is also right invariant; the G is said to be unimodular. It is very important to be able to integrate over a Lie group using Haar measure.

We next define a representation of a Lie group G on a finite dimensional vector space V. This is a smooth map

$$\pi : G \longrightarrow \operatorname{End}(V) \tag{H.9}$$

such that

$$\pi(e) = I, \quad \pi(gg') = \pi(g)\pi(g'), \quad g, g' \in G. \tag{H.10}$$

If $F \in C_0(G)$, i.e., F is continuous with compact support, we can define $\pi(F) \in \operatorname{End}(V)$ by

$$\pi(F)v = \int_G F(g)\pi(g)v \, dg. \tag{H.11}$$

We get different results depending on whether left or right Haar measure is used. Right now, let us use *right* Haar measure. Then, for $g \in G$, we have

$$\pi(F)\pi(g)v = \int_G F(x)\pi(xg)v \, dx = \int_G F(xg^{-1})\pi(x)v \, dx. \tag{H.12}$$

We also define the derived representation

$$d\pi : \mathfrak{g} \longrightarrow \operatorname{End}(V) \tag{H.13}$$

by

$$d\pi = D\pi(e) : T_e G \longrightarrow \operatorname{End}(V), \tag{H.14}$$

using the identification $\mathfrak{g} \approx T_e G$. Thus, for $X \in \mathfrak{g}$,

$$d\pi(X)v = \lim_{t \to 0} \frac{1}{t} \big[\pi(\operatorname{Exp} tX)v - v\big]. \tag{H.15}$$

The following result states that $d\pi$ is a Lie algebra homomorphism.

Proposition H.2. *For $X, Y \in \mathfrak{g}$, we have*

$$\big[d\pi(X), d\pi(Y)\big] = d\pi\big([X,Y]\big). \tag{H.16}$$

Proof. We will first produce a formula for $\pi(F)d\pi(X)$, given $F \in C_0^\infty(G)$. In fact, making use of (H.12), we have

$$\begin{aligned} \pi(F)d\pi(X)v &= \lim_{t\to 0} \frac{1}{t} \int\limits_G \big[F(g)\pi(g)\pi(\mathrm{Exp}\ tX) - F(g)\pi(g)\big] v \ dg \\ &= \lim_{t\to 0} \frac{1}{t} \int\limits_G \big[F\big(g\cdot \mathrm{Exp}(-tX)\big) - F(g)\big]\pi(g) v \ dg \\ &= -\pi(XF)v, \end{aligned} \tag{H.17}$$

where XF denotes the left invariant vector field X applied to F. It follows that

$$\pi(F)\big[d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)\big]v = \pi(YXF - XYF)v = -\pi\big([X,Y]F\big)v, \tag{H.18}$$

which by (H.17) is equal to $\pi(F)\, d\pi\big([X,Y]\big)v$. Now, if F is supported near $e \in G$ and integrates to 1, is is easily seen that $\pi(F)$ is close to the identity I, so this implies (H.16).

There is a representation of G on $\mathfrak{g}$, called the *adjoint representation*, defined as follows. Consider

$$K_g : G \longrightarrow G, \quad K_g(h) = ghg^{-1}. \tag{H.19}$$

Then $K_g(e) = e$, and we set

$$Ad(g) = DK_g(e) : T_eG \longrightarrow T_eG, \tag{H.20}$$

identifying $T_eG \approx \mathfrak{g}$. Note that $K_g \circ K_{g'} = K_{gg'}$, so the chain rule implies $Ad(g)Ad(g') = Ad(gg')$.

Note that $\gamma(t) = g\ \mathrm{Exp}(tX)g^{-1}$ is a one-parameter subgroup of G satisfying $\gamma'(0) = Ad(g)X$. Hence

$$\mathrm{Exp}(t\ Ad(g)X) = g\ \mathrm{Exp}(tX)\ g^{-1}. \tag{H.21}$$

In particular,

$$\mathrm{Exp}\big((Ad\ \mathrm{Exp}\ sY)tX\big) = \ \mathrm{Exp}(sY)\ \mathrm{Exp}(tX)\ \mathrm{Exp}(-sY). \tag{H.22}$$

Now, the right side of (H.22) is equal to $\mathcal{F}_Y^{-s} \circ \mathcal{F}_X^t \circ \mathcal{F}_Y^s(e)$, so by (3.1)-(3.3) we have

$$Ad(\mathrm{Exp}\ sY)X = \mathcal{F}^s_{Y\#}X. \tag{H.23}$$

If we take the s-derivative at $s = 0$, we get a formula for the derived representation of Ad, which is denoted ad, rather than $d\ Ad$. Using (3.3)–(3.5), we have

$$ad(Y)X = [Y,X]. \tag{H.24}$$

In other words, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is given by the Lie bracket. We mention that Jacobi's identity for Lie algebras is equivalent to the statement that

$$ad\big([X,Y]\big) = \big[ad(X), ad(Y)\big], \quad \forall\ X, Y \in \mathfrak{g}. \tag{H.25}$$

I. Frobenius' theorem

Let $G : U \to V$ be a diffeomorphism. Recall from §3 the action on vector fields:

$$G_\# Y(x) = DG(y)^{-1} Y(y), \quad y = G(x). \tag{I.1}$$

As noted there, an alternative characterization of $G_\# Y$ is given in terms of the flow it generates. One has

$$\mathcal{F}_Y^t \circ G = G \circ \mathcal{F}_{G_\# Y}^t. \tag{I.2}$$

The proof of this is a direct consequence of the chain rule. As a special case, we have the following

Proposition I.1. *If $G_\# Y = Y$, then $\mathcal{F}_Y^t \circ G = G \circ \mathcal{F}_Y^t$.*

From this, we derive the following condition for a pair of flows to commute. Let X and Y be vector fields on U.

Proposition I.2. *If X and Y commute as differential operators, i.e.,*

$$[X, Y] = 0, \tag{I.3}$$

then locally $\mathcal{F}_X^s$ and $\mathcal{F}_Y^t$ commute, i.e., for any $p_0 \in U$, there exists $\delta > 0$ such that, for $|s|, |t| < \delta$,

$$\mathcal{F}_X^s \mathcal{F}_Y^t p_0 = \mathcal{F}_Y^t \mathcal{F}_X^s p_0. \tag{I.4}$$

Proof. By Proposition I.1, it suffices to show that $\mathcal{F}_{X\#}^s Y = Y$. Clearly this holds at $s = 0$. But by (3.6), we have

$$\frac{d}{ds} \mathcal{F}_{X\#}^s Y = \mathcal{F}_{X\#}^s [X, Y],$$

which vanishes if (I.3) holds. This finishes the proof.

We have stated that, given (I.3), then (I.4) holds locally. If the flows generated by X and Y are not complete, this can break down globally. For example, consider $X = \partial/\partial x_1$, $Y = \partial/\partial x_2$ on $\mathbb{R}^2$, which satisfy (I.3) and generate commuting flows. These vector fields lift to vector fields on the universal covering surface $\tilde{M}$ of $\mathbb{R}^2 \setminus (0,0)$, which continue to satisfy (I.3). The flows on $\tilde{M}$ do not commute globally. This phenomenon does not arise, for example, for vector fields on a compact manifold.

We now consider when a family of vector fields has a multidimensional integral manifold. Suppose $X_1, \dots, X_k$ are smooth vector fields on U which are linearly independent at each point of a $k-$dimensional surface $\Sigma \subset U$. If each X_j is tangent to Σ at each point, Σ is said to be an integral manifold of $(X_1, \dots, X_k)$.

Proposition I.3. *Suppose $X_1, \dots, X_k$ are linearly independent at each point of U and $[X_j, X_\ell] = 0$ for all j, ℓ. Then, for each $x_0 \in U$, there is a k-dimensional integral manifold of $(X_1, \dots, X_k)$ containing x_0.*

Proof. We define a map $F : V \to U,\ V$ a neighborhood of 0 in $\mathbb{R}^k$, by

$$F(t_1, \dots, t_k) = \mathcal{F}^{t_1}_{X_1} \cdots \mathcal{F}^{t_k}_{X_k} x_0. \tag{I.5}$$

Clearly $(\partial/\partial t_1)F = X_1(F)$. Similarly, since $\mathcal{F}^{t_j}_{X_j}$ all commute, we can put any $\mathcal{F}^{t_j}_{X_j}$ first and get $(\partial/\partial t_j)F = X_j(F)$. This shows that the image of V under F is an integral manifold containing x_0.

We now derive a more general condition guaranteeing the existence of integral submanifolds. This important result is due to Frobenius. We say $(X_1, \dots, X_k)$ is *involutive* provided that, for each j, ℓ, there are smooth $b^{j\ell}_m(x)$ such that

$$[X_j, X_\ell] = \sum_{m=1}^{k} b^{j\ell}_m(x) X_m. \tag{I.6}$$

The following is Frobenius' Theorem.

Theorem I.4. *If $(X_1, \dots, X_k)$ are C^∞ vector fields on U, linearly independent at each point, and the involutivity condition (I.6) holds, then through each x_0 there is, locally, a unique integral manifold Σ, of dimension k.*

We will give two proofs of this result. First, let us restate the conclusion as follows. There exist local coordinates $(y_1, \dots, y_n)$ centered at x_0 such that

$$\text{span } (X_1, \dots, X_k) = \text{ span } \Big(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k}\Big). \tag{I.7}$$

First proof. The result is clear for $k = 1$. We will use induction on k. So let the set of vector fields $X_1, \dots, X_{k+1}$ be linearly independent at each point and involutive. Choose a local coordinate system so that $X_{k+1} = \partial/\partial u_1$. Now let

$$Y_j = X_j - (X_j u_1)\frac{\partial}{\partial u_1} \quad \text{for} \quad 1 \le j \le k, \quad Y_{k+1} = \frac{\partial}{\partial u_1}. \tag{I.8}$$

Since, in $(u_1, \dots, u_n)$ coordinates, no $Y_1, \dots, Y_k$ involves $\partial/\partial u_1$, neither does any Lie bracket, so

$$[Y_j, Y_\ell] \in \text{ span } (Y_1, \dots, Y_k), \quad j, \ell \le k.$$

Thus $(Y_1, \dots, Y_k)$ is involutive. The induction hypothesis implies there exist local coordinates $(y_1, \dots, y_n)$ such that

$$\text{span } (Y_1, \dots, Y_k) = \text{ span } \Big(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k}\Big).$$

Now let

$$(I.9) \qquad Z = Y_{k+1} - \sum_{\ell=1}^{k} \big(Y_{k+1} y_\ell\big) \frac{\partial}{\partial y_\ell} = \sum_{\ell > k} (Y_{k+1} y_\ell) \frac{\partial}{\partial y_\ell}.$$

Since, in the $(u_1, \dots, u_n)$ coordinates, $Y_1, \dots, Y_k$ do not involve $\partial/\partial u_1$, we have

$$[Y_{k+1}, Y_j] \in \text{ span } (Y_1, \dots, Y_k).$$

Thus $[Z, Y_j] \in \text{span } (Y_1, \dots, Y_k)$ for $j \le k$, while (I.9) implies that $[Z, \partial/\partial y_j]$ belongs to the span of $(\partial/\partial y_{k+1}, \dots, \partial/\partial y_n)$, for $j \le k$. Thus we have

$$\Big[Z, \frac{\partial}{\partial y_j}\Big] = 0, \quad j \le k.$$

Proposition I.3 implies span $(\partial/\partial y_1, \dots, \partial/\partial y_k, Z)$ has an integral manifold through each point, and since this span is equal to the span of $X_1, \dots, X_{k+1}$, the first proof is complete.

Second proof. Let $X_1, \dots, X_k$ be C^∞ vector fields, linearly independent at each point, and satisfying the condition (I.6). Choose an $n-k$ dimensional surface $\mathcal{O} \subset U$, transverse to $X_1, \dots, X_k$. For V a neighborhood of the origin in $\mathbb{R}^k$, define $\Phi : V \times \mathcal{O} \to U$ by

$$(I.10) \qquad \Phi(t_1, \dots, t_k, x) = \mathcal{F}_{X_1}^{t_1} \cdots \mathcal{F}_{X_k}^{t_k} x.$$

We claim that, for x fixed, the image of V in U is a k dimensional surface Σ tangent to each X_j, at each point of Σ. Note that, since $\Phi(0, \dots, t_j, \dots, 0, x) = \mathcal{F}_{X_j}^{t_j} x$, we have

$$(I.11) \qquad \frac{\partial}{\partial t_j} \Phi(0, \dots, 0, x) = X_j(x), \quad x \in \mathcal{O}.$$

To establish the claim, it suffices to show that $\mathcal{F}^t_{X_j \#} X_\ell$ is a linear combination with coefficients in $C^\infty(U)$ of $X_1, \dots, X_k$. This is accomplished by the following:

Lemma I.5. *Suppose* $[Y, X_j] = \sum_\ell \lambda_{j\ell}(x) X_\ell$, *with smooth coefficients* $\lambda_{j\ell}(x)$. *Then* $\mathcal{F}^t_{Y\#} X_j$ *is a linear combination of* $X_1, \dots, X_k$, *with coefficients in* $C^\infty(U)$.

Proof. Denote by Λ the matrix $(\lambda_{j\ell})$ and let $\Lambda(t) = \Lambda(t, x) = (\lambda_{j\ell}(\mathcal{F}^t_Y x))$. Now let $A(t) = A(t, x)$ be the unique solution to the ODE

$$(I.12) \qquad \frac{d}{dt} A(t) = \Lambda(t) A(t), \quad A(0) = I.$$

Write $A = (\alpha_{j\ell})$. We claim that

$$\mathcal{F}^t_{Y\#} X_j = \sum_\ell \alpha_{j\ell}(t,x) X_\ell. \tag{I.13}$$

This formula will prove the lemma. Indeed, we have

$$\begin{aligned}
\frac{d}{dt}(\mathcal{F}^t_Y)_\# X_j &= (\mathcal{F}^t_Y)_\#[Y, X_j] \\
&= (\mathcal{F}^t_Y)_\# \sum_\ell \lambda_{j\ell} X_\ell \\
&= \sum_\ell (\lambda_{j\ell} \circ \mathcal{F}^t_Y)(\mathcal{F}^t_{Y\#} X_\ell).
\end{aligned}$$

Uniqueness of the solution to (I.12) gives (I.13), and we are done.

This completes the second proof of Frobenius' Theorem.

There are a number of related results which go under the rubric of "Frobenius' Theorem." Here we mention one, needed for the proof of Proposition 13.2.

Proposition I.6. *Let A_j be smooth $m \times m$ matrix functions on $\mathcal{O} \subset \mathbb{R}^n$. Suppose the operators $L_j = \partial/\partial x_j + A_j(x)$, acting on functions with values in $\mathbb{R}^m$, all commute, $1 \le j \le n$. If $p \in \mathcal{O}$, there is a solution in a neighborhood of p to*

$$L_j u = 0, \quad 1 \le j \le n, \tag{I.14}$$

with $u(p) \in \mathbb{R}^m$ prescribed.

Proof. Assume $p = 0$, and $\mathcal{O}$ is a ball centered at 0. Let $u_0 \in \mathbb{R}^m$ be given. First solve

$$L_1 u = 0 \ \text{ on } \ \mathbb{R}^1 \cap \mathcal{O}, \quad u(0) = u_0, \tag{I.15}$$

which is just an ODE. Then solve

$$L_2 u = 0 \ \text{ on } \ \mathbb{R}^2 \cap \mathcal{O}, \quad u\big|_{\mathbb{R}^1 \cap \mathcal{O}} \text{ given by (I.15).} \tag{I.16}$$

Inductively, having u on $\mathbb{R}^j \cap \mathcal{O}$, solve

$$L_{j+1} u = 0 \ \text{ on } \ \mathbb{R}^{j+1} \cap \mathcal{O}, \quad u\big|_{\mathbb{R}^j \cap \mathcal{O}} \text{ obtained in previous step.} \tag{I.17}$$

At $j + 1 = n$, you have u on $\mathcal{O}$. Clearly $u(0) = u_0$ and $L_n u = 0$ on $\mathcal{O}$. To see that $v_{n-1} = L_{n-1} u = 0$ on $\mathcal{O}$, note that

$$L_n v_{n-1} = L_n L_{n-1} u = L_{n-1} L_n u = 0,$$

by commutativity, and

$$v_{n-1}\big|_{\mathbb{R}^{n-1} \cap \mathcal{O}} = 0.$$

This implies $v_{n-1} = 0$ on $\mathcal{O}$, i.e., $L_{n-1} u = 0$ on $\mathcal{O}$. Similarly one establishes $L_j u = 0$ on $\mathcal{O}$ for all $j < n$.

J. Exercises on determinants and cross products

If $M_{n\times n}$ denotes the space of $n \times n$ complex matrices, we want to show that there is a map

$$\det : M_{n\times n} \to \mathbb{C} \tag{J.1}$$

which is *uniquely* specified as a function $\vartheta : M_{n\times n} \to \mathbb{C}$ satisfying:
(a) ϑ is *linear* in each column a_j of A,
(b) $\vartheta(\widetilde{A}) = -\vartheta(A)$ if $\widetilde{A}$ is obtained from A by interchanging two columns.
(c) $\vartheta(I) = 1$.

1. Let $A = (a_1, \dots, a_n)$, where a_j are column vectors; $a_j = (a_{1j}, \dots, a_{nj})^t$. Show that, if (a) holds, we have the expansion

$$\begin{aligned} \det A &= \sum_j a_{j1} \det(e_j, a_2, \dots, a_n) = \cdots \\ &= \sum_{j_1,\cdots,j_n} a_{j_1 1} \cdots a_{j_n n} \det(e_{j_1}, e_{j_2}, \dots, e_{j_n}), \end{aligned} \tag{J.2}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$.

2. Show that, if (b) and (c) also hold, then

$$\det A = \sum_{\sigma\in S_n} (\operatorname{sgn} \sigma)\ a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}, \tag{J.3}$$

where S_n is the set of *permutations* of $\{1, \dots, n\}$, and

$$\operatorname{sgn} \sigma = \det(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \pm 1. \tag{J.4}$$

To define sgn σ, the "sign" of a permutation σ, we note that every permutation σ can be written as a product of transpositions: $\sigma = \tau_1 \cdots \tau_\nu$, where a transposition of $\{1, \dots, n\}$ interchanges two elements and leaves the rest fixed. We say sgn $\sigma = 1$ if ν is even and sgn $\sigma = -1$ if ν is odd. It is necessary to show that sgn σ is independent of the choice of such a product representation. (Referring to (J.4) begs the question until we know that det is well defined.)

3. Let $\sigma \in S_n$ act on a function of n variables by

$$(\sigma f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{J.5}$$

Let P be the polynomial

$$P(x_1, \dots, x_n) = \prod_{1 \le j < k \le n} (x_j - x_k). \tag{J.6}$$

Show that

$$(\sigma P)(x) = (\operatorname{sgn} \sigma)\ P(x), \tag{J.7}$$

and that this implies that sgn σ is well defined. (This argument is due to Cauchy.)

4. Deduce that there is a unique determinant satisfying (a)–(c), and that it is given by (J.3).

5. Show that (J.3) implies

$$\det A = \det A^t. \tag{J.8}$$

Conclude that one can replace columns by rows in the characterization (a)–(c) of determinants.
Hint. $a_{\sigma(j)j} = a_{\ell\tau(\ell)}$ with $\ell = \sigma(j),\ \tau = \sigma^{-1}$. Also, sgn $\sigma =$ sgn τ.

6. Show that, if (a)–(c) hold (for rows), it follows that
(d) $\vartheta(\widetilde{A}) = \vartheta(A)$ if $\widetilde{A}$ is obtained from A by adding $c\rho_\ell$ to ρ_k, for some $c \in \mathbb{C}$, where $\rho_1, \dots, \rho_n$ are the rows of A.
Re-prove the uniqueness of ϑ satisfying (a)–(d) (for rows) by applying row operations to A until either some row vanishes or A is converted to I.

7. Show that

$$\det(AB) = (\det A)(\det B). \tag{J.9}$$

Hint. For fixed $B \in M_{n\times n}$, compare $\vartheta_1(A) = \det(AB)$ and $\vartheta_2(A) = (\det A)(\det B)$. For uniqueness, use an argument from Exercise 6.

8. Show that

$$\text{(J.10)} \qquad \det \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det A_{11}$$

where $A_{11} = (a_{jk})_{2 \le j,k \le n}$.
Hint. Do the first identity by the analogue of (d), for columns. Then exploit uniqueness for det on $M_{(n-1)\times(n-1)}$.

9. Deduce that $\det(e_j, a_2, \dots, a_n) = (-1)^{j-1} \det A_{1j}$ where A_{kj} is formed by deleting the kth column and the jth row from A.

10. Deduce from the first sum in (J.2) that

$$\text{(J.11)} \qquad \det A = \sum_{j=1}^{n} (-1)^{j-1} a_{j1} \det A_{1j}.$$

More generally, for any $k \in \{1, \dots, n\}$,

$$\text{(J.12)} \qquad \det A = \sum_{j=1}^{n} (-1)^{j-k} a_{jk} \det A_{kj}.$$

This is called an expansion of det A by minors, down the kth column.

11. Show that

$$\text{(J.13)} \qquad \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} = a_{11} a_{22} \cdots a_{nn}.$$

Hint. Use (J.10) and induction.

The following exercises deal with cross products of vectors in $\mathbb{R}^3$.

12. If $u, v \in \mathbb{R}^3$, show that the formula

$$w \cdot (u \times v) = \det \begin{pmatrix} w_1 & u_1 & v_1 \\ w_2 & u_2 & v_2 \\ w_3 & u_3 & v_3 \end{pmatrix} \tag{J.14}$$

for $u \times v = \kappa(u, v)$ defines uniquely a bilinear map $\kappa : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Show that it satisfies

$$i \times j = k, \quad j \times k = i, \quad k \times i = j,$$

where $\{i, j, k\}$ is the standard basis of $\mathbb{R}^3$.

13. We say $T \in SO(3)$ provided that T is a real 3×3 matrix satisfying $T^t T = I$ and $\det T > 0$, (hence $\det T = 1$). Show that

$$T \in SO(3) \Longrightarrow Tu \times Tv = T(u \times v). \tag{J.15}$$

Hint. Multiply the 3×3 matrix in Exercise 12 on the left by T.

14. Show that, if θ is the angle between u and v in $\mathbb{R}^3$, then

$$|u \times v| = |u|\ |v|\ |\sin \theta|. \tag{J.16}$$

Hint. Check this for $u = i,\ v = ai + bj$, and use Exercise 13 to show this suffices.

15. Show that $\kappa : \mathbb{R}^3 \to$ Skew(3), the set of antisymmetric real 3×3 matrices, given by

$$\kappa(y_1, y_2, y_3) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \tag{J.17}$$

satisfies

$$Kx = y \times x, \quad K = \kappa(y). \tag{J.18}$$

Show that, with $[A, B] = AB - BA$,

$$\kappa(x \times y) = \big[\kappa(x), \kappa(y)\big], \quad \mathrm{Tr}\,\big(\kappa(x)\kappa(y)^t\big) = 2x \cdot y. \tag{J.19}$$

K. Exercises on the Frenet-Serret formulas

1. Let $x(t)$ be a smooth curve in $\mathbb{R}^3$; assume it is parametrized by arclength, so $T(t) = x'(t)$ has unit length; $T(t) \cdot T(t) = 1$. Differentiating, we have $T'(t) \perp T(t)$. The *curvature* is defined to be $\kappa(t) = \|T'(t)\|$. If $\kappa(t) \neq 0$, we set $N(t) = T'/\|T'\|$, so

$$T' = \kappa N,$$

and N is a unit vector orthogonal to T. We define $B(t)$ by

$$B = T \times N. \tag{K.1}$$

Note that (T, N, B) form an orthonormal basis of $\mathbb{R}^3$ for each t, and

$$T = N \times B, \quad N = B \times T. \tag{K.2}$$

By (K.1) we have $B' = T \times N'$. Deduce that B' is orthogonal to both T and B, hence parallel to N. We set

$$B' = -\tau N,$$

for smooth $\tau(t)$, called the *torsion*.

2. From $N' = B' \times T + B \times T'$ and the formulas for T' and B' above, deduce the following system, called the *Frenet-Serret formula*:

$$\begin{aligned} T' &= \qquad\quad \kappa N \\ N' &= -\kappa T \qquad\qquad + \tau B \\ B' &= \qquad\; - \tau N \end{aligned} \tag{K.3}$$

Form the 3×3 matrix

$$A(t) = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \tag{K.4}$$

and deduce that the 3×3 matrix $F(t)$ whose columns are T, N, B :

$$F = (T, N, B)$$

satisfies the ODE

$$\frac{dF}{dt} = F\,A(t).$$

3. Derive the following *converse* to the Frenet-Serret formula. Let $T(0), N(0), B(0)$ be an orthonormal set in $\mathbb{R}^3$, such that $B(0) = T(0) \times N(0)$, let $\kappa(t)$ and $\tau(t)$ be given smooth functions, and solve the system (K.3). Show that there is a unique curve $x(t)$ such that $x(0) = 0$ and $T(t), N(t), B(t)$ are associated to $x(t)$ by the construction in Exercise 1, so in particular the curve has curvature $\kappa(t)$ and torsion $\tau(t)$.
Hint. To prove that (K.1)–(K.2) hold for all t, consider the next exercise.

4. Let $A(t)$ be a smooth $n \times n$ real matrix function which is *skew adjoint* for all t (of which (K.4) is an example). Suppose $F(t)$ is a real $n \times n$ matrix function satisfying

$$\frac{dF}{dt} = F\,A(t).$$

If $F(0)$ is an orthogonal matrix, show that $F(t)$ is orthogonal for all t.
Hint. Set $J(t) = F(t)^* F(t)$. Show that $J(t)$ and $J_0(t) = I$ both solve the initial value problem

$$\frac{dJ}{dt} = [J, A(t)], \quad J(0) = I.$$

5. Let $U_1 = T,\ U_2 = N,\ U_3 = B$, and set

$$\omega(t) = \tau T + \kappa B.$$

Show that (K.3) is equivalent to

$$U_j' = \omega \times U_j,\ 1 \le j \le 3.$$

6. Suppose τ and κ are constant. Show that ω is constant, so $T(t)$ satisfies the constant coefficient ODE

$$T'(t) = \omega \times T(t).$$

Note that $\omega \cdot T(0) = \tau$. Show that, after a translation and rotation, $x(t)$ takes the form

$$\gamma(t) = \Bigl(\frac{\kappa}{\lambda^2}\cos\lambda t, \frac{\kappa}{\lambda^2}\sin\lambda t, \frac{\tau}{\lambda}t\Bigr), \quad \lambda^2 = \kappa^2 + \tau^2.$$

7. Suppose $x(t)$, parametrized by arclength, lies in the sphere $S_R = \{x \in \mathbb{R}^3 : |x| = R\}$, for all t. If $\rho(t) = \kappa(t)^{-1}$ denotes the radius of curvature of this curve, show that

$$\rho(t)^2 + \Bigl(\frac{\rho'(t)}{\tau(t)}\Bigr)^2 = R^2.$$

Hint. Differentiate the identity $x(t)\cdot x(t) = R^2$ repeatedly, and substitute in various parts of (K.3) for derivatives of T, etc.

8. In this problem, do *not* assume that $x(t)$ is parametrized by arclength. Define the arclength parameter s by $ds/dt = |x'(t)|$, and set $T(t) = x'(t)/|x'(t)|$, so $dT/ds = \kappa N$. Show that

$$x''(t) = \frac{d^2 s}{dt^2}\, T(t) + \Bigl(\frac{ds}{dt}\Bigr)^2 \kappa(t)\, N(t). \tag{K.5}$$

Taking the cross product of both sides with $T(t)$, deduce that

$$\kappa(t)\, B(t) = \frac{x'(t)\times x''(t)}{|x'(t)|^3}. \tag{K.6}$$

Hence

$$\kappa(t) = \frac{|x'(t)\times x''(t)|}{|x'(t)|^3}. \tag{K.7}$$

Hint. Differentiate the identity $x'(t) = (ds/dt)T(t)$ to get (K.5).

9. In the setting of Exercise 8, show that

$$x'''(t) = \bigl\{s''' - (s')^3\kappa^2\bigr\}T + \bigl\{3s's''\kappa + (s')^2\kappa'\bigr\}N + (s')^3\kappa\tau B. \tag{K.8}$$

Deduce that

$$x'(t)\times x''(t)\cdot x'''(t) = (s')^6\kappa^2\tau, \tag{K.9}$$

and hence that the torsion is given by

$$\tau(t) = \frac{x'(t)\times x''(t)\cdot x'''(t)}{|x'(t)\times x''(t)|^2}. \tag{K.10}$$

10. Let $x(t)$ be a unit speed curve, with T, κ, τ, etc. as in Exercise 1. Consider the curve $y(t) = T(t)$, a curve which is perhaps not of unit speed. Show that its curvature and torsion are given by

$$\tilde{\kappa} = \sqrt{1 + \Bigl(\frac{\tau}{\kappa}\Bigr)^2}, \quad \tilde{\tau} = \frac{\frac{d}{ds}\bigl(\frac{\tau}{\kappa}\bigr)}{\kappa\bigl[1 + \bigl(\frac{\tau}{\kappa}\bigr)^2\bigr]}. \tag{K.11}$$

Hint. Apply the results of Exercises 8–9 to $y(t) = T(t)$. Use (K.3) to derive

$$T'' = -\kappa^2 T + \kappa' N + \kappa\tau B, \quad T'\times T'' = \kappa^2\tau T + \kappa^3 B.$$

Then produce a formula for T'''.

L. Exercises on exponential and trigonometric functions

1. Let $a \in \mathbb{R}$. Show that the unique solution to $u'(t) = au(t),\ u(0) = 1$ is given by

$$u(t) = \sum_{j=0}^{\infty} \frac{a^j}{j!} t^j. \tag{L.1}$$

We denote this function by $u(t) = e^{at}$, the exponential function. We also write $\exp(t) = e^t$.
Hint. Integrate the series term by term and use the fundamental theorem of calculus.
Alternative. Setting $u_0(t) = 1$, and using the Picard iteration method (G.4) to define the sequence $u_k(t)$, show that $u_k(t) = \sum_{j=0}^{k} a^j t^j / j!$

2. Show that, for all $s, t \in \mathbb{R}$,

$$e^{a(s+t)} = e^{as} e^{at}. \tag{L.2}$$

Hint. Show that $u_1(t) = e^{a(s+t)}$ and $u_2(t) = e^{as} e^{at}$ solve the same initial value problem.
Alternative. Differentiate $e^{a(s+t)} e^{-at}$.

3. Show that $\exp : \mathbb{R} \to (0, \infty)$ is a diffeomorphism. We denote the inverse by

$$\log : (0, \infty) \longrightarrow \mathbb{R}.$$

Show that $v(x) = \log x$ solves the ODE $dv/dx = 1/x,\ v(1) = 0$, and deduce that

$$\int_1^x \frac{1}{y}\, dy = \log x. \tag{L.3}$$

4. Let $a \in \mathbb{R},\ i = \sqrt{-1}$. Show that the unique solution to $f'(t) = iaf,\ f(0) = 1$ is given by

$$f(t) = \sum_{j=0}^{\infty} \frac{(ia)^j}{j!} t^j. \tag{L.4}$$

We denote this function by $f(t) = e^{iat}$. Show that, for all $t \in \mathbb{R}$,

$$e^{ia(s+t)} = e^{ias} e^{iat}. \tag{L.5}$$

5. Write

$$e^{it} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} t^{2j} + i \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} t^{2j+1} = u(t) + iv(t). \tag{L.6}$$

Show that

$$u'(t) = -v(t), \quad v'(t) = u(t).$$

We denote these functions by $u(t) = \cos t,\ v(t) = \sin t$. The identity

$$e^{it} = \cos t + i \sin t \tag{L.7}$$

is called Euler's formula.

6. Use (L.5) to derive the identities

$$\begin{aligned} \sin(x+y) &= \sin x \ \cos y + \cos x \ \sin y \\ \cos(x+y) &= \cos x \ \cos y - \sin x \ \sin y. \end{aligned} \tag{L.8}$$

7. Use (L.7)–(L.8) to show that

$$\sin^2 t + \cos^2 t = 1, \quad \cos^2 t = \frac{1}{2}\bigl(1 + \cos 2t\bigr). \tag{L.9}$$

8. Show that

$$\gamma(t) = (\cos t, \sin t)$$

is a map of $\mathbb{R}$ onto the unit circle $S^1 \subset \mathbb{R}^2$ with non-vanishing derivative, and, as t increases, $\gamma(t)$ moves monotonically, counterclockwise.
We *define* π to be the smallest number $t_1 \in (0, \infty)$ such that $\gamma(t_1) = (-1, 0)$, so

$$\cos \pi = -1, \quad \sin \pi = 0.$$

Show that 2π is the smallest number $t_2 \in (0, \infty)$ such that $\gamma(t_2) = (1, 0)$, so

$$\cos 2\pi = 1, \quad \sin 2\pi = 0.$$

Show that

$$\cos(t+2\pi) = \cos t, \quad \sin(t+2\pi) = \sin t$$
$$\cos(t+\pi) = -\cos t, \quad \sin(t+\pi) = -\sin t.$$

Show that $\gamma(\pi/2) = (0,1)$, and that

$$\cos\Bigl(t+\frac{\pi}{2}\Bigr) = -\sin t, \quad \sin\Bigl(t+\frac{\pi}{2}\Bigr) = \cos t.$$

9. Show that $\sin : (-\pi/2, \pi/2) \to (-1,1)$ is a diffeomorphism. We denote its inverse by

$$\arcsin : (-1,1) \longrightarrow \Bigl(-\frac{\pi}{2}, \frac{\pi}{2}\Bigr).$$

Show that $u(t) =$ arcsin t solves the ODE

$$\frac{du}{dt} = \frac{1}{\sqrt{1-t^2}}, \quad u(0) = 0.$$

Hint. Apply the chain rule to $\sin\bigl(u(t)\bigr) = t$.
Deduce that, for $t \in (-1,1)$,

$$\arcsin t = \int_0^t \frac{dx}{\sqrt{1-x^2}}. \tag{L.10}$$

10. Show that

$$e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

Hint. First compute $\bigl(1/2+\sqrt{3}i/2\bigr)^3$ and use Exercise 8. Then compute $e^{\pi i/2}e^{-\pi i/3}$.
For intuition behind these formulas, look at Fig. L.1.

11. Show that $\sin \pi/6 = 1/2$, and hence that

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{a_n}{2n+1}\Bigl(\frac{1}{2}\Bigr)^{2n+1},$$

where

$$a_0 = 1, \quad a_{n+1} = \frac{2n+1}{2n+2}a_n.$$

Show that

$$\frac{\pi}{6} - \sum_{n=0}^{k} \frac{a_n}{2n+1}\Bigl(\frac{1}{2}\Bigr)^{2n+1} < \frac{4^{-k}}{3(2k+3)}.$$

Using a calculator, sum the series over $0 \le n \le 20$, and verify that

$$\pi \approx 3.141592653589 \cdots$$

12. For $x \neq (k+1/2)\pi,\ k \in \mathbb{Z}$, set

$$\tan x = \frac{\sin x}{\cos x}.$$

Show that $1 + \tan^2 x = 1/\cos^2 x$. Show that $w(x) = \tan x$ satisfies the ODE

$$\frac{dw}{dx} = 1 + w^2, \quad w(0) = 0.$$

13. Show that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a diffeomorphism. Denote the inverse by

$$\arctan : \mathbb{R} \longrightarrow \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

Show that

$$\arctan y = \int_0^y \frac{dx}{1+x^2}. \tag{L.11}$$

M. Exponentiation of matrices

Let A be an $n \times n$ matrix, real or complex. We consider the linear ODE

$$\frac{dy}{dt} = Ay; \quad y(0) = y_0. \tag{M.1}$$

In analogy to the scalar case, as treated in §L, we can produce the solution in the form

$$y(t) = e^{tA} y_0, \tag{M.2}$$

where we define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \tag{M.3}$$

We will establish estimates implying the convergence of this infinite series for all real t, indeed for all complex t. Then term by term differentiation is valid, and gives (M.1). To discuss convergence of (M.3), we need the notion of the *norm* of a matrix.

If $u = (u_1, \dots, u_n)$ belongs to $\mathbb{R}^n$ or to $\mathbb{C}^n$, set

$$\|u\| = \bigl(|u_1|^2 + \cdots + |u_n|^2\bigr)^{1/2}. \tag{M.4}$$

Then, if A is an $n \times n$ matrix, set

$$\|A\| = \sup\{\|Au\| : \|u\| \le 1\}. \tag{M.5}$$

The norm (M.4) possesses the following properties:

$$\|u\| \ge 0, \quad \|u\| = 0 \Longleftrightarrow u = 0, \tag{M.6}$$

$$\|cu\| = |c|\ \|u\|, \text{ for real or complex } c, \tag{M.7}$$

$$\|u + v\| \le \|u\| + \|v\|. \tag{M.8}$$

The last, known as the triangle inequality, follows from Cauchy's inequality:

$$|(u, v)| \le \|u\| \cdot \|v\|, \tag{M.9}$$

where the inner product is $(u, v) = u_1\overline{v}_1 + \cdots + u_n\overline{v}_n$. To deduce (M.8) from (M.9), just square both sides of (M.8). To prove (M.9), use $(u - v, u - v) \geq 0$ to get

$$2 \ \mathrm{Re}\ (u, v) \leq \|u\|^2 + \|v\|^2.$$

Then replace u by $e^{i\theta}u$ to deduce

$$2|(u, v)| \leq \|u\|^2 + \|v\|^2.$$

Next, replace u by tu and v by $t^{-1}v$, to get

$$2|(u, v)| \leq t^2\|u\|^2 + t^{-2}\|v\|^2,$$

for any $t > 0$. Picking t so that $t^2 = \|v\|/\|u\|$, we have Cauchy's inequality (M.9).

Granted (M.6)–(M.8), we easily get

$$\begin{aligned} \|A\| &\geq 0, \\ \|cA\| &= |c|\ \|A\|, \\ \|A + B\| &\leq \|A\| + \|B\|. \end{aligned} \tag{M.10}$$

Also, $\|A\| = 0$ if and only if $A = 0$. The fact that $\|A\|$ is the smallest constant K such that $\|Au\| \leq K\|u\|$ gives

$$\|AB\| \leq \|A\| \cdot \|B\|. \tag{M.11}$$

In particular,

$$\|A^k\| \leq \|A\|^k. \tag{M.12}$$

This makes it easy to check convergence of the power series (M.3).

Power series manipulations can be used to establish the identity

$$e^{sA}e^{tA} = e^{(s+t)A}. \tag{M.13}$$

Another way to prove this is the following. Regard t as fixed; denote the left side of (M.13) as $X(s)$ and the right side as $Y(s)$. Then differentiation with respect to s gives, respectively

$$\begin{aligned} X'(s) &= AX(s), \quad X(0) = e^{tA}, \\ Y'(s) &= AY(s), \quad Y(0) = e^{tA}, \end{aligned} \tag{M.14}$$

so uniqueness of solutions to the ODE implies $X(s) = Y(s)$ for all s. We note that (M.13) is a special case of the following.

Proposition M.1. *$e^{t(A+B)} = e^{tA}e^{tB}$ for all t, if and only if A and B commute.*

Proof. Let

$$Y(t) = e^{t(A+B)}, \quad Z(t) = e^{tA}e^{tB}. \tag{M.15}$$

Note that $Y(0) = Z(0) = I$, so it suffices to show that $Y(t)$ and $Z(t)$ satisfy the same ODE, to deduce that they coincide. Clearly

$$Y'(t) = (A+B)Y(t). \tag{M.16}$$

Meanwhile

$$Z'(t) = Ae^{tA}e^{tB} + e^{tA}Be^{tB}. \tag{M.17}$$

Thus we get the equation (4.16) for $Z(t)$ provided we know that

$$e^{tA}B = Be^{tA} \ \text{ if } \ AB = BA. \tag{M.18}$$

This follows from the power series expansion for e^{tA}, together with the fact that

$$A^kB = BA^k \text{ for all } k \geq 0, \text{ if } AB = BA. \tag{M.19}$$

For the converse, if $Y(t) = Z(t)$ for all t, then $e^{tA}B = Be^{tA}$, by (M.17), and hence, taking the t-derivative, $e^{tA}AB = BAe^{tA}$; setting $t = 0$ gives $AB = BA$.

If A is in diagonal form

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \tag{M.20}$$

then clearly

$$e^{tA} = \begin{pmatrix} e^{ta_1} & & \\ & \ddots & \\ & & e^{ta_n} \end{pmatrix} \tag{M.21}$$

The following result makes it useful to diagonalize A in order to compute e^{tA}.

Proposition M.2. *If K is an invertible matrix and $B = KAK^{-1}$, then*

$$e^{tB} = K\, e^{tA}\, K^{-1}. \tag{M.22}$$

Proof. This follows from the power series expansion (4.3), given the observation that

$$B^k = K\, A^k\, K^{-1}. \tag{M.23}$$

In view of (M.20)–(M.22), it is convenient to record a few standard results about eigenvalues and eigenvectors here. Let A be an $n \times n$ matrix over F, $F = \mathbb{R}$ or $\mathbb{C}$. An eigenvector of A is a nonzero $u \in F^n$ such that

$$Au = \lambda u \tag{M.24}$$

for some $\lambda \in F$. Such an eigenvector exists if and only if $A - \lambda I : F^n \to F^n$ is not invertible, i.e., if and only if

$$\det(A - \lambda I) = 0. \tag{M.25}$$

Now (M.25) is a polynomial equation, so it always has a complex root. This proves the following.

Proposition M.3. *Given an $n \times n$ matrix A, there exists at least one (complex) eigenvector u.*

Of course, if A is real and we know there is a real root of (M.25) (e.g., if n is odd) then a real eigenvector exists. One important class of matrices guaranteed to have real eigenvalues is the class of self adjoint matrices. The adjoint of an $n \times n$ complex matrix is specified by the identity $(Au, v) = (u, A^*v)$.

Proposition M.4. *If $A = A^*$, then all eigenvalues of A are real.*

Proof. $Au = \lambda u$ implies

$$\lambda \|u\|^2 = (\lambda u, u) = (Au, u) = (u, Au) = (u, \lambda u) = \overline{\lambda} \|u\|^2. \tag{M.26}$$

Hence $\lambda = \overline{\lambda}$, if $u \neq 0$.

We now establish the following important result.

Theorem M.5. *If $A = A^*$, then there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.*

Proof. Let u_1 be one unit eigenvector; $Au_1 = \lambda u_1$. Existence is guaranteed by Proposition 4.3. Let $V = (u_1)^\perp$ be the orthogonal complement of the linear span of u_1. Then dim V is $n - 1$ and

$$A : V \to V, \text{ if } A = A^*. \tag{M.27}$$

The result follows by induction on n.

Corollary M.6. *If $A = A^t$ is a real symmetric matrix, then there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.*

Proof. By Proposition M.4 and the remarks following Proposition M.3, there is one unit eigenvector $u_1 \in \mathbb{R}^n$. The rest of the proof is as above.

The proof of the last four results rests on the fact that every nonconstant polynomial has a complex root. This is the Fundamental Theorem of Algebra. A proof is given in §9 (Exercise 5).

Given an ODE in upper triangular form,

$$\frac{dy}{dt} = \begin{pmatrix} a_{11} & * & * \\ & \ddots & * \\ & & a_{nn} \end{pmatrix} y \tag{M.28}$$

you can solve the last ODE for y_n, as it is just $dy_n/dt = a_{nn}y_n$. Then you get a single inhomogeneous ODE for y_{n-1}, which can be solved as demonstrated in (G.16)-(G.19), and you can continue inductively to solve. Thus it is often useful to be able to put an $n \times n$ matrix A in upper triangular form, with respect to a convenient choice of basis. We will establish two results along these lines. The first is due to Schur.

Theorem M.7. *For any $n \times n$ matrix A, there is an orthonormal basis $u_1, \dots, u_n$ of $\mathbb{C}^n$ with respect to which A is in upper triangular form.*

This result is equivalent to:

Proposition M.8. *For any A, there is a sequence of vector spaces V_j of dimension j, contained in $\mathbb{C}^n$, with*

$$V_n \supset V_{n-1} \supset \cdots \supset V_1 \tag{M.29}$$

and

$$A : V_j \longrightarrow V_j. \tag{M.30}$$

To see the equivalence, if we are granted (M.29)–(M.30), pick $u_n \perp V_{n-1}$, a unit vector, then pick $u_{n-1} \in V_{n-1}$ such that $u_{n-1} \perp V_{n-2}$, and so forth. Meanwhile, Proposition M.8 is a simple inductive consequence of the following result.

Lemma M.9. *For any matrix A acting on V_n, there is a linear subspace V_{n-1}, of codimension 1, such that $A : V_{n-1} \to V_{n-1}$.*

Proof. Use Proposition M.3, applied to A^*. There is a vector v_1 such that $A^*v_1 = \lambda v_1$. Let $V_{n-1} = (v_1)^\perp$. This completes the proof of the Lemma, hence of Theorem M.7.

Let's look more closely at what you can say about solutions to an ODE which has been put in the form (M.28). As mentioned, we can obtain y_j inductively by solving nonhomogeneous scalar ODEs

$$\frac{dy_j}{dt} = a_{jj}y_j + b_j(t) \tag{M.31}$$

where $b_j(t)$ is a linear combination of $y_{j+1}(t), \dots, y_n(t)$, and the formula (G.19) applies, with $A(t) = a_{jj}t$. We have $y_n(t) = Ce^{a_{nn}t}$, so $b_{n-1}(t)$ is a multiple of $e^{a_{nn}t}$. If $a_{n-1,n-1} \neq a_{nn}$, $y_{n-1}(t)$ will be a linear combination of $e^{a_{nn}t}$ and $e^{a_{n-1,n-1}t}$, but if $a_{n-n,n-1} = a_{nn}$, $y_{n-1}(t)$ may be a linear combination of $e^{a_{nn}t}$ and $te^{a_{nn}t}$. Further integration will involve $\int p(t)e^{\alpha t}\, dt$ where $p(t)$ is a polynomial. That no other sort of function will arise is guaranteed by the following result.

Lemma M.10. *If $p(t) \in \mathcal{P}_n$, the space of polynomials of degree $\le n$, and $\alpha \neq 0$, then*

$$\int p(t)e^{\alpha t}\; dt = q(t)e^{\alpha t} + C \tag{M.32}$$

for some $q(t) \in \mathcal{P}_n$.

Proof. The map $p = Tq$ defined by $(d/dt)(q(t)e^{\alpha t}) = p(t)e^{\alpha t}$ is a map on $\mathcal{P}_n$; in fact we have

$$Tq(t) = \alpha q(t) + q'(t). \tag{M.33}$$

It suffices to show $T : \mathcal{P}_n \to \mathcal{P}_n$ is invertible. But $D = d/dt$ is *nilpotent* on $\mathcal{P}_n$; $D^{n+1} = 0$. Hence

$$T^{-1} = \alpha^{-1}(I + \alpha^{-1}D)^{-1} = \alpha^{-1}\big(I - \alpha^{-1}D + \cdots + \alpha^{-n}(-D)^n\big).$$

Note that this gives a neat formula for the integral (M.32). For example,

$$\begin{aligned}\int t^n e^{-t}\; dt &= -(t^n + nt^{n-1} + \cdots + n!)e^{-t} + C\\ &= -n!\Big(1 + t + \frac{1}{2}t^2 + \cdots + \frac{1}{n!}t^n\Big)e^{-t} + C.\end{aligned} \tag{M.34}$$

This could also be established by integration by parts and induction. Of course, when $\alpha = 0$ in (M.32) the result is different; $q(t)$ is a polynomial of degree $n+1$.

Now the implication for the solution to (M.28) is that all the components of $y(t)$ are products of polynomials and exponentials. By Theorem M.7, we can draw the same conclusion about the solution to $dy/dt = Ay$ for any $n \times n$ matrix A. We can formally state the result as follows.

Proposition M.11. *For any $n \times n$ matrix A,*

$$e^{tA}v = \sum e^{\lambda_j t}v_j(t), \tag{M.35}$$

where $\{\lambda_j\}$ is the set of eigenvalues of A and $v_j(t)$ are $\mathbb{C}^n$-valued polynomials. All the $v_j(t)$ are constant when A is diagonalizable.

To see that the λ_j are the eigenvalues of A, note that, in the upper triangular case, only the exponentials $e^{a_{jj}t}$ arise, and in that case the eigenvalues are precisely the diagonal elements.

If we let $\mathcal{E}_\lambda$ denote the space of $\mathbb{C}^n$-valued functions of the form $V(t) = e^{\lambda t}v(t)$, where $v(t)$ is a $\mathbb{C}^n$-valued polynomial, then $\mathcal{E}_\lambda$ is invariant under the action of both d/dt and A, hence of $d/dt - A$. Hence, if a sum $V_1(t) + \cdots + V_k(t),\ V_j(t) \in \mathcal{E}_{\lambda_j}$ (with λ_js distinct) is annihilated by $d/dt - A$, so is each term in this sum.

Therefore, if (M.35) is a sum over the distinct eigenvalues λ_j of A, it follows that each term $e^{\lambda_j t}v_j(t)$ is annihilated by $d/dt - A$, or equivalently is of the form $e^{tA}w_j$ where $w_j = v_j(0)$. This leads to the following conclusion. Set

$$G_\lambda = \{v \in \mathbb{C}^n : e^{tA}v = e^{t\lambda}v(t),\ v(t) \text{ polynomial}\}. \tag{M.36}$$

Then $\mathbb{C}^n$ has a direct sum decomposition

$$\mathbb{C}^n = G_{\lambda_1} + \cdots + G_{\lambda_k} \tag{M.37}$$

where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A. Furthermore, each G_{λ_j} is invariant under A, and

$$A_j = A|_{G_{\lambda_j}} \text{ has exactly one eigenvalue, } \lambda_j. \tag{M.38}$$

This last statement holds because, when $v \in G_{\lambda_j},\ e^{tA}v$ involves only the exponential $e^{\lambda_j t}$. We say G_{λ_j} is the *generalized eigenspace* of A, with eigenvalue λ_j. Of course, G_{λ_j} contains ker $(A - \lambda_j I)$. Now $B_j = A_j - \lambda_j I$ has only 0 as an eigenvalue. It is subject to the following result.

Lemma M.12. *If $B : \mathbb{C}^k \to \mathbb{C}^k$ has only 0 as an eigenvalue, then B is nilpotent, i.e.,*

$$B^m = 0 \textit{ for some } m. \tag{M.39}$$

Proof. Let $W_j = B^j(\mathbb{C}^k)$; then $\mathbb{C}^k \supset W_1 \supset W_2 \supset \cdots$ is a sequence of finite dimensional vector spaces, each invariant under B. This sequence must stabilize, so for some $m,\ B : W_m \to W_m$ bijectively. If $W_m \neq 0,\ B$ has a nonzero eigenvalue.

We next discuss the famous Jordan normal form of a complex $n \times n$ matrix. The result is the following.

Theorem M.13. *If A is an $n \times n$ matrix, then there is a basis of $\mathbb{C}^n$ with respect to which A becomes a direct sum of blocks of the form*

$$\begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} \tag{M.40}$$

In light of the decomposition (M.37) and Lemma M.12, it suffices to establish the Jordan normal form for a nilpotent matrix B. Given $v_0 \in \mathbb{C}^k$, let m be the smallest integer such that $B^m v_0 = 0$; $m \le k$. If $m = k$, then $\{v_0, Bv_0, \dots, B^{m-1}v_0\}$ gives a basis of $\mathbb{C}^k$ putting B in Jordan normal form. We then say v_0 is a *cyclic* vector for B, and $\mathbb{C}^k$ is generated by v_0. We call $\{v_0, \dots, B^{m-1}v_0\}$ a *string*.

We will have a Jordan normal form precisely if we can write $\mathbb{C}^k$ as a direct sum of cyclic subspaces. We establish that this can be done by induction on the dimension.

Thus, inductively, we can suppose $W_1 = B(\mathbb{C}^k)$ is a direct sum of cyclic subspaces, so W_1 has a basis that is a union of strings, let's say a union of d strings $\{v_j, Bv_j, \dots, B^{\ell_j}v_j\}$, $1 \le j \le d$. In this case, $\ker B \cap W_1 = N_1$ has dimension d, and the vectors $B^{\ell_j}v_j$, $1 \le j \le d$, span N_1. Furthermore, each v_j has the form $v_j = Bw_j$ for some $w_j \in \mathbb{C}^k$.

Now $\dim \ker B = k - r \ge d$, where $r = \dim W_1$. Let $\{z_1, \dots, z_{k-r-d}\}$ span a subspace of $\ker B$ complementary to N_1. Then the strings $\{w_j, v_j = Bw_j, \dots, B^{\ell_j}v_j\}$, $1 \le j \le d$, and $\{z_1\}, \dots, \{z_{k-r-d}\}$ generate cyclic subspaces whose direct sum is $\mathbb{C}^k$, giving the Jordan normal form.

The argument above is part of an argument of Filippov. In fact, Filippov's proof contains a further clever twist, enabling one to prove Theorem M.13 without using the decomposition (M.37). See Strang [Stra] for Filippov's proof.

We have seen how constructing e^{tA} solves the equation (M.1). We can also use it to solve an inhomogeneous equation, of the form

$$\frac{dy}{dt} = Ay + b(t); \quad y(0) = y_0. \tag{M.41}$$

Direct calculation shows that the solution is given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}\, b(s)\, ds. \tag{M.42}$$

Note how this partially generalizes the formula (G.19).

N. Isothermal coordinates

Let M be an oriented manifold of dimension 2, endowed with a Riemannian metric g. We aim to sketch a proof of the following result.

Proposition N.1. *There exists a covering U_j of M and coordinate maps*

$$\varphi_j : U_j \longrightarrow \mathcal{O}_j \subset \mathbb{R}^2 \tag{N.1}$$

which are conformal (and orientation preserving).

By definition, a map $\varphi : U \to \mathcal{O}$ between two manifolds, with Riemannian metrics g and g_0 is *conformal* provided

$$\varphi^* g_0 = \lambda g \tag{N.2}$$

for some positive $\lambda \in C^\infty(U)$. In (N.1), $\mathcal{O}_j$ is of course given the flat metric $dx^2 + dy^2$. Coordinates (N.1) which are conformal are also called "isothermal coordinates." It is clear that the composition of conformal maps is conformal, so if Proposition N.1 holds then the transition maps

$$\psi_{jk} = \varphi_j \circ \varphi_k^{-1} : \mathcal{O}_{jk} \longrightarrow \mathcal{O}_{kj} \tag{N.3}$$

are conformal, where $\mathcal{O}_{jk} = \varphi_k(U_j \cap U_k)$. This is particularly significant, in view of the following fact:

Proposition N.2. *An orientation preserving conformal map*

$$\psi : \mathcal{O} \longrightarrow \mathcal{O}' \tag{N.4}$$

between two open domains in $\mathbb{R}^2 = \mathbb{C}$ is a holomorphic map.

One way to see this is with the aid of the Hodge star operator $*$, introduced in §22, which maps $\Lambda^1(M)$ to $\Lambda^1(M)$ if $\dim M = 2$. Note that, for $M = \mathbb{R}^2$, with its standard orientation and flat metric,

$$*dx = dy, \quad *dy = -dx. \tag{N.5}$$

Since the action of a map (N.4) on 1-forms is given by

$$\begin{aligned} \psi^* dx &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df, \\ \psi^* dy &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy = dg, \end{aligned} \tag{N.6}$$

if $\psi(x, y) = (f, g)$, then the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial y} \quad (\text{i.e., } *df = dg) \tag{N.7}$$

are readily seen to be equivalent to the commutativity relation

$$* \circ \big(\psi^*\big) = \big(\psi^*\big) \circ * \ \text{ on } \ 1\text{-forms}. \tag{N.8}$$

Thus Proposition N.2 is a consequence of the following:

Proposition N.3. *If M is oriented and of dimension 2, then the Hodge star operator $* : T_p^*M \to T_p^*M$ is conformally invariant.*

In fact, in this case, $*$ can be simply characterized as counterclockwise rotation by 90°, as can be seen by picking a coordinate system centered at $p \in M$ such that $g_{jk} = \delta_{jk}$ at p and using (10.5). This characterization of $*$ is clearly conformally invariant.

Thus Proposition N.1 implies that an oriented two-dimensional Riemannian manifold has an associated complex structure. A manifold of (real) dimension two with a complex structure is called a Riemann surface.

To begin the proof of Proposition N.1, we note that it suffices to show that, for any $p \in M$, there exists a neighborhood U of p and a coordinate map

$$\psi = (f, g) : U \to \mathcal{O} \subset \mathbb{R}^2 \tag{N.9}$$

which is conformal. If $df(p)$ and $dg(p)$ are linearly independent, the map (f, g) will be a coordinate map on some neighborhood of p, and (f, g) will be conformal provided

$$*df = dg. \tag{N.10}$$

Note that, if $df(p) \neq 0$, then $df(p)$ and $dg(p)$ are linearly independent. Suppose $f \in C^\infty(U)$ is given. Then, by the Poincaré lemma, if U is diffeomorphic to a disk, there will exist a $g \in C^\infty(U)$ satisfying (N.10) precisely when

$$d * df = 0. \tag{N.11}$$

Now, as we saw in §22, the Laplace operator on $C^\infty(M)$ is given by

$$\Delta f = -\delta df = - * d * df, \tag{N.12}$$

when $\dim M = 2$, so (N.11) is simply the statement that f is a *harmonic* function on U. Thus Proposition N.1 is a consequence of the following.

Proposition N.4. *There is a neighborhood U of p and a function $f \in C^\infty(U)$ such that $\Delta f = 0$ on U and $df(p) \neq 0$.*

Sketch of Proof. In a coordinate system $x = (x_1, x_2)$, we have

$$\begin{aligned}\Delta f(x) &= g(x)^{-1/2} \partial_j \big(g^{jk}(x) g(x)^{1/2} \partial_k f \big) \\ &= g^{jk}(x) \partial_j \partial_k f + b^k(x) \partial_k f.\end{aligned}$$

Pick some coordinate system centered at p, identifying the unit disk $\mathcal{D} \subset \mathbb{R}^2$ with some neighborhood U_1 of p. Now dilate the variables by a factor ε, to map the small neighborhood U_ε of p (the image of the disk $\mathcal{D}_\varepsilon$ of radius ε in the original coordinate system) onto the unit disk $\mathcal{D}$. In this dilated coordinate system, we have

$$\Delta f(x) = g^{jk}(\varepsilon x)\partial_j\partial_k f + \varepsilon b^k(\varepsilon x)\partial_k f. \tag{N.13}$$

Now we define $f = f_\varepsilon$ to be the harmonic function on U_ε equal to x_1/ε on ∂U_ε (in the original coordinate system), i.e., to x_1 on $\partial\mathcal{D}$ in the dilated coordinate system. We need only show that, for $\varepsilon > 0$ sufficiently small, we can guarantee that $df_\varepsilon(p) \neq 0$.

To see this note that, in the dilated coordinate system, we can write

$$f_\varepsilon = x_1 - \varepsilon v_\varepsilon \text{ on } \mathcal{D}, \tag{N.14}$$

where v_ε is defined by

$$\Delta_\varepsilon v_\varepsilon = b^1(\varepsilon x) \text{ on } \mathcal{D}, \quad v_\varepsilon\big|_{\partial\mathcal{D}} = 0, \tag{N.15}$$

Δ_ε being given by (N.13). Now the regularity estimates for elliptic PDE hold uniformly in $\varepsilon \in (0,1]$ in this case; see Chapter 5 of [T1] for details. Thus we have uniform estimates on v_ε in $C^1(\overline{\mathcal{D}})$ as $\varepsilon \to 0$. This shows $df_\varepsilon(p) \neq 0$ for ε small, and completes the proof.

O. Sard's theorem

Let $F : \Omega \to \mathbb{R}^n$ be a C^1 map, with Ω open in $\mathbb{R}^n$. If $p \in \Omega$ and $DF(p) : \mathbb{R}^n \to \mathbb{R}^n$ is not surjective, then p is said to be a *critical point*, and $F(p)$ a *critical value.* The set C of critical points can be a large subset of Ω, even all of it, but the set of critical values $F(C)$ must be small in $\mathbb{R}^n$. This is part of Sard's Theorem.

Theorem O.1. *If $F : \Omega \to \mathbb{R}^n$ is a C^1 map, then the set of critical values of F has measure* 0 *in* $\mathbb{R}^n$.

Proof. If $K \subset \Omega$ is compact, cover $K \cap C$ with m-dimensional cubes Q_j, with disjoint interiors, of side δ_j. Pick $p_j \in C \cap Q_j$, so $L_j = DF(p_j)$ has rank $\leq n-1$. Then, for $x \in Q_j$,

$$F(p_j + x) = F(p_j) + L_j x + R_j(x), \quad \|R_j(x)\| \leq \rho_j = \eta_j \delta_j,$$

where $\eta_j \to 0$ as $\delta_j \to 0$. Now $L_j(Q_j)$ is certainly contained in an $(n-1)$-dimensional cube of side $C_0\delta_j$, where C_0 is an upper bound for $\sqrt{m}\|DF\|$ on K. Since all points of $F(Q_j)$ are a distance $\leq \rho_j$ from (a translate of) $L_j(Q_j)$, this implies

$$\text{meas } F(Q_j) \leq 2\rho_j(C_0\delta_j + 2\rho_j)^{n-1} \leq C_1 \eta_j \delta_j^n,$$

provided δ_j is sufficiently small that $\rho_j \leq \delta_j$. Now $\sum_j \delta_j^n$ is the volume of the cover of $K \cap C$. For fixed K this can be assumed to be bounded. Hence

$$\text{meas } F(C \cap K) \leq C_K \ \eta,$$

where $\eta = \max\ \{\eta_j\}$. Picking a cover by small cubes, we make η arbitrarily small, so meas $F(C \cap K) = 0$. Letting $K_j \nearrow \Omega$, we complete the proof.

Sard's theorem also treats the more difficult case when Ω is open in $\mathbb{R}^m, m > n$. Then a more elaborate argument is needed, and one requires more differentiability, namely that F is class C^k, with $k = m - n + 1$. A proof can be found in Sternberg [Stb]. The theorem also clearly extends to smooth mappings between separable manifolds.

Theorem O.1 is applied in §9, in the study of degree theory. We give another application of Theorem O.1, to the existence of lots of Morse functions. This application gives the typical flavor of how one uses Sard's theorem, and it is used in a Morse theory argument in §20. We begin with a special case:

Proposition O.2. *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in C^\infty(\Omega)$. For $a \in \mathbb{R}^n$, set $f_a(x) = f(x) - a \cdot x$. Then, for almost every $a \in \mathbb{R}^n$, f_a is a Morse function, i.e., it has only nondegenerate critical points.*

Proof. Consider $F(x) = \nabla f(x); F : \Omega \to \mathbb{R}^n$. A point $x \in \Omega$ is a critical point of f_a if and only if $F(x) = a$, and this critical point is degenerate only if, in addition, a is a critical value of F. Hence the desired conclusion holds for all $a \in \mathbb{R}^n$ that are not critical values of F.

Now for the result on manifolds:

Proposition O.3. *Let M be an n-dimensional manifold, imbedded in $\mathbb{R}^K$. Let $f \in C^\infty(M)$, and, for $a \in \mathbb{R}^K$, let $f_a(x) = f(x) - a \cdot x$, for $x \in M \subset \mathbb{R}^K$. Then, for almost all $a \in \mathbb{R}^K$, f_a is a Morse function.*

Proof. Each $p \in M$ has a neighborhood Ω_p such that some n of the coordinates x_ν on $\mathbb{R}^K$ produce coordinates on Ω_p. Let's say $x_1, \dots, x_n$ do it. Let $(a_{n+1}, \dots, a_K)$ be fixed, but arbitrary. Then, by Proposition O.2, for almost every $(a_1, \dots, a_n) \in \mathbb{R}^n$, f_a has only nondegenerate critical points on Ω_p. By Fubini's theorem, we deduce that, for almost every $a \in \mathbb{R}^K$, f_a has only nondegenerate critical points on Ω_p. (The set of bad $a \in \mathbb{R}^K$ is readily seen to be a countable union of closed sets, hence measurable.) Covering M by a countable family of such sets Ω_p, we finish the proof.

P. Variational property of the Einstein tensor

In this appendix, we calculate the variation of the integral of scalar curvature, with respect to the metric.

Theorem P.1. *If M is a manifold with nondegenerate metric tensor (g_{jk}), associated Einstein tensor $G_{jk} = Ric_{jk} - (1/2)Sg_{jk}$, scalar curvature S, and volume element dV, then, with respect to a compactly supported variation of the metric we have*

$$\delta \int S\, dV = \int G_{jk}\ \delta g^{jk}\ dV = -\int G^{jk}\ \delta g_{jk}\ dV. \tag{P.1}$$

To establish this, we first obtain formulas for the variation of the Riemann curvature tensor, then of the Ricci tensor and the scalar curvature. Let $\Gamma^i{}_{jk}$ be the connection coefficients. Then $\delta\Gamma^i{}_{jk}$ is a tensor field. The formula (15.62) states that, if $\widetilde{R}$ and R are the curvatures of the connections $\widetilde{\nabla}$ and $\nabla = \widetilde{\nabla} + \varepsilon C$, then

$$(R - \widetilde{R})(X, Y)u = \varepsilon(\widetilde{\nabla}_X C)(Y, u) - \varepsilon(\widetilde{\nabla}_Y C)(X, u) + \varepsilon^2 [C_X, C_Y]u. \tag{P.2}$$

It follows that

$$\delta R^i{}_{jk\ell} = \delta\Gamma^i{}_{j\ell;k} - \delta\Gamma^i{}_{jk;\ell}. \tag{P.3}$$

Contracting, we obtain

$$\delta\ \mathrm{Ric}_{jk} = \delta\Gamma^i{}_{ji;k} - \delta\Gamma^i{}_{jk;i}. \tag{P.4}$$

Another contraction yields

$$g^{jk}\ \delta\ \mathrm{Ric}_{jk} = \bigl(g^{jk}\ \delta\Gamma^\ell{}_{j\ell}\bigr)_{;k} - \bigl(g^{jk}\ \delta\Gamma^\ell{}_{jk}\bigr)_{;\ell} \tag{P.5}$$

since the metric tensor has vanishing covariant derivative. The identities (P.3)–(P.5) are called "Palatini identities."

Note that the right side of (P.5) is the divergence of a vector field. This will be significant for our calculation of (P.1). By the Divergence Theorem, it implies that

$$\int g^{jk}\bigl(\delta\ \mathrm{Ric}_{jk}\bigr)\ dV = 0, \tag{P.6}$$

as long as δg_{jk} (hence $\delta\ \mathrm{Ric}_{jk}$) is compactly supported.

We now compute the left side of (P.1). Since $S = g^{jk}\ \mathrm{Ric}_{jk}$, we have

$$\delta S = \mathrm{Ric}_{jk}\ \delta g^{jk} + g^{jk}\ \delta\ \mathrm{Ric}_{jk}. \tag{P.7}$$

Now we have

$$dV = \sqrt{|g|}\ dx \Longrightarrow \delta(dV) = -\tfrac{1}{2} g_{jk}\ \delta g^{jk}\ dV, \tag{P.8}$$

since $\det(A + \varepsilon B) = \det(A)\ \det(I + \varepsilon A^{-1}B) = \det(A)\big(1 + \varepsilon\ \mathrm{Tr}(A^{-1}B) + O(\varepsilon^2)\big)$. Hence

$$\begin{aligned} \delta(S\ dV) &= \mathrm{Ric}_{jk}\ \delta g^{jk}\ dV + g^{jk}(\delta\ \mathrm{Ric}_{jk})\ dV + S\ \delta(dV) \\ &= \big(\mathrm{Ric}_{jk} - \tfrac{1}{2} S g_{jk}\big)\delta g^{jk}\ dV + g^{jk}(\delta\ \mathrm{Ric}_{jk})\ dV. \end{aligned} \tag{P.9}$$

The last term integrates to zero, by (P.6), so we have (P.1).

Note that verifying (P.1) did not require computation of $\delta\Gamma^i{}_{jk}$ in terms of δg_{jk}, though this can be done explicitly. Indeed, formula (15.63) implies

$$\delta\Gamma_{\ell jk} = \tfrac{1}{2}\Big[\delta g_{\ell j;k} - \delta g_{\ell k;j} + \delta g_{jk;\ell}\Big]. \tag{P.10}$$

If $\dim M = 2$, then, as shown by (15.26), G^{jk} is identically zero. Thus (P.1) has the following implication (since $S = 2K$ in this case).

Corollary P.2. *If M is a compact Riemannian manifold of dimension 2, then the integrated Gauss curvature*

$$\int_M K\ dV = C(M) \tag{P.11}$$

is independent of the choice of Riemannian metric tensor on M.

This is a proof of (17.26), different from the other proofs given in §17. It thus leads to another proof of the Gauss-Bonnet theorem for compact, orientable 2-dimensional Riemannian manifolds, for example by following (P.11) with reasoning used in (17.27)–(17.31), or alternatively, giving M a metric arising from an embedding in $\mathbb{R}^3$, and using (17.48)–(17.49).

Q. A generalized Gauss map

Let $j : M \hookrightarrow \mathbb{R}^{n+1}$ be a compact, connected hypersurface, with outward pointing normal N. Let $k : M \hookrightarrow \mathbb{R}^{n+1}$ be given by $k(x) = j(x) + sN(x)$, for fixed small s. Define $\tau : M \times M \to S^n$ by

$$\tau(x, y) = \frac{k(x) - j(y)}{|k(x) - j(y)|}. \tag{Q.1}$$

Thus $\tau(x, x) = N(x)$. Let $\omega \in \Lambda^n S^n$ be a normalized volume element, so $\int_{S^n} \omega = 1$. Then we have

$$[\tau^*\omega] \in H^n(M \times M) \approx \bigoplus_{j=0}^{n} H^j(M) \otimes H^{n-j}(M), \tag{Q.2}$$

via the Kunneth formula. We want to tackle the following:

Problem. *Identify the components*

$$(\tau^*\omega)_{j,n-j} \in H^j(M) \otimes H^{n-j}(M). \tag{Q.3}$$

To begin the analysis, and also to highlight the significance of making such an explicit identification, we note that, if $p \in M$ is fixed and

$$i_\nu : M \to M \times M, \quad i_1(x) = (x, p),\ i_2(x) = (p, x), \tag{Q.4}$$

then, by degree theory (compare (9.18)), we have:

$$i_1^*(\tau^*\omega)[M] = 1, \quad i_2^*(\tau^*\omega) = 0. \tag{Q.5}$$

Here and below, $\beta[M]$ denotes $\int_M \beta$. To restate (Q.5), we have

$$(\tau^*\omega)_{n,0} = \mu \otimes 1 \in H^n(M) \otimes H^0(M), \text{ with } \mu[M] = 1, \tag{Q.6}$$

and

$$(\tau^*\omega)_{0,n} = 0 \in H^0(M) \otimes H^n(M). \tag{Q.7}$$

Next, note that, if $\delta : M \to M \times M$ is the diagonal map, $\delta(x) = (x, x)$, then $\tau \circ \delta : M \to S^n$ is the Gauss map. Thus, by Hopf's special case of the generalized Gauss-Bonnet theorem, discussed in the early part of §20,

$$n \text{ even} \Longrightarrow \delta^*(\tau^*\omega)[M] = \frac{1}{2}\chi(M). \tag{Q.8}$$

Now $\delta^* : H^n(M \times M) \to H^n(M)$ is given by the cup product on each part of the Kunneth decomposition:

$$\cup : H^j(M) \otimes H^{n-j}(M) \longrightarrow H^n(M). \tag{Q.9}$$

Thus, when n is even, we have

$$\sum_{j=0}^{n} \cup (\tau^* \omega)_{j,n-j}[M] = \frac{1}{2}\chi(M). \tag{Q.10}$$

Another consequence of (Q.6)–(Q.7) is that

$$n = 1 \Longrightarrow \delta^*(\tau^*\omega)[M] = 1. \tag{Q.11}$$

This is equivalent to a result of Hopf regarding the index of a vector field with a closed orbit. We can prove it by noting that, when $n = 1$, then

$$\delta^*(\tau^*\omega)[M] = (\tau^*\omega)_{1,0}[M] + (\tau^*\omega)_{0,1}[M] = 1 + 0 = 1. \tag{Q.12}$$

This result was established in §9 by a different argument; see Proposition 9.12.

Let's look a little closer at the case dim $M = 2$. Then (Q.10) is a sum of three terms, two of which are specified by (Q.6)–(Q.7). Hence

$$\cup(\tau^*\omega)_{1,1}[M] = \frac{1}{2}\chi(M) - 1 = -\frac{1}{2}\ \dim\, H^1(M). \tag{Q.13}$$

This gives partial information on $(\tau^*\omega)_{1,1} \in H^1(M) \otimes H^1(M)$, in this case. Note that, if c_1 and c_2 are closed curves in M, defining cycles for $H_1(M)$, then

$$(\tau^*\omega)_{1,1}(c_1 \otimes c_2) = \int\limits_{c_1 \times c_2} \tau^*\omega = \text{ linking } \#\big(k(c_1), j(c_2)\big), \tag{Q.14}$$

where the last identity defines the *linking number* of the curves in $k(c_1)$ and $j(c_2)$ in $\mathbb{R}^3$. For short, let us denote this quantity by $\lambda(c_1 \otimes c_2)$.

While the quantity (Q.13) is independent of the imbedding of M, $(\tau^*\omega)_{1,1}$ itself can depend on the imbedding, as illustrated by two surfaces of genus 2 in $\mathbb{R}^3$, in Figures Q.1 and Q.2. The quantities $\lambda(b_1 \otimes b_2)$ and $\lambda(b_2 \otimes b_1)$ are different in the two cases.

One might consider further generalizations, such as

$$\tau : M_1 \times M_2 \longrightarrow S^n$$

given by (Q.1), for general $M_j \subset \mathbb{R}^{n+1}$ such that $M_1 \cap M_2 = \emptyset$.

R. Moser's area preservation result

Given two Riemannian manifolds which are diffeomorphic, typically there is no isometry taking one to another. On the other hand, as long as the two volumes are the same (and the manifolds are compact, connected, and orientable) the following result of J.Moser shows that there is a *volume preserving* diffeomorphism from one to another.

We define a "volume form" on an oriented smooth manifold M of dimension n to be a nowhere vanishing n-form on M, determining its orientation.

Proposition R.1. *Let ω_0 and ω_1 be two volume forms on a compact, connected, oriented manifold M. If*

$$\int_M \omega_0 = \int_M \omega_1 \tag{R.1}$$

then there is a diffeomorphism $F : M \to M$ such that

$$F^*\omega_1 = \omega_0. \tag{R.2}$$

Proof. Taking convex linear combinations, we have a smooth family ω_t of nowhere vanishing n-forms $\omega_t,\ 0 \le t \le 1$, all with the same integral. We will construct a 1-parameter family F_t of diffeomorphisms on M, such that $F_t^*\omega_t = \omega_0$ for each t; then taking $F = F_1$ gives the desired result.

We will obtain F_t as the flow of a t-dependent family of vector fields X_t on M, i.e.,

$$\frac{d}{dt}F_t(x) = X_t\bigl(F_t(x)\bigr), \quad F_0(x) = x. \tag{R.3}$$

It remains to construct the family X_t in such a fashion that $F_t^*\omega_t$ is independent of t. Now,

$$\frac{d}{dt}\bigl(F_t^*\omega_t\bigr) = F_t^*\Bigl(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\Bigr), \tag{R.4}$$

so we want to find X_t with the property that

$$\mathcal{L}_{X_t}\omega_t = -\omega_t', \tag{R.5}$$

where $\omega_t' = d\omega_t/dt$. Note that

$$\int_M \omega_t' = 0, \quad \forall\ t. \tag{R.6}$$

Thus we can set

$$\omega_t' = d\alpha_t, \quad \alpha_t = \delta G \omega_t', \tag{R.7}$$

using the Hodge decomposition (22.19) (for any metric on M, not necessarily related to ω_t). Meanwhile, the left side of (R.5) is given by

$$\mathcal{L}_{X_t}\omega_t = (d\omega_t)\rfloor X_t + d\big(\omega_t \rfloor X_t\big) = d\big(\omega_t \rfloor X_t\big). \tag{R.8}$$

Thus (R.5) holds if X_t has the property

$$\omega_t \rfloor X_t = -\alpha_t. \tag{R.9}$$

In (R.9), we are given the nowhere vanishing n-form ω_t and the $(n-1)$-form α_t. Thus X_t is uniquely specified by (R.9), and this makes (R.4) vanish. This proves the proposition.

We remark that Hodge theory can be avoided here. A careful examination of the proof of Proposition 9.5 enables one to construct an operator $T : \Lambda^n(M) \to \Lambda^{n-1}(M)$ such that $\int \beta = 0$ implies $\beta = d(T\beta)$, and one can use this T instead of δG in (R.7).

There is an analogous result for compact manifolds with boundary.

Proposition R.2. *Let ω_0 and ω_1 be two volume forms on a compact, connected, oriented manifold with boundary $\overline{M}$. If (R.1) holds, then there is a diffeomorphism $F : \overline{M} \to \overline{M}$ satisfying (R.2).*

Proof. Of course, if $\partial M \neq \emptyset$, we want $F : \partial M \to \partial M$. We produce F_t as before, by specifying a smooth family of vector fields X_t on $\overline{M}$ such that (R.4) vanishes. The only extra condition we have to impose is that each X_t be *tangent* to ∂M.

We get this using the Hodge decomposition (23.21) for *relative cohomology.* As noted in (23.49), under our hypotheses on $\overline{M}$,

$$\mathcal{H}^n(\overline{M}, \partial M) \approx \mathbb{R}, \tag{R.10}$$

if $n = \dim M$. We fix a Riemannian metric on $\overline{M}$ and then set

$$\alpha_t = \delta G^R \omega_t', \tag{R.11}$$

which implies $d\alpha_t = \omega_t'$ since $\int_M \omega_t' = 0$. Once we have α_t, then as above X_t is uniquely specified by the identity (R.9). By the characterization (23.7) of relative boundary conditions, we see that

$$j^*\alpha_t = 0, \quad j : \partial M \hookrightarrow \overline{M}. \tag{R.12}$$

It follows that the vector field X_t defined by (R.12) is tangent to ∂M, so the proof is done.

S. The Poincaré disc and Ahlfors' inequality

The Poincaré disc is the disc $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, with metric

$$g_{ij} = \frac{4}{(1-r^2)^2}\delta_{ij}. \tag{S.1}$$

This has Gauss curvature -1, as a consequence of the formula (15.41), which says

$$g_{ij} = e^{2v}\delta_{ij} \Longrightarrow K(x) = -(\Delta v)e^{-2v}, \tag{S.2}$$

for any 2-dimensional surface, where

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \tag{S.3}$$

One reason this metric is important is that it is invariant under all "Möbius transformations," i.e., maps $F : D \to D$ of the form

$$F(z) = \frac{az + b}{\overline{b}z + \overline{a}}, \quad |a|^2 - |b|^2 = 1, \tag{S.4}$$

where we identify (x, y) with $z = x + iy$.

Proposition S.1. *When the metric tensor g is given by (S.1),*

$$F^* g = g \tag{S.5}$$

for all maps $F : D \to D$ of the form (S.4).

We leave as an exercise the proof of this result, and also of the next result, to the effect that the set $\mathcal{M}$ of Möbius transformations acts *transitively* on D.

Proposition S.2. *For each pair $p, q \in D$, there exists $F \in \mathcal{M}$ such that $F(p) = q$.*

Note that, together, Propositions S.1 and S.2 imply that the Gauss curvature of the Poincaré disc must be constant.

The following result, discovered by L. Ahlfors, is of great utility in complex function theory. Let Ω be a planar region with a metric tensor (h_{ij}) which is also a conformal multiple of the Euclidean metric:

$$h_{ij} = e^{2w}\delta_{ij}. \tag{S.6}$$

Theorem S.3. *Assume that (Ω, h) has Gauss curvature $K(x) \le -1$ everywhere. If $F : D \to \Omega$ is a conformal map (in particular, if F is holomorphic) then it is distance decreasing, i.e.,*

$$(F^*h)_{ij} = e^{2u}\delta_{ij} \Longrightarrow e^{2u} \le e^{2v} = \frac{4}{(1-r^2)^2}. \tag{S.7}$$

Proof. Replacing F by $F_\rho(z) = F(\rho z)$, $\rho \nearrow 1$, we see that it suffices to show that $F_\rho^* h$ is dominated by the Poincaré metric for all such ρ. Thus, it suffices to show that (S.7) holds when u is *bounded* on D. Since $e^{2v} \to \infty$ on ∂D, it follows that, in such a case, we have $e^{2u}/e^{2v} \to 0$ on ∂D, so this quotient has a maximum inside D, say at a point z_0.

Thus, we will have (S.7) if we show that $e^{2u(z_0)} \le e^{2v(z_0)}$. To establish this, note that

$$0 \ge \Delta \log\Big(\frac{e^{2u}}{e^{2v}}\Big)(z_0) = 2\Delta u(z_0) - 2\Delta v(z_0). \tag{S.8}$$

Now, our curvature hypothesis is equivalent to $-(\Delta u)e^{-2u} \le -1$, while the computation that the Poincaré metric has curvature -1 is equivalent to $(-\Delta v)e^{-2v} = -1$, so

$$\Delta v = e^{2v}, \quad \Delta u \ge e^{2u}. \tag{S.9}$$

Combining (S.8) and (S.9), we have $e^{2u(z_0)} \le \Delta u(z_0) \le \Delta v(z_0) = e^{2v(z_0)}$, so Theorem S.3 is proved.

One of the most important examples of such (Ω, h) as in Theorem S.3 is

$$\Omega = \mathbb{C} \setminus \{0,1\} = \mathbb{R}^2 \setminus \{(0,0),(1,0)\}. \tag{S.10}$$

given a metric of the form (S.6), with

$$e^{2w(z)} = A\ \frac{1+|z|^{1/3}}{|z|^{5/3}} \cdot \frac{1+|z-1|^{1/3}}{|z-1|^{5/3}}. \tag{S.11}$$

A calculation using (S.2) shows that, for any $A > 0$, the Gauss curvature of Ω satisfies $K_A(x) \le -\alpha < 0$, and if A is small enough then $K_A(x) \le -1$. Using this, we can establish the following result, known as Picard's Little Theorem:

Theorem S.4. *If $F : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ is an entire holomorphic function, then F is constant.*

Proof. Consider the functions $f_r : D \to \mathbb{C} \setminus \{0,1\}$ given by $f_r(z) = F(rz)\big|_D$. Each f_r is distance decreasing, so $|f_r'(0)|$ must be uniformly bounded as $r \to \infty$. Since $f_r'(0) = rF'(0)$, this implies $F'(0) = 0$. Similarly considering $F(rz+\zeta)$, we get $F'(\zeta) = 0$ for all $\zeta \in \mathbb{C}$, and the proof is complete.

Further discussion of Ahlfors' inequality, including a proof of Picard's Big Theorem, can be found in [Kran].

We remark that, using the results of §N, we can extend Theorem S.3 to the case where Ω is any 2-dimensional surface with Gauss curvature ≤ -1, and hence to the case where Ω is any Riemannian manifold with sectional curvature ≤ -1.

T. Rigid body motion in $\mathbb{R}^n$ and geodesics on $SO(n)$

Suppose there is a rigid body in $\mathbb{R}^n$, with a mass distribution at $t = 0$ given by a function $\rho(x)$, which we will assume is piecewise continuous and has compact support. Suppose the body moves, subject to no external forces, only the constraint of being rigid; we want to describe the motion of the body. According to the Lagrangian approach to mechanics, we seek an extremum of the integrated kinetic energy, subject to this constraint. For more on this approach to the equations of classical mechanics, see [AbM] and [Ar].

If $\xi(t,x)$ is the position in $\mathbb{R}^n$ at time t of the point on the body whose position at time 0 is x, then we can write the "Lagrangian" as

$$I(\xi) = \frac{1}{2}\int_{t_0}^{t_1}\int\limits_{\mathbb{R}^n} \rho\bigl(\xi(t,x)\bigr)|\dot{\xi}(t,x)|^2\; dx\; dt. \tag{T.1}$$

Here, $\dot{\xi}(t,x) = \partial\xi/\partial t$.

Using center of mass coordinates, we will assume that the center of mass of the body is at the origin, and its total linear momentum is zero, so

$$\xi(t,x) = W(t)x, \quad W(t) \in SO(n), \tag{T.2}$$

where $SO(n)$ is the group of rotations of $\mathbb{R}^n$. Thus, describing the motion of the body becomes the problem of specifying the curve $W(t)$ in $SO(n)$. We can write (T.1) as

$$\begin{aligned} I(\xi) &= \frac{1}{2}\int_{t_0}^{t_1}\int\limits_{\mathbb{R}^n} \rho\bigl(W(t)x\bigr)|W'(t)x|^2\; dx\; dt \\ &= \frac{1}{2}\int_{t_0}^{t_1}\int\limits_{\mathbb{R}^n} \rho(y)|W'(t)W(t)^{-1}y|^2\; dy\; dt \\ &= J(W). \end{aligned} \tag{T.3}$$

We look for an extremum, or other critical point, where we vary the family of paths $W : [t_0,t_1] \to SO(n)$ (keeping the endpoints fixed).

Let us reduce the formula (T.3) for $J(W)$ to a single integral, over t. In fact, we have the following.

Lemma T.1. *If A and B are real $n \times n$ matrices, i.e., belong to $M(n,\mathbb{R})$, then*

$$\int \rho(y)\;(Ay,By)\; dy = Tr(B^t A\mathcal{I}_\rho) = Tr(A\mathcal{I}_\rho B^t), \tag{T.4}$$

where

$$\mathcal{I}_\rho = \int \rho(y)\ y \otimes y\ dy \in \bigotimes^2 \mathbb{R}^n \approx M(n, \mathbb{R}). \tag{T.5}$$

Proof. Both sides of (T.4) are linear in A and in B, and the formula is easily verified for $A = E_{ij},\ B = E_{k\ell}$, where E_{ij} has a 1 in the ith column and jth row, and zeros elsewhere.

Recall that we are assuming that the body's center of mass is at 0 and its total linear momentum vanishes (at $t = 0$), i.e.,

$$\int \rho(y) y\ dy = 0. \tag{T.6}$$

By (T.4), we can write the Lagrangian (T.3) as

$$\begin{aligned} J(W) &= \frac{1}{2} \int_{t_0}^{t_1} \operatorname{Tr} \left(W'(t) W(t)^{-1} \mathcal{I}_\rho (W'(t) W(t)^{-1})^t \right)\ dt \\ &= \frac{1}{2} \int_{t_0}^{t_1} \operatorname{Tr} \left(Z(t) \mathcal{I}_\rho Z(t)^t \right) dt, \end{aligned} \tag{T.7}$$

where

$$Z(t) = W'(t) W(t)^{-1}. \tag{T.8}$$

Note that, if $W : (t_0, t_1) \to SO(n)$ is smooth, then

$$Z(t) \in so(n), \quad \forall\ t \in (t_0, t_1). \tag{T.9}$$

where $so(n)$ is the set of real antisymmetric $n \times n$ matrices, the Lie algebra of $SO(n)$; in particular $so(n)$ is the tangent space to $SO(n) \subset M(n, \mathbb{R})$ at the identity element.

Now we can define an inner product on $T_g SO(n)$ for general $g \in SO(n) = G$ by

$$\begin{aligned} \langle U, V \rangle_g &= \langle U g^{-1}, V g^{-1} \rangle_I \\ &= \operatorname{Tr} \left(U g^{-1} \mathcal{I}_\rho (V g^{-1})^t \right) \\ &= \operatorname{Tr} \left(U g^{-1} \mathcal{I}_\rho g V^t \right), \end{aligned} \tag{T.10}$$

where, as in (T.9), $V \in T_g SO(n) \Rightarrow V g^{-1} \in so(n)$. Note that

$$\lambda_g V = gV, \quad \rho_g V = V g^{-1}, \quad g \in SO(n) \tag{T.11}$$

define actions of $G = SO(n)$:

$$\lambda_g : T_{\tilde{g}} G \to T_{g\tilde{g}} G, \quad \rho_g : T_{\tilde{g}} \to T_{\tilde{g} g^{-1}} G, \tag{T.12}$$

satisfying $\lambda_{g_1 g_2} = \lambda_{g_1}\lambda_{g_1}$ and $\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}$. The inner product (T.10) satisfies

$$\langle U, V\rangle_g = \langle \rho(h)U, \rho(h)V\rangle_{gh^{-1}}, \quad \forall\ g, h \in G,\ U, V \in T_g G. \tag{T.13}$$

Thus we have a *right invariant* Riemannian metric on $SO(n)$, defined by (T.10). If $\mathcal{I}_\rho$ is a scalar multiple of the identity, it will also be left invariant, since, for $U, V \in T_g G$,

$$\langle \lambda(g^{-1})U, \lambda(g^{-1})V\rangle_I = \operatorname{Tr}\left(g^{-1}U\mathcal{I}_\rho V^t g\right) = \operatorname{Tr}\left(U\mathcal{I}_\rho V^t\right), \tag{T.14}$$

which is equal to (T.10) provided $g^{-1}\mathcal{I}_\rho g = \mathcal{I}_\rho$.

Returning to (T.7), we see that

$$J(W) = \frac{1}{2}\int_{t_0}^{t_1} \big\langle W'(t), W'(t)\big\rangle_{W(t)}\ dt. \tag{T.15}$$

The discussion in (11.25)–(11.27) shows that the stationary condition for (T.15), for a family of curves in $SO(n)$, is precisely the condition that $W(t)$ be a *geodesic* in $SO(n)$, endowed with the right invariant Riemannian matric defined in (T.10), via the tensor $\mathcal{I}_\rho$.

We now pursue the stationary condition for (T.7) directly, independently of its connection with geodesic motion. Let W_s be a one parameter family of curves in $SO(n)$, with endpoints (at t_1 and t_2) fixed, such that $W_0 = W$. The variation

$$\partial_s W_s(t)\big|_{s=0} = X(t) \tag{T.16}$$

is a curve in $M(n, \mathbb{R})$, belonging to $T_{W(t)}G$ for each t, so

$$X(t)W(t)^{-1} = Y(t) \in so(n). \tag{T.17}$$

We have

$$\begin{aligned} \frac{d}{ds}J(W_s)\big|_{s=0} = \frac{1}{2}\int_{t_0}^{t_1} \operatorname{Tr}\Big(&X'W^{-1}\mathcal{I}_\rho(W'W^{-1})^t \\ &- W'W^{-1}XW^{-1}\mathcal{I}_\rho(W'W^{-1})^t \\ &+ W'W^{-1}\mathcal{I}_\rho X W'^t + W'W^{-1}\mathcal{I}_\rho W X'^t\Big)\ dt. \end{aligned} \tag{T.18}$$

Recalling that $W'W^{-1} = Z$, we note that $X'W^{-1} = Y' + YZ$; hence

$$\frac{d}{ds}J(W_s)\big|_{s=0} = \frac{1}{2}\int_{t_0}^{t_1} \operatorname{Tr}\Big(-2(Y' + YZ)\mathcal{I}_\rho Z + Y\mathcal{I}_\rho Z^2 - YZ^2\mathcal{I}_\rho\Big)\ dt. \tag{T.19}$$

If we integrate by parts, we see that the stationary condition is

$$\operatorname{Tr}\Big(Y(2\mathcal{I}_\rho Z' - 2Z\mathcal{I}_\rho Z + \mathcal{I}_\rho Z^2 - Z^2\mathcal{I}_\rho)\Big) = 0 \tag{T.20}$$

for all $t \in (t_0, t_1)$, $Y \in so(n)$. Now, given $A \in M(n, \mathbb{R})$, $\operatorname{Tr}(YA) = 0$ for all $Y \in so(n)$ if and only if $A = A^t$. Thus the stationary condition is

$$\mathcal{A} - \mathcal{A}^t = 0, \tag{T.21}$$

where

$$\mathcal{A} = 2\mathcal{I}_\rho Z' - 2Z\mathcal{I}_\rho Z + \mathcal{I}_\rho Z^2 - Z^2\mathcal{I}_\rho, \tag{T.22}$$

so

$$\tfrac{1}{2}(\mathcal{A} - \mathcal{A}^t) = \mathcal{I}_\rho Z' + Z'\mathcal{I}_\rho + [\mathcal{I}_\rho, Z^2], \tag{T.23}$$

i.e., the stationary condition is that Z satisfy the evolution equation

$$\mathcal{I}_\rho Z' + Z'\mathcal{I}_\rho = -[\mathcal{I}_\rho, Z^2]. \tag{T.24}$$

Note that, if we solve the first order nonlinear system (T.24), with initial condition $Z(0) = W'(0)$, then we can solve for $W(t)$ the linear system $W'(t) = Z(t)W(t)$, arising from (T.8), with initial condition $W(0) = I$.

The equation (T.24) makes it natural to bring in the quantity

$$M(t) = \mathcal{I}_\rho Z(t) + Z(t)\mathcal{I}_\rho. \tag{T.25}$$

We have $M' = \mathcal{I}_\rho Z' + Z'\mathcal{I}_\rho$, and

$$[M, Z] = [\mathcal{I}_\rho, Z^2]. \tag{T.26}$$

Hence the stationary condition is equivalent to

$$\frac{dM}{dt} = -[M, Z]. \tag{T.27}$$

Note that the inner product $\langle U, V\rangle_I = \operatorname{Tr}(U\mathcal{I}_\rho V^t)$ in (T.10) defines a linear transformation

$$\mathcal{L}_\rho : so(n) \longrightarrow so(n) \tag{T.28}$$

by

$$\operatorname{Tr}\big(\mathcal{L}_\rho(U)V^t\big) = \operatorname{Tr}\big(U\mathcal{I}_\rho V^t\big), \quad U, V \in so(n), \tag{T.29}$$

i.e., $\mathcal{L}_\rho(U)$ is the skew-adjoint part of $U\mathcal{I}_\rho$:

$$\mathcal{L}_\rho(U) = \frac{1}{2}\big(U\mathcal{I}_\rho + \mathcal{I}_\rho U\big). \tag{T.30}$$

Hence, in (T.25), $M(t) = 2\mathcal{L}_\rho\bigl(Z(t)\bigr)$.

In case $n = 3$, we have an isomorphism $\kappa : \mathbb{R}^3 \to so(3)$ given by

$$\kappa(\omega_1, \omega_2, \omega_3) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{T.31}$$

as in (J.17), so that the cross product on $\mathbb{R}^3$ satisfies $\omega \times x = Ax,\ A = \kappa(\omega)$. Then the vector-valued function

$$\omega(t) = -\kappa^{-1} Z(t) \tag{T.32}$$

is called the angular velocity of the body. Note that

$$(\omega \times y, \omega \times y) = (A_y^t A_y \omega, \omega), \tag{T.33}$$

where $A_y = \kappa(y)$, and a calculation gives

$$A_y^t A_y = |y|^2 I - y \otimes y. \tag{T.34}$$

Consequently, the Lagrangian integral (T.7) is equal to

$$J(W) = \frac{1}{2} \int_{t_0}^{t_1} \bigl(\mathcal{J}_\rho \omega(t), \omega(t)\bigr)\, dt, \tag{T.35}$$

where

$$\mathcal{J}_\rho = \int \rho(y) \bigl[|y|^2 I - y \otimes y\bigr]\, dy = (\mathrm{Tr}\ \mathcal{I}_\rho) I - \mathcal{I}_\rho. \tag{T.36}$$

As noted in (J.18)–(J.19), the isomorphism κ has the properties

$$\kappa(x \times y) = [\kappa(x), \kappa(y)], \quad \mathrm{Tr}\,\bigl(\kappa(x)\kappa(y)^t\bigr) = 2x \cdot y. \tag{T.37}$$

Furthermore, a calculation shows that, if $\mathcal{L}_\rho$ is given by (T.29)–(T.30), then

$$\kappa^{-1} \mathcal{L}_\rho \kappa(\omega) = \frac{1}{2} \mathcal{J}_\rho \omega, \tag{T.38}$$

so, with $M(t) \in so(3)$ defined by (T.25), we have

$$\mu(t) = -\kappa^{-1} M(t) = \mathcal{J}_\rho \omega(t), \tag{T.39}$$

the first identity defining $\mu(t) \in \mathbb{R}^3$. The equation (T.27) is then equivalent to

$$\frac{d\mu}{dt} = -\omega \times \mu. \tag{T.40}$$

The vector $\mu(t)$ is called the angular momentum of the body, and $\mathcal{J}_\rho$ is called the inertia tensor. The equation (T.40) is the standard form of Euler's equation for the free motion of a rigid body in $\mathbb{R}^3$.

Note that $\mathcal{J}_\rho$ is a positive definite 3×3 matrix. Let us choose a positively oriented orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $\mathcal{J}_\rho$, say $\mathcal{J}_\rho e_j = J_j e_j$. Then, if $\omega = (\omega_1, \omega_2, \omega_3)$, we have $\mu = (J_1\omega_1, J_2\omega_2, J_3\omega_3)$, and

$$\omega \times \mu = \big((J_3 - J_2)\omega_2\omega_3, (J_1 - J_3)\omega_1\omega_3, (J_2 - J_1)\omega_1\omega_2\big).$$

Hence, (T.40) takes the form

$$\begin{aligned} J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 &= 0 \\ J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3 &= 0 \\ J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 &= 0. \end{aligned} \tag{T.41}$$

If we multiply the ℓth line in (T.41) by $\dot{\omega}_\ell$ and sum over ℓ, we get $(d/dt)(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2) = 0$, while if instead we multiply by $J_\ell\dot{\omega}_\ell$ and sum, we get $(d/dt)(J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2) = 0$. Thus we have the conserved quantities

$$\begin{aligned} J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2 &= C_1, \\ J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 &= C_2. \end{aligned} \tag{T.42}$$

If any of the quantities J_ℓ coincide, the system (T.41) simplifies. If, on the other hand, we assume that $J_1 < J_2 < J_3$, then we can write the system (T.41) as

$$\dot{\omega}_2 = \beta^2\omega_1\omega_3, \quad \dot{\omega}_1 = -\alpha^2\omega_2\omega_3, \quad \dot{\omega}_3 = -\gamma^2\omega_1\omega_2, \tag{T.43}$$

where

$$\alpha^2 = \frac{J_3 - J_2}{J_1}, \quad \beta^2 = \frac{J_3 - J_1}{J_2}, \quad \gamma^2 = \frac{J_2 - J_1}{J_3}. \tag{T.44}$$

If we then set

$$\zeta_1 = \alpha\omega_2, \quad \zeta_2 = \beta\omega_1, \quad \zeta_3 = \alpha\beta\omega_3, \tag{T.45}$$

this system becomes

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2\zeta_3 \\ \dot{\zeta}_2 &= -\zeta_1\zeta_3 \\ \dot{\zeta}_3 &= -\gamma^2\zeta_1\zeta_2. \end{aligned} \tag{T.46}$$

For this system we have conserved quantities

$$\zeta_1^2 + \zeta_2^2 = c_1, \quad \gamma^2\zeta_1^2 + \zeta_3^2 = c_2, \tag{T.47}$$

a fact which is equivalent to (T.42). (We mention that arranging that $J_1 < J_2 < J_3$ might change the orientation, hence the sign in (T.40).)

Note that we can use (T.47) to decouple the system (T.46), obtaining

$$\begin{aligned}
\dot{\zeta}_1 &= \bigl[(c_1 - \zeta_1^2)(c_2 - \gamma^2\zeta_1^2)\bigr]^{1/2} \\
\dot{\zeta}_2 &= -\bigl[(c_1 - \zeta_2^2)(c_2 - \gamma^2 c_1 + \zeta_2^2)\bigr]^{1/2} \\
\dot{\zeta}_3 &= -\bigl[(c_2 - \zeta_3^2)(c_1 - \gamma^{-2} c_2 + \gamma^{-2}\zeta_3^2)\bigr]^{1/2}
\end{aligned} \tag{T.48}$$

Thus ζ_j are given by elliptic integrals; cf. [Lawd].

U. Adiabatic limit and parallel transport.

Let $H(t), t \in I$, be a smooth family of self adjoint operators on a Hilbert space $\mathcal{H}$, with a smoothly varying family of eigenspaces $E(t)$, of constant dimension k, with eigenvalues $\lambda(t)$. Assume the spectrum of $H(t)$ on $E(t)^{\perp}$ is bounded away from $\lambda(t)$. Making a trivial adjustment, we will assume $\lambda(t) = 0$. For simplicity, assume all the operators $H(t)$ have the same domain.

Consider the solution operator $S(t,s)$ to the "Schrödinger equation"

$$\frac{\partial u}{\partial t} = iH(t)u, \tag{U.1}$$

taking $u(s)$ to $u(t)$. Now slow down the rate of change of H, and consider the solution operators $S_n(t,s)$ to

$$\frac{\partial u}{\partial t} = iH\Big(\frac{t}{n}\Big)u. \tag{U.2}$$

The claim is that, if $u(0) = u_0 \in E(0)$, then $S_n(nt, 0)u_0 \to w(t) \in E(t)$ as $n \to \infty$, and there is a simple geometrical description of $w(t)$.

This was established by T. Kato [Kat], and rediscovered by M. Berry [Ber1], the geometrical content brought out by B. Simon [Si]. Berry worked with the case dim $E(t) = 1$, but that restriction is not necessary. The more general case was already dealt with by Kato; that the argument can be so extended was rediscovered by F. Wilczek and A. Zee [WZ]. A collection of subsequent literature can be found in [SW]. Curiously, Kato's work has been ignored, despite the fact that it was cited by [Si].

The geometrical structure is the following. The family $E(t)$ gives a vector bundle $E \to I$, a subbundle of the product bundle $I \times \mathcal{H}$. If $P(t)$ denotes the orthogonal projection of $\mathcal{H}$ on $E(t)$, we have a covariant derivative on sections of E defined by

$$\nabla_T u(t) = P(t) D_T u(t), \tag{U.3}$$

where D_T is the standard componentwise derivative of $\mathcal{H}$-valued functions. Parallel transport is defined by $\nabla_T u = 0$. It is easily seen, using

$$P'P = (I-P)P', \tag{U.4}$$

that parallel transport is also characterized by

$$\frac{dw}{dt} = P'(t)w, \quad \text{if} \quad w(0) \in E(0). \tag{U.5}$$

See Exercise 3 of §13. The claim is that the adiabatic limit $w(t)$ mentioned above exists and is equal to the solution to (U.5), with $w(0) = u_0 \in E(0)$.

To prove this, we will rescale the t-variable in the equation (U.2). Thus we compare the solutions u and w to

$$\begin{aligned} \frac{\partial u}{\partial t} &= inH(t)u \\ \frac{\partial w}{\partial t} &= P'(t)w, \end{aligned} \tag{U.6}$$

given $u(0) = w(0) = u_0 \in E(0)$. Then we know $w(t) \in E(t)$ for each t, so $H(t)w(t) = 0$. Also, by (U.4), $P'(t)w = (I-P)P'w = (I-P)w'$.

Let $v(t) = u(t) - w(t)$. Then $v(0) = 0$ and

$$\frac{\partial v}{\partial t} - inH(t)v = -P^{\perp}(t)w' = f(t). \tag{U.7}$$

Here we have used (U.4) and set $P^{\perp} = I - P$. Thus $f(t) \perp E(t)$ for each t. Now let $\mathcal{S}_n(t,s)$ denote the solution operator to $\partial u/\partial t = inH(t)u$. Then the solution $v(t)$ to (U.7) is given by

$$v(t) = \int_0^t \mathcal{S}_n(t,s)f(s)\,ds. \tag{U.8}$$

We know that

$$\frac{\partial}{\partial s}\mathcal{S}_n(t,s) = -in\mathcal{S}_n(t,s)H(s). \tag{U.9}$$

Now the spectral hypothesis on $H(t)$ implies we can set

$$f(t) = H(t)g(t), \tag{U.10}$$

with $g(t)$ a smooth family of elements of $\mathcal{H}$. Hence

$$\begin{aligned} v(t) &= \int_0^t \mathcal{S}_n(t,s)H(s)g(s)\,ds \\ &= -\frac{1}{in}\int_0^t \frac{\partial}{\partial s}\mathcal{S}_n(t,s)\; g(s)\,ds \\ &= -\frac{1}{in}\bigl[g(t) - \mathcal{S}_n(t,0)g(0)\bigr] + \frac{1}{in}\int_0^t \mathcal{S}_n(t,s)g'(s)\,ds. \end{aligned} \tag{U.11}$$

Hence

$$\|v(t)\|_{\mathcal{H}} \le \frac{K}{n}. \tag{U.12}$$

This proves our assertion:

$$\mathcal{S}_n(t,0)u_0 \longrightarrow w(t) \quad \text{as} \quad n\to\infty \quad \text{if} \quad u_0\in E(0), \tag{U.13}$$

where $w(t)$ is obtained from u_0 by parallel translation.

Note that if $H(1)=H(0)$, so $E(1)=E(0)$, then $w(1)$ will typically differ from u_0 by the application of a unitary operator on E_0, since the connection (U.3) on E is typically not flat. In the case Berry considered, where $\dim_{\mathbb{C}} E(0)=1$, this could only be multiplication by $e^{i\theta}$, θ being called Berry's phase.

If we assume that $H(t)$ has purely distinct spectrum $\lambda_1(t)<\lambda_2(t)<\cdots$, of constant multiplicity, and no crossings, then we can analyze the behavior of solutions to $\partial u/\partial t = inH(t)u$ via superpositions. Let $P_j(t)$ denote the orthogonal projection of $\mathcal{H}$ onto the $\lambda_j(t)$-eigenspace of $H(t)$. Write $u(0)=\sum u_j(0)$, with $u_j(0)\in\mathcal{R}(P_j(0))=E_j(0)$. Let $\mathcal{T}_j(t)$ denote parallel translation in the vector bundle $\mathcal{R}(P_j(t))=E_j(t)$, i.e., the solution operator to

$$\frac{dw_j}{dt} = P_j'(t)w_j, \quad w_j(0)\in E_j(0). \tag{U.14}$$

Then we can compare $w_j(t)=\mathcal{T}_j(t)u_j(0)$ to the solution $u_j(t)$ to

$$\frac{\partial u_j}{\partial t} = inH(t)u_j, \quad u_j(0)=P_j(0)u(0), \tag{U.15}$$

i.e., to $\mathcal{S}_n(t,0)P_j(0)u(0)$. By (U.11), we have

(U.16)

$$u_j(t) = e^{in\Lambda_j(t)}\Big\{w_j(t) - \frac{1}{in}\big[g_j(t)-\mathcal{S}_{jn}(t,0)g_j(0)\big] - \frac{1}{in}\int_0^t \mathcal{S}_{jn}(t,s)g_j'(s)\,ds\Big\},$$

where $\mathcal{S}_{jn}(t,s)$ denotes the solution operator for $\partial/\partial t - in\big(H(t)-\lambda_j(t)\big)$, and $g_j(t)$ is obtained in a fashion similar to $g(t)$ in (U.11). Also, we have set $\Lambda_j(t)=\int_0^t \lambda_j(s)\,ds$.

If all the spectral gaps are bounded below:

$$\lambda_{j+1}(t)-\lambda_j(t)\ge C>0, \tag{U.17}$$

then we can decompose any $u(0)\in\mathcal{H}$ and sum over j, obtaining

(U.18)

$$\begin{aligned}\mathcal{S}_n(t,0)u(0) = &\sum_j e^{in\Lambda_j(t)}\mathcal{T}_j(t)u(0)\\ &+\frac{1}{in}\sum_j e^{in\Lambda_j(t)}\Big[g_j(t)-\mathcal{S}_{jn}(t,0)g_j(0)+\int_0^t \mathcal{S}_{jn}(t,s)g_j'(s)\,ds\Big].\end{aligned}$$

Similar approximations, for $\mathcal{S}_{jn}(t,0)$ and $\mathcal{S}_{jn}(t,s)$, can be made on the right, and this process iterated, to obtain higher order asymptotic expansions.

Of course, the hypothesis (U.17) is rather restrictive. If one weakens it to

$$|\lambda_{j+1}(t)-\lambda_j(t)|\ge C\langle\lambda_j(t)\rangle^{-K}>0, \tag{U.19}$$

then one can iterate (U.18), at least a finite number of times, provided $u(0)$ belongs to the domain of some power of $H(0)$.

V. Grassmannians (symmetric spaces, and Kähler manifolds)

Given $k \le n$, we let $G_{k,n}$ denote the space of k-dimensional linear subspaces of $\mathbb{R}^n$. This space has a natural structure of a smooth manifold and a natural Riemannian metric, which we proceed to construct. It will be helpful to consider also an associated set of projections. Given a k-dimensional linear space $V \subset \mathbb{R}^n$, let P_V be the orthogonal projection of $\mathbb{R}^n$ onto V. The correspondence $V \mapsto P_V$ sets up a bijection

$$G_{k,n} \approx \Pi_{k,n} = \{P \in \operatorname{End}(\mathbb{R}^n) : P = P^t = P^2,\ \operatorname{Rank} P = k\}. \tag{V.1}$$

Let $V_0 \in G_{k,n}$ denote the space $V_0 = \{(x_1, \dots, x_k, 0, \dots, 0) : x_\nu \in \mathbb{R}\}$. We have a natural identification $V_0 = \mathbb{R}^k$ and $V_0^\perp = \mathbb{R}^{n-k}$. An element $V \in G_{k,n}$ close to V_0 can be described uniquely as the graph of a map $A \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$:

$$V = \{(x, Ax) : x \in \mathbb{R}^k\}, \tag{V.2}$$

where we take $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$. This provides a coordinate system for a neighborhood of V_0 in $G_{k,n}$:

$$\varphi : \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k}) \longrightarrow G_{k,n}. \tag{V.3}$$

Alternatively,

$$\varphi(A) = \mathcal{R}(B), \quad B = I \oplus A : \mathbb{R}^k \to \mathbb{R}^n. \tag{V.4}$$

Note that $G_{k,n} \setminus \operatorname{image}(\varphi)$ consists of the collection of k-dimensional subspaces V of $\mathbb{R}^n$ such that $V \cap (V_0)^\perp \neq \emptyset$.

The associated map

$$\psi : \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k}) \longrightarrow \Pi_{k,n} \tag{V.5}$$

can be described as follows; $\psi(A)$ is the orthogonal projection of $\mathbb{R}^n$ onto the space (V.2), i.e., onto $\mathcal{R}(B) = \mathcal{R}(BB^t)$. Hence

$$\psi(A) = B(B^tB)^{-1}B^t. \tag{V.6}$$

Recalling that $B = (I, A)^t$, we have

$$\psi(A) = \begin{pmatrix} (I + A^tA)^{-1} & (I + A^tA)^{-1}A^t \\ A(I + A^tA)^{-1} & A(I + A^tA)^{-1}A^t \end{pmatrix}. \tag{V.7}$$

Note that

$$D\psi(0)X = \begin{pmatrix} 0 & X^t \\ X & 0 \end{pmatrix}. \tag{V.8}$$

Thus, ψ is a diffeomorphism of a neighborhood of 0 in $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$ onto a $k(n-k)$-dimensional manifold in $\mathrm{End}(\mathbb{R}^n)$, more precisely onto a neighborhood in $\Pi_{k,n}$ of P_0, the orthogonal projection of $\mathbb{R}^n$ on V_0. We have

$$T_{P_0}\Pi_{k,n} = \left\{ \begin{pmatrix} 0 & X^t \\ X & 0 \end{pmatrix} : X \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k}) \right\}. \tag{V.9}$$

The space $\Pi_{k,n}$ gets a natural Riemannian metric as a submanifold of $\mathrm{End}(\mathbb{R}^n)$, endowed with the inner product

$$\langle S, T\rangle = \mathrm{Tr}\,(S^t T). \tag{V.10}$$

Note that the inner product induced on $T_{P_0}\Pi_{k,n}$ is

$$\langle X, Y\rangle = 2\ \mathrm{Tr}\,(X^t Y). \tag{V.11}$$

since

$$\big(D\psi(0)X\big)^t\big(D\psi(0)Y\big) = \begin{pmatrix} 0 & X^t \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & Y^t \\ Y & 0 \end{pmatrix} = \begin{pmatrix} X^tY & 0 \\ 0 & XY^t \end{pmatrix}.$$

The group $O(n)$ acts transitively on $G_{k,n}$. The subgroup fixing V_0 is $O(k) \times O(n-k)$, so

$$G_{k,n} \approx O(n)/\big(O(k) \times O(n-k)\big). \tag{V.12}$$

The way $O(n)$ acts on $\Pi_{k,n}$ is by restriction to $\Pi_{k,n}$ of the action by conjugation on $\mathrm{End}(\mathbb{R}^n)$:

$$g \cdot P = gPg^{-1}. \tag{V.13}$$

Clearly this action preserves the inner product (V.10) on $\mathrm{End}(\mathbb{R}^n)$, so $O(n)$ acts as a group of isometries on $\Pi_{k,n}$. Note that the derived action of $O(k) \times O(n-k)$ on $T_{P_0}\Pi_{k,n}$ is given by

$$\begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} 0 & X^t \\ X & 0 \end{pmatrix}\begin{pmatrix} h^{-1} & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 0 & hX^tg^{-1} \\ gXh^{-1} & 0 \end{pmatrix}. \tag{V.14}$$

Now, one can show that the action $(h, g) \cdot X = gXh^{-1}$ of $O(k) \times O(n-k)$ on $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$ is *irreducible*, i.e., there are no proper invariant linear subspaces. It follows that, up to a constant multiple, the inner product (V.11) on $T_{P_0}\Pi_{k,n}$ is the unique inner product which is invariant under the action of $O(k)\times O(n-k)$. Thus the Riemannian metric on $\Pi_{k,n}$ constructed above is, up to a constant factor, the unique metric invariant under the action of $O(n)$.

We next consider the space $G_{k,n}(\mathbb{C})$ of k-dimensional complex linear subspaces of $\mathbb{C}^n$. This has a description similar to $G_{k,n}$. Here and below, "linear" means $\mathbb{C}$-linear, unless otherwise stated. We have

$$G_{k,n}(\mathbb{C}) \approx \Pi_{k,n}(\mathbb{C}) = \{P \in \operatorname{End}(\mathbb{C}^n) : P = P^* = P^2, \text{ Rank } P = k\}. \tag{V.15}$$

Let $V_0 = \mathbb{C}^k = \{(z_1,\dots,z_k,0,\dots,0) : z_\nu \in \mathbb{C}\}$, P_0 the orthogonal projection of $\mathbb{C}^n$ on V_0. We have a map

$$\psi : \mathcal{L}(\mathbb{C}^k,\mathbb{C}^{n-k}) \longrightarrow \Pi_{k,n}(\mathbb{C}) \tag{V.16}$$

given by (V.7), with A^t replaced by A^*. The natural (real) inner product on $\operatorname{End}(\mathbb{C}^n)$ is

$$\langle S,T\rangle = \operatorname{Re}\,\operatorname{Tr}(S^*T). \tag{V.17}$$

We have

$$T_{P_0}\Pi_{k,n}(\mathbb{C}) = \left\{ \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} : X \in \mathcal{L}(\mathbb{C}^k,\mathbb{C}^{n-k}) \right\}, \tag{V.18}$$

and the inner product induced on $\mathcal{L}(\mathbb{C}^k,\mathbb{C}^{n-k})$ from that on $T_{P_0}\Pi_{k,n}(\mathbb{C})$ satisfies

$$\langle X,Y\rangle = 2\,\operatorname{Re}\,\operatorname{Tr}(X^*Y). \tag{V.19}$$

The group $U(n)$ acts transitively on $G_{k,n}(\mathbb{C})$. The subgroup fixing V_0 is $U(k)\times U(n-k)$, so

$$G_{k,n}(\mathbb{C}) \approx U(n)/\big(U(k)\times U(n-k)\big). \tag{V.20}$$

The action of $U(n)$ on $\Pi_{k,n}(\mathbb{C})$ is given by (V.13), so this is an action by isometries.

The action of $U(k)\times U(n-k)$ on $T_{P_0}\Pi_{k,n}(\mathbb{C})$ is given by (V.14), with X^t replaced by X^*. Note that $T_{P_0}\Pi_{k,n}(\mathbb{C})$, given by (V.18), has a natural *complex* structure, via $X \mapsto iX \in \mathcal{L}(\mathbb{C}^k,\mathbb{C}^{n-k})$. Denote this by

$$J : T_P\Pi_{k,n}(\mathbb{C}) \longrightarrow T_P\Pi_{k,n}(\mathbb{C}), \tag{V.21}$$

where for now $P = P_0$. In such a case, $U(k)\times U(n-k)$ acts as a group of *complex linear* transformations on $T_{P_0}\Pi_{k,n}(\mathbb{C})$. Also, J is an isometry for the inner product (V.19). Thus

$$J = -J^t = -J^{-1} \tag{V.22}$$

on $T_{P_0}\Pi_{k,n}(\mathbb{C})$. It follows that there is a unique complex structure (V.21) on $T_P\Pi_{k,n}(\mathbb{C})$ for each P, coinciding with that described above if $P = P_0$, and also having the property that, if $g \in U(n)$ and $g \cdot P = P'$, then

$$g_* : T_P\Pi_{k,n}(\mathbb{C}) \longrightarrow T_{P'}\Pi_{k,n}(\mathbb{C}) \tag{V.23}$$

is complex linear. Also, J satisfies (V.22) at each $P \in \Pi_{k,n}(\mathbb{C})$. One says $\Pi_{k,n}(\mathbb{C})$ has an *almost complex* structure. Note that, if $\mathcal{R}(P) = V$,

$$T_P\Pi_{k,n}(\mathbb{C}) = \{X + X^* : X \in \mathcal{L}(V, V^\perp)\}, \tag{V.24}$$

where $X + X^*$ is shorthand for $XP + X^*(I - P) = PXP + (I - P)X^*(I - P)$. We have

$$J(X + X^*) = iX - iX^*. \tag{V.25}$$

There are some other algebraic structures that arise naturally. First, we can define an $\mathbb{R}$-bilinear form

$$(X, Y) = \langle X, Y\rangle + i\langle X, JY\rangle, \quad X, Y \in T_P\Pi_{k,n}(\mathbb{C}), \tag{V.26}$$

where $\langle X, Y\rangle$ is the real inner product given by the Riemannian metric discussed above. Note that

$$(X, X) = \langle X, X\rangle, \quad (JX, Y) = i(X, Y), \quad (X, Y) = \overline{(Y, X)}. \tag{V.27}$$

Thus, $(\ ,\ )$ is a Hermitian inner product on the complex vector space (T_P, J), and

$$\langle X, Y\rangle = \text{ Re } (X, Y). \tag{V.28}$$

Note that, in parallel with (V.19), we have

$$(X, Y) = 2 \operatorname{Tr}(X^*Y) \tag{V.29}$$

as the Hermitian inner product induced on $\mathcal{L}(\mathbb{C}^k, \mathbb{C}^{n-k})$ from that on $T_{P_0}\Pi_{k,n}(\mathbb{C})$. Also, the current analogue of (V.14) implies that $U(k) \times U(n - k)$ acts as a group of unitary operators on $T_{P_0}\Pi_{k,n}(\mathbb{C})$, with this Hermitian inner product. Now, this action of $U(k) \times U(n - k)$ is irreducible, so it follows that this is (up to a positive scalar) the only Hermitian inner product on $T_{P_0}\Pi_{k,n}(\mathbb{C})$ invariant under this group action. Consequently, (V.26) gives $T\Pi_{k,n}(\mathbb{C})$ the structure of a Hermitian complex vector bundle, and this is the unique Hermitian structure (up to a positive constant factor) which is invariant under the action of $U(n)$. Generally, a Riemannian manifold M with an almost complex structure satisfying (V.22) is called a Hermitian (almost complex) manifold.

We now look at the action on $\Pi_{k,n}(\mathbb{C})$ of a special element of $U(n)$, namely

$$g_0 = \begin{pmatrix} -I & \\ & I \end{pmatrix}, \tag{V.30}$$

which acts as $-I$ on V_0 and as I on $V_0^\perp$. Denote $P \mapsto g_0 \cdot P = g_0 P g_0^{-1}$ by

$$\iota : \Pi_{k,n}(\mathbb{C}) \longrightarrow \Pi_{k,n}(\mathbb{C}). \tag{V.31}$$

Clearly ι fixes P_0. As for its action on $T_{P_0}\Pi_{k,n}(\mathbb{C})$, this is given by

(V.32)
$$D\iota(P_0)\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & -X^* \\ -X & 0 \end{pmatrix},$$

i.e.,

$$D\iota(P_0) = -I. \tag{V.33}$$

When a Riemannian manifold M has a transitive group G of isometries, it is called a *homogeneous* space. If furthermore, there is a point $p_0 \in M$ and an isometry $\iota \in G$ such that $\iota(p_0) = p_0$ and $D\iota(p_0) = -I$ on $T_{p_0}M$, one calls M a *symmetric* space. We have just seen that $G_{k,n}(\mathbb{C}) \approx \Pi_{k,n}(\mathbb{C})$ is a symmetric space. A similar argument shows that the real Grassmannian $G_{k,n}$ is also a symmetric space.

If a symmetric space M has an almost complex structure J, satisfying (V.22), which is invariant under the action of G, we say M is a *Hermitian* symmetric space. The following result is useful.

Proposition V.1. *If M is a Hermitian symmetric space with almost complex structure J (a tensor field of type (1,1)), then, for all vector fields X on M,*

$$\nabla_X J = 0. \tag{V.34}$$

Proof. Consider $F = \nabla J$, a tensor field of type (1,2). It is invariant under the action of G. However, since $D\iota(p_0) = -I$, we must have $\iota^* F = -F$ at p_0; hence $F = -F = 0$ at p_0. Hence $F = 0$ everywhere on M.

A Hermitian manifold whose almost complex structure J satisfies (V.34) is called a *Kähler* manifold. We have just seen that $G_{k,n}(\mathbb{C}) \approx \Pi_{k,n}(\mathbb{C})$ is a Kähler manifold. Note that (V.34) is equivalent to

$$\nabla_X(JY) = J(\nabla_X Y) \tag{V.35}$$

for all vector fields Y on M. Equivalently, on a Kähler manifold, parallel transport along any smooth curve (say with endpoints at $p, q \in M$) gives a $\mathbb{C}$-linear map from T_pM to T_qM.

The almost complex structure of a Kähler manifold M is always *integrable*, i.e., there is a holomorphic coordinate system on M, so M is actually a *complex* manifold. We will not prove this here though below we will say more about the integrability condition. However, we note that the map $\psi : \mathcal{L}(\mathbb{C}^k, \mathbb{C}^{n-k}) \to \Pi_{k,n}(\mathbb{C})$ given by (V.7), with A^t replaced by A^*, is actually holomorphic, i.e., given $X \in \mathcal{L}(\mathbb{C}^k, \mathbb{C}^{n-k})$,

$$D\psi(A)(iX) = J\ D\psi(A)(X), \tag{V.36}$$

where J acts on $T_P\Pi_{k,n}(\mathbb{C})$, $P = \psi(A)$, in the manner specified in (V.24)-(V.25). We leave this calculation to the reader. It follows that $G_{k,n}(\mathbb{C}) \approx \Pi_{k,n}(\mathbb{C})$ is a complex manifold.

A Kähler manifold M has another structure, a 2-form ω, called the Kähler form, defined by

$$\omega(X, Y) = \langle X, JY \rangle. \tag{V.37}$$

That $\omega(X, Y) = -\omega(Y, X)$ follows from (V.22). Since $\nabla J = 0$, it follows that

$$\nabla \omega = 0. \tag{V.38}$$

The formula (12.41) for $d\omega$ then gives

$$d\omega = 0, \tag{V.39}$$

i.e., ω is a closed 2-form on M. Since J is invertible, ω is *nondegenerate*, i.e., if $\omega(X, Y) = 0$ for all $Y \in T_pM$, then $X = 0$ at p. (A nondegenerate closed 2-form is called a symplectic form). It follows that, if M has complex dimension m (hence real dimension $2m$), then, for $1 \le j \le m$,

$$\omega^j \in \Lambda^j(M), \quad \omega^j \neq 0, \quad d\omega^j = 0. \tag{V.40}$$

We claim that, for $1 \le j \le m$, ω^j is *not* cohomologous to zero if M is compact. Clearly ω^m is not; it is nowhere vanishing and so $\int_M \omega^m \neq 0$. On the other hand, if $j < m$,

$$\omega^j = d\beta \Longrightarrow \omega^m = \omega^{m-j} \wedge d\beta = \pm d(\omega^{m-j} \wedge \beta), \tag{V.41}$$

giving a contradiction. Thus

$$M \text{ compact Kähler } \Longrightarrow \mathcal{H}^{2j}(M) \neq 0, \quad 1 \le j \le m. \tag{V.42}$$

Note that (V.37) defines a nondegenerate 2-form whenever M is a Hermitian almost complex manifold, whether or not M is Kähler. The following is a useful characterization of Kähler manifolds.

Proposition V.2. *If M is a Hermitian complex manifold (i.e., the almost complex structure is integrable) and ω is defined by (V.37), then*

$$d\omega = 0 \Longrightarrow M \text{ is Kähler.} \tag{V.43}$$

The formula (12.41) shows that $d\omega = 0$ is equivalent to

$$(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(Z, X) + (\nabla_Z \omega)(X, Y) = 0 \tag{V.44}$$

for all vector fields X, Y, Z on M. We want to show that this implies $\nabla_X \omega = 0$, which is equivalent to $\nabla_X J = 0$, since

$$\langle Y, (\nabla_X J)Z\rangle = (\nabla_X \omega)(Y, Z). \tag{V.45}$$

Before proving this, we need to investigate further the integrability condition on J that holds if M has a holomorphic coordinate system. The condition that a function $u : \mathcal{O} \to \mathbb{C}$ be holomorphic, given $\mathcal{O} \subset M$ open, is that

$$(X + iJX)u = 0 \tag{V.46}$$

for every (real) vector field X on M. If M has a holomorphic coordinate system, it is readily verified that, for any X and Y,

$$[X + iJX, Y + iJY] = [X, Y] + i[JX, Y] + i[X, JY] - [JX, JY] \tag{V.47}$$

must also have the same form, i.e., that

$$\mathcal{N}(X, Y) = J\big([X, Y] - [JX, JY]\big) - \big([JX, Y] + [X, JY]\big) \tag{V.48}$$

must vanish. It is easily verified that $\mathcal{N}(fX, gY) = fg\mathcal{N}(X, Y)$ for any smooth (real valued) f and g, so $\mathcal{N}$ defines a tensor field of type (1,2), on any almost complex manifold M. In the literature one frequently sees $N(X, Y) = 2J\mathcal{N}(X, Y)$, called the Nijenhuis tensor.

The result that an almost complex manifold for which $\mathcal{N} = 0$ has a holomorphic coordinate system is the Newlander-Nirenberg theorem. A proof can be found in [FoKo], or in [T1].

On a general Hermitian almost complex manifold, one has the identity

$$2\langle(\nabla_X J)Y, Z\rangle = (d\omega)(X, JY, JZ) - (d\omega)(X, Y, Z) + \langle \mathcal{N}(Y, Z), X\rangle. \tag{V.49}$$

The reader can verify this from the definitions. Thus, if M is a Hermitian almost complex manifold, we have

$$d\omega = 0 \Longrightarrow 2\langle(\nabla_X J)Y, Z\rangle = \langle \mathcal{N}(Y, Z), X\rangle. \tag{V.50}$$

Since the integrability condition is $\mathcal{N} = 0$, we have Proposition V.2.

There is the following important consequence.

Corollary V.3. *If M is a Kähler manifold and $X \subset M$ is a complex submanifold, then X is also Kähler.*

Proof. If $j : X \hookrightarrow M$ denotes the inclusion, then $\omega_X = j^*\omega_M$, so $d\omega_X = j^* d\omega_M$.

The study of Kähler manifolds is an important area, involving both differential geometry and algebraic geometry. For more on this topic, see [Weil] and [Wel]. The study of symmetric spaces is closely tied to the study of Lie groups; [Hel] has a great deal of information on this.

W. The Hopf invariant

Assume M and N are compact oriented manifolds, such that

$$\dim N = n, \quad \dim M = 2n - 1. \tag{W.1}$$

Also assume that

$$N \ \text{ connected}, \quad \mathcal{H}^n(M) = 0. \tag{W.2}$$

For example, one could take $M = S^{2n-1}$, provided $n \geq 2$. Let $f : M \to N$ be a smooth map. We define the Hopf invariant of f as follows.

Pick $\omega \in \Lambda^n(N)$ such that $\int_N \omega = 1$. Let $B = f^*\omega$. Pick $A \in \Lambda^{n-1}(M)$ such that $dA = B$. The hypothesis (W.2) implies this can be done. Set

$$\mathfrak{H}(f) = \int\limits_M A \wedge dA. \tag{W.3}$$

Note that, if n is odd, then $d(A \wedge A) = 2A \wedge dA$, so (W.3) vanishes. Thus n must be even for the Hopf invariant to be nontrivial.

We establish independence of choices. Suppose we pick another $\omega' \in \Lambda^n(N)$ such that $\int_N \omega' = 1$, so

$$\omega' = \omega + d\beta \quad \text{on} \ \ N. \tag{W.4}$$

Then $f^*\omega' = d(A + f^*\beta)$, and

$$\begin{aligned} \int\limits_M (A + f^*\beta) \wedge (dA + df^*\beta) = \int\limits_M A \wedge dA + \int\limits_M (f^*\beta) \wedge dA + \int\limits_M A \wedge df^*\beta \\ + \int\limits_M f^*\beta \wedge df^*\beta. \end{aligned} \tag{W.5}$$

Consider the four integrals on the right side of (W.5). The second integrand is equal to $f^*(\beta \wedge \omega) = 0$. The third integrand is equal to $(-1)^{n-1} d(A \wedge f^*\beta) + (-1)^n dA \wedge f^*\beta$, and as just seen the last term here is zero, so the second integral on the right side of (W.5) vanishes. The last integrand on the right side of (W.5) is equal to $f^*(\beta \wedge d\beta) = 0$. Hence the left side of (W.5) is equal to (W.3), so (W.3) is independent of the choice of ω on N.

Next suppose $dA = dA'$, so $A' - A = \alpha \in \Lambda^{n-1}(M)$ is closed. Then $A' \wedge dA' = A \wedge dA + \alpha \wedge dA = A \wedge dA + (-1)^{n-1} d(\alpha \wedge A)$, so (W.3) is unchanged upon replacing A by A'.

The following result asserts the homotopy invariance of the Hopf invariant.

Proposition W.1. *If $f_0 : M \to N$ and $f_1 : M \to N$ are smoothly homotopic, then*

$$\mathfrak{H}(f_0) = \mathfrak{H}(f_1). \tag{W.6}$$

This result is a special case of the following restricted cobordism invariance, when $\overline{\Omega} = M \times [0,1]$.

Proposition W.2. *Let M and N be as above, and suppose $M = \partial\Omega$, where $\overline{\Omega}$ is a compact, oriented, $2n$-dimensional manifold. Also assume that*

$$\mathcal{H}^n(\overline{\Omega}) = 0. \tag{W.7}$$

Then

$$F : \overline{\Omega} \longrightarrow N, \quad f = F\big|_{\partial\Omega} \Longrightarrow \mathfrak{H}(f) = 0. \tag{W.8}$$

Proof. This time, let $B = F^*\omega \in \Lambda^n(\overline{\Omega})$, and take $A \in \Lambda^{n-1}(\overline{\Omega})$ such that $dA = B$. Apply Stokes' formula to $A \wedge dA \in \Lambda^{2n-1}(\overline{\Omega})$, to get

$$\int_{\Omega} dA \wedge dA = \int_{\partial\Omega} A \wedge dA. \tag{W.9}$$

The right side of (W.9) is equal to $\mathfrak{H}(f)$. On the other hand, the integrand on the left side is equal to $F^*(\omega \wedge \omega) = 0$.

We mention the following additional flexibility in the formula for the Hopf invariant.

Proposition W.3. *Let ω and A be as in the definition (W.3). Let also $\int_N \omega' = 1$ and $f^*\omega' = dA'$. Then*

$$\mathfrak{H}(f) = \int_M A' \wedge dA. \tag{W.10}$$

Proof. Write $\omega - \omega' = d\alpha$. Then $dA - dA' = df^*\alpha$, so $A - A' - f^*\alpha = \beta$ is closed. We have

$$\int_M (A - A') \wedge dA = \int (f^*\alpha + \beta) \wedge f^*\omega. \tag{W.11}$$

Now $f^*(\alpha \wedge \omega) = 0$ and $\beta \wedge f^*\omega = \beta \wedge dA = \pm d(\beta \wedge A)$, so the right side of (W.11) vanishes.

There is the following relationship between the Hopf invariant and degree.

Proposition W.4. *Let M and N be as above. Suppose also M' is a compact oriented manifold of dimension $2n-1$, with $\mathcal{H}^n(M')=0$, and $\varphi : M' \to M$. Then*

$$\mathfrak{H}(f\circ\varphi) = (Deg\,\varphi)\mathfrak{H}(f). \tag{W.12}$$

Proof. If $A' = \varphi^* A$, we have $dA' = \varphi^* dA = (f\circ\varphi)^*\omega$, and $A'\wedge dA' = \varphi^*(A\wedge dA)$, so

$$\begin{aligned} \mathfrak{H}(f\circ\varphi) &= \int_{M'} A'\wedge dA' = \int_{M'} \varphi^*(A\wedge dA) \\ &= (\operatorname{Deg}\varphi)\int_M A\wedge dA. \end{aligned} \tag{W.13}$$

We now compute an important example of the Hopf invariant. Let $N=S^2$ and $M=SO(3)$. The group $SO(3)$ has Lie algebra $so(3)$, spanned by

$$J_1 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & & -1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix}, \tag{W.14}$$

satisfying commutation relations

$$[J_1,J_2]=J_3, \quad [J_2,J_3]=J_1, \quad [J_3,J_1]=J_2. \tag{W.15}$$

The group $SO(3)$ acts transitively in $S^2\subset\mathbb{R}^3$, and the subgroup fixing the "north pole" $p=(0,0,1)$ is the group generated by J_3. This defines a map

$$\varphi : SO(3) \longrightarrow S^2. \tag{W.16}$$

Let us define a 1-form α on $SO(3)$ as follows. If J_ν are extended as left-invariant vector fields on $SO(3)$, set

$$\alpha(J_1)=\alpha(J_2)=0, \quad \alpha(J_3)=1. \tag{W.17}$$

Then the formula $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$, together with the commutation relations (W.15) yields

$$d\alpha(J_1,J_2)=-1, \quad d\alpha(J_2,J_3)=0, \quad d\alpha(J_3,J_1)=0, \tag{W.18}$$

and hence

$$\alpha\wedge d\alpha(J_1,J_2,J_3) = -1. \tag{W.19}$$

It follows from (W.18) that, up to sign, $d\alpha$ is the pull-back via φ of the volume form ω_{S^2} on S^2 given by its standard metric, so $\int_{S^2} \omega_{S^2} = 4\pi$.

Before completing the computation of $\mathfrak{H}(\varphi)$, let us bring in $M' = SU(2)$. We have a two-fold covering homomorphism

$$\tau : SU(2) \longrightarrow SO(3), \tag{W.20}$$

obtained by mapping X_ν to J_ν, where X_ν are the following basis of $su(2)$,

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{W.21}$$

with the same commutation relations as in (W.15), i.e.,

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \tag{W.22}$$

Thus $\alpha' = \tau^*\alpha$ satisfies analogues of (W.17)–(W.19), with J_ν replaced by X_ν, extended to be left-invariant vector fields on $SU(2)$. Now $SU(2)$ acts simply transitively on $S^3 \subset \mathbb{C}^2$. Note that

$$e^{4\pi X_\nu} = I, \tag{W.23}$$

and 4π is the smallest positive real number for which this holds. Hence, if $SU(2)$ is given the Riemannian metric induced from S^3, we see that $\|X_\nu\| = 1/2$. Also, the X_ν are mutually orthogonal at each point of $SU(2)$. Thus $\alpha' \wedge d\alpha'$ is eight times the volume element on $SU(2)$ induced by this metric. By (4.30) we know that $\mathrm{Vol}(S^3) = 2\pi^2$, so

$$\int_{SU(2)} \alpha' \wedge d\alpha' = 16\pi^2, \tag{W.24}$$

provided we give $SU(2)$ the orientation so that $(-X_1, -X_2, -X_3)$ is an oriented basis of the tangent space. Now, let us set

$$\omega = \frac{1}{4\pi}\omega_{S^2}, \quad A = \frac{1}{4\pi}\alpha, \quad A' = \frac{1}{4\pi}\alpha'. \tag{W.25}$$

Then

$$dA = \varphi^*\omega, \quad dA' = \psi^*\omega, \tag{W.26}$$

where

$$\psi = \varphi \circ \tau : SU(2) \longrightarrow S^2, \tag{W.27}$$

and we have

$$A' \wedge dA' = \frac{1}{16\pi^2}\, \alpha' \wedge d\alpha'. \tag{W.28}$$

By (W.24), we obtain the following:

Conclusion. *For the maps φ and ψ given by (W.16) and (W.27), we have*

$$\mathfrak{H}(\varphi) = \frac{1}{2}, \quad \mathfrak{H}(\psi) = 1. \tag{W.29}$$

The map (W.27) is essentially equivalent to the classical Hopf map

$$f : S^3 \longrightarrow S^2, \quad \mathfrak{H}(f) = 1. \tag{W.30}$$

Note that, upon composing f with a map $g_\nu : S^3 \to S^3$ of degree ν and using Proposition W.4, we obtain, for each $\nu \in \mathbb{Z}$, a map $f_\nu : S^3 \to S^2$ with $\mathfrak{H}(f_\nu) = \nu$.

It is known that, whenever $M = S^{2n-1}$ and $f : S^{2n-1} \to N$, then $\mathfrak{H}(f)$ is an integer. See [BTu] for a detailed discussion of the case $n = 2$. In fact, the Hopf invariant of $f : M \to N$ is usually studied only for $M = S^{2n-1}$. We claim that generally, if M and N satisfy our standing hypotheses and $f : M \to N$, then $\mathfrak{H}(f)$ is a rational number. In fact, this is a consequence of formula (W.35) below, established by C. Lebrun.

Let $p \in N$ be a regular value of f; almost all points in N are regular, by Sard's theorem. Let ω be a (delta-function type distributional) n-form supported at p, such that $\int_N \omega = 1$, and let $B = f^*\omega$. This is a distributional n-form on M, supported on $\Gamma = f^{-1}(p)$, a smooth $(n-1)$-dimensional submanifold of M. A key fact is that Γ inherits a natural orientation such that

$$\int_M \psi \wedge B = \int_\Gamma \psi \tag{W.31}$$

for any smooth $(n-1)$-form ψ on M. This is readily established if ψ has support on a small neighborhood of a point $q \in \Gamma$ on which f can be put in the normal form $(x_1, \dots, x_{2n-1}) \mapsto (x_1, \dots, x_n)$. Then a partition of unity argument treats the general case.

In particular, if ω' is an n-form on N integrating to 1 and $f^*\omega' = B' = dA'$, then

$$\mathfrak{H}(f) = \int_\Gamma A'. \tag{W.32}$$

To proceed further, we bring in some consequences of the fact that the compact, oriented manifold M is assumed to have $\mathcal{H}^n(M) = 0$. By Poincaré duality we have for the real homology group $\mathcal{H}_{n-1}(M) = 0$. Hence the integral homology group $\mathcal{H}_{n-1}(M, \mathbb{Z})$ is a finite abelian group. This implies that there is a nonzero $m \in \mathbb{Z}$ and an integral chain X such that

$$m\Gamma = \partial X. \tag{W.33}$$

Using this and applying Stokes' theorem to (W.32), we have

$$\mathfrak{H}(f) = \frac{1}{m} \int\limits_X dA' = \frac{1}{m} \int\limits_X f^* \omega'. \tag{W.34}$$

Consequently, as is easily seen by picking ω' with support disjoint from $\{p\}$,

$$\mathfrak{H}(f) = \frac{1}{m} \deg\big[f : (X, \Gamma) \to (N, p)\big]. \tag{W.35}$$

Of course, if $\mathcal{H}^n(M, \mathbb{Z}) = 0$, then we can take $m = 1$ in (W.33), and hence in (W.35), and see that $\mathfrak{H}(f)$ is an integer. This occurs in the classical case of $f : S^{2n-1} \to N$.

X. Jacobi fields and conjugate points

Jacobi's equation and the concept of a Jacobi field arose in Exercises 8–11 of §15. We recall that a Jacobi field is a vector field J along a unit speed geodesic γ that is a solution to Jacobi's equation

$$\nabla_T \nabla_T J - R(T,J)T = 0, \tag{X.1}$$

where $T(t) = \gamma'(t)$. We have

$$D\,\mathrm{Exp}_p(tv)w = \frac{1}{t} J_w(t), \tag{X.2}$$

where $J_w(t)$ is the Jacobi field along $\gamma(t) = \mathrm{Exp}_p(tv)$ such that

$$J_w(0) = 0, \quad \nabla_T J_w(0) = w. \tag{X.3}$$

Also if γ_s is a 1-parameter family of curves such that $\gamma_s(a) = p,\ \gamma_s(b) = q$, and if γ_0 is a geodesic, then the second derivative at $s = 0$ of the energy integral

$$E(s) = \frac{1}{2}\int_a^b \langle \gamma_s'(t), \gamma_s'(t)\rangle\, dt \tag{X.4}$$

is given by

$$\begin{aligned} E''(0) &= \int_a^b \big[\langle R(V,T)V,T\rangle + \langle \nabla_T V, \nabla_T V\rangle\big]\, dt \\ &= \int_a^b \big[\langle R(V,T)V,T\rangle - \langle V, \nabla_T\nabla_T V\rangle\big]\, dt, \end{aligned} \tag{X.5}$$

where $V = \partial_s \gamma_s(t)|_{s=0}$. Note that the right side of (X.5) vanishes if V is a Jacobi field along γ that vanishes at the endpoints.

We say two points p and q on a geodesic γ are *conjugate* if there exists a nontrivial Jacobi field J along γ that vanishes at p and q. An equivalent condition is that $D\,\mathrm{Exp}_p$ is singular at $v \in T_pM$, if $\gamma(t) = \mathrm{Exp}_p(tv),\ p = \gamma(0)$, and $q = \gamma(1)$.

The theory of Jacobi fields and conjugate points contains many important results, and we touch on only a few here. For more material, one can consult [CE], [Mil]. We begin with a result of Jacobi.

Proposition X.1. *If there is no $t \in (a,b]$ such that $\gamma(t)$ is conjugate to $p = \gamma(a)$, then $E''(0) > 0$ for all nontrivial V along γ that vanish at $\gamma(a)$ and $\gamma(b)$.*

Proof. We assume $\gamma(t)$ is a unit speed geodesic. By (X.2) we see that we can write $\gamma(a+t) = \mathrm{Exp}_p(tv)$ for some unit $v \in T_pM$, and Exp_p gives a local diffeomorphism from a neighborhood Ω of $\{tv : 0 \le t \le b-a\}$ into M. We now make an argument similar to that in the proof of Corollary 11.3.

Given a smooth 1-parameter family $\gamma_s(t)$ with $\gamma_0(t) = \gamma(t)$, $\gamma_s(a) = p$, $\gamma_s(b) = q$, we can write, for $\gamma_s(t) \ne p$,

$$\gamma_s(t) = \operatorname{Exp}_p\bigl(r_s(t)\omega_s(t)\bigr), \tag{X.6}$$

at least for s close to 0, with $\omega_s(t)$ in the unit sphere in T_pM and $r_s(t) > 0$. We have

$$|\gamma_s'(t)|^2 = r_s'(t)^2 + |r_s(t)\omega_s'(t)|^2, \tag{X.7}$$

using the metric tensor on Ω induced from M via Exp_p, and applying the Gauss lemma. Hence

$$E(s) = \frac{1}{2}\int_a^b r_s'(t)^2\,dt + \frac{1}{2}\int_a^b |r_s(t)\omega_s'(t)|^2\,dt. \tag{X.8}$$

Comparing the first term on the right to $\int_a^b |r_s'(t)|\,dt$, as in the proof of Corollary 11.3, we see that

$$\frac{1}{2}\int_a^b r_s'(t)^2\,dt \ge E(0), \tag{X.9}$$

so under our hypotheses we have

$$E(s) \ge E(0) + \frac{1}{2}\int_a^b |r_s(t)\omega_s'(t)|^2\,dt, \tag{X.10}$$

so

$$E''(0) \ge \int_a^b (t-a)^2 \bigl|\partial_t\partial_s\omega_s(t)|_{s=0}\bigr|^2\,dt. \tag{X.11}$$

Note that $\partial_s\omega_s(b) = 0$. Thus this last inequality implies that $E''(0) > 0$ if $V(t)$ is transverse to $\gamma(t)$ anywhere. On the other hand, if $V(t)$ is parallel to $T(t)$ everywhere, the first identity in (X.5) gives this result.

Sometimes one can guarantee that there are no conjugate points:

Proposition X.2. *Suppose the sectional curvature of M is ≤ 0 everywhere. Then no two points of M are conjugate along any geodesic.*

Proof. Let γ be a unit speed geodesic, $T = \gamma'(t)$. Suppose J is a Jacobi field along γ, so (X.1) holds. A computation gives

$$\begin{aligned}\frac{d}{dt}\langle \nabla_T J, J\rangle &= |\nabla_T J|^2 + \langle \nabla_T\nabla_T J, J\rangle \\ &= |\nabla_T J|^2 - \langle R(J,T)T, J\rangle.\end{aligned} \tag{X.12}$$

If the sectional curvature is ≤ 0, then this is $\ge |\nabla_T J|^2$. Thus if $J(a) = 0$ and $J(b) = 0$ for some $b > a$, we deduce that $\int_a^b |\nabla_T J(t)|^2\, dt = 0$, so $J(t)$ vanishes along γ.

An alternative proof follows from examining (X.5). If $V = J$ is a Jacobi field that vanishes at the endpoints $t = a$ and $t = b$, the second integral on the right side of (X.5) must vanish, but under the curvature hypotheses the first integral there is $\ge \int_a^b |\nabla_T J|^2\, dt$, yielding the result.

Curvature conditions of the opposite flavor can guarantee the existence of conjugate points. The following is a result of Bonnet and Myers.

Proposition X.3. *Let $\gamma : [a, b] \to M$ be a unit speed geodesic, and suppose that*

$$Ric\,(T, T) \ge (n-1)\kappa > 0, \tag{X.13}$$

where $T = \gamma'$ and $n = dim\, M$. If $b - a \ge \pi/\sqrt{\kappa}$, then there is a point $q = \gamma(t)$ conjugate to $p = \gamma(a)$, for some $t \in (a, b]$.

Proof. Let $\{T, e_1, \dots, e_{n-1}\}$ be an orthonormal basis of T_pM. Extend e_j along γ by parallel translation, and set $V_j(t) = \sin(\pi(t-a)/\ell)\, e_j(t)$, with $\ell = b - a$. If we compute the second variation of energy (X.5) using $V = V_j$, and denote the result by $E_j''(0)$, we obtain

$$\begin{aligned}\sum_j E_j''(0) &= \int_0^\ell \Big\{(n-1)\frac{\pi^2}{\ell^2}\cos^2\frac{\pi t}{\ell} - \mathrm{Ric}\,(T,T)\sin^2\frac{\pi t}{\ell}\Big\}\, dt \\ &\le (n-1)\int_0^\ell \Big\{\frac{\pi^2}{\ell^2}\cos^2\frac{\pi t}{\ell} - \kappa\sin^2\frac{\pi t}{\ell}\Big\}\, dt \\ &= (n-1)\frac{\ell}{2}\Big(\frac{\pi^2}{\ell^2} - \kappa\Big),\end{aligned} \tag{X.14}$$

which is ≤ 0 if $\ell \ge \pi/\sqrt{\kappa}$. The conclusion then follows from Proposition X.1.

When conjugate points exist, there can be important geometric consequences.

Proposition X.4. *If γ is a unit speed geodesic and $p = \gamma(a)$ and $q = \gamma(b)$ are conjugate along γ, then $d(p, \gamma(t)) < t - a$ for $t > b > a$.*

Proof. It suffices to show that there is a variation for which $\partial_s^2 E(0) < 0$, when $E(s)$ is given by (X.4), with b replaced by $t_1 > b$. To establish this, let J be a nontrivial Jacobi field vanishing at $t = a$ and $t = b$, so $\nabla_T J(b) \ne 0$. Let $\tilde{J}(t) = J(t)$ on $[a, b]$, 0 on $[b, t_1]$. Let Z be a smooth vector field along γ, vanishing at $t = a$ and $t = t_1$, such that $Z(b) = -\nabla_T J(b)$. Then set

$$V = \tilde{J} + \varepsilon Z. \tag{X.15}$$

Using the first identity in (X.5) (valid for V Lipschitz), we obtain

$$\begin{aligned} E''(0) &= 2\varepsilon \int_a^b \bigl\{ \langle \nabla_T J, \nabla_T Z\rangle + \langle R(Z,T)Z, T\rangle \bigr\}\, dt + O(\varepsilon^2) \\ &= 2\varepsilon \int_a^b T\langle \nabla_T J, Z\rangle \, dt + O(\varepsilon^2) \\ &= -2\varepsilon |\nabla_T J(b)|^2 + O(\varepsilon^2). \end{aligned} \tag{X.16}$$

Thus, for sufficiently small $\varepsilon > 0$, use of (X.15) yields $E''(0) < 0$, and we are done.

Proposition X.4 is due to Jacobi. M. Morse established a much more precise result, which has had a great impact on modern geometry and topology. See [Mil] for a treatment.

Together, Propositions X.3 and X.4 imply the following:

Corollary X.5. *If M is a complete Riemannian manifold of dimension n and $Ric\,(X,X) \geq (n-1)\kappa|X|^2$ for some $\kappa > 0$, then M is compact, with diameter $\leq \pi/\sqrt{\kappa}$.*

Alternatively, one can deduce this directly from the calculation (X.14), without bringing in the notion of Jacobi fields.

Y. Isometric imbedding of Riemannian manifolds

In this appendix we will establish the following result.

Theorem Y.1. *If M is a compact Riemannian manifold, there exists a C^∞ map*

$$\Phi : M \longrightarrow \mathbb{R}^N \tag{Y.1}$$

which is an isometric imbedding.

This was first proved by J. Nash [Na], but the proof was vastly simplified by M. Günther [Gu1]–[Gu3]. These works also dealt with noncompact Riemannian manifolds, and derived good bounds for N, but to keep the exposition simple we will not cover these results. The proof will make use of some results on elliptic PDE, which can be found in Chapters 13–14 of [T1].

To prove Theorem Y.1, we can suppose without loss of generality that M is a torus $\mathbb{T}^k$. In fact, imbed M smoothly in some Euclidean space $\mathbb{R}^k$; M will sit inside some box; identify opposite faces to have $M \subset \mathbb{T}^k$. Then smoothly extend the Riemannian metric on M to one on $\mathbb{T}^k$.

If $\mathcal{R}$ denotes the set of smooth Riemannian metrics on $\mathbb{T}^k$ and $\mathcal{E}$ is the set of such metrics arising from smooth imbeddings of $\mathbb{T}^k$ into some Euclidean space, our goal is to prove

$$\mathcal{E} = \mathcal{R}. \tag{Y.2}$$

Now $\mathcal{R}$ is clearly an open convex cone in the Frechet space $V = C^\infty(\mathbb{T}^k, S^2T^*)$ of smooth second order symmetric covariant tensor fields. As a preliminary to demonstrating (Y.2), we show that the subset $\mathcal{E}$ shares some of these properties.

Lemma Y.2. *$\mathcal{E}$ is a convex cone in V.*

Proof. If $g_0 \in \mathcal{E}$, it is obvious from scaling the imbedding producing g_0 that $\alpha g_0 \in \mathcal{E}$, for any $\alpha \in (0, \infty)$. Suppose also $g_1 \in \mathcal{E}$. If these metrics g_j arise from imbeddings $\varphi_j : \mathbb{T}^k \to \mathbb{R}^{\nu_j}$, then $g_0 + g_1$ is a metric arising from the imbedding $\varphi_0 \oplus \varphi_1 : \mathbb{T}^k \to \mathbb{R}^{\nu_0+\nu_1}$. This proves the lemma.

Using Lemma Y.2 plus some functional analysis, we will proceed to establish that any Riemannian metric on $\mathbb{T}^k$ can be *approximated* by one in $\mathcal{E}$. First, we define some more useful objects. If $u : \mathbb{T}^k \to \mathbb{R}^m$ is any smooth map, let γ_u denote the symmetric tensor field on $\mathbb{T}^k$ obtained by pulling back the Euclidean metric on $\mathbb{R}^m$. In a natural local coordinate system on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, arising from standard coordinates $(x_1, \dots, x_k)$ on $\mathbb{R}^k$,

$$\gamma_u = \sum_{i,j,\ell} \frac{\partial u_\ell}{\partial x_i} \frac{\partial u_\ell}{\partial x_j}\ dx_i \otimes dx_j. \tag{Y.3}$$

Whenever u is an immersion, γ_u is a Riemannian metric, and if u is an imbedding, then γ_u is of course an element of $\mathcal{E}$. Denote by $\mathcal{C}$ the set of tensor fields on $\mathbb{T}^k$ of the form γ_u. By the same reasoning as in Lemma Y.2, $\mathcal{C}$ is a convex cone in V.

Lemma Y.3. *$\mathcal{E}$ is a dense subset of $\mathcal{R}$.*

Proof. If not, take $g \in \mathcal{R}$ such that $g \notin \overline{\mathcal{E}}$, the closure of $\mathcal{E}$ in V. Now $\overline{\mathcal{E}}$ is a closed convex subset of V, so the Hahn-Banach theorem implies there is a continuous linear functional $\ell : V \to \mathbb{R}$ such that $\ell(\overline{\mathcal{E}}) \le 0$ while $\ell(g) = a > 0$.

Let us note that $\mathcal{C} \subset \overline{\mathcal{E}}$ (and hence $\overline{\mathcal{C}} = \overline{\mathcal{E}}$). In fact, if $u : \mathbb{T}^k \to \mathbb{R}^m$ is any smooth map and $\varphi : \mathbb{T}^k \to \mathbb{R}^n$ is an imbedding, then, for any $\varepsilon > 0$, $\varepsilon\varphi \oplus u : \mathbb{T}^k \to \mathbb{R}^{n+m}$ is an imbedding, and $\gamma_{\varepsilon\varphi\oplus u} = \varepsilon^2\gamma_\varphi + \gamma_u \in \mathcal{E}$. Taking $\varepsilon \searrow 0$, we have $\gamma_u \in \overline{\mathcal{E}}$.

Consequently, the linear functional ℓ produced above has the property $\ell(\overline{\mathcal{C}}) \le 0$. Now we can represent ℓ as a $k \times k$ symmetric matrix of distributions ℓ_{ij} on $\mathbb{T}^k$, and we deduce that

$$\sum_{i,j} \langle \partial_i f \partial_j f, \ell_{ij} \rangle \le 0, \quad \forall f \in C^\infty(\mathbb{T}^k). \tag{Y.4}$$

If we apply a Friedrichs mollifier J_ε, in the form of a convolution operator on $\mathbb{T}^k$, it follows easily that (Y.4) holds with $\ell_{ij} \in \mathcal{D}'(\mathbb{T}^k)$ replaced by $\lambda_{ij} = J_\varepsilon \ell_{ij} \in C^\infty(\mathbb{T}^k)$. Now it is an exercise to show that, if $\lambda_{ij} \in C^\infty(\mathbb{T}^k)$ satisfies $\lambda_{ij} = \lambda_{ji}$ and the analogue of (5.4), then $\Lambda = (\lambda_{ij})$ is a negative semidefinite matrix valued function on $\mathbb{T}^k$, and hence, for any positive definite $G = (g_{ij}) \in C^\infty(\mathbb{T}^k, S^2T^*)$,

$$\sum_{i,j} \langle g_{ij}, \lambda_{ij} \rangle \le 0. \tag{Y.5}$$

Taking $\lambda_{ij} = J_\varepsilon \ell_{ij}$ and passing to the limit $\varepsilon \to 0$, we have

$$\sum_{i,j} \langle g_{ij}, \ell_{ij} \rangle \le 0, \tag{Y.6}$$

for any Riemannian metric tensor (g_{ij}) on $\mathbb{T}^k$. This contradicts the hypothesis that we can take $g \notin \overline{\mathcal{E}}$, so Lemma Y.3 is proved.

The following result, to the effect that $\mathcal{E}$ has nonempty interior, is the analytical heart of the proof of Theorem Y.1.

Lemma Y.4. *There exists a Riemannian metric $g_0 \in \mathcal{E}$ and a neighborhood U of 0 in V, such that $g_0 + h \in \mathcal{E}$ whenever $h \in U$.*

We now prove (Y.2), hence Theorem Y.1, granted this result. Let $g \in \mathcal{R}$, and take $g_0 \in \mathcal{E}$, given by Lemma Y.4. Then set $g_1 = g + \alpha(g - g_0)$, where $\alpha > 0$ is picked sufficiently small that $g_1 \in \mathcal{R}$. It follows that g is a convex combination of g_0 and g_1, i.e., $g = ag_0 + (1-a)g_1$ for some $a \in (0, 1)$. By Lemma Y.4, we have an open set $U \subset V$ such that $g_0 + h \in \mathcal{E}$ whenever $h \in U$. But by Lemma Y.3, there exists $h \in U$ such that $g_1 - bh \in \mathcal{E}$, $b = a/(1-a)$. Thus $g = a(g_0 + h) + (1-a)(g_1 - bh)$ is a convex combination of elements of $\mathcal{E}$, so by Lemma Y.1, $g \in \mathcal{E}$, as desired.

We turn now to a proof of Lemma Y.4. The metric g_0 will be one arising from a *free* imbedding

$$u : \mathbb{T}^k \longrightarrow \mathbb{R}^\mu, \tag{Y.7}$$

defined as follows.

Definition. *An imbedding (Y.7) is free provided that the $k + k(k+1)/2$ vectors*

$$\partial_j u(x), \quad \partial_j \partial_k u(x) \tag{Y.8}$$

are linearly independent in $\mathbb{R}^\mu$, for each $x \in \mathbb{T}^k$.

Here, we regard $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, so $u : \mathbb{R}^k \to \mathbb{R}^\mu$, invariant under the translation action of $\mathbb{Z}^k$ on $\mathbb{R}^k$, and $(x_1, \dots x_k)$ are the standard coordinates on $\mathbb{R}^k$. It is not hard to establish the existence of free imbeddings; see the exercises at the end of this appendix.

Now, given that u is a free imbedding, and that (h_{ij}) is a smooth symmetric tensor field which is small in some norm (stronger than the C^2-norm), we want to find $v \in C^\infty(\mathbb{T}^k, \mathbb{R}^\mu)$, small in a norm at least as strong as the C^1-norm, such that, with $g_0 = \gamma_u$,

$$\sum_\ell \partial_i(u_\ell + v_\ell)\partial_j(u_\ell + v_\ell) = g_{0ij} + h_{ij}, \tag{Y.9}$$

or equivalently, using the dot product on $\mathbb{R}^\mu$,

$$\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v = h_{ij}. \tag{Y.10}$$

We want to solve for v. Now, such a system turns out to be highly underdetermined, and the key to success is to append convenient side conditions. Following [Gu3], we apply $\Delta - 1$ to (Y.10), where $\Delta = \sum \partial_j^2$, obtaining

$$\begin{aligned} \partial_i\Big\{(\Delta - 1)(\partial_j u \cdot v) + \Delta v \cdot \partial_j v\Big\} + \partial_j\Big\{(\Delta - 1)(\partial_i u \cdot v) + \Delta v \cdot \partial_i v\Big\} \\ -2\Big\{(\Delta - 1)(\partial_i\partial_j u \cdot v) + \tfrac{1}{2}\partial_i v \cdot \partial_j v - \partial_i\partial_\ell v \cdot \partial_j\partial_\ell v \\ +\Delta v \cdot \partial_i\partial_j v + \tfrac{1}{2}(\Delta - 1)h_{ij}\Big\} = 0, \end{aligned} \tag{Y.11}$$

where we sum over ℓ. Thus (Y.10) will hold whenever v satisfies the new system (Y.12)

$$\begin{aligned} (\Delta - 1)\big(\zeta_i(x) \cdot v\big) &= -\Delta v \cdot \partial_i v \\ (\Delta - 1)\big(\zeta_{ij}(x) \cdot v\big) &= -\tfrac{1}{2}(\Delta - 1)h_{ij} + \Big(\partial_i\partial_\ell v \cdot \partial_j\partial_\ell v - \Delta v \cdot \partial_i\partial_j v - \tfrac{1}{2}\partial_i v \cdot \partial_j v\Big). \end{aligned}$$

Here we have set $\zeta_i(x) = \partial_i u(x)$, $\zeta_{ij}(x) = \partial_i\partial_j u(x)$, smooth $\mathbb{R}^\mu$-valued functions on $\mathbb{T}^k$.

Now (Y.12) is a system of $k(k+3)/2 = \kappa$ equations in μ unknowns, and it has the form

$$(\Delta - 1)\big(\xi(x)v\big) + Q(D^2v, D^2v) = H = \big(0, -\tfrac{1}{2}(\Delta - 1)h_{ij}\big), \tag{Y.13}$$

where $\xi(x) : \mathbb{R}^\mu \to \mathbb{R}^\kappa$ is *surjective* for each x, by the linear independence hypothesis on (Y.8), and Q is a bilinear function of its arguments $D^2 v = \{D^\alpha v : |\alpha| \le 2\}$. This is hence an underdetermined system for v. We can obtain a determined system for a function w on $\mathbb{T}^k$ with values in $\mathbb{R}^\kappa$, by setting

$$v = \xi(x)^t w, \tag{Y.14}$$

namely

$$(\Delta - 1)\big(A(x)w\big) + \widetilde{Q}(D^2 w, D^2 w) = H, \tag{Y.15}$$

where, for each $x \in \mathbb{T}^k$,

$$A(x) = \xi(x)\xi(x)^t \in \ \mathrm{End}(\mathbb{R}^\kappa) \text{ is invertible.} \tag{Y.16}$$

If we denote the left side of (Y.15) by $F(w)$, the operator F is a nonlinear differential operator of order 2, and we have

$$DF(w)f = (\Delta - 1)\big(A(x)f\big) + B(D^2 w, D^2 f), \tag{Y.17}$$

where B is a bilinear function of its arguments. In particular,

$$DF(0)f = (\Delta - 1)\big(A(x)f\big). \tag{Y.18}$$

Elliptic estimates, which can be found in Chapter 13, §8 of [T1], imply that, for any $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$,

$$DF(0) : C^{r+2}(\mathbb{T}^k, \mathbb{R}^\kappa) \longrightarrow C^r(\mathbb{T}^k, \mathbb{R}^\kappa) \text{ is invertible.} \tag{Y.19}$$

Consequently, if we fix $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, and if $H \in C^r(\mathbb{T}^k, \mathbb{R}^\kappa)$ has sufficiently small norm, i.e., if $(h_{ij}) \in C^{r+2}(\mathbb{T}^k, S^2 T^*)$ has sufficiently small norm, then (Y.15) has a unique solution $w \in C^{r+2}(\mathbb{T}^k, \mathbb{R}^\kappa)$, with small norm, and via (Y.14) we get a solution $v \in C^{r+2}(\mathbb{T}^k, \mathbb{R}^\mu)$, with small norm, to (Y.13). If the norm of v is small enough, then of course $u + v$ is also an imbedding.

Furthermore, if the C^{r+2}-norm of w is small enough, then (Y.15) is an *elliptic* system for w. By regularity results established, e.g., in Chapter 14, §4 of [T1], we can deduce that w is C^∞ (hence v is C^∞), if h is C^∞. This concludes the proof of Lemma Y.4, hence of Nash's imbedding theorem.

Exercises

In Exercises 1–3, let B be the unit ball in $\mathbb{R}^k$, centered at 0. Let (λ_{ij}) be a smooth symmetric matrix-valued function on B such that

$$\sum_{i,j} \int (\partial_i f)(x)\ (\partial_j f)(x)\ \lambda_{ij}(x)\ dx \le 0, \quad \forall f \in C_0^\infty(B). \tag{Y.20}$$

1. Taking $f_\varepsilon \in C_0^\infty(B)$ of the form

$$f_\varepsilon(x) = f(\varepsilon^{-2}x_1, \varepsilon^{-1}x'), \quad 0 < \varepsilon < 1,$$

examine the behavior as $\varepsilon \searrow 0$ of (Y.20), with f replaced by f_ε. Establish that $\lambda_{11}(0) \le 0$.

2. Show that the condition (Y.20) is invariant under rotations of $\mathbb{R}^k$, and deduce that $\big(\lambda_{ij}(0)\big)$ is a negative semidefinite matrix.

3. Deduce that $\big(\lambda_{ij}(x)\big)$ is negative semidefinite for all $x \in B$.

4. Using the results above, demonstrate the implication (Y.4) $\Rightarrow$ (Y.5), used in the proof of Lemma Y.3.

5. Suppose we have a C^∞ imbedding $\varphi : \mathbb{T}^k \to \mathbb{R}^n$. Take $\varepsilon > 0$. Define a map

$$\psi : \mathbb{T}^k \longrightarrow \mathbb{R}^n \oplus S^2\mathbb{R}^n \approx \mathbb{R}^\mu, \quad \mu = n + \tfrac{1}{2}n(n+1),$$

to have components

$$\varphi_j(x),\ 1 \le j \le n, \quad \varepsilon\varphi_i(x)\varphi_j(x),\ 1 \le i \le j \le n.$$

Show that ψ is a *free* imbedding.

6. Using Leibniz' rule to expand derivatives of products, verify that (Y.10) and (Y.11) are equivalent, for $v \in C^\infty(\mathbb{T}^k, \mathbb{R}^\mu)$.

7. In [Na], the system (Y.10) was augmented with $\partial_i u \cdot v = 0$, yielding, instead of (Y.12), the system

$$\begin{aligned} \zeta_i(x) \cdot v &= 0, \\ \zeta_{ij}(x) \cdot v &= \tfrac{1}{2}\big(\partial_i v \cdot \partial_j v - h_{ij}\big). \end{aligned} \tag{Y.21}$$

What makes this system more difficult to solve than (Y.12)?

Z. DeRham Cohomology of Compact Symmetric Spaces

A Riemannian manifold M is a homogeneous space provided its isometry group acts transitively on M. A connected homogeneous Riemannian manifold is a symmetric space if, in addition, given $p \in M$, there exists an isometry $\iota_p : M \to M$ such that

$$\iota_p(p) = p, \quad \text{and} \ \ d\iota_p(p) = -I \ \text{ on } \ T_pM. \tag{Z.1}$$

Let M be a compact symmetric space, and let G be the connected component of the identity in its isometry group. (Sometimes $\iota_p \in G$ and sometimes $\iota_p \notin G$.) For $0 \le k \le n = \dim M$, let

$$\mathcal{B}_k = \{\beta \in \Lambda^k(M) : g^*\beta = \beta, \ \forall\, g \in G\}. \tag{Z.2}$$

Let $\mathcal{H}_k$ denote the space of harmonic k-forms on M. Basic Hodge theory gives a natural isomorphism

$$\mathcal{H}_k \approx \mathcal{H}^k(M), \tag{Z.3}$$

where the right side of (Z.3) is the kth deRham cohomology group of M. We aim to prove the following.

Theorem Z.1. *If M is a compact, connected symmetric space,*

$$\mathcal{B}_k = \mathcal{H}_k. \tag{Z.4}$$

To begin, we note that, since G is connected, $\beta \in \mathcal{H}_k \Rightarrow g^*\beta$ is both harmonic and cohomologous to β, hence equal to β, so clearly

$$\mathcal{H}_k \subset \mathcal{B}_k. \tag{Z.5}$$

It remains to prove that $\mathcal{B}_k \subset \mathcal{H}_k$. We bring in some lemmas.

Lemma Z.2. *Given the isometry ι_p as in (Z.1),*

$$g \in G \Longrightarrow \iota_p g \iota_p^{-1} \in G. \tag{Z.6}$$

Proof. Take a continuous $\gamma : [0,1] \to G,\ \gamma(0) = e$ (the identity element), $\gamma(1) = g$. Then $\iota_p\gamma(t)\iota_p^{-1} \in G$ for all $t \in [0,1]$.

Lemma Z.3. *We have*

$$\beta \in \mathcal{B}_k \Longrightarrow \iota_p^* \beta \in \mathcal{B}_k. \tag{Z.7}$$

Proof. Given $g \in G$, Lemma Z.2 implies $\iota_p g \iota_p^{-1} = \tilde{g} \in G$. Now, for $\beta \in \mathcal{B}_k$,

$$g^* \iota_p^* \beta = (\iota_p g)^* \beta = (\tilde{g} \iota_p)^* \beta = \iota_p^* \tilde{g}^* \beta = \iota_p^* \beta, \tag{Z.8}$$

so (Z.7) holds.

Lemma Z.4. *We have*

$$\beta \in \mathcal{B}_k \Longrightarrow d\beta = 0. \tag{Z.9}$$

Proof. If $\beta \in \mathcal{B}_k$, we have that $\iota_p^* \beta \in \mathcal{B}_k$, and, for each $p \in M$,,

$$\iota_p^* \beta(p) = (-1)^k \beta(p). \tag{Z.10}$$

Since β and $\iota_p^* \beta$ belong to $\mathcal{B}_k$, (Z.10) plus the transitivity of G on M imply

$$\iota_p^* \beta = (-1)^k \beta \quad \text{on} \quad M. \tag{Z.11}$$

Also $d\beta \in \mathcal{B}_{k+1}$, so

$$\iota_p^* d\beta = (-1)^{k+1} d\beta \quad \text{on} \quad M. \tag{Z.12}$$

But $\iota_p^* d\beta = d\iota_p^* \beta = (-1)^k d\beta$ on M, by (Z.11), so $(-1)^{k+1} d\beta = (-1)^k d\beta$ on M, which forces $d\beta = 0$.

The following complement to Lemma Z.4 establishes the reverse inclusion $\mathcal{B}_k \subset \mathcal{H}_k$, and completes the proof of Theorem Z.1.

Lemma Z.5. *For $\delta = d^*$, we have*

$$\beta \in \mathcal{B}_k \Longrightarrow \delta\beta = 0. \tag{Z.13}$$

Proof. If M is oriented, it has a Hodge $*$-operator $* : \Lambda^k(M) \to \Lambda^{n-k}(M)$. Since each $g \in G$ is an orientation-preserving isometry, $*$ commutes with the action of such g on forms, so

$$* : \mathcal{B}_k \longrightarrow \mathcal{B}_{n-k}, \tag{Z.14}$$

and, since $\delta = \pm * d *$, (Z.13) follows from (Z.9).

If M is not orientable, we use the following argument (thanks to S. Kumar). While d commutes with all pull-backs φ^* for smooth $\varphi : M \to M$, $\delta = d^*$ commutes with φ^* as long as φ is an isometry. In such a case, the action of φ^* on forms is unitary (say U), and we have

$$dU = Ud \Rightarrow U^{-1}d^* = d^*U^{-1} \Rightarrow d^*U = Ud^*. \tag{Z.15}$$

Thus the proof of Lemma Z.4 extends as follows. If $\beta \in \mathcal{B}_k$, then for each $p \in M$, $\iota_p^*\beta \in \mathcal{B}_k$, and (11) holds. Also, since $g^*\delta = \delta g^*$ for $g \in G$, $\delta\beta \in \mathcal{B}_{k-1}$. Hence $\iota_p^*\delta\beta = \delta\iota_p^*\beta$ equals both $(-1)^k\delta\beta$ and $(-1)^{k-1}\delta\beta$, forcing $\delta\beta = 0$.

An alternative endgame for the proof of Theorem Z.1 (noted by S. Kumar) is to use $\beta \in \mathcal{B}_k \Rightarrow \delta\beta \in \mathcal{B}_k$ and apply Lemma Z.4 to deduce that $d\delta\beta = 0$, hence $\Delta\beta = -(d\delta\beta + \delta d\beta) = 0$.

It is desirable to say some more about $\mathcal{B}_k$. So take $p \in M$, and let

$$K = \{g \in G : gp = p\}, \tag{Z.16}$$

so K is a closed subgroup of G and

$$M = G/K. \tag{Z.17}$$

Given $g \in K$, we have the representation

$$\pi(g) = Dg(p) : T_pM \longrightarrow T_pM, \tag{Z.18}$$

of K on T_pM, yielding representations $\Lambda^k\pi$ of K on Λ^kT_pM.

Proposition Z.6. *Let M be a compact symmetric space, G the connected component of the identity in its isometry group, $p \in M$, and K as in (Z.16). Take $\mathcal{B}_k$ as in (Z.2). Then, for $0 \le k \le n$, $\mathcal{B}_k$ is isomorphic to*

$$\{v \in \Lambda^kT_pM : \Lambda^k\pi(g)v = v, \ \forall\, g \in K\}. \tag{Z.19}$$

Hence such a linear space is isomorphic to $\mathcal{H}^k(M)$.

We can denote the space (Z.19) as $\mathcal{I}^k(T_pM, K)$, where, generally, if V is a finite-dimensional real inner-product space and K a closed subgroup of $\mathrm{O}(V)$,

$$\mathcal{I}^k(V, K) = \{v \in \Lambda^kV : \Lambda^kg\, v = v, \ \forall\, g \in K\}. \tag{Z.20}$$

Note that

$$\mathcal{I}^*(V, K) = \bigoplus_k \mathcal{I}^k(V, K) \tag{Z.21}$$

has a natural ring structure, and, in the setting of Proposition Z.6, $\mathcal{I}^*(T_pM, K)$ is isomorphic to the cohomology ring $\mathcal{H}^*(M)$.

Z2. Spheres, and real and complex projective spaces

For a straightforward application of Proposition Z.6, we consider the n-dimensional sphere,

$$S^n = SO(n+1)/SO(n), \quad n \geq 2. \tag{Z2.1}$$

Here $\mathcal{B}_k$ is isomorphic to (Z.19) with $K = SO(n)$, acting in the canonical fashion on $\mathbb{R}^n \approx T_p S^n$. Clearly $\Lambda^0 \mathbb{R}^n$ and $\Lambda^n \mathbb{R}^n$ are one-dimensional spaces on which $SO(n)$ acts trivially. We claim that

$$\Lambda^\ell \mathbb{R}^n \text{ has no nonzero } SO(n) \text{ invariant elements, for } 1 \leq \ell \leq n-1. \tag{Z2.2}$$

To see this, take $\alpha \in \Lambda^\ell \mathbb{R}^n$ and write it as a linear combination of monomomials

$$e_{j_1} \wedge \cdots \wedge e_{j_\ell}, \quad 1 \leq j_1 < \cdots < j_\ell \leq n, \tag{Z2.3}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$. If $i, j \in \{1, \dots, n\}$, $i \neq j$, then let K_{ij} denote the group of rotations in the (e_i, e_j)-plane. Given a monomial (Z2.3) that contains one of the factors $\{e_i, e_j\}$ but not the other, the average of the action of K_{ij} applied to such a monomial is 0. All other monomials, i.e., those containing either both e_i and e_j or neither of these factors, are unaffected by such an averaging process, but each such monomial will be annihilated by averaging the action of some such subgroup K_{ab}. It follows that successively averaging any $\alpha \in \Lambda^\ell \mathbb{R}^n$ over all these various group actions yields 0. Hence, if α is $SO(n)$-invariant, it must be 0. Applying Proposition Z.6 and (Z2.2), we have

$$\begin{aligned} \mathcal{H}^\ell(S^n) \approx \mathbb{R}, &\quad \text{for } \ \ell = 0, n, \\ 0, &\quad \text{for } \ 1 \leq \ell \leq n-1. \end{aligned} \tag{Z2.4}$$

As a variant, consider real projective space,

$$\mathbb{RP}^n = SO(n+1)/(S(O(1) \times O(n)), \quad n \geq 2. \tag{Z2.5}$$

The group $K = S(O(1) \times O(n))$ sits in $SO(n+1)$ as

$$\Big\{ \begin{pmatrix} T & \\ & a \end{pmatrix} : a \in \{\pm 1\},\ T \in O(n),\ a \det T = 1 \Big\}. \tag{Z2.6}$$

The action of K on $T_p \mathbb{RP}^n = \mathbb{R}^n$ is given by

$$\pi \left(\begin{pmatrix} T & \\ & a \end{pmatrix} \right) v = aTv. \tag{Z2.7}$$

We have

$$\text{(Z2.8)} \qquad \begin{aligned} \pi(K) = SO(n), &\quad \text{if } n \text{ is odd,} \\ O(n), &\quad \text{if } n \text{ is even.} \end{aligned}$$

We deduce from (Z2.2) that

$$\text{(Z2.9)} \qquad \mathcal{H}^\ell(\mathbb{RP}^n) = 0 \quad \text{for} \ \ 1 \le \ell \le n-1.$$

Clearly $\Lambda^0\mathbb{R}^n$ is a one-dimensional space on which $O(n)$ acts trivially, so we have $H^0(\mathbb{RP}^n) \approx \mathbb{R}$, as we know, since $\mathbb{RP}^n$ is connected. Now $\Lambda^n\mathbb{R}^n$ is one dimensional, and as noted above, $SO(n)$ acts trivially on this space. However, $O(n)$ does not, so

$$\text{(Z2.10)} \qquad \Lambda^n\mathbb{R}^n \ \ \text{has no nonzero elements invariant under} \ \ O(n).$$

It follows from (Z2.8) that

$$\text{(Z2.11)} \qquad \begin{aligned} \mathcal{H}^n(\mathbb{RP}^n) \approx \mathbb{R}, &\quad \text{if } n \text{ is odd,} \\ 0, &\quad \text{if } n \text{ is even.} \end{aligned}$$

This reflects the fact that $\mathbb{RP}^n$ is orientable if n is odd, but not orientable if n is even.

Before taking up further specific cases, we record a general result that can be deduced from Proposition Z.6.

Proposition Z2.1. *In the setting of Proposition Z.6,*

$$\text{(Z2.12)} \qquad -I \in \pi(K) \Longrightarrow \mathcal{H}^\ell(M) = 0, \quad \textit{for all odd} \ \ \ell \in \mathbb{N}.$$

Proof. The transformation $-I \in \mathcal{L}(T_pM)$ acts as $(-1)^\ell$ on $\Lambda^\ell T_pM$.

Note. In light of (Z.1), the hypothesis $-I \in \pi(K)$ is equivalent to the hypothesis that ι_p belongs to the conected component of the identity in the isometry group of M. As a check, note that the conclusion in (Z2.12) holds for the sphere S^n when n is even, but fails (at $\ell = n$) when n is odd.

Let us see how Proposition Z2.1 applies to Grassmannians. Recall from Appendix V that the Grassmannian manifolds $G_{k,n}(\mathbb{F})$ of k-planes through the origin in $\mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) are given by

$$\text{(Z2.13)} \qquad G_{k,n}(\mathbb{R}) = SO(n)/S(O(k) \times O(n-k)),$$

and

$$\text{(Z2.14)} \qquad G_{k,n}(\mathbb{C}) = U(n)/(U(k) \times U(n-k)).$$

With base point given by $p = V_0 = \{(x_1, \dots, x_k, 0, \dots, 0) : x_j \in \mathbb{F}\}$, we have

$$
\begin{aligned}
T_p G_{k,n}(\mathbb{R}) &\approx \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k}), \\
T_p G_{k,n}(\mathbb{C}) &\approx \mathcal{L}(\mathbb{C}^k, \mathbb{C}^{n-k}).
\end{aligned} \tag{Z2.15}
$$

Regarding the latter identification, which is a vector space with a natural complex structure, keep in mind that the exterior products in (Z.19) are taken with T_pM regarded as a *real* vector space. Now, given $(A, B) \in K$, where, respectively, $A \in O(k), B \in O(n-k)$ and $(\det A)(\det B) = 1$, or $A \in U(k),\ B \in U(n-k)$, and given $T \in \mathcal{L}(\mathbb{F}^k, \mathbb{F}^{n-k})$, the action of $\pi(A, B)$ on T takes the form

$$
\pi(A, B)T = BTA^t. \tag{Z2.16}
$$

We see that $\pi(I_k, -I_{n-k}) = \pi(-I_k, I_{n-k}) = -I$ on $T_pG_{k,n}(\mathbb{F})$ provided $\mathbb{F} = \mathbb{C}$. For $\mathbb{F} = \mathbb{R}$, $(I_k, -I_{n-k}) \in K$ if $n - k$ is even, and $(-I_k, I_{n-k}) \in K$ if k is even. As a result, (Z2.12) applies to $M = G_{k,n}(\mathbb{C})$ for all $n \in \mathbb{N},\ 1 \le k \le n-1$, and it applies to $G_{k,n}(\mathbb{R})$ provided either k or $n - k$ is even.

Note that

$$
\mathbb{RP}^n = G_{1,n+1}(\mathbb{R}), \quad \mathbb{CP}^n = G_{1,n+1}(\mathbb{C}), \tag{Z2.17}
$$

so (Z2.12) applies to $\mathbb{CP}^n$ for all $n \in \mathbb{N}$, and it applies to $\mathbb{RP}^n$ if k is even (since $k = 1 \Rightarrow (n+1) - k = n$). Of course, this last observation merely recovers part of (Z2.9) and (Z2.11).

Results on $\mathcal{H}^\ell(\mathbb{CP}^n)$ are somewhat different from those on $\mathcal{H}^\ell(\mathbb{RP}^n)$. For one thing, as seen in Appendix V, the complex projective spaces $\mathbb{CP}^n$, and indeed all the complex Grassmannians $G_{k,n}(\mathbb{C})$, are Hermitian symmetric spaces, and hence, by Proposition V.1, Kähler manifolds. Thus, by (V.42),

$$
\mathcal{H}^{2j}(\mathbb{CP}^n) \neq 0, \quad \text{for} \ \ 1 \le j \le n. \tag{Z2.18}
$$

Before identifying these spaces more precisely, we will bring in another general result, Proposition Z2.2, whose proof will use some basic representation theory. To wit, if K is a compact Lie group and π a unitary representation of K on an inner product space (real or complex), then

$$
P = \int_K \pi(g)\, dg \tag{Z2.19}
$$

is the orthogonal projection of V onto the linear space V_0 on which π acts trivially. Hence

$$
\dim V_0 = \operatorname{Tr}\, P = \int_K \operatorname{Tr} \pi(g)\, dg. \tag{Z2.20}
$$

Here and below, dg denotes Haar measure on K. For a proof of (Z2.20), see §5 of [T2].

Proposition Z2.2. *In the setting of Proposition Z.6, if $n = \dim M$, then, for $\lambda \in \mathbb{R}$,*

$$\sum_{j=0}^{n} \lambda^{n-j} \dim \mathcal{H}^j(M) = \int_K \det_{\mathbb{R}}(\lambda I + \pi(g))\, dg. \tag{Z2.21}$$

Proof. We have

$$\begin{aligned} \dim \mathcal{H}^j(M) &= \dim \mathcal{B}_j \\ &= \int_K \operatorname{Tr}_{\mathbb{R}} \Lambda^j \pi(g)\, dg, \end{aligned} \tag{Z2.22}$$

by (Z2.20), so the left side of (Z2.21) is equal to

$$\int_K \sum_{j=0}^{n} \lambda^{n-j} \operatorname{Tr}_{\mathbb{R}} \Lambda^j \pi(g)\, dg. \tag{Z2.23}$$

But

$$\det_{\mathbb{R}}(\lambda I + \pi(g)) = \sum_{j=0}^{n} \lambda^{n-j} \operatorname{Tr}_{\mathbb{R}} \Lambda^j \pi(g), \tag{Z2.24}$$

so we have (Z2.21).

Using Proposition Z2.2, together with some more representation theory, we will establish the following.

Proposition Z2.3. *The deRham cohomology of complex projective space satisfies*

$$\mathcal{H}^{2j}(\mathbb{CP}^n) \approx \mathbb{R}, \quad \text{for } \ 1 \le j \le n. \tag{Z2.25}$$

Consequently, by (V.37)–(V.42), for each $j \in \{1, \dots, n\}$, $\mathcal{H}^{2j}(\mathbb{CP}^n)$ is spanned by ω^j, where ω is the Kähler form, given by (V.37).

Proof. By Proposition Z2.2, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \sum_{j=0}^{2n} \lambda^{2n-j} \dim \mathcal{H}^j(\mathbb{CP}^n) &= \int_{U(1)} \int_{U(n)} \det_{\mathbb{R}}(\lambda I + ag)\, dg\, da \\ &= \int_{U(n)} \det_{\mathbb{R}}(\lambda I + g)\, dg. \end{aligned} \tag{Z2.26}$$

Here, the representation π of $U(1) \times U(n)$ on $T_p\mathbb{CP}^n$ gives the standard action of ag on $\mathbb{C}^n$. The second identity in (Z2.26) uses left invariance of Haar measure on $U(n)$. Now we use the identity

$$A \in \mathcal{L}(\mathbb{C}^n) \Longrightarrow \det_{\mathbb{R}} A = |\det_{\mathbb{C}} A|^2 \tag{Z2.27}$$

to write the right side of (Z2.26) as

$$\int_{U(n)} |\det_{\mathbb{C}}(\lambda I + g)|^2 \, dg. \tag{Z2.28}$$

Now, parallel to (Z2.24), we have

$$\det_{\mathbb{C}}(\lambda I + g) = \sum_{j=0}^{n} \lambda^{n-j} \operatorname{Tr}_{\mathbb{C}} \Lambda^j_{\mathbb{C}} g, \tag{Z2.29}$$

so (Z2.28) is equal to

$$\sum_{i,j=0}^{n} \lambda^{2n-i-j} \int_{U(n)} \big(\operatorname{Tr} \Lambda^j_{\mathbb{C}} g\big) \overline{\big(\operatorname{Tr} \Lambda^i_{\mathbb{C}} g\big)} \, dg. \tag{Z2.30}$$

Now we bring in the fact that

$$\begin{aligned} \Lambda^j_{\mathbb{C}} g, \quad 0 \le j \le n, \ \ &\text{are mutually inequivalent irreducible} \\ &\text{unitary representations of } U(n) \text{ on } \ \Lambda^j_{\mathbb{C}}\mathbb{C}^n. \end{aligned} \tag{Z2.31}$$

See [T2], §20. This together with the Weyl orthogonality relations (cf. [T2], §6) gives

$$\int_{U(n)} \big(\operatorname{Tr} \Lambda^j_{\mathbb{C}} g\big) \overline{\big(\operatorname{Tr} \Lambda^i_{\mathbb{C}} g\big)} \, dg = \delta_{ij}, \tag{Z2.32}$$

and plugging this into (Z2.30), which is equal to (Z2.26), yields

$$\sum_{j=0}^{2n} \lambda^{2n-j} \dim \mathcal{H}^j(\mathbb{CP}^n) = \sum_{\ell=0}^{n} \lambda^{2n-2\ell}, \tag{Z2.33}$$

proving (Z2.25).

Z3. Real Grassmannians

We now take a closer look at the real Grassmannians $G_{k,n}(\mathbb{R})$, starting with the issue of orientability.

Proposition Z3.1. *Given $n \geq 2$, $1 \leq k \leq n-1$, $G_{k,n}(\mathbb{R})$ is orientable if and only if n is even.*

Proof. The manifold $G_{k,n}(\mathbb{R})$ has dimension $d = k(n-k)$, by (Z2.15). This manifold is orientable if and only if $\mathcal{H}_d \neq 0$, hence, by (Z.4), if and only if $\mathcal{B}_d \neq 0$. In this case, by (Z2.13),

$$K = S(O(k) \times O(n-k)) \tag{Z3.1}$$

acts on $\Lambda^\ell \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$, and the invariant elements of this space span $\mathcal{B}_\ell$. If $(A, B) \in S(O(k) \times O(n-k))$, with $A \in O(k)$, $B \in O(n-k)$, and $T \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$, then $\pi(A, B)T$ is given by (Z2.16). It follows that the action of $\Lambda^d \pi(A, B)$ on the one-dimensional space $\Lambda^d \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$ is given by multiplication by

$$\det \pi(A, B) = (\det A)^{n-k} (\det B)^k. \tag{Z3.2}$$

On the other hand, given $A \in O(k)$, $B \in O(n-k)$,

$$(A, B) \in S(O(k) \times O(n-k)) \Longrightarrow (\det A)(\det B) = 1. \tag{Z3.3}$$

Hence (Z3.2) is equal to $(\det A)^{n-2k}$, so $\mathcal{B}_d \neq 0$ if and only if n is even.

It is natural to look at the space $\widetilde{G}_{k,n}(\mathbb{R})$ of oriented k-planes through the origin in $\mathbb{R}^n$. This is a 2-fold cover of $G_{k,n}(\mathbb{R})$, given by

$$\widetilde{G}_{k,n}(\mathbb{R}) = SO(n)/SO(k) \times SO(n-k). \tag{Z3.4}$$

This time we have

$$\mathcal{H}^\ell(\widetilde{G}_{k,n}(\mathbb{R})) \approx \mathcal{B}_\ell, \tag{Z3.5}$$

with

$$\begin{aligned} \mathcal{B}_\ell = \{ v \in \Lambda^\ell \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k}) : \Lambda^\ell \pi(A, B) v = v, \\ \forall\, A \in SO(k),\ B \in SO(n-k) \}, \end{aligned} \tag{Z3.6}$$

where again the action $\pi(A, B)$ on $T \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{n-k})$ is given by (Z2.16). An argument parallel to the proof of Proposition Z3.1 shows that $\widetilde{G}_{k,n}(\mathbb{R})$ is always orientable. A variant of the analysis below (Z2.15) shows that (Z2.12) applies to $\widetilde{G}_{k,n}(\mathbb{R})$ provided either k or $n-k$ is even, so

$$k(n-k) \text{ even} \Longrightarrow \mathcal{H}^\ell(\widetilde{G}_{k,n}(\mathbb{R})) = 0, \quad \text{for all odd } \ell. \tag{Z3.7}$$

The algebraic problem of specifying the spaces (Z3.6) has some intricacy. Let us specialize to the case $k = 2$, and use

$$\mathcal{L}(\mathbb{R}^2, \mathbb{R}^{n-2}) \approx \mathbb{R}^2 \otimes V \approx V \oplus V, \quad V = \mathbb{R}^{n-2}, \tag{Z3.8}$$

together with

$$\text{(Z3.9)} \qquad \Lambda^\ell(V \oplus V) \approx \bigoplus_{j_1+j_2=\ell, j_\nu \ge 0} \Lambda^{j_1} V \otimes \Lambda^{j_2} V.$$

To start, let us specialize to the case $\ell = 2$, so we are looking at

$$\text{(Z3.10)} \qquad \Lambda^2(V \oplus V) \approx (\Lambda^0 V \otimes \Lambda^2 V) \oplus (V \otimes V) \oplus (\Lambda^2 V \otimes \Lambda^0 V),$$

carrying a natural $SO(2) \times SO(V)$ action. Each of the three summands on the right side of (Z3.10) is invariant under the $SO(V)$ action. The $SO(2)$ action on $V \oplus V$ is given by

$$\text{(Z3.11)} \qquad e^{tJ}, \quad J \in \mathcal{L}(V \oplus V), \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

An element $\beta \in \Lambda^2(V \oplus V)$ is fixed by the $SO(2)$ action if and only if

$$\text{(Z3.11A)} \qquad d\Lambda^2 J\, \beta = \frac{d}{dt} \Lambda^2 e^{tJ} \beta \big|_{t=0} = 0.$$

To see the action of $d\Lambda^2 J$, take an orthonormal basis of $V \oplus V$ of the form $\{e_j, f_j : 1 \le j \le n-2\}$, where $\{e_j\}$ is an orthonormal basis of the first summand V, and $\{f_j\}$ an orthonormal basis of the second. Then orthonormal bases of the three spaces on the right side of (Z3.10) are, respectively,

$$\text{(Z3.12)} \qquad \begin{gathered} \{f_j \wedge f_k : 1 \le j < k \le m\}, \quad \{e_j \wedge f_k : 1 \le j, k \le m\}, \\ \{e_j \wedge e_k : 1 \le j < k \le m\}, \end{gathered}$$

where $m = n - 2$. Since $Je_j = f_j$ and $Jf_j = -e_j$, the action of $d\Lambda^2 J$ is

$$\text{(Z3.13)} \qquad \begin{aligned} d\Lambda^2 J(f_j \wedge f_k) &= -e_j \wedge f_k - f_j \wedge e_k, \\ d\Lambda^2 J(e_j \wedge f_k) &= f_j \wedge f_k - e_j \wedge e_k, \\ d\Lambda^2 J(e_j \wedge e_k) &= f_j \wedge e_k + e_j \wedge f_k. \end{aligned}$$

Note in particular that

$$\text{(Z3.13A)} \qquad d\Lambda^2 J(e_j \wedge f_j) = 0, \quad d\Lambda^2 J(e_j \wedge e_k + f_j \wedge f_k) = 0.$$

We are now ready to specify the $SO(2) \times SO(V)$-fixed subspaces of $\Lambda^2(V \oplus V)$. We start with the $SO(V)$-fixed subspaces. As mentioned, $SO(V)$ leaves each summand on the right side of (Z3.10) invariant. If $\dim V = 2$, then $SO(V)$ acts trivially on the one-dimensional space $\Lambda^2 V$. If $\dim V > 2$, then, by (Z2.2), $SO(V)$ acts on $\Lambda^2 V$, with no nonzero invariant elements. As for the action on $V \otimes V$, the isomorphism $V \otimes V \approx \mathcal{L}(V)$, via the inner product on V, shows that there is a one-dimensional

$SO(V)$-fixed subspace, spanned by the identity $I \in \mathcal{L}(V)$, hence, in terms of the second basis in (Z3.12), by

$$\sigma = e_1 \wedge f_1 + \cdots + e_m \wedge f_m. \tag{Z3.14}$$

This element is also invariant under the action of $\Lambda^2 SO(2)$, as (Z3.13A) shows. In fact, σ is the symplectic form on $V \oplus V$, and all its exterior powers are invariant under the $SO(2) \times SO(V)$ action.

When $\dim V > 2$, $SO(V)$ acts irreducibly on $\mathbb{C} \otimes V$, and the subspace of $V \otimes V$ of elements on which $SO(V)$ acts trivially is one-dimensional, so the only contributions to $SO(V)$-fixed elements in $\Lambda^2(V \oplus V)$ from $V \otimes V$ are scalar multiples of (Z3.14). When $\dim V = 2$, there is a two-dimensional subspace of $V \otimes V \approx \mathcal{L}(V)$ on which $SO(V)$ acts trivially, spanned by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The image of the latter element in $\Lambda^2(V \oplus V)$ is given by

$$e_1 \wedge f_2 - e_2 \wedge f_1. \tag{Z.3.14A}$$

This element is not annihilated by the action of $d\Lambda^2 J$, as one sees from (Z.3.13). Hence this does not yield an additional $SO(2) \times SO(V)$-fixed element of $\Lambda^2(V \oplus V)$.

In case $\dim V = 2$, we also have the $SO(2) \times SO(V)$-invariant element of $(\Lambda^0 V \otimes \Lambda^2 V) \oplus (\Lambda^2 V \otimes \Lambda^0 V)$, given by

$$e_1 \wedge e_2 + f_1 \wedge f_2, \tag{Z3.15}$$

the $SO(2)$ invariance following from (Z3.13A).

These arguments establish the following.

Proposition Z3.2. *For $n \geq 4$, we have*

$$\begin{aligned} \mathcal{H}^2(\widetilde{G}_{2,n}(\mathbb{R})) \approx \mathbb{R}^2, &\quad \textit{if} \;\; n = 4, \\ \mathbb{R}, &\quad \textit{if} \;\; n > 4. \end{aligned} \tag{Z3.16}$$

From here, we can identify all the deRham cohomology groups $\mathcal{H}^\ell(\widetilde{G}_{2,n}(\mathbb{R}))$ for $n = 4$ and 5. Note that $\dim \widetilde{G}_{k,n}(\mathbb{R}) = 2(n-2)$, which is 4 for $n = 4$ and 6 for $n = 5$. We see that

$$\begin{aligned} \mathcal{H}^\ell(\widetilde{G}_{2,4}(\mathbb{R})) \approx \mathbb{R}, &\quad \text{if} \;\; \ell = 0 \;\text{ or }\; 4, \\ \mathbb{R}^2, &\quad \text{if} \;\; \ell = 2, \\ 0, &\quad \text{otherwise}, \end{aligned} \tag{Z3.17}$$

and

$$\mathcal{H}^\ell(\widetilde{G}_{2,5}(\mathbb{R})) \approx \begin{array}{ll} \mathbb{R}, & \text{if } \ell = 0,2,4,6, \\ 0, & \text{otherwise,} \end{array} \tag{Z3.18}$$

this by (Z3.16) for $\ell = 2$, and by Poincaré duality for $\ell = 4$.

We move on to an examination of $\mathcal{H}^\ell(\widetilde{G}_{2,n}(\mathbb{R}))$ for general ℓ. By (Z3.7), we can restrict attention to even ℓ, i.e., $\ell = 2\nu$, so we specialize (Z3.9) to

$$\begin{aligned} &\Lambda^{2\nu}(V \oplus V) \\ &\approx \bigoplus_{j_1+j_2=2\nu, j_\mu \geq 0} \Lambda^{j_1} V \otimes \Lambda^{j_2} V \\ &= (\Lambda^0 V \otimes \Lambda^{2\nu} V) \oplus (V \otimes \Lambda^{2\nu-1} V) \oplus \cdots \oplus (\Lambda^\nu V \otimes \Lambda^\nu V) \oplus \cdots \\ &\quad \cdots \oplus (\Lambda^{2\nu-1} V \otimes V) \oplus (\Lambda^{2\nu} V \otimes \Lambda^0 V). \end{aligned} \tag{Z3.19}$$

Again each summand is invariant under the $SO(V)$ action. We divide the study into two cases.

Case A. $\dim V \neq 2\nu$.

Then none of the summands on the right side of (Z3.19) can have $SO(V)$-fixed elements except perhaps

$$\Lambda^\nu V \otimes \Lambda^\nu V.$$

If $\nu > \dim V$, this space is 0. If $\nu \leq \dim V$ (but $2\nu \neq \dim V$), then $SO(V)$ acts irreducibly on the complexification of $\Lambda^\nu V$, so there is a one-dimensional subspace of $\Lambda^\nu V \otimes \Lambda^\nu V \approx \mathcal{L}(\Lambda^\nu V)$ consisting of $SO(V)$-fixed elements, namely the span of $I \in \mathcal{L}(\Lambda^\nu V)$. The image of this in $\Lambda^{2\nu}(V \oplus V)$ is

$$\sum_{1 \leq i_1 < \cdots < i_\nu \leq m} e_{i_1} \wedge \cdots \wedge e_{i_\nu} \wedge f_{i_1} \wedge \cdots \wedge f_{i_\nu}, \tag{Z3.20}$$

where $m = \dim V = n - 2$. This is a constant times σ^ν, with σ as in (23.14). One sees that this is annihilated by $d\Lambda^{2\nu} J$, and is fixed by the $SO(2)$-action. Thus $\Lambda^{2\nu}(V \oplus V)$ has a one-dimensional space of elements fixed by the $SO(2) \times SO(V)$ action, provided $\dim V \neq 2\nu$ and $\nu \leq \dim V$. This establishes the following.

Proposition Z3.3. *Assume $\nu \leq n - 2$, i.e., $2\nu \leq \dim \widetilde{G}_{2,n}(\mathbb{R})$. Then*

$$\mathcal{H}^{2\nu}(\widetilde{G}_{2,n}(\mathbb{R})) \approx \mathbb{R}, \tag{Z3.21}$$

provided

$$2\nu \neq n - 2, \quad \textit{i.e., provided } \ 2\nu \neq \frac{1}{2} \dim \widetilde{G}_{2,n}(\mathbb{R}). \tag{Z3.22}$$

Case B. $\dim V = 2\nu$.

The following result completes the description of the deRham cohomology of $\widetilde{G}_{2,n}(\mathbb{R})$. Here we also make use of basic results on representation theory for $SO(V)$, which can be found in [T2].

Proposition Z3.4. *Assume* $2\nu = n - 2$, *i.e.,*

$$2\nu = \frac{1}{2}\dim \widetilde{G}_{2,n}(\mathbb{R}). \tag{Z3.23}$$

Then

$$\mathcal{H}^{2\nu}(\widetilde{G}_{2,n}(\mathbb{R})) \approx \mathbb{R}^2. \tag{Z3.24}$$

Proof. In (Z3.19), we decompose $\Lambda^{2\nu}(V \oplus V)$ into $2\nu + 1$ pieces $\Lambda^\mu V \otimes \Lambda^{2\nu-\mu}V$, $\mu \in \{0, \dots, 2\nu\}$. For each such μ, this is a tensor product of two equivalent representations of $SO(V)$, intertwined by the Hodge star operator. If $\mu \neq \nu$, the representation of $SO(V)$ on the complexification of $\Lambda^\mu V$ is irreducible, so $\Lambda^\mu V \otimes \Lambda^{2\nu-\mu}V$ has a one dimensional subspace on which $SO(V)$ acts trivially, and it is spanned by

$$\alpha_\mu = \sum_{1\le i_1<\cdots<i_\mu\le 2\nu} e_{i_1} \wedge \cdots \wedge e_{i_\mu} \wedge *(f_{i_1} \wedge \cdots \wedge f_{i_\mu}). \tag{Z3.25}$$

If $\mu = \nu$, we have the following modification. The action of $SO(V)$ on the complexification of $\Lambda^\nu V$ has two irreducible components, and we have a two-dimensional space of elements of $\Lambda^\nu V \otimes \Lambda^\nu V$ on which $SO(V)$ acts trivially, namely the span of α_μ in (Z3.25), with $\mu = \nu$, together with the element (Z3.20), which, recall, is a nonzero multiple of σ^ν. Our remaining task is to take the space E_ν of $SO(V)$-invariant elements of $\Lambda^{2\nu}(V \oplus V)$, just described, and examine the behavior of $d\Lambda^{2\nu}J$ on this space (which is mapped to itself since $d\Lambda^{2\nu}J$ commutes with the $SO(V)$ action).

Note that since $\Lambda^{2\nu}e^{tJ}$ is a group of isometries on $\Lambda^{2\nu}(V \oplus V)$, $d\Lambda^{2\nu}J$ is skew adjoint on this space, hence on the subspace E_ν. As we have seen, $d\Lambda^{2\nu}J$ annihilates σ^ν. Hence is acts as a skew-adjoint transformation on

$$F_\nu = \mathrm{Span}\{\alpha_\mu : 0 \le \mu \le 2\nu\}, \tag{Z3.26}$$

a real vector space of dimension $2\nu + 1$. since

$$d\Lambda^{2\nu}J : F_\nu \longrightarrow F_\nu \tag{Z3.27}$$

is skew adjoint, its range must have even dimension, hence its null space must have odd dimension. The content of (Z3.24) is that this null space has dimension 1. To see this, we make the following observations. First,

$$d\Lambda^{2\nu}J : \mathrm{Span}(\alpha_\mu) \longrightarrow \mathrm{Span}(\alpha_{\mu-1}, \alpha_{\mu+1}), \quad 0 \le \mu \le 2\nu, \tag{Z3.28}$$

where, by convention, we set $\alpha_{-1} = \alpha_{2\nu+1} = 0$. Next

(Z3.29) $d\Lambda^{2\nu}J(\alpha_\mu)$ has a nonzero component in $\mathrm{Span}(\alpha_{\mu+1})$, for $0 \le \mu \le 2\nu-1$.

It follows that the range of $d\Lambda^{2\nu}J$ in (Z3.27) has dimension 2ν. Thus the null space of $d\Lambda^{2\nu}J$ on F_ν is one dimensional, and we have the proof of Proposition Z3.4.

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Index

A

B

C

D

E

F

G

H

I

N

O

P

R

S

T

U

V

W