Pseudodifferential Operators and K-Homology, II

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1. Introduction

This work is a continuation of [15], particularly §§3–4 of that paper. The first two sections of [15] recalled work of the author with P. Baum and R. Douglas [5] on relative K-homology classes in $K_j(\Omega, \partial\Omega)$ defined by elliptic operators on a manifold with boundary. Central to that work were identities in relative K-homology, which, via the boundary map $\partial : K_j(\Omega, \partial\Omega) \to K_{j-1}(\partial\Omega)$, led to identities in $K_{j-1}(\partial\Omega)$. Of particular interest is the following result, proved in [5]. Let Ω be a strongly pseudoconvex complex manifold. Then there is the Szegö projector S acting on $L^2(\partial\Omega)$, and also $\partial\Omega$ has a natural spin^c-structure, with associated Dirac operator D. The identity is

$$[D] = [S] \quad \text{in} \quad K_1(\partial\Omega). \tag{1.1}$$

As explained in [5], this provides a refinement of Boutet de Monvel's index theorem (and also an extension, as this result generalizes to a large class of weakly pseudoconvex manifolds). The identity (1.1) has also further implications for index theory.

In particular (1.1) applies to $\Omega = B^*M$, the ball bundle of a smooth compact manifold M. With the identity (1.1) in hand, it is possible to prove

$$[D] = \mathcal{P}_M \quad \text{in} \quad K_1(S^*M), \tag{1.2}$$

where $\mathcal{P}_M \in K_1(S^*M)$ is the "pseudodifferential operator extension," which associates to each $p \in C(S^*M)$ the pseudodifferential operator with principal symbol p, which is well defined modulo the ideal $\mathcal{K}$ of compact operators on $L^2(M)$, and hence gives a $*$-homomorphism $C(S^*M) \to \mathcal{Q}(L^2(M))$, the Calkin algebra. One way in which the identity (1.2) is useful is the following. There is the commutative diagram, analyzed in [3]

$$\begin{tikzcd} K^0(S^*M) \arrow[r, "\rho"] \arrow[d, "\cap\mathcal{P}_M"', "\approx"] & K_1(M) \arrow[d, "\text{id.}"] \\ K_1(S^*M) \arrow[r, "\pi_*"'] & K_1(M) \end{tikzcd} \tag{1.3}$$

with the following property. If $A : C^\infty(M, E_0) \to C^\infty(M, E_0)$ is an elliptic self adjoint pseudodifferential operator defining a class $[A] \in K_1(M)$, and if $E_+ \to S^*M$ is the vector subbundle of $\pi^*E_0 \to S^*M$ which is the direct sum of the positive eigenspaces of the symbol of A, then

$$\pi_*(E_+ \cap \mathcal{P}_M) = [A], \quad \text{and} \quad \rho([E_+]) = [A]. \tag{1.4}$$

By (1.2) we can replace $\mathcal{P}_M$ by $[D]$ in (1.3). Using the Bott map, it is then established in [5] that there is a commutative diagram of the form:

$$\begin{tikzcd} K^0(\widehat{M}) \arrow[r, "i_a"] \arrow[d, "\cap[D]"', "\approx"] & K_0(M) \arrow[d, "\text{id.}"] \\ K_0(\widehat{M}) \arrow[r, "\pi_*"'] & K_0(M) \end{tikzcd} \tag{1.5}$$

Here $\widehat{M}$ is the double of the ball bundle in T^*M. A vector bundle E over $\widehat{M}$ is obtained from a pair of bundles $E_0, E_1 \to M$ via the clutching construction, given a symbol over S^*M of an elliptic pseudodifferential operator $P : L^2(M, E_0) \to L^2(M, E_1)$, and

$$i_a([E]) = [P] \quad \text{in} \quad K_0(M). \tag{1.6}$$

Every elliptic pseudodifferential operator P is obtained, and amongst other things (1.5) implies that the index of every such pseudodifferential operator is equal to the index of an explicitly described twisted Dirac operator.

We note here another implication of (1.5); namely this can be used to prove that every element of $K_0(M)$ is represented by an elliptic pseudodifferential operator. Indeed, since the double of the ball bundle $\widehat{M} \to M$ has a section, the map $\pi_* : K_0(\widehat{M}) \to K_0(M)$ is surjective. Since the operation $\cap[D]$ is an isomorphism, i_a is also surjective, so the result is established.

It is interesting to compare this with the following commutative diagram, a variant of (1.3):

$$\begin{tikzcd} K^1(S^*M) \arrow[r, "\rho"] \arrow[d, "\cap\mathcal{P}_M"', "\approx"] & K_0(M) \arrow[d, "\text{id.}"] \\ K_0(S^*M) \arrow[r, "\pi_*"'] & K_0(M) \end{tikzcd} \tag{1.7}$$

Here, if $\Phi : S^*M \to \mathrm{Gl}(k, \mathbb{C})$ is a smooth map, defining a class $[\Phi] \in K^1(S^*M)$, then $\rho([\Phi])$ is the class of a pseudodifferential operator on M whose principal symbol is given by Φ. Thus the image of

$$\pi_* : K_0(S^*M) \longrightarrow K_0(M) \tag{1.8}$$

consists precisely of classes in $K_0(M)$ that can be represented by elliptic operators acting on *trivial* bundles, $P : L^2(M,\mathbb{C}^k) \to L^2(M,\mathbb{C}^k)$. When M has a nonvanishing vector field, the map (1.8) is surjective, but it is not always surjective. For example, with $M = S^2$, one has

$$K_0(S^2) = \mathbb{Z}\oplus\mathbb{Z}, \quad K_0(S^*S^2) = K^1(\mathbb{RP}^3) = \mathbb{Z}, \tag{1.9}$$

so π_* is certainly not surjective in this case.

Sections 3 and 4 of [15], upon which we will build in this paper, dealt with the K-cohomology of the C^*-algebra $\Psi^0(M)$, the L^2-operator norm closure of the algebra $OPS^0(M)$ of classical scalar pseudodifferential operators on order 0 on a compact manifold M. The K-cohomology groups $K^j(\Psi^0(M))$ are closely related to the K-homology groups of the sphere bundle S^*M, $K_j(S^*M)$. As noted in §3 of [15], the general K-cohomology exact sequence easily yields the following, with exact top rows and commuting squares:

$$\begin{tikzcd} 0 \arrow[r] & K_0(S^*M) \arrow[r, "\sigma^*"] \arrow[d, "\pi_*"'] & K^0(\Psi^0(M)) \arrow[r] \arrow[d, "\mu^*"] & 0 \\ & K_0(M) \arrow[r, "\mathrm{id.}"'] & K_0(M) & \end{tikzcd} \tag{1.10}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot\mathcal{P}_M"] & K_1(S^*M) \arrow[r, "\sigma^*"] \arrow[d, "\pi_*"'] & K^1(\Psi^0(M)) \arrow[r] \arrow[d, "\mu^*"] & 0 \\ & & K_1(M) \arrow[r, "\mathrm{id.}"'] & K_1(M) & \end{tikzcd} \tag{1.11}$$

In concert with Poincaré duality, we hence have isomorphisms

$$K^0(\Psi^0(M)) \approx K^1(S^*M), \quad K^1(\Psi^0(M)) \approx \widetilde{K}^0(S^*M). \tag{1.12}$$

One of our motivations for looking at $K^j(\Psi^0(M))$ is that elliptic pseudodifferential operators, which define cycles for classes in $K_j(M)$, may also define classes in $K^j(\Psi^0(M))$ that have more structure. For example, if $A : C^\infty(M,E) \to C^\infty(M,E)$ is a self adjoint elliptic operator in $OPS^m(M)$, let Q denote the orthogonal projection onto the positive spectral space of A, $Q \in OPS^0(M)$, acting on $L^2(M,E)$. Let $\mathcal{R}$ denote its range. Then

$$f \mapsto QM_f\big|_{\mathcal{R}} \pmod{\mathcal{K}} \tag{1.13}$$

defines a $*$-homomorphism $C(M) \to \mathcal{Q}(\mathcal{R})$, hence an element $[A] \in K_1(M)$. (Generally, $\mathcal{K}$ will denote the space of compact operators on a Hilbert space, and $\mathcal{Q}$ the Calkin algebra.) Now also

$$P \mapsto Q\widetilde{P}\big|_{\mathcal{R}} \pmod{\mathcal{K}}, \tag{1.14}$$

where $P \in OPS^0(M)$ is scalar and $\widetilde{P}$ is any operator on $L^2(M,E)$ with the same (scalar) principal symbol as P, defines a *-homomorphism $\Psi^0(M) \to \mathcal{Q}(\mathcal{R})$, hence an element $\{A\} \in K^1(\Psi^0(M))$, satisfying

$$\mu^*\{A\} = [A]. \tag{1.15}$$

The extra structure in $\{A\}$ arises because its construction uses the fact that pseudodifferential operators of order 0 (such as Q) have compact commutator with general 0 order pseudodifferential operators with scalar principal symbol, not just with multiplication operators, and it is reflected in the fact that the map $\mu^* : K^1(\Psi^0(M)) \to K_1(M)$ is not generally injective, though it is always surjective.

In case $M = \partial\Omega$ is the boundary of a strongly pseudoconvex manifold, the Szegö projector S belongs to $OPS^0_{1/2,1/2}(M)$, so it has compact commutator with elements of $OPS^0(M)$. It follows that the map

$$P \mapsto SP\big|_{\mathrm{Range}\, S} \pmod{\mathcal{K}} \tag{1.16}$$

gives a *-homomorphism $\Psi^0(M) \to \mathcal{Q}(\mathrm{Range}\, S)$, hence an element $\{S\} \in K^1(\Psi^0(\partial\Omega))$, as noted in [15]. Since the Dirac operator D on $\partial\Omega$ also gives an element $\{D\} \in K^1(\Psi^0(M))$, it is natural to ask if we can specify a relation between these two elements, refining the result (1.1). We will show in §3 of this paper that in some cases the identity (1.1) can be strengthened to $\{D\} = \{S\}$ in $K^1(\Psi^0(\partial\Omega))$.

An elliptic pseudodifferential operator $P : C^\infty(M, E_0) \to C^\infty(M, E_1)$ gives rise to an element $[P]$ of $K_0(M)$, but not necessarily an element of $K^0(\Psi^0(M))$. To see this, we recall how $[P]$ arises. If P has order 0, it intertwines mod $\mathcal{K}$ the *-representations of $C(M)$ on $L^2(M, E_0)$ and $L^2(M, E_1)$, given by scalar multiplication. Now there are not such *-representations of $\Psi^0(M)$ on $L^2(M, E_j)$ in general. If E_j is trivial there is, but as noted in §3 of [15], an obstruction to this for nontrivial E_j is given by a map

$$\tau : K^0(M) \longrightarrow K^1(\Psi^0(M)), \tag{1.17}$$

characterized as follows. If $E \to M$ is a vector bundle, $P \in OPS^0(M)$ is scalar, then $\widetilde{P} : L^2(M,E) \to L^2(M,E)$ with scalar principal symbol equal to the symbol of P is well defined mod $OPS^{-1}(M)$, and this gives rise to a *-homomorphism $\Psi^0(M) \to \mathcal{Q}(L^2(M,E))$, defining $\tau(E) \in K^1(\Psi^0(M))$. Now $\tau(E) = 0$ when E is trivial, so (1.17) factors through a map

$$\tilde{\tau} : \widetilde{K}^0(M) \longrightarrow K^1(\Psi^0(M)). \tag{1.18}$$

As shown in §3 of [15], ker τ consists of $[E] \in K^0(M)$ such that $\pi^*[E]$ is stably trivial over S^*M, where $\pi^* : K^0(M) \to K^0(S^*M)$ is the natural map. Whenever π^* is injective, for example when M has a nonvanishing vector field, the map $\tilde{\tau}$ in (1.18) is injective.

The map τ also helps measure the failure of μ^* in (1.11) to be injective, as one clearly has $\mu^*\tau = 0$. In §3 of [15] it was shown that the sequence

$$\widetilde{K}^0(M) \xrightarrow{\tilde{\tau}} K^1(\Psi^0(M)) \xrightarrow{\mu^*} K_1(M) \longrightarrow 0 \tag{1.19}$$

is exact whenever M has a trivial tangent bundle. There are indications that this exactness holds more generally, though I have not been able to establish the exactness for all M. We recall from (3.33) of [15] that this is equivalent to exactness of

$$K^0(M) \xrightarrow{\pi^*} K^0(S^*M) \xrightarrow{\rho} K_1(M) \longrightarrow 0, \tag{1.20}$$

with ρ as in (1.3).

Returning to $K^0(\Psi^0(M))$, we conclude that an elliptic pseudodifferential operator $P : C^\infty(M, E_0) \to C^\infty(M, E_1)$ determines an element $\{P\} \in K^0(\Psi^0(M))$ provided E_j are trivial bundles. There is an essential converse to this result, which is derived as follows. The commutative diagram (1.7) can be enlarged to the diagram

$$\begin{tikzcd} K^1(S^*M) \arrow[r, "\rho"] \arrow[d, "\cap \mathcal{P}_M"', "\approx"] & K_0(M) \\ K_0(S^*M) \arrow[r, "\approx", "\sigma^*"'] & K^0(\Psi^0(M)) \arrow[u, "\mu^*"'] \end{tikzcd} \tag{1.21}$$

satisfying the property that, if $P \in OPS^0(M, \mathbb{C}^k)$ is elliptic, with symbol $\Phi : S^*M \to \mathrm{Gl}(k, \mathbb{C})$, defining $\{\Phi\} \in K^1(S^*M)$, then

$$\sigma^*([\Phi] \cap \mathcal{P}_M) = \{P\} \ \text{ in } \ K^0(\Psi^0(M)). \tag{1.22}$$

Thus, complementing the conclusions drawn after (1.7), we see that every element of $K^0(\Psi^0(M))$ is of the form $\{P\}$ where P is an elliptic operator on a trivial bundle over M.

We remark that if P is an elliptic operator between sections of trivial bundles E_0 and E_1, the class of P depends on a choice of trivializations of these bundles. Different trivializations might yield different K-cohomology classes.

In addition to (1.21), we also have the following enlargement of the diagram (1.3):

$$\begin{tikzcd} K^0(S^*M) \arrow[r, "\text{id.}"] & K^0(S^*M) \arrow[r, "\rho"] \arrow[d, "\approx"', "\cap \mathcal{P}_M"] & K_1(M) \arrow[d, "\mu^*"] \\ 0 \to \mathbb{Z} \arrow[u] \arrow[r] & K_1(S^*M) \arrow[r, "\sigma^*"'] & K^1(\Psi^0(M)) \to 0 \end{tikzcd} \tag{1.23}$$

the bottom row being exact. In particular, the identity (1.4) has the refinement

$$\sigma^*(E_+ \cap \mathcal{P}_M) = \{A\} \quad \text{in} \quad K^1(\Psi^0(M)). \tag{1.24}$$

Indeed, when the bundle $E_0 \to M$ on whose sections A acts is trivial, the proof of (1.4) given in Lemma 4.13 of [3] actually implies (1.24). To establish (1.24) in general it remains to note that when $A = I$ on $L^2(M, E)$, so $E_+ = \pi^* E$, then

$$\sigma^*(\pi^* E \cap \mathcal{P}_M) = \tau(E), \tag{1.25}$$

which is easily verified; compare Proposition 3.1 of [15].

As we stated above, one of the main goals of this paper is to derive identities in $K^j(\Psi^0(M))$ refining previously obtained identities in $K_j(M)$. The identities (1.22) and (1.24) are elementary examples of this. In the next section we set up some machinery to derive more subtle identities. As in [5], these are obtained by applying the (co)boundary map to certain relative cohomology classes. In §2 we set up a C^*-algebra Ξ of operators on a compact manifold Ω with boundary $\partial\Omega = M$, and a *-ideal Ξ_0 such that $\Xi/\Xi_0 \approx \Psi^0(M)$. The pair Ξ, Ξ_0 also has other technically useful properties, required to carry out the necessary program. In §3 we derive extensions of (1.1), for certain types of pseudoconvex domains. We see this as the first look into a phenomenon that might be quite subtle.

Identities in K-(co)homology lead to index formulas, via the intersection product. In §4 we make note of some properties of the intersection product $K_j(\Psi^0(M)) \times K^j(\Psi^0(M)) \to \mathbb{Z}$, and how this relates to identities obtained in §3.

As results of this paper focus on K-cohomology and K-homology of algebras of pseudodifferential operators, we note that papers of Brylinski-Getzler [9] and Wodzicki [17] deal with related subjects. These papers focus on rather different phenomena from those considered here, and there seems to be almost no overlap with the present paper, though future developments might incorporate these various perspectives.

2. Approach to identities in $K^1(\Psi^0)$ via the (co)boundary map

Here we develop further the technique of using the coboundary map

$$\delta : K^0(\mathfrak{A}, \mathfrak{I}) \longrightarrow K^1(\mathfrak{A}/\mathfrak{I}) \tag{2.1}$$

to produce identities in $K^1(\mathfrak{A}/\mathfrak{I})$, which figured in the analysis in [5]. First we recall the general set-up. If $\mathfrak{A}$ is a C^*-algebra with unit and $\mathfrak{I}$ a *-ideal, a cycle defining a class in $K^0(\mathfrak{A}, \mathfrak{I})$ consists of a pair (σ, T). Here $\sigma = \sigma_0 \oplus \sigma_1$ is a representation of $\mathfrak{A}$ on a sum $H_0 \oplus H_1$ of Hilbert spaces, and $T : H_0 \to H_1$ is a bounded map, assumed to have closed range and be a partial isometry mod $\mathcal{K}$, the space of compact operators. We require in addition the following two conditions:

$$\sigma_1(f)T = T\sigma_0(f) \mod \mathcal{K}, \quad \forall\, f \in \mathfrak{A}, \tag{2.2}$$

and

$$\sigma_j(f)P_j \in \mathcal{K}, \quad \forall\, f \in \mathfrak{J}, \tag{2.3}$$

where P_j is the orthogonal projection onto Ker T or Ker T^*, for $j = 0$ or 1. Given such a cycle, the image $\delta[(\sigma, T)] \in K^1(\mathfrak{A}/\mathfrak{J})$ is defined as follows. We have *-homomorphisms

$$\tau_j : \mathfrak{A}/\mathfrak{J} \longrightarrow \mathcal{Q}(\mathrm{Range}\, P_j), \tag{2.4}$$

given by

$$\tau_j(f) = P_j \sigma_j(\tilde{f}) P_j \pmod{\mathcal{K}}, \tag{2.5}$$

$\tilde{f} \in \mathfrak{A}$ being any preimage of $f \in \mathfrak{A}/\mathfrak{J}$. The hypothesis (2.3) guarantees that (2.5) is well defined mod $\mathcal{K}$. Thus we have classes $[\tau_j] \in K^1(\mathfrak{A}/\mathfrak{J})$, and

$$\delta[(\sigma, T)] = [\tau_0] - [\tau_1] \in K^1(\mathfrak{A}/\mathfrak{J}). \tag{2.6}$$

A pair (σ, T) as above satisfies more than enough conditions to define also a class in the Kasparov group $KK(\mathfrak{J}, \mathbb{C})$. In [4] it was proved that there is an isomorphism $KK(\mathfrak{J}, \mathbb{C}) \approx K^0(\mathfrak{A}, \mathfrak{J})$, which makes (2.1) consistent with Kasparov's coboundary map $\delta : KK(\mathfrak{J}, \mathbb{C}) \to K^1(\mathfrak{A}/\mathfrak{J})$. One consequence of this isomorphism is that the rich set of equivalence relations established amongst cycles in Kasparov K-theory can to a large degree be transferred to $K^0(\mathfrak{A}, \mathfrak{J})$. In particular one has the following result. Let (σ, T') define another cycle for $K^0(\mathfrak{A}, \mathfrak{J})$, with $\sigma = \sigma_0 \oplus \sigma_1$ as before. Then, provided

$$\begin{aligned} T\sigma_0(f) &= T'\sigma_0(f) \mod \mathcal{K}, \\ \sigma_1(f)T &= \sigma_1(f)T' \mod \mathcal{K}, \end{aligned} \tag{2.7}$$

for all $f \in \mathfrak{J}$, we have

$$[(\sigma, T)] = [(\sigma, T')] \in K^0(\mathfrak{A}, \mathfrak{J}). \tag{2.8}$$

Applications of the formula (2.6) for δ to identities of the form (2.8) yield identities in $K^1(\mathfrak{A}/\mathfrak{J})$ that are nontrivial. As mentioned in the introduction, the identity (1.1) is an example of this, established in [5].

We produce identities (2.8) by taking different closed extensions of an unbounded operator D from H_0 to H_1, with densely defined domain in H_0. Suppose D_e is one such closed extension. We set

$$T = D_e(D_e^* D_e + I)^{-1/2}, \tag{2.9}$$

a bounded operator from H_0 to H_1. In §1 of [5] we established a sufficient condition that (σ, T) define a cycle for $K^0(\mathfrak{A}, \mathfrak{J})$, a variant of a result of Baaj-Julg. We suppose there is a dense *-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ such that

$$\sigma_0(f) \ \text{ preserves } \ \mathcal{D}(D_e), \quad \forall\, f \in \mathfrak{A}_0, \tag{2.10}$$

and

$$\sigma_1(f)D_e - D_e\sigma_0(f) \ \text{ extends from } \mathcal{D}(D_e) \ \text{ to a bounded operator } H_0 \to H_1. \tag{2.11}$$

Suppose furthermore that

$$\begin{aligned} &\text{Either } (D_e^*D_e + I)^{-1} \in \mathcal{K}(H_0), \\ &\text{or } (D_eD_e^* + I)^{-1} \in \mathcal{K}(H_1). \end{aligned} \tag{2.12}$$

Our result is the following.

Lemma 2.1. *Under hypotheses (2.10)–(2.12), T is a partial isometry mod $\mathcal{K}$ with closed range, satisfying the condition (2.2). Thus (σ, T) defines a cycle for $K^0(\mathfrak{A}, \mathfrak{J})$ provided also (2.3) holds, or equivalently provided*

$$\sigma_0(f)(D_e^*D_e + I)^{-1} \ \textit{ and } \ \sigma_1(f)(D_eD_e^* + I)^{-1} \ \textit{ are compact, } \forall\, f \in \mathfrak{J}. \tag{2.13}$$

This result was applied in [5] in the following situation. With $\overline{\Omega}$ a compact manifold with boundary, we took $\mathfrak{A} = C(\overline{\Omega})$, $\mathfrak{J} = C_0(\Omega)$, the space of functions $f \in C(\overline{\Omega})$ vanishing on the boundary $\partial\Omega$. Thus $\mathfrak{A}/\mathfrak{J} = C(M)$, with $M = \partial\Omega$. The operator D was a first order elliptic differential operator between sections of vector bundles, and we took $\mathfrak{A}_0 = C^\infty(\overline{\Omega})$. The maps σ_j gave the usual scalar action on sections of vector bundles. In that case, we saw that (2.10)–(2.13) hold for certain closed extensions D_e of D; for example they hold for the maximal extension $D_{\max}$. For a pair of closed extensions of D satisfying the hypotheses of Lemma 2.1, identities of the form (2.8) were proved in [5] via a finite propagation speed argument, of a sort that will be extended in the proof of Theorem 2.4 below.

We now construct a new algebra of operators on $L^2(\overline{\Omega})$, when $\overline{\Omega}$ is such a compact manifold with boundary $\partial\Omega = M$. Impose a collaring $\mathcal{C} = [0,1) \times M$ on a neighborhood of $\partial\Omega$ in $\overline{\Omega}$; $\{0\} \times M = \partial\Omega$. Give $\overline{\Omega}$ a Riemannian metric, of product type on $\mathcal{C}$. We say an operator P belongs to $\mathfrak{P}$ if it is of the following form:

$$Pu(x) = p(x)u(x), \quad x \in \overline{\Omega} \setminus \mathcal{C}, \tag{2.14}$$

with $p \in C^\infty(\overline{\Omega} \setminus \mathcal{C})$, and, on $\mathcal{C}$, with $(s, y) \in [0,1) \times M$,

$$Pu(s, y) = P(s)u(s, y), \tag{2.15}$$

with $P(s)=p(s,y,D_y)$ a smooth function of $s\in[0,1]$ with values in $OPS^0(M)$, having the property that for $s\in(1/2,1)$, $P(s)$ is a multiplication operator, matching up smoothly with $p(x)$ in (2.14) at $s=1$. Note that the L^2-operator norm of such P is given by

$$\|P\|=\sup\{|p(x)|,\ x\in\overline{\Omega}\setminus\mathcal{C},\ \|P(s)\|,\ s\in[0,1]\},\tag{2.16}$$

where $\|P(s)\|$ denotes the operator norm of $P(s)$ on $L^2(M)$. Let $\mathfrak{P}_0$ denote the subset of $\mathfrak{P}$ consisting of operators such as just described such that $P(0)=0$, and let $\mathfrak{P}_{00}$ denote the subset consisting of those operators such that $P(s)=0$ for s sufficiently close to 0.

Let Ξ be the L^2-operator norm closure of $\mathfrak{P}$, and let Ξ_0 be the closure of $\mathfrak{P}_0$. From (2.16) it easily follows that

$$\Xi/\Xi_0\approx\Psi^0(M),\tag{2.17}$$

and that

$$\mathfrak{P}_{00}\ \text{ is dense in }\ \Xi_0.\tag{2.18}$$

From (2.17), we have $\delta:K^0(\Xi,\Xi_0)\rightarrow K^1(\Psi^0(M))$.

Now let $D:C^\infty(\overline{\Omega},E_0)\rightarrow C^\infty(\overline{\Omega},E_1)$ be an elliptic first order differential operator between sections of vector bundles $E_j\rightarrow\overline{\Omega}$. Let D_e be a closed extension such that $\mathcal{D}(D_{\min})\subset\mathcal{D}(D_e)\subset\mathcal{D}(D_{\max})$. We make the assumption

$$E_0\ \text{ and }\ E_1\ \text{ are trivial bundles over }\ \mathcal{C},\tag{2.19}$$

which holds provided E_0 and E_1 are trivial over $\partial\Omega$. In fact, it suffices to assume that E_0 is trivial over $\partial\Omega$, since the symbol of D evaluated at the conormal vector gives an isomorphism of E_0 and E_1 over $\partial\Omega$. Given trivializations of E_j over $\mathcal{C}$, the algebra Ξ acts on $L^2(\Omega,E_j)$ in a natural way; denote these actions by σ_j. Let us furthermore assume that

$$\sigma_0(P)\ \text{ presrves }\ \mathcal{D}(D_e),\quad\forall\,P\in\mathfrak{P}.\tag{2.20}$$

It follows that

$$\sigma_1(P)D_e-D_e\sigma_0(P)\ \text{ is bounded from }\ L^2(\Omega,E_0)\ \text{ to }\ L^2(\Omega,E_1),\tag{2.21}$$

for each $P\in\mathfrak{P}$, since this commutator is multiplication by a smooth $\operatorname{Hom}(E_0,E_1)$ valued function on $\overline{\Omega}\setminus\mathcal{C}$, and on $\overline{\mathcal{C}}=[0,1]\times M$ we have

$$[p(s,y,D_y),D_{y_j}]=q_j(s,y,D_y),\tag{2.22}$$

a smooth function of s with values in $OPS^0(M)$, and

$$[p(s,y,D_y),D_s]=\frac{\partial p}{\partial s}(s,y,D_y).\tag{2.23}$$

We can hence apply Lemma 2.1 and deduce the following.

Proposition 2.2. *If hypotheses (2.19)–(2.20) hold and if either* $D_e^*D_e$ *or* $D_eD_e^*$ *has compact resolvent, then* T*, given by (2.9), is a partial isometry, mod* $\mathcal{K}$*, with closed range, and the pair* (σ, T) *defines a cycle for* $K^0(\Xi, \Xi_0)$*.*

Proof. It remains only to verify (2.13) in this case, with $f = P \in \mathfrak{P}_{00}$, in view of (2.18). Since $(D_e^*D_e + I)^{-1}$ and $(D_eD_e^* + I)^{-1}$ both map $L^2(\Omega, E_j)$ to $H^2_{\mathrm{loc}}(\Omega, E_j)$, this follows from Rellich's theorem.

We denote the relative cohomology class by

$$\{D_e\} \in K^0(\Xi, \Xi_0). \tag{2.24}$$

We note that as long as (2.19) holds, (2.20) is always valid for $D_e = D_{\max}$, so we have $\{D_{\max}\} \in K^0(\Xi, \Xi_0)$ in such a case. We consider more generally the following class of extensions. Let F be a vector bundle over $\partial\Omega$, let $B \in C^\infty(\partial\Omega, \mathrm{Hom}(E_0, F))$ have constant rank, and let D_B be the closure of the restriction of D to

$$\{u \in C^\infty(\overline{\Omega}, E_0) : Bu = 0 \ \text{ on } \ \partial\Omega\}. \tag{2.25}$$

(Note that $D_B = D_{\max}$ if $B = 0$.) Then an action σ_0 of $\mathfrak{P}$ on $C^\infty(\overline{\Omega}, E_0)$ can be defined so that $\mathcal{D}(D_B)$ is preserved provided the following condition is satisfied:

$$\begin{array}{l}\text{Ker } B \text{ is a trivial subbundle of } \ E_0\big|_{\partial\Omega}, \\ \text{with a trivial complementary bundle.}\end{array} \tag{2.26}$$

Thus Proposition 2.2 yields the following.

Corollary 2.3. *Let* $D : C^\infty(\overline{\Omega}, E_0) \to C^\infty(\overline{\Omega}, E_1)$ *be a first order elliptic differential operator between sections of bundles that are trivial over* $\partial\Omega$*, and let* D_B *be the closure of* D *acting on the space (2.25). Assume that (2.26) holds. Then, provided either* $D_B^*D_B$ *or* $D_BD_B^*$ *has compact resolvent, the conclusion of Proposition 2.2 holds, and we have a class*

$$\{D_B\} \in K^0(\Xi, \Xi_0). \tag{2.27}$$

We are now ready to state the main result of this section, giving rise to identities in $K^1(\Psi^0(M))$.

Theorem 2.4. *Let* D_B *and* D_C *be two closed extensions of an elliptic operator* D*, satisfying all the hypotheses of Corollary 2.3. Then*

$$\{D_B\} = \{D_C\} \ \ \textit{in} \ \ K^0(\Xi, \Xi_0). \tag{2.28}$$

Hence

$$\delta\{D_B\} = \delta\{D_C\} \ \ \textit{in} \ \ K^1(\Psi^0(M)). \tag{2.29}$$

Proof. The identity (2.28) is established by an argument similar to the proof of the identity (2.8) in [5]; compare the discussion following Proposition 1.1 in [15]. In detail, we set

$$B = \begin{pmatrix} 0 & D_B^* \\ D_B & 0 \end{pmatrix}, \tag{2.30}$$

and define B' similarly, with D_B replaced by D_C. Then B and B' are self adjoint on $L^2(\Omega, E_0 \oplus E_1)$. It suffices to show that

$$\varphi(B)\sigma(P) = \varphi(B')\sigma(P) \mod \mathcal{K}, \quad \forall\, P \in \mathfrak{P}_{00}, \tag{2.31}$$

where

$$\varphi(\lambda) = \frac{\lambda}{\sqrt{\lambda^2+1}}. \tag{2.32}$$

The operator $\varphi(B)$, defined by the Spectral Theorem, can be analyzed as

$$\varphi(B) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{itB}\, dt. \tag{2.33}$$

Here $\hat{\varphi} \in \mathcal{S}'(\mathbb{R})$ is singular only at $t = 0$; on $\mathbb{R}\setminus 0$ it is smooth, and rapidly decreasing as $|t| \to \infty$. For each $\varepsilon > 0$, we can write $\varphi = \varphi_1 + \varphi_2$ with supp $\varphi_1 \subset (-\varepsilon, \varepsilon)$ and $\varphi_2 \in \mathcal{S}(\mathbb{R})$. Thus, for any given $P \in \mathfrak{P}_{00}$, we can write the difference of the two sides of (2.31) as

$$[\varphi_1(B) - \varphi_1(B')]\sigma(P) + [\varphi_2(B) - \varphi_2(B')]\sigma(P). \tag{2.34}$$

We can analyze the first of these two terms by exploiting finite propagation speed for e^{itB} and $e^{itB'}$, solution operators to symmetric hyperbolic systems. Thus, for any given $P \in \mathfrak{P}_{00}$, if ε is picked small enough, the first term in (2.34) *vanishes.* The second term is the product of the compactly supported factor $\sigma(P)$ by a smoothing operator, and hence is compact. This proves the theorem.

We can calculate $\delta\{D_{\max}\}$ by the same argument used to prove Proposition 4.4 of [5], in case D is an operator of "Dirac type," i.e., the symbol $\sigma_D(x,\xi) : E_{0x} \to E_{1x}$ is proportional to an isometry, $\|\sigma_D(x,\xi)v\| = \|\xi\| \cdot \|v\|$. In this case there is a self adjoint first order differential operator $D^\#$ on $\partial\Omega = M$, whose principal symbol is of the form

$$\tau(x)^{-1}\sigma_D(x,\xi) : E_{0x} \to E_{0x}, \quad \tau(x) = \frac{1}{i}\sigma(x,\nu), \tag{2.35}$$

for $x \in \partial\Omega$, ν denoting the conormal to $\partial\Omega$. The operator $D^\#$ is of Dirac type. Precisely as in Proposition 4.4 of [5], we have the following result:

Proposition 2.5. *Let* $D : C^\infty(\overline{\Omega}, E_0) \to C^\infty(\overline{\Omega}, E_1)$ *be a first order elliptic differential operator between sections of bundles that are trivial over* $\partial\Omega = M$*. Then, with* $D^\#$ *specified above*

$$\delta\{D_{\max}\} = \{D^\#\} \quad \text{in} \quad K^1(\Psi^0(M)). \tag{2.36}$$

Consequently, if D_B *is any other closed extension of* D *satisfying the hypotheses of Corollary 2.3,*

$$\{D^\#\} = \delta\{D_B\} \quad \text{in} \quad K^1(\Psi^0(M)). \tag{2.37}$$

At this point we can extend the scope of the discussion following Proposition 4.4 of [5], as follows. If $\overline{\Omega}$ is an even dimensional spin^c-manifold with boundary, $E \to \overline{\Omega}$ a smooth Hermitian vector bundle, and D_E an associated twisted Dirac operator, then, provided $E \otimes S_+$ and $E \otimes S_-$ are trivial over $\partial\Omega = M$ ($S_\pm$ being the bundles of spinors), we have $\{D_{E,\max}\} \in K^0(\Xi, \Xi_0)$, and the operator $(D_E)^\#$ described above is D_F, the twisted Dirac operator on $\partial\Omega$, with its induced spin^c-structure, and with $F = E|_{\partial\Omega}$. In this case, (2.36) becomes

$$\delta\{D_{E,\max}\} = \{D_F\} \quad \text{in} \quad K^1(\Psi^0(M)). \tag{2.38}$$

3. Identities involving the Szegö projector

Let Ω be a strongly pseudoconvex manifold. There is a standard $\overline{\partial}$-operator, $\overline{\partial} : \Lambda^{0,\text{even}}(\overline{\Omega}) \to \Lambda^{0,\text{odd}}(\overline{\Omega})$, and the operator

$$D = \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\text{even}}(\overline{\Omega}) \longrightarrow \Lambda^{0,\text{odd}}(\overline{\Omega}) \tag{3.1}$$

is eliptic. Furthermore, it is the Dirac operator on Ω, with spin^c-structure induced from its complex structure. Throughout this section we will make the following hypothesis:

$$\Lambda^{0,\text{even}} \quad \text{is trivial over} \quad \partial\Omega. \tag{3.2}$$

Note that this holds provided

$$\Lambda^{0,1} \quad \text{is trivial over} \quad \partial\Omega; \tag{3.3}$$

of course, this holds whenever Ω is a domain in $\mathbb{C}^n$. Granted (3.2), it follows that $D_{\max}$ defines a class in $K^0(\Xi, \Xi_0)$ and, by Proposition 2.5 and the discussion following it,

$$\delta\{D_{\max}\} = \{D_{\partial\Omega}\} \quad \text{in} \quad K^1(\Psi^0(\partial\Omega)), \tag{3.4}$$

where $D_{\partial\Omega}$ is the Dirac operator on $\partial\Omega$, with its induced spin^c-structure.

We now recall the closed extension used in [5] to obtain the identity (1.1), defined by the zero-order part of the $\overline{\partial}$-Neumann boundary condition. The operator D_N is the closure of the restriction of D in (3.1) to

$$\{u \in \Lambda^{0,\text{even}}(\overline{\Omega}) : \sigma_{\overline{\partial}}(x, dr)^* u = 0 \text{ on } \partial\Omega\}, \tag{3.5}$$

where $\{r = 0\}$ defined $\partial\Omega$. In any case where the condition (2.26) holds, since $D_N D_N^*$ has compact resolvent by the subelliptic estimates for the $\overline{\partial}$-Neumann problem, Corollary 2.3 applies. In such a case, the same calculations used to establish Proposition 4.6 in [5] yield the identity

$$\delta\{D_N\} = \{S\} \ \text{ in } \ K^1(\Psi^0(\partial\Omega)), \tag{3.6}$$

where S is the Szegö projector, the orthogonal projection of $L^2(\partial\Omega)$ onto the space of L^2-boundary values of functions holomorphic in Ω. Whenever this holds, then, comparing (3.4) with (3.6) and appealing to Theorem 2.4, we have

$$\{D_{\partial\Omega}\} = \{S\} \ \text{ in } \ K^1(\Psi^0(\partial\Omega)). \tag{3.7}$$

These calculations apply whenever Ω is a strongly pseudoconvex manifold that has the property

$$\begin{array}{l} \text{Ker } \ \sigma_{\overline{\partial}}(x, dr)^* \ \text{ is a trivial subbundle of } \ \Lambda^{0,\text{even}}\big|_{\partial\Omega}, \\ \text{with trivial complementary bundle.} \end{array} \tag{3.8}$$

We now look at some cases where this holds, in case Ω has complex dimension two. In this case, we have

$$\sigma_{\overline{\partial}}(x, dr)^* : \Lambda^{0,0}\big|_{\partial\Omega} \oplus \Lambda^{0,2}\big|_{\partial\Omega} \longrightarrow \Lambda^{0,1}\big|_{\partial\Omega}, \tag{3.9}$$

and the kernel consists precisely of the bundle $\Lambda^{0,0}|_{\partial\Omega}$, which of course is trivial. We therefore have the following.

Proposition 3.1. *If Ω is a strongly pseudoconvex manifold of complex dimension 2, with the property that the line bundle*

$$\Lambda^{0,2}\big|_{\partial\Omega} \ \textit{ is trivial,} \tag{3.10}$$

then D_N defines a class in $K^0(\Xi, \Xi_0)$, and the identity (3.7) relating the Szegö projector and the Dirac operator holds.

Clearly (3.10) holds whenever (3.3) holds, in particluar whenever Ω is a strongly pseudoconvex domain in $\mathbb{C}^2$. In fact, since $\Lambda^{0,0}|_{\partial\Omega} \oplus \Lambda^{0,2}|_{\partial\Omega}$ and $\Lambda^{0,1}|_{\partial\Omega}$ are equivalent in this case, (3.10) is actually equivalent to (3.3). The following gives another class of manifolds for which the identity (3.7) can be deduced.

Proposition 3.2. *Let M be a compact orientable surface, $\Omega = B^*M$. Then (3.3) holds. Hence Proposition 3.1 applies to $\Omega = B^*M$, and thus (3.7) holds with $\partial\Omega = S^*M$.*

Proof. For $p \in S^*M$, let $T_p\Omega$ denote the tangent space, as a real vector space of dimension 4, and $\mathcal{T}_p\Omega$ the tangent space as a complex vector space, of complex dimension 2. Endow M with a Riemannian metric. Then $H_p\Omega$, the horizontal space determined by the Levi-Civita connection on M, is a totally real subspace of $\mathcal{T}_p\Omega$ (of real dimension 2). There is a tautological section s_1 of $H\Omega$ over S^*M; an orientation of M then gives an orthogonal section s_2 of $H\Omega$ over S^*M. This gives a splitting $H\Omega|_{S^*M} = \Sigma_1 \oplus \Sigma_2$, a sum of two trivial line bundles over S^*M. Complexifying Σ_j gives a splitting

$$\mathcal{T}\Omega\big|_{S^*M} = \Sigma_1^\# \oplus \Sigma_2^\#, \tag{3.11}$$

a sum of two trivial complex line bundles over S^*M. From this observation it is easy to construct a trivialization of $\Lambda^{0,1}\Omega$ over S^*M.

Remark. For $\Omega = B^*M$ as above, the tangent bundle of $\partial\Omega = S^*M$ is of the form $(s_1) + \mathcal{L}$, where (s_1) is the real linear span of s_1 and $\mathcal{L}$ is a complex line bundle, defining the CR-structure of $\partial\Omega$. Using s_2, we can construct a nonvanishing section, hence a trivialization, of $\mathcal{L}$. Thus the results of §4 in [15], given there for the case where Ω is a strongly pseudoconvex domain in $\mathbb{C}^2$, also apply to $\Omega = B^*M$.

We next consider the extent to which Propositions 3.1 and 3.2 refine the identity (1.1), i.e.,

$$[D_{\partial\Omega}] = [S] \text{ in } K_1(\partial\Omega). \tag{3.12}$$

Of course, the identity (3.7) refines (3.12) precisely when the natural map $\mu^* : K^1(\Psi^0(\partial\Omega)) \to K_1(\partial\Omega)$ is not injective. Since $\partial\Omega$ has a nonvanishing vector field, we see from the discussion following (1.18) that μ^* is not injective whenever $\widetilde{K}^0(\partial\Omega) \neq 0$.

When $\overline{\Omega}$ is diffeomorphic to a ball in $\mathbb{C}^2$, $\partial\Omega$ is diffeomorphic to S^3. Now $\widetilde{K}^0(S^3) = 0$; moreover, as shown in (3.20) of [15], we have

$$K^1(\Psi^0(S^3)) = K_1(S^3) = \mathbb{Z}, \tag{3.13}$$

so (3.7) gives no improvement over (3.12) in this case.

On the other hand, we claim that, for $\Omega = B^*M$, $\partial\Omega = S^*M$, as in Proposition 3.2, the result (3.7) is always stronger than (3.12). For example, when $M = \mathbb{T}^2$, we have $\partial\Omega = \mathbb{T}^3$, and as shown in (3.11) of [15],

$$K_1(\mathbb{T}^3) = \mathbb{Z}^4, \quad K^1(\Psi^0(\mathbb{T}^3)) = \mathbb{Z}^7. \tag{3.14}$$

In this case, $\widetilde{K}^0(\mathbb{T}^3)=\mathbb{Z}^3$.

In the case $M=S^2$, we have $S^*M=\mathbb{RP}^3$, and it is a well known calculation (cf. [1]; compare the use of (3.38) in [15]) that $\widetilde{K}^0(\mathbb{RP}^3)=\mathbb{Z}_2$. To compare $K_1(S^*S^2)$ and $K^1(\Psi^0(S^*S^2))$, we first note that, by Poincaré duality,

$$K_1(S^*S^2)=K^0(S^*S^2)=\mathbb{Z}\oplus\mathbb{Z}_2. \tag{3.15}$$

Since S^*S^2 has trivial tangent bundle, Proposition 3.2 of [15] implies that the sequence

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{K}^0(S^*S^2) \arrow[r, "\tilde{\tau}"] \arrow[d, equal] & K^1(\Psi^0(S^*S^2)) \arrow[r, "\mu^*"] & K_1(S^*S^2) \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z}_2 & & \mathbb{Z}\oplus\mathbb{Z}_2 & \end{tikzcd} \tag{3.16}$$

is exact.

Now consider a surface M of genus $g\geq 2$. Rather than calculate $\widetilde{K}^0(S^*M)$ directly to show it is nonzero (we suspect this calculation is well known but have not seen it), we bring in some connections with K-homology of M. Recall from (1.12) that $\widetilde{K}^0(S^*M)$ is isomorphic to $K^1(\Psi^0(M))$, which surjects by μ^* onto $K_1(M)$, so it suffices to show that $K_1(M)\neq 0$ in this case. Since M has a spin^c-structure, we have $K_1(M)\approx K^1(M)$. But the Chern character produces an isomorphism

$$\mathrm{Ch}: K^1(M)\otimes\mathbb{Q}\xrightarrow{\approx} H^1(M,\mathbb{Q})=\mathbb{Q}^{2g}, \tag{3.17}$$

so this point is established.

We discuss briefly the condition (3.8) for Ω of higher dimension. First, using exactness of the symbol sequence for $\overline{\partial}^*$, we have (3.8) if and only if the analogous condition holds, with $\Lambda^{0,\mathrm{even}}|_{\partial\Omega}$ replaced by $\Lambda^{0,\mathrm{odd}}|_{\partial\Omega}$. Then by duality and another use of exactness, we deduce that (3.8) is equivalent to the following condition on

$$\sigma_{\overline{\partial}}(x,dr):\Lambda^{0,\mathrm{odd}}\big|_{\partial\Omega}\longrightarrow\Lambda^{0,\mathrm{even}}\big|_{\partial\Omega}, \tag{3.18}$$

namely

$$\begin{gathered}\text{Ker }\ \sigma_{\overline{\partial}}(x,dr)\ \text{ is a trivial subbundle of }\ \Lambda^{0,\mathrm{odd}}\big|_{\partial\Omega},\\ \text{with trivial complementary bundle.}\end{gathered} \tag{3.19}$$

This is also equivalent to the same sort of condition with $\Lambda^{0,\mathrm{odd}}|_{\partial\Omega}$ replaced by $\Lambda^{0,\mathrm{even}}|_{\partial\Omega}$. Furthermore, condition (3.19) holds provided

$$(\overline{\partial}r)\ \text{ has a trivial complementary bundle in }\ \Lambda^{0,1}\big|_{\partial\Omega}, \tag{3.20}$$

where $(\overline{\partial}r)$ denotes the complex line bundle spanned by $\overline{\partial}r$, over $\partial\Omega$. We formalize this:

Proposition 3.3. *The identity (3.7) holds whenever Ω is a strongly pseudoconvex manifold satisfying the condition (3.20).*

This condition is a strong topological resriction in higher domensions. A family of examples is provided by the following straightforward extension of Proposition 3.2.

Proposition 3.4. *Let M be a compact manifold, $\mathcal{H} \to S^*M$ the pull-back of T^*M. Suppose $\mathcal{H}$ is trivial and furthermore that the tautological line subbundle of $\mathcal{H}$ has a trivial complementary bundle. Then $\Omega = B^*M$ satisfies the condition (3.20), and hence the identity (3.7) holds.*

An example of a class of manifolds M to which Proposition 3.4 applies is the following. Let M be a conpact 4-dimensional manifold with a quaternionic action on each cotangent space T^*_xM, so that $S^*M \to M$ gets the structure of a principal SU(2)-bundle. Then applying the quaternions i, j, k to an arbitrary $v \in S^*_xM$ provides a trivialization of the complementary bundle to the tautological bundle in $\mathcal{H}$, so (3.7) applies to $\Omega = B^*M,\ \partial\Omega = S^*M$, in such a case.

We end this section by noting a special property possessed by the class $\{S\}$ of the Szegö projector in $K^1(\Psi^0(\partial\Omega))$. This is connected to the special section of $S^*(\partial\Omega)$, corresponding to the natural contact structure on $\partial\Omega$, and the fact that the wave front relation of S is contained in this contact ray bundle.

More generally, suppose M is a compact manifold such that S^*M has a section χ. This gives rise to a map

$$\chi_* : K_1(M) \longrightarrow K_1(S^*M), \tag{3.21}$$

which then fits into the diagram (1.11). If $S \in OPS^0_{1/2,1/2}(M)$ is a projection, defining

$$[S] \in K_1(M), \quad \{S\} \in K^1(\Psi^0(M)), \tag{3.22}$$

and if the wave front relation of S is contained in the cone generated by χ, then

$$\{S\} = \sigma^*\chi_*[S] \ \text{ in } \ K^1(\Psi^0(M)). \tag{3.23}$$

We can restate this as $\{S\} = \sigma^*\chi_*\mu^*\{S\}$. Now we have $\mu^*(\sigma^*\chi_*) = \pi_*\chi_* = \text{id.}$ on $K_1(M)$, so

$$\Pi_\chi = \sigma^*\chi_*\mu^* \ \text{ is a projection on } \ K^1(\Psi^0(M)). \tag{3.24}$$

Note that Id. $- \Pi_\chi$ is a projection onto the kernel of $\mu^* : K^1(\Psi^0(M)) \to K_1(M)$. Then (3.23) is equivalent to the relation

$$\{S\} = \Pi_\chi\{S\} \ \text{ in } \ K^1(\Psi^0(M)). \tag{3.25}$$

Now, for $M = \partial\Omega$, the boundary of a strongly pseudoconvex manifold, since (3.12) holds generally, we see that the refinement (3.7) holds if and only if

$$\{D_{\partial\Omega}\} = \sigma^* \chi_* [D_{\partial\Omega}], \tag{3.26}$$

equivalently if and only if

$$\{D_{\partial\Omega}\} = \Pi_\chi \{D_{\partial\Omega}\}, \tag{3.27}$$

χ being a section of $S^*(\partial\Omega)$ defining the contact structure on $\partial\Omega$.

4. The intersection product $K_1(\Psi^0) \times K^1(\Psi^0) \to \mathbb{Z}$

One way to understand how identities in $K^\ell(\Psi^0(M))$ refine identities in $K_\ell(M)$ is to study the intersection product

$$K_j(\Psi^0(M)) \times K^\ell(\Psi^0(M)) \longrightarrow K^{j+\ell}(\Psi^0(M)), \tag{4.1}$$

followed by the index map $K^0(\Psi^0(M)) \to \mathbb{Z}$ if $j + \ell = 0 \bmod 2$. We begin this study by deriving a few elementary facts about the K-homology geoups $K_j(\Psi^0(M))$, parallel to (1.10)–(1.12).

We start with a more general situation. Let $\mathfrak{A}$ be a C^*-algebra, acting on a Hilbert space H, containing the set $\mathcal{K}$ of compact operators. From the exact sequence $0 \to \mathcal{K} \to \mathfrak{A} \to \mathfrak{A}/\mathcal{K} \to 0$ there arises the K-homology exact sequence. Since $K_0(\mathcal{K}) = \mathbb{Z}$ and $K_1(\mathcal{K}) = 0$, this takes the form

$$0 \to K_1(\mathfrak{A}) \to K_1(\mathfrak{A}/\mathcal{K}) \xrightarrow{\delta} \mathbb{Z} \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}/\mathcal{K}) \to 0. \tag{4.2}$$

Here $\delta : K_1(\mathfrak{A}/\mathcal{K}) \to \mathbb{Z}$ is given by Kasparov product with the element $[\mathfrak{A}/\mathcal{K}] \in K^1(\mathfrak{A}/\mathcal{K})$ defined by the natural homomorphism $\mathfrak{A}/\mathcal{K} \to \mathcal{Q}(H)$ induced by the action of $\mathfrak{A}$ on H. *Provided* this map δ is surjective, we can break this exact sequence into two pieces:

$$\begin{aligned} 0 \longrightarrow K_1(\mathfrak{A}) \longrightarrow K_1(\mathfrak{A}/\mathcal{K}) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0 \\ 0 \longrightarrow K_0(\mathfrak{A}) \longrightarrow K_0(\mathfrak{A}/\mathcal{K}) \longrightarrow 0. \end{aligned} \tag{4.3}$$

Now, in the case $\mathfrak{A} = \Psi^0(M)$, $\mathfrak{A}/\mathcal{K} = C(S^*M)$, we have the commutative diagram (which can be compared with (1.21)):

(4.4)
$$\begin{tikzcd} K^1(S^*M) \arrow[r, "\delta"] \arrow[d, "\cap\mathcal{P}_M"', "\approx"] & \mathbb{Z} \\ K_0(S^*M) \arrow[r, "\pi_*"'] & K_0(M) \arrow[u, "\text{ind.}"'] \end{tikzcd}$$

which makes it clear that $\delta : K^1(S^*M) \to \mathbb{Z}$ is surjective. Thus we have the following commutative diagrams, with exact top rows:

$$\begin{tikzcd} 0 \arrow[r] & K_0(\Psi^0(M)) \arrow[r, "\sigma_*"] & K^0(S^*M) \arrow[r] & 0 \\ & K^0(M) \arrow[u, "\mu_*"] \arrow[r, "\mathrm{id.}"'] & K^0(M) \arrow[u, "\pi^*"'] & \end{tikzcd} \tag{4.5}$$

and

$$\begin{tikzcd} 0 \arrow[r] & K_1(\Psi^0(M)) \arrow[r, "\sigma_*"] & K^1(S^*M) \arrow[r, "\delta"] & \mathbb{Z} \arrow[r] & 0 \\ & K^1(M) \arrow[u, "\mu_*"] \arrow[r, "\mathrm{id.}"'] & K^1(M) \arrow[u, "\pi^*"'] & & \end{tikzcd} \tag{4.6}$$

In particular, we have the isomorphisms

$$K_0(\Psi^0(M)) \approx K^0(S^*M), \quad K_1(\Psi^0(M)) \approx \operatorname{Ker}\delta \subset K^1(S^*M). \tag{4.7}$$

Compare these with the isomorphisms implied by (1.10)–(1.11):

$$K^0(\Psi^0(M)) \approx K_0(S^*M), \quad K^1(\Psi^0(M)) \approx K_1(S^*M)/(\mathcal{P}_M). \tag{4.8}$$

It is useful to record the following fact about δ, whose proof is routine.

Proposition 4.1. *Let $\Phi : S^*M \to Gl(k,\mathbb{C})$ be a smooth map, defining a class $[\Phi] \in K^1(S^*M)$. Let $P \in OPS^0(M,\mathbb{C}^k)$ have principal symbol on S^*M given by Φ. Then*

$$\rho([\Phi]) = [P] \ \ \text{in} \ \ K_0(M). \tag{4.9}$$

Thus $\delta([\Phi])$ is equal to the index of P.

In light of (4.7)–(4.8), it is reasonable to analyze the product $K_1(\Psi^0(M)) \times K^1(\Psi^0(M)) \to \mathbb{Z}$ in terms of a product

$$[\operatorname{Ker}\delta \subset K^1(S^*M)] \times K^1(\Psi^0(M)) \longrightarrow \mathbb{Z}. \tag{4.10}$$

This product has the following description.

Proposition 4.2. *Let $\Phi : S^*M \to Gl(k,\mathbb{C})$ be a symbol of a pseudodifferential operator of index 0, defining a class $[\Phi] \in Ker\delta \subset K^1(S^*M)$. Consider $\{A\} \in K^1(\Psi^0(M))$, where $A \in OPS^m(M,E)$ is elliptic and self adjoint. Let Q denote the orthogonal projection of $L^2(M)$ onto the positive spectral space of A. Let $\Phi_E^{op} \in OPS^0(M, E\otimes\mathbb{C}^k)$ be an arbitrary pseudodifferential operator with principal symbol*

$I\otimes Q$, and let $Q_{(k)}\in OPS^0(M,E\otimes\mathbb{C}^k)$ be the direct sum of k copies of Q. Then the product (4.10) satisfies

$$[\Phi]\cdot\{A\}=\mathit{Index}\ Q_{(k)}\,\Phi_E^{op}\big|_{\mathcal{R}}, \tag{4.11}$$

where $\mathcal{R}$ is the range of $Q_{(k)}$.

Note that a special case of this involves $A=I_E$, the identity operator on $C^\infty(M,E)$, where $E\to M$ is a nontrivial vector bundle. In that case we have $\{I_E\}=\tau(E)$, and, by (4.11),

$$[\Phi]\cdot\tau(E)=\text{Index}\ \Phi_E^{op}. \tag{4.12}$$

This last identity also follows from the commutative diagram

$$\begin{tikzcd} K^0(M) \arrow[r, "\tau"] \arrow[d, "\pi^*"'] & K^1(\Psi^0(M)) \\ K^0(S^*M) \arrow[r, "\cap\mathcal{P}_M", "\approx"'] & K_1(S^*M) \arrow[u, "\sigma^*"'] \end{tikzcd} \tag{4.13}$$

discussed in §3 of [15], which implies

$$[\Phi]\cdot\tau(E)=[\Phi]\cap(\pi^*(E)\cap\mathcal{P}_M). \tag{4.14}$$

We now discuss intersection products involving the class $\{S\}$ of the Szegö projector on $M=\partial\Omega$, where Ω is a strongly pseudoconvex manifold. Recall $\{S\}\in K^1(\Psi^0(M))$. In analogy with (4.11), we have

$$[\Phi]\cdot\{S\}=\text{Index}\ S_{(k)}\Phi^{op}\big|_{\mathcal{H}^{(k)}}, \tag{4.15}$$

where $S_{(k)}=S\otimes I$ on $L^2(M,\mathbb{C}^k)$, Φ^{op} is as above in the case of the trivial line bundle E, and $\mathcal{H}^{(k)}$ denotes the orthogonal sum of k copies of the range of S.

The fact that the wave front set of S is contained in the image of the section χ of S^*M defining the contact structure on $M=\partial\Omega$ has the following implication for the index calculation (4.15). Given $\Phi:S^*M\to\text{Gl}(k,\mathbb{C})$, define Φ_χ^b to be $\Phi\circ\chi$, so $\Phi_\chi^b:M\to\text{Gl}(k,\mathbb{C})$. We also regard this map as $\Phi_\chi:S^*M\to\text{Gl}(k,\mathbb{C})$, constant on the fibers. We denote by Φ_χ^{op} the operator on $L^2(M,\mathbb{C}^k)$ obtained by matrix multiplication by Φ_χ^b. Then

$$S_{(k)}(\Phi^{op}-\Phi_\chi^{op})\in OPS^{-1/2}_{1/2,1/2}(M),$$

hence is compact on $L^2(M)$. We conclude that if Φ^{op} has index zero,

$$[\Phi]\cdot\{S\}=[\Phi_\chi]\cdot\{S\}. \tag{4.16}$$

This identity is also related to properties of $\{S\}$ discussed at the end of §3.

In fact, since $[\Phi_\chi]=\pi^*\chi^*[\Phi]$ in $L^1(S^*M)$, functoriality of the intersection product gives

$$[\Phi_\chi]\cdot\alpha=[\Phi]\cdot\Pi_\chi\alpha \tag{4.17}$$

for all $\alpha\in K^1(\Psi^0(M))$, where Π_χ is the projection given by (3.24). In view of this, (4.16) then follows from the identity (3.25).

We furthermore note that whenever the sequence (1.19) is exact, for each $\alpha\in K^1(\Psi^0(M))$ we can write

$$\alpha=\Pi_\chi\alpha+\tau(E),\quad \text{for some}\ \ E\in K^0(M). \tag{4.18}$$

When this holds, then (4.17) implies

$$\begin{aligned}([\Phi]-[\Phi_\chi])\cdot\alpha&=[\Phi]\cdot(\alpha-\Pi_\chi\alpha)\\&=[\Phi]\cdot\tau(E)\\&=\text{Index }\Phi_E^{op},\end{aligned} \tag{4.19}$$

the latter identity following from (4.12).

Finally, let us relate these computations to the question of whether $\{S\}=\{D_{\partial\Omega}\}$ in $K^1(\Psi^0(\partial\Omega))$, when $\partial\Omega$ is the boundary of a strongly pseudoconvex manifold. In view of (4.16), this identity would yield the following identity, also implied by (3.27):

$$[\Phi]\cdot\{D_{\partial\Omega}\}=[\Phi_\chi]\cdot\{D_{\partial\Omega}\}, \tag{4.20}$$

for each $\Phi : S^*(\partial\Omega)\to \mathrm{Gl}(k,\mathbb{C})$ such that Φ^{op} has index zero. Now each side of (4.20) can in principle be calculated by the Atiyah-Singer index theorem, in light of the formula (4.11).

We close with the following speculative note. It seems rather likely that in general $\{S\}$ and $\{D_{\partial\Omega}\}$ are not indentical in $K^1(\Psi^0(\partial\Omega))$, i.e., $(\mathrm{Id.}-\Pi_\chi)\{D_{\partial\Omega}\}\neq 0$. It is tempting to guess that a further general refinement of Boutet de Monvel's index theorem would involve specifying this difference, in terms of the map τ.

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References

1. M. Atiyah, K-Theory, Benjamin, New York, 1967.
2. P. Baum and R. Douglas, K-homology and index theory, Proc. Symp. Pure Math. 38, Part 1 (1982), 117–173.

3. P. Baum and R. Douglas, Toeplitz operators and Poincare duality, Proc. Toeplitz Memorial Conf., Tel Aviv, Birkhauser, Basel, 1982, 137–166.
4. P. Baum and R. Douglas, Relative K-homology and C^*-algebras, K-Theory 5 (1991), 1–46.
5. P. Baum, R. Douglas, and M. Taylor, Cycles and relative cycles in analytic K-homology, J. Diff. Geom. 30 (1989), 761–804.
6. B. Blackadar, K-Theory for Operator Algebras, Springer-Verlag, New York, 1986.
7. L. Boutet de Monvel, On the index of Toeplitz operators of several complex variables, Invent. Math. 50 (1979), 249–272.
8. L. Brown, R. Douglas, and P. Fillmore, Extensions of C^*-algebras and K-homology, Annals of Math. 105 (1977), 265–324.
9. J.-L. Brylinski and E. Getzler, The homology of algebras of pseudodifferenaitl symbols and the noncommutative residue, K-Theory 1 (1987), 385–403.
10. G. Folland and J. J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex, Princeton Univ. Press, 1972.
11. L. Hörmander, The Analysis of Linear Partial Differential Operators, Vols. 3–4, Springer-Verlag, New York, 1985.
12. G. Kasparov, Topological invariants of elliptic operators I: K-homology, Math. USSR Izv. 9 (1975), 751–792.
13. G. Kasparov, The operator K-functor and extensions of C^*-algebras, Math. USSR Izv. 16 (1981), 513–572.
14. G. Kasparov, Operator K-theory and its applications: elliptic operators, group representations, higher signatures, C^*-extensions, Chernogolovka, 1982.
15. M. Taylor, Pseudodifferential operators and K-homology, Proc. Symp. Pure Math. 51 (Part 1) (1990), 561–583.
16. M. Taylor, Pseudodifferential operators, Princeton Univ. Press, 1981.
17. M. Wodzicki, Cyclic homology of differential operators, Duke Math. J. 54 (1987), 641–647.

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Addendum

This is a TeX version of [T], which was the product of ball point pen and old fashion typewriter. I was motivated to prepare this somewhat more readable version by the appearance of [E], citing [T]. Other than producing neater displayed formulas and updating some references, I made very few changes.

[E] C. Epstein, Subelliptic Spin$_{\mathbb{C}}$ Dirac Operators, Ann. Math. 166 (2007), 183–214, 723–777.

[T] M. Taylor, Pseudodifferential operators and K-homology, II, Contemp. Math. 105 (1990), 245–269.