Notes on Integration on Lie Groups

Michael Taylor

Contents

Abstract. This is a very informal set of notes on integration on Lie groups and connections with basic representation theory. We give some constructions of the Lie group integral, show how some integrals can be computed by using simple symmetry considerations, and present some cases where more earnest efforts are required to compute integrals.

1. Construction of Haar measure

For our first construction, assume G is a compact subgroup of the unitary group $U(n)$, sitting in $M_n(\mathbb{C})$, the space of complex $n \times n$ matrices. The space $M_n(\mathbb{C})$ has a Hermitian inner product,

$$(A, B) = \operatorname{Tr} AB^* = \operatorname{Tr} B^* A, \tag{1.1}$$

giving a real inner product $\langle A, B \rangle = \operatorname{Re}(A, B)$. This induces a Riemannian metric on G. Let us define, for $g \in G$,

$$L_g, R_g : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}), \quad L_g X = gX, \quad R_g X = Xg. \tag{1.2}$$

Clearly each such map is a linear isometry on $M_n(\mathbb{C})$, and we have isometries L_g and R_g on G.

A Riemannian metric tensor on a smooth manifold induces a volume element on M, as follows. In local coordinates $(x_1, \dots, x_N)$ on $U \subset M$, say the metric tensor has components $h_{jk}(x)$. Then, on U,

$$dV(x) = \sqrt{\det(h_{jk})}\, dx_1 \cdots dx_N. \tag{1.3}$$

In such a way we get a volume element on a compact group $G \subset U(n)$, and since L_g and R_g are isometries, they also preserve the volume element. We normalize this volume element to define normaluized Haar measure on G:

$$\int_G f(g)\, dg = \frac{1}{V(G)} \int_G f\, dV. \tag{1.4}$$

We have left invariance

$$\int_G f(hg)\, dg = \int_G f(g)\, dg \tag{1.5}$$

and right invariance

$$\int_G f(gh)\, dg = \int_G f(g)\, dg, \tag{1.6}$$

for all $h \in G$, in such a situation.

We now give a second construction of Haar measure, valid in much greater generality. Let G be any Lie group, say of dimension N. Pick any nonzero $\omega_e \in$

$\Lambda^N T_e^* G$, where e denotes the identity element of G. Then there is a unique N form ω_ℓ on G such that

$$\omega_\ell(e) = \omega_e, \quad L_g^* \omega_\ell = \omega_\ell, \quad \forall\, g \in G, \tag{1.7}$$

and a unique N-form ω_r on G such that

$$\omega_r(e) = \omega_e, \quad R_g^* \omega_r = \omega_r, \quad \forall\, g \in G. \tag{1.8}$$

In fact $\omega_e = L_g^* \omega_\ell(g)$ and $\omega_e = R_g^* \omega_r(g)$. If we use ω_ℓ (or ω_r) to define an orientation on G, then we have volume elements, which we denote dV_ℓ and dV_r. Note that, for all $h \in G$,

$$\int_G f(hg)\, dV_\ell(g) = \int_G f(g)\, dV_\ell(g), \quad \int_G f(gh)\, dV_r(g) = \int_G f(g)\, dV_r(g). \tag{1.9}$$

Since $\Lambda^N T_e^* G$ is 1-dimensional, it is clear that both dV_ℓ and dV_r are unique, up to a constant positive multiple.

Note that L_g^* and R_h^* commute for each $g, h \in G$. Hence $R_g^* \omega_\ell$ is left-invariant and $L_g^* \omega_r$ is right-invariant for each $g, h \in G$. The uniqueness mentioned above implies

$$R_h^* \omega_\ell = \alpha(h) \omega_\ell, \quad L_g^* \omega_r = \beta(g) \omega_r, \tag{1.10}$$

for all $g, h \in G$, with $\alpha, \beta : G \to (0, \infty)$. It is clear that α and β are homomorphisms:

$$\alpha(gh) = \alpha(g)\alpha(h), \quad \beta(gh) = \beta(g)\beta(h). \tag{1.11}$$

We say G is unimodular if $\alpha \equiv 1$ (equivalently, $\beta \equiv 1$). In such a case, the left invariant Haar measure is also right invariant; we say Haar measure is bi-invariant on G, and that G is unimodular. The Haar measure constructed on a compact group $G \subset U(n)$ at the begining of this section is bi-invariant. From another perspective, note that the image of G under α is a subgroup of $(0, \infty)$; if G is compact this must be a compact subgroup, hence $\{1\}$.

Lots of noncompact Lie groups are also unimodular, but some are not unimodular.

2. Integrating a representation

Let G be a compact Lie group, π a unitary representation of G on V, a finite-dimensional vector space with an inner product. We set

$$Pv = \int_G \pi(g)v\, dg.$$

Claim. *P is the orthogonal projection of V on the space where π acts trivially.*

The proof consists of four easy pieces:

$$\pi(g)Pv = Pv, \quad \forall\, g \in G, \tag{2.1}$$

$$P^* = \int \pi(g^{-1})\, dg = P, \tag{2.2}$$

$$P^2 = \iint \pi(g)\pi(h)\, dg\, dh = \iint \pi(gh)\, dg\, dh = P, \tag{2.3}$$

$$\pi(g)v = v\ \forall\, g \Longrightarrow Pv = v. \tag{2.4}$$

3. Weyl orthogonality

Let G be a compact Lie group. Assume π is an irreducible unitary representation of G on V and λ an irreducible unitary representation of G on W. Define P acting on $\mathrm{Hom}(V,W)$ as follows. If $A : V \to W$, set

$$P(A) = \int_G \lambda(g) A \pi(g)^{-1}\, dg. \tag{3.1}$$

It is readily verified that

$$\lambda(g) P(A) \pi(g)^{-1} = P(A), \quad \forall\, g \in G. \tag{3.2}$$

In other words, $P(A)$ intertwines π and λ. Now Schur's lemma gives the following:

$$\begin{aligned} \pi \text{ not} \approx \lambda &\Longrightarrow P(A) = 0, \quad \forall A, \\ \pi = \lambda &\Longrightarrow P(A) = c_\pi(A) I, \end{aligned} \tag{3.3}$$

where $c_\pi(A)$ is scalar and I the identity operator on $V = W$. In the latter case, taking the trace yields $d_\pi\, c_\pi(A) = \mathrm{Tr}\, A$ (where $d_\pi = \dim\, V$), hence $c_\pi(A) = d_\pi^{-1}\ \mathrm{Tr}\, A$, so

$$\int_G \pi(g) A \pi(g)^*\, dg = d_\pi^{-1} (\mathrm{Tr}\, A) I. \tag{3.4}$$

If matrix entries are denoted $\pi(g)_{jk},\ A_{jk}$, etc., we have

$$\begin{aligned} \sum_{k,\ell} \int_G \pi(g)_{jk} A_{k\ell} \overline{\pi(g)}_{m\ell}\, dg &= d_\pi^{-1}\, \delta_{jm}\ \mathrm{Tr}\, A \\ &= d_\pi^{-1}\, \delta_{jm} \sum_{k,\ell} \delta_{k\ell} A_{k\ell}, \end{aligned} \tag{3.5}$$

hence

$$\int_G \pi(g)_{jk} \overline{\pi(g)}_{m\ell}\, dg = d_\pi^{-1}\, \delta_{jm}\, \delta_{k\ell}. \tag{3.6}$$

These are Weyl orthogonality relations. They are complemented by

$$\int_G \pi(g)_{jk} \overline{\lambda(g)}_{m\ell}\, dg = 0, \quad \pi \text{ not} \approx \lambda, \tag{3.7}$$

which follows from the first part of (3.3).

4. The adjoint representation

The adjoint representation of a Lie group G is a representation of G on its Lie algebra $\mathfrak{g}$. We recall that $\mathfrak{g}$ consists of left invariant vector fields on G. Such $X \in \mathfrak{g}$ is uniquely determined by $X(e) \in T_eG$, so $\mathfrak{g} \approx T_eG$. A vector field X on G is left invariant if and only if the flow $\mathcal{F}^t_X$ it generates commutes with L_g for all $g \in G$, that is, $g(\mathcal{F}^t_X h) = \mathcal{F}^t_X(gh)$ for all $g, h \in G$. If we set

$$\gamma_X(t) = \mathcal{F}^t_X e, \tag{4.1}$$

we obtain $\gamma_X(t+s) = \mathcal{F}^s_X(\mathcal{F}^t_X e) \cdot e = (\mathcal{F}^t_X e)(\mathcal{F}^s_X e)$, and hence $\gamma_X(s+t) = \gamma_X(s)\gamma_X(t)$, for $s, t \in \mathbb{R}$. Clearly $\gamma'_X(0) = X(e)$. The exponential map

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow G \tag{4.2}$$

is defined by

$$\mathrm{Exp}(X) = \gamma_X(1). \tag{4.3}$$

If G is a Lie subgroup of $Gl(n, \mathbb{C})$, then T_eG is a subspace of $M_n(\mathbb{C})$, and (4.2) coincides with the matrix exponential e^X.

To define the adjoint representation of G on $\mathfrak{g}$, consider

$$K_g : G \longrightarrow G, \quad K_g(h) = ghg^{-1}. \tag{4.4}$$

Then $K_g(e) = e$, and we set

$$\mathrm{Ad}(g) = DK_g(e) : T_eG \longrightarrow T_eG, \tag{4.5}$$

identifying $T_eG \approx \mathfrak{g}$. Since $K_{gh} = K_g \circ K_h$, the chain rule implies

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\,\mathrm{Ad}(h). \tag{4.6}$$

Note that $\gamma(t) = g\,\mathrm{Exp}(tX)\,g^{-1}$ is a 1-parameter subgroup of G satisfying $\gamma'(0) = \mathrm{Ad}(g)X$. Hence

$$\mathrm{Exp}(t\ \mathrm{Ad}(g)X) = g\ \mathrm{Exp}(tX)\,g^{-1}. \tag{4.7}$$

In particular,

$$\mathrm{Exp}\big(\mathrm{Ad}(\mathrm{Exp}\,sY)tX\big) = \mathrm{Exp}(sY)\ \mathrm{Exp}(tX)\ \mathrm{Exp}(-sY). \tag{4.8}$$

The right side of (4.8) is equal to $\mathcal{F}^s_Y \circ \mathcal{F}^t_X \circ \mathcal{F}^{-s}_Y(e)$.

In general a representation π of G on V yields a representation $d\pi$ of $\mathfrak{g}$ on V by

$$d\pi(X) = D\pi(e)X, \quad D\pi(e) : T_e G \to V. \tag{4.9}$$

One shows that, for $X, Y \in \mathfrak{g}$,

$$[d\pi(X), d\pi(Y)] = d\pi([X, Y]), \tag{4.10}$$

where $[X, Y]$ denotes the Lie bracket of vector fields. (See [T2], Appendix B, for more details on this, and on the material below.) From (4.8) it can be deduced that $D\,\mathrm{Ad}(X) = \mathrm{ad}\,X$, given by

$$\mathrm{ad}\,X(Y) = [X, Y]. \tag{4.11}$$

We mention another useful identity:

$$e^{\mathrm{ad}\,X} = \mathrm{Ad}(\mathrm{Exp}\,X), \tag{4.12}$$

a special case of the more general identity

$$e^{t\,d\pi(X)} = \pi(\mathrm{Exp}\,tX), \tag{4.13}$$

valid when π is a representation of G on V.

Finally we tie in with the question of whether G is unimodular. A comparison of (1.10) and (4.4) shows that

$$\alpha(g) = \det \mathrm{Ad}(g). \tag{4.14}$$

In other words, G is unimodular if and only if $\det \mathrm{Ad}(g) \equiv 1$.

5. Haar measure in exponential coordinates

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. We assume $G \subset \mathrm{Gl}(\mathbb{C}^n)$, so $\mathfrak{g} \subset \mathrm{End}(\mathbb{C}^n)$, and $\mathrm{Exp} : \mathfrak{g} \to G$ is given by $\mathrm{Exp}(X) = e^X$. We have

$$D\,\mathrm{Exp}(X)Y = e^X\,\Xi(\mathrm{ad}\,X)Y, \quad \Xi(z) = \frac{1-e^{-z}}{z}. \tag{5.1}$$

Here $D\,\mathrm{Exp}(X) : \mathfrak{g} \to T_gG$, $g = \mathrm{Exp}(X)$, and also left multiplication by e^X of $\Xi(\mathrm{ad}\,X)Y \in \mathfrak{g}$ yields an element of $T_gG \subset \mathrm{End}(\mathbb{C}^n)$. Using the left-invariant volume form on G, we have

$$\det D\,\mathrm{Exp}(X) = \det \Xi(\mathrm{ad}\,X). \tag{5.2}$$

Thus Haar measure pulled back to $\mathfrak{g}$ is given by $H(X)\,dX$, with dX Lebesgue measure on $\mathfrak{g}$ and $H(X) = |\det \Xi(\mathrm{ad}\,X)|$. If G is unimodular, i.e., if $\det \mathrm{Ad}\,g = 1$ for all $g \in G$, then

$$H(X) = |\det \mathcal{S}(\mathrm{ad}\,X)|, \quad \mathcal{S}(z) = \frac{\sinh(z/2)}{z/2}. \tag{5.3}$$

Derivative of the exponential map

We sketch a proof of (5.1), which is equivalent to

$$\frac{d}{dt}e^{X+tY}\big|_{t=0} = e^X\,\Xi(\mathrm{ad}\,X)Y, \tag{5.4}$$

(at least when G is a matrix group). To get this, look at

$$U(s,t) = e^{s(X+tY)}. \tag{5.5}$$

Then $\partial_t U(s,t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial s}\partial_t U(s,t) &= \frac{\partial}{\partial t}(X+tY)U(s,t) \\ &= (X+tY)\partial_t U(s,t) + YU(s,t), \end{aligned} \tag{5.6}$$

and $\partial_t U(0,t) = 0$. The unique solution to this ODE is

$$\partial_t U(s,t) = \int_0^s e^{(s-\sigma)(X+tY)}Ye^{\sigma(X+tY)}\,d\sigma. \tag{5.7}$$

Taking $s = 1$, $t = 0$ gives

$$\begin{aligned} \frac{d}{dt}e^{X+tY}\big|_{t=0} &= \int_0^1 e^{(1-\sigma)X}Ye^{\sigma X}\,d\sigma \\ &= e^X \int_0^1 \mathrm{Ad}(e^{-\sigma X})Y\,d\sigma, \end{aligned} \tag{5.8}$$

and since $\mathrm{Ad}(e^{-\sigma X}) = e^{-\sigma\,\mathrm{ad}\,X}$, this gives (5.1).

6. The Weyl integration formula

Say G is a compact, connected Lie group, $T \subset G$ a maximal torus. We derive Weyl's formula:

$$\int\limits_G f(x)\,dx = \frac{1}{W}\int\limits_T \Big(\int\limits_G f(g^{-1}kg)\,dg\Big)|\det(I - \operatorname{Ad} k)_{\mathfrak{g}/\mathfrak{t}}|\,dk, \tag{6.1}$$

using a variant of an argument from [DK]. Here W is the order of the Weyl group. We get this formula from a study of

$$F : G \times T \longrightarrow G, \quad F(g,h) = ghg^{-1}, \tag{6.2}$$

and its induced action

$$\widetilde{F} : (G/T) \times T \longrightarrow G. \tag{6.3}$$

Since there are natural volume elements on $(G/T)\times T$ and on G, we need to compute $\det D\widetilde{F}$. Note that $DF(g,h) : T_gG \oplus T_hT \to T_{ghg^{-1}}G$; it is convenient to produce a linear map that takes $T_eG \oplus T_eT \to T_eG$. That would be

$$DL_{gh^{-1}g^{-1}}(ghg^{-1}) \circ DF(g,h) \circ (DL_g(e) \times DL_h(e)), \tag{6.4}$$

where $L_g(x) = gx$. Note that (6.4) is equal to $DG(e,e)$, where

$$\begin{aligned} G(x,z) &= L_{gh^{-1}g^{-1}} \circ F \circ (L_g \times L_h)(x,z) \\ &= gh^{-1}xhzx^{-1}g^{-1}. \end{aligned} \tag{6.5}$$

Note that $G(e,e) = e$; we compute

$$DG(e,e) : \mathfrak{g} \oplus \mathfrak{t} \longrightarrow \mathfrak{g}, \tag{6.6}$$

where $\mathfrak{t}$ denotes the Lie algebra of T.

First, with $Z \in \mathfrak{t},\ z(t)$ a curve in T such that $z(0) = e,\ z'(0) = Z$, we have

$$\begin{aligned} D_2G(e,e)Z &= \frac{d}{dt}\, gz(t)g^{-1}\big|_{t=0} \\ &= \operatorname{Ad} g(Z). \end{aligned} \tag{6.7}$$

Next, with $X \in \mathfrak{g},\ x(t)$ a curve in G such that $x(0) = e,\ x'(0) = X$, we have

$$\begin{aligned} D_1G(e,e)X &= \frac{d}{dt}\, gh^{-1}x(t)hx(t)^{-1}g^{-1}\big|_{t=0} \\ &= \operatorname{Ad} g\, DK(e)X, \end{aligned} \tag{6.8}$$

where

$$K(x) = h^{-1}xhx^{-1}, \tag{6.9}$$

so

$$\begin{aligned} DK(e)X &= \frac{d}{dt}\, h^{-1}x(t)hx(t)^{-1}\big|_{t=0} \\ &= h^{-1}Xh - X \\ &= (\operatorname{Ad} h^{-1} - I)X. \end{aligned} \tag{6.10}$$

(Here we take $G \subset \operatorname{End}(\mathbb{C}^n)$, to simplify the calculation.) Putting together (6.7), (6.8), and (6.10), we have

$$DG(e,e)(X,Z) = \operatorname{Ad} g(\operatorname{Ad} h^{-1} - I)X + \operatorname{Ad} g\, Z. \tag{6.11}$$

Now we can take $X \in \mathfrak{g}/\mathfrak{t}$. Thus we have

$$\det D\widetilde{F}(g,h) = \det(\operatorname{Ad} h^{-1} - I)_{\mathfrak{g}/\mathfrak{t}} = \det(I - \operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{t}}. \tag{6.12}$$

The formula (6.1) now follows upon noting that $\widetilde{F}$ in (6.3) is onto and, for generic $g \in G$, $\widetilde{F}^{-1}(g) \subset (G/T) \times T$ has W elements.

In the case $G = U(n)$, we take T to be the set of diagonal matrices with diagonal entries in $S^1 \subset \mathbb{C}$. The surjectivity of $\widetilde{F}$ is equivalent to the statement that every unitary matrix yields an orthonormal basis of eigenvectors. If $g \in U(n)$ has distinct eigenvalues, then the eigenspaces are all 1-dimensional, and the diagonalized form is determined up to ordering of the eigenvalues, so such a matrix has $n!$ pre-images in $(G/T) \times T$.

We give an explicit formula for the right side of (6.12) when $G = U(n)$. In such a case, $\mathfrak{g}_{\mathbb{C}} = \operatorname{End}(\mathbb{C}^n)$. Let e_{jk} be the matrix with 1 at row j, column k, 0 elsewhere, and set $e_j = ie_{jj}$. Then $\mathfrak{t}$ is the real linear span of $\{e_j : 1 \le j \le n\}$, and

$$H = \sum t_j e_j \Longrightarrow [H, e_{jk}] = i(t_j - t_k)e_{jk}. \tag{6.13}$$

Using this, we have that, when $G = U(n)$, $h = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \in T$,

$$\operatorname{Ad} h(e_{jk}) = e^{i(\theta_j - \theta_k)}\, e_{jk}. \tag{6.14}$$

Thus

$$\begin{aligned} \det(I - \operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{t}} &= \det(I - \operatorname{Ad} h)_{\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}} \\ &= \prod_{j \ne k} (1 - e^{i(\theta_j - \theta_k)}) \\ &= \prod_{j \ne k} e^{-i\theta_k}(e^{i\theta_k} - e^{i\theta_j}), \end{aligned} \tag{6.15}$$

and hence

$$|\det(I - \operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{t}}| = \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2. \tag{6.16}$$

7. Ensembles of Hermitian matrices

Let H be the space of self-adjoint operators on $\mathbb{C}^n$; H has a Lebesgue measure. We derive the formula

$$\int_H f(x)\,dx = C_n \int_D \Bigl(\int_G f(g^{-1}hg)\,dg\Bigr) D(h)\,dh. \tag{7.1}$$

Here $G = U(n)$, with Haar measure dg, D is the space of diagonal matrices with real diagonal entries, with Lebesgue measure dh, and C_n is a constant, depending on the normalizations of these measures. The factor $D(h)$ is the discriminant of h:

$$h = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \Longrightarrow D(h) = \prod_{j<k} (\lambda_j - \lambda_k)^2. \tag{7.2}$$

This is somewhat similar to Weyl's formula for integration on $U(n)$.

We will get (7.1) from a study of

$$F : G \times D \longrightarrow H, \quad F(g, h) = ghg^{-1}, \tag{7.3}$$

and its induced action

$$\widetilde{F} : (G/T) \times D \longrightarrow H, \tag{7.4}$$

where $T \subset G$ is the group of diagonal unitary matrices; note that both domain and range in (7.4) have real dimension $2n^2$. Since there are natural volume elements on $(G/T)\times D$ and on H, we need to compute $\det D\widetilde{F}$. Note that $DF(g,h) : T_g G \oplus D \to H$. It is convenient to produce a linear map that takes $T_e G \oplus D \to H$. We take

$$G(x,h) = F \circ (L_g \times I)(x,h) = F(gx, h) = gxhx^{-1}g^{-1}, \tag{7.5}$$

with $L_g x = gx$, so we want to study $DG(e,h) : \mathfrak{u} \oplus D \to H$, where $\mathfrak{u}$ is the Lie algebra of G, i.e., $\mathfrak{u}$ is the space of skew-adjoint operators on $\mathbb{C}^n$, or equivalently $\mathfrak{u} = iH$. Note that, given $Z \in D$, and taking $z(t)$ a curve in D with $z(0) = h, z'(0) = Z$, we have

$$D_2 G(e,h) Z = \frac{d}{dt}\, g z(t) g^{-1}\big|_{t=0} = \operatorname{Ad} g(Z). \tag{7.6}$$

Next, given $X \in \mathfrak{u}$ and taking $x(t)$ a curve in G with $x(0) = e, x'(0) = X$, we have

$$\begin{aligned} D_1 G(e,h) X &= \frac{d}{dt}\, g x(t) h x(t)^{-1} g^{-1}\big|_{t=0} \\ &= \operatorname{Ad} g(Xh - hX). \end{aligned} \tag{7.7}$$

Hence, with G given by (7.5),

$$DG(e,h)(X,Z) = \operatorname{Ad} g(\operatorname{ad} h(X) + Z). \tag{7.8}$$

Recall $g \in G, h \in D, X \in \mathfrak{u}, Z \in D$. Note that the right side of (7.8) is well defined for $X \in \mathfrak{u}/\mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of T, so $\mathfrak{t} = iD$. It follows that

$$\det D\widetilde{F}(g,h) = \det \operatorname{ad} ih\big|_{\mathfrak{u}/\mathfrak{t}}. \tag{7.9}$$

The demonstration that this equals $D(h)$, given by (7.2), is given in §8. The formula (7.1) now follows, upon noting that $\widetilde{F}$ in (7.3) is onto and, for generic $x \in H$, $\widetilde{F}^{-1}(x) \subset (G/T) \times D$ has $n!$ elements.

8. The discriminant of a matrix

Take $A \in \mathrm{End}(\mathbb{C}^n)$. Say $\mathrm{Spec}\, A = \{\lambda_1, \dots, \lambda_n\}$, counting multiplicities. Then

$$L_A, R_A : \mathrm{End}(\mathbb{C}^n) \longrightarrow \mathrm{End}(\mathbb{C}^n), \quad L_A X = AX, \quad R_A X = XA, \tag{8.1}$$

have the same spectrum, with n-fold increases in multiplicity. Since L_A and R_A commute, we can say about $\mathrm{ad}\, A = L_A - R_A$ that

$$\mathrm{Spec}\,\mathrm{ad}\, A = \{\lambda_j - \lambda_k : 1 \le j, k \le n\}. \tag{8.2}$$

We thus have

$$\begin{aligned} \det(sI - \mathrm{ad}\, A) &= \prod_{j,k} \bigl[s - (\lambda_j - \lambda_k)\bigr] \\ &= s^n \prod_{j<k} \bigl[s^2 - (\lambda_j - \lambda_k)^2\bigr] \\ &= (-1)^{n(n-1)/2}\, s^n D(A) + O(s^{n+1}), \end{aligned} \tag{8.3}$$

as $s \to 0$, where $D(A)$ is the discriminant of A:

$$D(A) = \prod_{j<k} (\lambda_j - \lambda_k)^2. \tag{8.4}$$

It follows that

$$D(A) = \frac{(-1)^{n(n-1)/2}}{n!} \frac{d^n}{ds^n} \det(sI - \mathrm{ad}\, A)\big|_{s=0}. \tag{8.5}$$

Suppose A is diagonal, say $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Let E_{jk} denote the $n \times n$ matrix with a 1 in row j, column k, zeroes elsewhere. We have

$$[A, E_{jk}] = (\lambda_j - \lambda_k) E_{jk}. \tag{8.6}$$

It follows readily from (8.6) that, when A is diagonal,

$$D(A) = \det \mathrm{ad}\, A\big|_{\mathrm{End}(\mathbb{C}^n)/\mathcal{D}}, \tag{8.7}$$

where $\mathcal{D}$ is the space of complex diagonal matrices. This yields

$$D(A) = \det \mathrm{ad}\, A\big|_{\mathfrak{u}/\mathfrak{t}}, \tag{8.8}$$

when $A \in \mathfrak{t}$, $\mathfrak{u} =$ set of skew-adjoint operators on $\mathbb{C}^n$, $\mathfrak{t} =$ space of diagonal matrices with purely imaginary diagonal entries.

9. The integral of $|\operatorname{Tr} M^j|^2$

Here we establish the following identity:

Proposition 9.1. *We have*

$$\int_{U(n)} |Tr\, M^j|^2\, dM = \begin{cases} j & \text{for } 1 \le j \le n, \\ n & \text{for } j \ge n. \end{cases} \tag{9.1}$$

This was proven in [BDK], using Dyson's formula. Another proof was given in [DE] (Theorem 2.1, part (b)), using an identity between power-sum symmetric functions and Schur functions. Here we give a "naive" proof, based on Weyl's integration formula, which implies that whenever $\varphi : U(n) \to \mathbb{C}$ is invariant under conjugation, then

$$\int_{U(n)} \varphi(M)\, dM = C_n (2\pi)^{-n} \int_{\mathbb{T}^n} \varphi(D(\theta)) J(\theta)\, d\theta_1 \cdots d\theta_n, \tag{9.2}$$

where $D(\theta)$ is the diagonal matrix with diagonal entries $e^{i\theta_1}, \dots, e^{i\theta_n}$, and

$$J(\theta) = \prod_{k<\ell} |e^{i\theta_k} - e^{i\theta_\ell}|^2. \tag{9.3}$$

We will verify in calculations below that

$$C_n = \frac{1}{n!}. \tag{9.4}$$

In particular, (9.2) gives

$$\int_{U(n)} |\operatorname{Tr} M^j|^2\, dM = C_n (2\pi)^{-n} \int_{\mathbb{T}^n} |e^{ij\theta_1} + \cdots + e^{ij\theta_n}|^2 J(\theta)\, d\theta. \tag{9.5}$$

We re-state this as follows. Set $\zeta_j = e^{i\theta_j}$, so

$$|e^{ij\theta_1} + \cdots + e^{ij\theta_n}|^2 = |\zeta_1^j + \cdots + \zeta_n^j|^2 = \sum_{\mu,\nu} \zeta_\mu^j \zeta_\nu^{-j}, \tag{9.6}$$

and

$$
\begin{aligned}
J(\theta) &= \prod_{k<\ell} |e^{i\theta_k} - e^{i\theta_\ell}|^2 = \prod_{k<\ell} |\zeta_k - \zeta_\ell|^2 \\
&= \prod_{k<\ell} (\zeta_k - \zeta_\ell)(\zeta_k^{-1} - \zeta_\ell^{-1}) \\
&= (\operatorname{sgn}\gamma)(\zeta_1 \cdots \zeta_n)^{-(n-1)} \prod_{k<\ell} (\zeta_k - \zeta_\ell)^2,
\end{aligned} \tag{9.7}
$$

where γ is the permutation on $\{1,\dots,n\}$ such that $\gamma(k) = n+1-k$. We see that $\int_{U(n)} |\operatorname{Tr} M^j|^2 \, dM$ is the constant term in

$$
C_n (\operatorname{sgn}\gamma)(\zeta_1 \cdots \zeta_n)^{-(n-1)} \Big(\sum_{\mu,\nu} \zeta_\mu^j \zeta_\nu^{-j} \Big) \prod_{k<\ell} (\zeta_k - \zeta_\ell)^2. \tag{9.8}
$$

Thus our task is to identify the constant term in this Laurent polynomial. To work on the last factor, we recognize

$$
V(\zeta) = \prod_{k<\ell} (\zeta_k - \zeta_\ell) \tag{9.9}
$$

as a Vandermonde determinant; hence

$$
V(\zeta) = \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) \zeta_1^{\sigma(1)-1} \cdots \zeta_n^{\sigma(n)-1}, \tag{9.10}
$$

where S_n denotes the group of permutations of $\{1,\dots,n\}$. Hence

$$
\prod_{k<\ell} (\zeta_k - \zeta_\ell)^2 = V(\zeta)^2 = \sum_{\sigma,\tau \in S_n} (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau) \zeta_1^{\sigma(1)+\tau(1)-2} \cdots \zeta_n^{\sigma(n)+\tau(n)-2}. \tag{9.11}
$$

Before getting back to (9.8), let us first identify the constant term in

$$
J(\theta) = (\operatorname{sgn}\gamma)(\zeta_1 \cdots \zeta_n)^{-(n-1)} V(\zeta)^2. \tag{9.12}
$$

We see this is

$$
(\operatorname{sgn}\gamma) \sum (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau), \tag{9.13}
$$

where the sum is over all $\sigma, \tau \in S_n$ such that $\sigma(k) + \tau(k) = n+1$, for each $k \in \{1,\dots,n\}$. In other words, we require $\tau = \gamma\sigma$, where γ is as specified in (9.7). Thus the sum in (9.13) is equal to $n!(\operatorname{sgn}\gamma)^2 = n!$, which establishes (9.4).

Back to (9.8), i.e., the study of the constant term in

$$
C_n (\operatorname{sgn}\gamma)(\zeta_1 \cdots \zeta_n)^{-(n-1)} \sum_{\mu,\nu,\sigma,\tau} (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau) \zeta_\mu^j \zeta_\nu^{-j} \zeta_1^{\sigma(1)+\tau(1)-2} \cdots \zeta_n^{\sigma(n)+\tau(n)-2}, \tag{9.14}
$$

which we write as

$$C_n(\operatorname{sgn}\gamma)(S_1+S_2), \tag{9.15}$$

where S_1 arises from the sum over $\mu=\nu$ and S_2 arises from the sum over $\mu\neq\nu$. Parallel to the analysis of (9.12)–(9.13), we have

$$S_1=n\cdot n!\,(\operatorname{sgn}\gamma), \tag{9.16}$$

or $C_n(\operatorname{sgn}\gamma)S_1=n$.

It remains to consider S_2. We see that, for a given $\mu\neq\nu$, a pair $\sigma,\tau\in S_n$ contributes to S_2 in the sum (9.14) if and only if $\sigma(k)+\tau(k)=n+1$ for all but two values of $k\in\{1,\dots,n\}$, namely $k=\mu$ and ν, and

$$\begin{aligned}\sigma(\mu)+\tau(\mu)&=n+1-j,\\ \sigma(\nu)+\tau(\nu)&=n+1+j.\end{aligned} \tag{9.17}$$

Equivalently, we require $\tau=\psi\gamma\sigma$ where γ is as in (9.7) and $\psi\in S_n$ has the property that $\psi(k)=k$ except for two values of $k\in\{1,\dots,n\}$, namely $k_1=\gamma\sigma(\mu)$ and $k_2=\gamma\sigma(\nu)$, and

$$\psi(k_1)=k_1-j,\quad \psi(k_2)=k_2+j.$$

This requires $\psi(k_1)=k_2,\ \psi(k_2)=k_2$, with

$$k_1=k_2+j. \tag{9.18}$$

Then

$$S_2=\sum(\operatorname{sgn}\gamma)(\operatorname{sgn}\psi), \tag{9.19}$$

the sum running over such allowable (μ,ν,σ,ψ). Note that (9.18) constrains k_1; we require $j+1\le k_1\le n$. Thus if $j\ge n$ the sum in (9.19) is empty and $S_2=0$. If $1\le j\le n-1$, then there are $(n-j)\cdot n!$ terms in the sum (9.19), and each $\operatorname{sgn}\psi=-1$. Hence

$$S_2=-(n-j)\cdot n!\,(\operatorname{sgn}\gamma),\quad 1\le j\le n-1, \tag{9.20}$$

and we have (9.1).

The formula (9.1) is useful for evaluating inner products of trace functions on $U(n)$, which arise as follows. If $f:S^1\to\mathbb{C}$ is a bounded Borel function, define $f(M)$ by the spectral representation of $M\in U(n)$. Set $X_f(M)=\operatorname{Tr}f(M)$. Using (9.1), one can show that

$$\int\limits_{U(n)} X_f(M)X_g(M)\,dM=\sum_{j=-\infty}^{\infty}a_{nj}\,\hat f(j)\hat g(-j), \tag{9.21}$$

where $\hat f(j)$ are the Fourier coefficients of f, $a_{n0}=n^2$, and $a_{nj}=\min(|j|,n)$ for $j\neq 0$.

10. The integral of $|\operatorname{Tr} M|^{2j}$

In this section we investigate

$$I_{nj} = \int\limits_{U(n)} |\operatorname{Tr} M|^{2j}\, dM. \tag{10.1}$$

This integral can be evaluated in representation-theoretic terms, using the identities

$$\begin{aligned} |\operatorname{Tr} M|^{2j} &= |\operatorname{Tr} \otimes^j M|^2 \\ &= \operatorname{Tr}\big[(\otimes^j M) \otimes (\otimes^j \overline{M})\big]. \end{aligned} \tag{10.2}$$

Here $\otimes^j M$ acts on $\otimes^j \mathbb{C}^n$ and $(\otimes^j M) \otimes (\otimes^j \overline{M})$ acts on $(\otimes^j \mathbb{C}^n) \otimes (\otimes^j \mathbb{C}^n)$. Thus we see that

$$I_{nj} = \operatorname{Tr} P_{nj}, \tag{10.3}$$

where

$$P_{nj} = \int\limits_{U(n)} (\otimes^j M) \otimes (\otimes^j \overline{M})\, dM \tag{10.4}$$

is the orthogonal projection of $(\otimes^j \mathbb{C}^n) \otimes (\otimes^j \mathbb{C}^n)$ onto the space E_{nj}, where $(\otimes^j M) \otimes (\otimes^j \overline{M})$ acts trivially for all $M \in U(n)$. Equivalently,

$$I_{nj} = \dim E_{nj}. \tag{10.5}$$

The first fundamental theorem of invariant theory specifies a spanning set of E_{nj}, consisting of $\{t_\pi : \pi \in S_j\}$, where

$$t_\pi(v_1 \otimes \cdots \otimes v_j, w_1 \otimes \cdots \otimes w_j) = \langle v_1, w_{\pi(1)} \rangle \cdots \langle v_j, w_{\pi(j)} \rangle. \tag{10.6}$$

See [GW]. However, this is not an orthonormal set typically, so it does not render the structure of P_{nj} obvious. (We examine below when this set is a basis of E_{nj}.) An equivalent formulation is obtained as follows. The representation $M \mapsto (\otimes^j M) \otimes (\otimes^j \overline{M})$ is equivalent to the representation ϑ_{nj} of $U(n)$ on $\operatorname{End}(\otimes^j \mathbb{C}^n)$ given by

$$\vartheta_{nj}(M)A = (\otimes^j M) A (\otimes^j M^{-1}). \tag{10.7}$$

Using this isomorphism, we identify E_{nj} with $\mathcal{E}_{nj}$, the space of operators on $\otimes^j \mathbb{C}^n$ that *commute* with the $U(n)$ action. A reformulation of the first fundamental theorem is that $\mathcal{E}_{nj}$ is spanned by the operators

$$\tau(\pi) \in \operatorname{End}(\otimes^j \mathbb{C}^n),\ \pi \in S_j, \quad \tau(\pi)(v_1 \otimes \cdots \otimes v_j) = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(j)}. \tag{10.8}$$

To elaborate, (10.8) yields a linear map

$$\tau^{\#}_{nj} : \ell^1(S_j) \longrightarrow \mathrm{End}(\otimes^j \mathbb{C}^n), \tag{10.9}$$

and $\mathcal{E}_{nj}$ is the range of $\tau^{\#}_{nj}$. In particular,

$$I_{nj} = \dim \mathcal{E}_{nj} = \frac{j!}{\dim\, \mathrm{Ker}\, \tau^{\#}_{nj}}. \tag{10.10}$$

Since it is not straightforward to specify $\dim E_{nj}$, the calculations given above are not the end of the story of evaluating (10.1). Let us turn to consequences of the first identity in (10.2). We make use of more detailed results on the representation theory of $U(n)$, and of S_j, which can be found in [S]. Let us note that the $U(n)$ action on $\otimes^j \mathbb{C}^n$ together with (10.8) produces a representation of $S_j \times U(n)$ on $\otimes \mathbb{C}^n$. A fundamental result on decomposing this into irreducibles yields

$$\mathrm{Tr}(\tau(\pi) \cdot \otimes^j M) = \sum_{\lambda \in F_{nj}} \chi^S_\lambda(\pi) s_\lambda(M). \tag{10.11}$$

Here F_{nj} is the set of Young diagrams, with rows of length $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_\ell$, such that $\lambda_1 + \cdots + \lambda_\ell = |\lambda| = j$ and $\ell \leq n$. Associated to each $\lambda \in F_{nj}$ is an irreducible representation of $U(n)$, with character $s_\lambda(M)$, and an irreducible representation of S_j, with character $\chi^S_\lambda(\pi)$. In particular, we have

$$(\mathrm{Tr}\, M)^j = \mathrm{Tr} \otimes^j M = \sum_{\lambda \in F_{nj}} f^\lambda s_\lambda(M), \quad f^\lambda = \chi^S_\lambda(id.). \tag{10.12}$$

It follows from §3 that the characters of irreducible representations of $U(n)$ form an orthonormal set, so

$$\int_{U(n)} |\,\mathrm{Tr}\, M|^{2j}\, dM = \sum_{\lambda \in F_{nj}} (f^\lambda)^2. \tag{10.13}$$

A further study of Young tableaux, involving something known as the Robinson-Schensted-Knuth correspondence, yields the following identity:

$$I_{nj} = \#\{\pi \in S_j : L_j(\pi) \leq n\}, \tag{10.14}$$

where $L_j : S_j \to \mathbb{C}^+$ is defined as follows. $L_j(\pi)$ is the length of the largest increasing subsequence of $(\pi(1), \dots, \pi(j))$. Note in particular that if $j \leq n$ then $L_j(\pi) \leq n$ for all π, so (10.14) implies

$$j \leq n \Longrightarrow I_{nj} = j!. \tag{10.15}$$

In other words, with $\tau^{\#}_{nj}$ given by (10.9),

$$j \leq n \Longrightarrow \mathrm{Ker}\, \tau^{\#}_{nj} = 0. \tag{10.16}$$

One can tackle the exercise of deducing this directly from (10.8)–(10.9). For a proof of (10.14), see, e.g., [VM].

In fact, the study of I_{nj} has motivated many recent papers; we mention [BDJ], [DE], [J], [TW], and various papers in [BI], and papers cited there.

References

[BDJ] J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. AMS 12 (1999), 1119–1178.

[BI] P. Bleher and A. Its (eds.), Random Matrix Models and Their Applications, MSRI Publications #40, Cambridge Univ. Press, 2001.

[BDK] D. Bump, P. Diaconis, and J. Keller, Unitary correlations and the Fejer kernel, Preprint, 2001.

[D] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach, Courant Lecture Notes #3, AMS, Providence, R.I., 1998.

[DE] P. Diaconis and S. Evans, Linear functionals of eigenvalues of random matrices, Trans. AMS 353 (2001), 2615–2633.

[DK] J.J. Duistermaat and J. Kolk, Lie Groups, Springer-Verlag, New York, 2000.

[GW] R. Goodman and N. Wallach, Representations and Invariants of the Classical Groups, Cambridge Univ. Press, 1998.

[Hel] G. Helminck (ed.), Geometric and Quantum Aspects of Integrable Systems, Lecture Notes in Physics #424, Springer-Verlag, New York, 1993.

[J] K. Johansson, Random permutations and the discrete Bessel kernel, pp. 259–269 in [BI].

[M] M. Mehta, Random Matrices, Academic Press, New York, 1991.

[S] B. Simon, Representations of Finite and Compact Groups, AMS, Providence, R.I., 1996.

[T1] M. Taylor, Noncommutative Harmonic Analysis, AMS, Providence, R.I., 1986.

[T2] M. Taylor, Partial Differential Equations, Vol. 1, Springer-Verlag, New York, 1996.

[TW] C. Tracy and H. Widom, Introduction to random matrices, pp. 103–130 in [Hel].

[VM] P. van Moerbeke, Integrable lattices, random matrices, and random permutations, pp. 321–406 in [BI].