ELEMENTARY DIFFERENTIAL GEOMETRY

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Introduction

These notes were produced to complement other material used by students in a first-year graduate course in elementary differential geometry at UNC. The primary texts used for the course were [E] and [O]. Also, the students received copies of the notes [T], which deal with the basic notions of multivariable calculus needed as background for differential geometry. This background material includes the notion of the derivative as a linear map, the inverse function theorem, existence and uniqueness of solutions to ODE, the multidimensional Riemann integral, including the change of variable formula, and an introduction to differential forms. Further material on differential geometry can be found in [T2].

The current notes contain 15 sections and a few appendices. Sections 1–3 are simply collections of exercises on determinants, cross products, trigonometric functions, and the elementary geometry of curves in Euclidean space. Section 4 establishes a special result about curves which we will not comment on here. Section 5 introduces the notion of a surface, and its metric tensor and surface measure. Much of this is done in n dimensions, but the focus of this course is on 2-dimensional surfaces in $\mathbb{R}^3$, and subsequent sections largely restrict attention to this case.

Sections 6–14 bear on the heart of the course. Section 6 discusses vector fields on a surface M, and defines $\nabla_V W$ when V is tangent to M (while W may or may not be tangent to M). Section 7 defines the shape operator on a surface $M \subset \mathbb{R}^3$ by $S(V) = -\nabla_V N$, where N is a unit normal field to M. The Gauss curvature and mean curvature are defined via the shape operator.

In §8 we define the covariant derivative on a surface M as $\nabla^M_V W = P\nabla_V W$, for V, W tangent to M, where $P(x)$ is the orthogonal projection of $\mathbb{R}^3$ onto the tangent space of M at x. The Riemann tensor R is defined in terms of ∇^M, and a formula for R is derived in terms of "connection coefficients" $\Gamma^j{}_{k\ell}$. Using this, we prove the Gauss *Theorema Egregium*, to the effect that the Gauss curvature of M is derivable from the metric tensor on M, without reference to the shape operator. A complementary result, known as the Codazzi equation, is treated in the exercises. This section differs most from the approaches in [E] and [O]; it makes closer contact with a variety of other treatments, including [DoC] and [Sp].

Sections 9–11 are devoted to geodesics, locally length-minimizing curves on a surface. These are characterized in terms of vanishing geodesic curvature, and also as solutions to certain systems of ODE. Material of §8 is used to treat the geodesic equations.

Section 12 connects the perspectives of §8 on curvature with those of [E] and [O], through material on frame fields and connection forms. At this point, I devote about 2 weeks to a treatment of differential forms, including Stokes' theorem. I used §§6–9 of [T] here.

In §13 the material of §12 and Stokes' formula are used to establish the Gauss-Bonnet formula, the pinnacle of this course. In §14 we discuss the Cartan structure equations associated with a frame field, again making close contact with [E] and [O]. We use these structure equations to re-derive Gauss-Codazzi equations of §8 and §12.

As a complement to the study of surfaces in Euclidean space $\mathbb{R}^3$, in §15 we look at surfaces in Minkowski space $\mathbb{R}^{2,1}$, particularly hyperbolic space, which has Gauss curvature $K = -1$.

There are several appendices. Appendix A discusses exponentiation of matrices, useful for solving constant-coefficient systems of ODE, and in particular useful for an exercise in §3. Appendix B derives an identity for the exterior derivative of a 1-form, used in §14. Appendices C and D provide proofs for some results on multivariable integrals used in the treatment of Crofton's formula in [E]. As this course has evolved, this topic is no longer covered in class, but we can still offer it to the interested reader.

1. Exercises on determinants and cross products

If $M_{n\times n}$ denotes the space of $n \times n$ complex matrices, we want to show that there is a map

$$\det : M_{n\times n} \to \mathbb{C} \tag{1.1}$$

which is *uniquely* specified as a function $\vartheta : M_{n\times n} \to \mathbb{C}$ satisfying:
(a) ϑ is *linear* in each column a_j of A,
(b) $\vartheta(\widetilde{A}) = -\vartheta(A)$ if $\widetilde{A}$ is obtained from A by interchanging two columns.
(c) $\vartheta(I) = 1$.

1. Let $A = (a_1, \dots, a_n)$, where a_j are column vectors; $a_j = (a_{1j}, \dots, a_{nj})^t$. Show that, if (a) holds, we have the expansion

$$\begin{aligned} \det A &= \sum_j a_{j1} \det(e_j, a_2, \dots, a_n) = \cdots \\ &= \sum_{j_1,\cdots,j_n} a_{j_1 1} \cdots a_{j_n n} \det(e_{j_1}, e_{j_2}, \dots, e_{j_n}), \end{aligned} \tag{1.2}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$.

2. Show that, if (b) and (c) also hold, then

$$\det A = \sum_{\sigma\in S_n} (\operatorname{sgn} \sigma)\, a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}, \tag{1.3}$$

where S_n is the set of *permutations* of $\{1, \dots, n\}$, and

$$\operatorname{sgn} \sigma = \det(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \pm 1. \tag{1.4}$$

To define sgn σ, the "sign" of a permutation σ, we note that every permutation σ can be written as a product of transpositions: $\sigma = \tau_1 \cdots \tau_\nu$, where a transposition of $\{1, \dots, n\}$ interchanges two elements and leaves the rest fixed. We say sgn $\sigma = 1$ if ν is even and sgn $\sigma = -1$ if ν is odd. It is necessary to show that sgn σ is independent of the choice of such a product representation. (Referring to (1.4) begs the question until we know that det is well defined.)

3. Let $\sigma \in S_n$ act on a function of n variables by

$$(\sigma f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{1.5}$$

Let P be the polynomial

$$P(x_1, \dots, x_n) = \prod_{1 \le j < k \le n} (x_j - x_k). \tag{1.6}$$

Show that

$$(\sigma P)(x) = (\mathrm{sgn}\ \sigma)\ P(x), \tag{1.7}$$

and that this implies that sgn σ is well defined.

4. Deduce that there is a unique determinant satisfying (a)–(c), and that it is given by (1.3). If (c) is replaced by $\vartheta(I) = r$, show that $\vartheta(A) = r\ \det A$.

5. Show that (1.3) implies

$$\det A = \det A^t. \tag{1.8}$$

Conclude that one can replace columns by rows in the characterization (a)–(c) of determinants.
Hint. $a_{\sigma(j)j} = a_{\ell\tau(\ell)}$ with $\ell = \sigma(j),\ \tau = \sigma^{-1}$. Also, sgn $\sigma =$ sgn τ.

6. Show that, if (a)–(c) hold (for rows), it follows that
(d) $\vartheta(\widetilde{A}) = \vartheta(A)$ if $\widetilde{A}$ is obtained from A by adding $c\rho_\ell$ to ρ_k, for some $c \in \mathbb{C}$, where $\rho_1, \dots, \rho_n$ are the rows of A.
Re-prove the uniqueness of ϑ satisfying (a)–(d) (for rows) by applying row operations to A until either some row vanishes or A is converted to I.

7. Show that

$$\det(AB) = (\det A)(\det B). \tag{1.9}$$

Hint. For fixed $B \in M_{n\times n}$, compare $\vartheta_1(A) = \det(AB)$ and $\vartheta_2(A) = (\det A)(\det B)$.

8. Show that

$$\det \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \det A_{11} \tag{1.10}$$

where $A_{11} = (a_{jk})_{2\le j,k\le n}$.
Hint. Do the first identity by the analogue of (d), for columns. Then exploit uniqueness for det on $M_{(n-1)\times(n-1)}$.

9. Deduce that $\det(e_j, a_2, \dots, a_n) = (-1)^{j-1} \det A_{1j}$ where A_{kj} is formed by deleting the kth column and the jth row from A.

10. Deduce from the first sum in (A.2) that

$$\det A = \sum_{j=1}^{n} (-1)^{j-1} a_{j1} \det A_{1j}. \tag{1.11}$$

More generally, for any $k \in \{1, \dots, n\}$,

$$\det A = \sum_{j=1}^{n} (-1)^{j-k} a_{jk} \det A_{kj}. \tag{1.12}$$

This is called an expansion of det A by minors, down the kth column.

11. Show that

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} = a_{11} a_{22} \cdots a_{nn}. \tag{1.13}$$

Hint. Use (1.10) and induction.

12. Given $A \in M_{n\times n}$, show that A is invertible if and only if det $A \neq 0$.
Hint. If $AB = I$, apply (1.9). On the other hand, if the rows of A are linearly dependent, apply Exercise 6.

The following exercises deal with the cross product of vectors in $\mathbb{R}^3$.

13. If $u, v \in \mathbb{R}^3$, show that the formula

$$w \cdot (u \times v) = \det \begin{pmatrix} w_1 & u_1 & v_1 \\ w_2 & u_2 & v_2 \\ w_3 & u_3 & v_3 \end{pmatrix} \tag{1.14}$$

for $u \times v = \kappa(u, v)$ defines uniquely a bilinear map $\kappa : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Show that it satisfies

$$i \times j = k, \quad j \times k = i, \quad k \times i = j,$$

where $\{i, j, k\}$ is the standard basis of $\mathbb{R}^3$.

14. We say $T \in SO(3)$ provided that T is a real 3×3 matrix satisfying $T^t T = I$ and $\det T > 0$, (hence $\det T = 1$). Show that

$$T \in SO(3) \Longrightarrow Tu \times Tv = T(u \times v). \tag{1.15}$$

Hint. Multiply the 3×3 matrix in Exercise 13 on the left by T.

15. Show that, if θ is the angle between u and v in $\mathbb{R}^3$, then

$$|u \times v| = |u| \; |v| \; |\sin \theta|. \tag{1.16}$$

More generally, show that for all $u, v, w, x \in \mathbb{R}^3$,

$$(u \times v) \cdot (w \times x) = (u \cdot w)(v \cdot x) - (u \cdot x)(v \cdot w).$$

Hint. Check this for $u = i, \ v = ai + bj$, and use Exercise 14 to show this suffices.

16. Show that $\kappa : \mathbb{R}^3 \to \mathrm{Skew}(3)$, the set of antisymmetric real 3×3 matrices, given by

$$\kappa(y_1, y_2, y_3) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \tag{1.17}$$

satisfies

$$Kx = y \times x, \quad K = \kappa(y). \tag{1.18}$$

Show that, with $[A, B] = AB - BA$,

$$\begin{aligned} \kappa(x \times y) &= \big[\kappa(x), \kappa(y)\big], \\ \mathrm{Tr}\left(\kappa(x)\kappa(y)^t\right) &= 2x \cdot y. \end{aligned} \tag{1.19}$$

2. Exercises on trigonometric functions

1. Let $\gamma(t)$ be a unit-speed parametrization of the unit circle $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, such that $\gamma(0) = (1, 0)$ and $\gamma'(0) = (0, 1)$. Define $\cos t$ and $\sin t$ by

$$\gamma(t) = (\cos t, \sin t). \tag{2.1}$$

Show that

$$\gamma'(t) = (-\sin t, \cos t), \tag{2.2}$$

without using previously acquired facts about sin and cos.
Hint. Use $|\gamma'(t)| = 1$ and the implication $\gamma(t) \cdot \gamma(t) = 1 \Rightarrow \gamma(t) \cdot \gamma'(t) = 0$.

2. Let

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2.3}$$

Show that, for each $s \in \mathbb{R}$,

$$\sigma_s(t) = R_t\, \gamma(s)^t \tag{2.4}$$

is a unit-speed curve lying on S^1, with $\sigma_s(0) = \gamma(s)^t$. Deduce that $R_t \gamma(s)^t = \gamma(s+t)^t$, and use this to show that

$$R_t R_s = R_{s+t}. \tag{2.5}$$

3. Use (2.5) to derive the identities

$$\begin{aligned} \sin(x+y) &= \sin x \ \cos y + \cos x \ \sin y \\ \cos(x+y) &= \cos x \ \cos y - \sin x \ \sin y. \end{aligned} \tag{2.6}$$

4. Use (2.1) and (2.6) to show that

$$\sin^2 t + \cos^2 t = 1, \quad \cos^2 t = \frac{1}{2}\big(1 + \cos 2t\big). \tag{2.7}$$

5. Note that $\gamma(t) = (\cos t, \sin t)$ is a map of $\mathbb{R}$ onto the unit circle $S^1 \subset \mathbb{R}^2$ with non-vanishing derivative, and, as t increases, $\gamma(t)$ moves monotonically, counter-clockwise.

We *define* π to be the smallest number $t_1 \in (0, \infty)$ such that $\gamma(t_1) = (-1, 0)$, so

$$\cos \pi = -1, \quad \sin \pi = 0. \tag{2.8}$$

Show that 2π is the smallest number $t_2 \in (0, \infty)$ such that $\gamma(t_2) = (1, 0)$, so

$$\cos 2\pi = 1, \quad \sin 2\pi = 0. \tag{2.9}$$

Show that

$$\begin{aligned} \cos(t + 2\pi) &= \cos t, \quad \cos(t + \pi) = -\cos t \\ \sin(t + 2\pi) &= \sin t, \quad \sin(t + \pi) = -\sin t. \end{aligned}$$

Show that $\gamma(\pi/2) = (0, 1)$, and that

$$\cos\Bigl(t + \frac{\pi}{2}\Bigr) = -\sin t, \quad \sin\Bigl(t + \frac{\pi}{2}\Bigr) = \cos t.$$

6. Show that $\sin : (-\pi/2, \pi/2) \to (-1, 1)$ is a diffeomorphism. We denote its inverse by

$$\text{arcsin} : (-1, 1) \longrightarrow \Bigl(-\frac{\pi}{2}, \frac{\pi}{2}\Bigr).$$

Show that $u(t) =$ arcsin t solves the ODE

$$\frac{du}{dt} = \frac{1}{\sqrt{1 - t^2}}, \quad u(0) = 0.$$

Hint. Apply the chain rule to $\sin\bigl(u(t)\bigr) = t$.
Deduce that, for $t \in (-1, 1)$,

$$\text{arcsin } t = \int_0^t \frac{dx}{\sqrt{1 - x^2}}. \tag{2.10}$$

7. Show that

$$\gamma\Bigl(\frac{\pi}{3}\Bigr) = \Bigl(\frac{1}{2}, \frac{\sqrt{3}}{2}\Bigr), \quad \gamma\Bigl(\frac{\pi}{6}\Bigr) = \Bigl(\frac{\sqrt{3}}{2}, \frac{1}{2}\Bigr).$$

Hint. Let

$$S = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Compute S^3 and show that $S = R_{\pi/3}$ Then compute $R_{\pi/2}R_{\pi/3}^{-1}$.
For intuition behind these formulas, look at Fig. 2.1.

8. Show that $\sin \pi/6 = 1/2$, and hence that

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} \Bigl(\frac{1}{2}\Bigr)^{2n+1},$$

where

$$a_0 = 1, \quad a_{n+1} = \frac{2n+1}{2n+2} a_n.$$

Show that

$$\frac{\pi}{6} - \sum_{n=0}^{k} \frac{a_n}{2n+1} \Bigl(\frac{1}{2}\Bigr)^{2n+1} < \frac{4^{-k}}{3(2k+3)}.$$

Using a calculator, sum the series over $0 \le n \le 20$, and verify that

$$\pi \approx 3.141592653589 \cdots$$

9. For $x \neq (k+1/2)\pi,\ k \in \mathbb{Z}$, set

$$\tan x = \frac{\sin x}{\cos x}.$$

Show that $1 + \tan^2 x = 1/\cos^2 x$. Show that $w(x) = \tan x$ satisfies the ODE

$$\frac{dw}{dx} = 1 + w^2, \quad w(0) = 0.$$

10. Show that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a diffeomorphism. Denote the inverse by

$$\arctan : \mathbb{R} \longrightarrow \Bigl(-\frac{\pi}{2}, \frac{\pi}{2}\Bigr).$$

Show that

$$\arctan y = \int_0^y \frac{dx}{1+x^2}. \tag{2.11}$$

3. Exercises on the Frenet-Serret formulas

1. Let $x(t)$ be a smooth curve in $\mathbb{R}^3$; assume it is parametrized by arclength, so $T(t) = x'(t)$ has unit length; $T(t) \cdot T(t) = 1$. Differentiating, we have $T'(t) \perp T(t)$. The *curvature* is defined to be $\kappa(t) = \|T'(t)\|$. If $\kappa(t) \neq 0$, we set $N(t) = T'/\|T'\|$, so

$$T' = \kappa N,$$

and N is a unit vector orthogonal to T. We define $B(t)$ by

$$B = T \times N. \tag{3.1}$$

Note that (T, N, B) form an orthonormal basis of $\mathbb{R}^3$ for each t, and

$$T = N \times B, \quad N = B \times T. \tag{3.2}$$

By (3.1) we have $B' = T \times N'$. Deduce that B' is orthogonal to both T and B, hence parallel to N. We set

$$B' = -\tau N,$$

for smooth $\tau(t)$, called the *torsion*.

2. From $N' = B' \times T + B \times T'$ and the formulas for T' and B' above, deduce the following system, called the *Frenet-Serret formula*:

$$\begin{aligned} T' &= \qquad \kappa N \\ N' &= -\kappa T \qquad + \tau B \\ B' &= \qquad -\tau N \end{aligned} \tag{3.3}$$

Form the 3×3 matrix

$$A(t) = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \tag{3.4}$$

and deduce that the 3×3 matrix $F(t)$ whose columns are T, N, B :

$$F = (T, N, B)$$

satisfies the ODE

$$\frac{dF}{dt} = F\,A(t).$$

3. Derive the following *converse* to the Frenet-Serret formula. Let $T(0), N(0), B(0)$ be an orthonormal set in $\mathbb{R}^3$, such that $B(0) = T(0) \times N(0)$, let $\kappa(t)$ and $\tau(t)$ be given smooth functions, and solve the system (3.3). Show that there is a unique curve $x(t)$ such that $x(0) = 0$ and $T(t), N(t), B(t)$ are associated to $x(t)$ by the construction in Exercise 1, so in particular the curve has curvature $\kappa(t)$ and torsion $\tau(t)$.
Hint. To prove that (3.1)–(3.2) hold for all t, consider the next exercise.

4. Let $A(t)$ be a smooth $n\times n$ real matrix function which is *skew adjoint* for all t (of which (3.4) is an example). Suppose $F(t)$ is a real $n \times n$ matrix function satisfying

$$\frac{dF}{dt} = F\,A(t).$$

If $F(0)$ is an orthogonal matrix, show that $F(t)$ is orthogonal for all t.
Hint. Set $K(t) = F(t)^*F(t)$. Show that $K(t)$ and $K_0(t) = I$ both solve the initial value problem

$$\frac{dK}{dt} = [K, A(t)], \quad K(0) = I.$$

5. Let $U_1 = T,\ U_2 = N,\ U_3 = B$, and set

$$\omega(t) = \tau T + \kappa B.$$

Show that (3.3) is equivalent to

$$U_j' = \omega \times U_j,\ 1 \le j \le 3.$$

6. Suppose τ and κ are constant. Show that ω is constant, so $T(t)$ satisfies the constant coefficient ODE

$$T'(t) = \omega \times T(t).$$

Note that $\omega \cdot T(0) = \tau$. Show that, after a translation and rotation, $x(t)$ takes the form

$$\gamma(t) = \Bigl(\frac{\kappa}{\lambda^2}\cos \lambda t, \frac{\kappa}{\lambda^2}\sin \lambda t, \frac{\tau}{\lambda}t\Bigr), \quad \lambda^2 = \kappa^2 + \tau^2.$$

7. Suppose $x(t)$, parametrized by arclength, lies in the sphere $S_R = \{x \in \mathbb{R}^3 : |x| = R\}$, for all t. If $\rho(t) = \kappa(t)^{-1}$ denotes the radius of curvature of this curve, show that

$$\rho(t)^2 + \Bigl(\frac{\rho'(t)}{\tau(t)}\Bigr)^2 = R^2.$$

Hint. Differentiate the identity $x(t) \cdot x(t) = R^2$ repeatedly, and substitute in various parts of (3.3) for derivatives of T, etc. Use this to show that

$$x \cdot T = 0, \quad x \cdot N = -\rho, \quad x \cdot B = -\rho'/\tau.$$

8. In this problem, do *not* assume that $x(t)$ is parametrized by arclength. Define the arclength parameter s by $ds/dt = |x'(t)|$, and set $T(t) = x'(t)/|x'(t)|$, so $dT/ds = \kappa N$. Show that

$$x''(t) = \frac{d^2s}{dt^2}\, T(t) + \Big(\frac{ds}{dt}\Big)^2 \kappa(t)\, N(t). \tag{3.5}$$

Taking the cross product of both sides with $T(t)$, deduce that

$$\kappa(t)\, B(t) = \frac{x'(t) \times x''(t)}{|x'(t)|^3}. \tag{3.6}$$

Hence

$$\kappa(t) = \frac{|x'(t) \times x''(t)|}{|x'(t)|^3}. \tag{3.7}$$

Hint. Differentiate the identity $x'(t) = (ds/dt)T(t)$ to get (3.5).

9. In the setting of Exercise 8, show that

$$x'''(t) = \big\{s''' - (s')^3\kappa^2\big\}T + \big\{3s's''\kappa + (s')^2\kappa'\big\}N + (s')^3\kappa\tau B. \tag{3.8}$$

Deduce that

$$x'(t) \times x''(t) \cdot x'''(t) = (s')^6\kappa^2\tau, \tag{3.9}$$

and hence that the torsion is given by

$$\tau(t) = \frac{x'(t) \times x''(t) \cdot x'''(t)}{|x'(t) \times x''(t)|^2}. \tag{3.10}$$

10. Let $x(t)$ be a unit speed curve, with T, κ, τ, etc. as in Exercise 1. Consider the curve $y(t) = T(t)$, a curve which is perhaps not of unit speed. Show that its curvature and torsion are given by

$$\tilde{\kappa} = \sqrt{1 + \Big(\frac{\tau}{\kappa}\Big)^2}, \quad \tilde{\tau} = \frac{\frac{d}{ds}\big(\frac{\tau}{\kappa}\big)}{\kappa\big[1 + \big(\frac{\tau}{\kappa}\big)^2\big]}. \tag{3.11}$$

Hint. Apply the results of Exercises 8–9 to $y(t) = T(t)$. Use (3.3) to derive

$$T'' = -\kappa^2 T + \kappa' N + \kappa\tau B, \quad T' \times T'' = \kappa^2 \tau T + \kappa^3 B.$$

Then produce a formula for T'''.

11. Consider a plane curve $(x_1(t), x_2(t))$. Show that its curvature at is given by

$$\kappa(t) = |x_1' x_2'' - x_2' x_1''|/|x'|^3.$$

Hint. Apply (3.7) to $\gamma(t) = (x_1(t), x_2(t), 0)$.

A unit speed curve $x(t)$ in $\mathbb{R}^3$ is called a cylindrical helix provided there is a unit $u \in \mathbb{R}^3$ such that $T(t) \cdot u = c$ is constant. Rotating, we can assume $u = k$ and write

$$x(t) = \big(x_1(t), x_2(t), ct\big), \quad (x_1')^2 + (x_2')^2 = 1 - c^2. \tag{3.12}$$

Exercises 12–16 deal with cylindrical helices.

12. Directly computing $x'' = T' = \kappa N$ from (3.12) and comparing the calculation of the quantities in (3.6), show that

$$\kappa = |x''|, \quad (1 - c^2)\kappa^2 = (x_1' x_2'' - x_2' x_1'')^2.$$

13. If $\gamma(t) = (x_1(t), x_2(t))$ is the planar projection of (3.12), show that its curvature $\tilde{\kappa}$ satisfies

$$\tilde{\kappa}(t) = \frac{\kappa(t)}{1 - c^2}.$$

14. Show that a unit-speed curve $x(t)$ in $\mathbb{R}^3$ is a cylindrical helix if and only if $\tau(t)/\kappa(t)$ is constant (equal to $\pm c/\sqrt{1 - c^2}$).
One approach. If $x(t)$ has the form (3.12), show that

$$k \perp N \Longrightarrow k = cT(t) \pm \sqrt{1 - c^2}\, B(t),$$

and differentiate this, using (3.3).
Another approach. Use (3.11).

15. Let $\kappa(t) > 0$ be a given smooth function and let $\beta \in \mathbb{R}$ be given. Show that a cylindrical helix with curvature $\kappa(t)$ and torsion $\tau(t) = \beta\kappa(t)$ is given by $x(t) = \int_0^t T(s)\, ds$, with

$$\big(T(t), N(t), B(t)\big) = e^{\sigma(t)K},$$

where

$$K = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\beta \\ 0 & \beta & 0 \end{pmatrix}, \quad \sigma(t) = \int_0^t \kappa(s)\, ds.$$

Hint. Take $T(0) = i, N(0) = j, B(0) = k$ and use Exercise 2.

16. Let K be the 3×3 matrix of Exercise 15. Let

$$v = (1 + \beta^2)^{-1/2}(\beta i + k), \quad w = (1 + \beta^2)^{-1/2}(i - \beta k).$$

Show that v, j, w form an orthonormal basis of $\mathbb{R}^3$ and

$$Kv = 0, \quad Kj = -(1 + \beta^2)^{1/2} w, \quad Kw = (1 + \beta^2)^{1/2} j.$$

Deduce that

$$\begin{aligned} e^{\sigma K} v &= v, \\ e^{\sigma K} j &= (\cos \eta) j - (\sin \eta) w, \\ e^{\sigma K} w &= (\sin \eta) j + (\cos \eta) w, \end{aligned}$$

where $\eta = (1 + \beta^2)^{1/2} \sigma$.

4. Curves with nonvanishing curvature

Proposition 4.1. *Let γ be a smooth curve in $\mathbb{R}^3$. Then γ can be smoothly approximated by a sequence of curves in $\mathbb{R}^3$ with nowhere vanishing curvature.*

Proof. If γ is a line, it can be approximated by helices. Otherwise, making a preliminary approximation, e.g., by a real analytic curve, we can assume the curvature has only isolated zeros.

Suppose for example that for t in a segment $[a,b]$, the curvature of $\gamma(t)$ vanishes at just one point, $t_0 \in (a,b)$. We can assume there exist $\alpha > 0$ and $c > 0$ such that $\kappa(t) \geq \alpha$ for $t \in [a, a+c]$ and for $t \in [b-c, b]$. We will perturb γ on a compact subset of (a,b) to get a curve whose curvature vanishes nowhere on $[a,b]$. Iteration of this procedure proves the proposition.

Since curvature is independent of parametrization, there is no loss of generality in assuming that $\gamma(t) = (x_1(t), x_2(t), t)$, for $t \in [a,b]$. Then

$$\gamma'(t) \times \gamma''(t) = \bigl(-x_2''(t), x_1''(t), x_1'(t)x_2''(t) - x_1''(t)x_2'(t)\bigr).$$

A similar calculation holds for a perturbed curve $\tilde{\gamma}(t) = (x_1(t) + \xi_1(t), x_2(t) + \xi_2(t), t)$. Hence it suffices to produce an arbitrarily small perturbation $(\xi_1(t), \xi_2(t))$, supported on $t \in [a + c/2, b - c/2]$, with the property that

$$\sigma(t) = \bigl(x_1''(t) + \xi_1''(t), x_2''(t) + \xi_2''(t)\bigr)$$

is nowhere vanishing for $t \in [a+c, b-c]$.

To get this, fix a bump function $\varphi \in C_0^\infty(a + c/2, b - c/2)$, equal to 1 on the interval $[a+c, b-c]$. Since the image of $(x_1''(t), x_2''(t))$ has empty interior in $\mathbb{R}^2$, one can choose a point $(p_1, p_2) \in \mathbb{R}^2$ arbitrarily close to $(0,0)$, not in this image. Now simply take

$$\bigl(\xi_1(t), \xi_2(t)\bigr) = -\varphi(t)(p_1, p_2).$$

5. Surfaces, metric tensors, and surface integrals

A smooth m-dimensional surface $M \subset \mathbb{R}^n$ is characterized by the following property. Given $p \in M$, there is a neighborhood U of p in M and a smooth map $\varphi : \mathcal{O} \to U$, from an open set $\mathcal{O} \subset \mathbb{R}^m$ bijectively to U, with injective derivative at each point. Such a map φ is called a *coordinate chart* on M. We call $U \subset M$ a coordinate patch. If all such maps φ are smooth of class C^k, we say M is a surface of class C^k. In §8 we will define analogous notions of a C^k surface with boundary, and of a C^k surface with corners.

There is an abstraction of the notion of a surface, namely the notion of a "manifold," which is useful in more advanced studies. A treatment of calculus on manifolds can be found in [Sp].

If $\varphi : \mathcal{O} \to U$ is a C^k coordinate chart, such as described above, and $\varphi(x_0) = p$, we set

$$T_pM = \operatorname{Range} D\varphi(x_0), \tag{5.1}$$

a linear subspace of $\mathbb{R}^n$ of dimension m, and we denote by N_pM its orthogonal complement. It is useful to consider the following map. Pick a linear isomorphism $A : \mathbb{R}^{n-m} \to N_pM$, and define

$$\Phi : \mathcal{O} \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n, \quad \Phi(x, z) = \varphi(x) + Az. \tag{5.2}$$

Thus Φ is a C^k map defined on an open subset of $\mathbb{R}^n$. Note that

$$D\Phi(x_0, 0) \begin{pmatrix} v \\ w \end{pmatrix} = D\varphi(x_0)v + Aw, \tag{5.3}$$

so $D\Phi(x_0, 0) : \mathbb{R}^n \to \mathbb{R}^n$ is surjective, hence bijective, so the Inverse Function Theorem applies; Φ maps some neighborhood of $(x_0, 0)$ diffeomorphically onto a neighborhood of $p \in \mathbb{R}^n$.

Suppose there is another C^k coordinate chart, $\psi : \Omega \to U$. Since φ and ψ are by hypothesis one-to-one and onto, it follows that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is a well defined map, which is one-to-one and onto. See Fig. 5.1. In fact, we can say more.

Lemma 5.1. *Under the hypotheses above, F is a C^k diffeomorphism.*

Proof. It suffices to show that F and F^{-1} are C^k on a neighborhood of x_0 and y_0, respectively, where $\varphi(x_0) = \psi(y_0) = p$. Let us define a map Ψ in a fashion similar to (5.2). To be precise, we set $\widetilde{T}_pM = \operatorname{Range} D\psi(y_0)$, and let $\widetilde{N}_pM$ be its orthogonal complement. (Shortly we will show that $\widetilde{T}_pM = T_pM$, but we are not quite ready for that.) Then pick a linear isomorphism $B : \mathbb{R}^{n-m} \to \widetilde{N}_pM$ and set

$\Psi(y,z)=\psi(y)+Bz$, for $(y,z)\in\Omega\times\mathbb{R}^{n-m}$. Again, Ψ is a C^k diffeomorphism from a neighborhood of $(y_0,0)$ onto a neighborhood of p.

It follows that $\Psi^{-1}\circ\Phi$ is a C^k diffeomeophism from a neighborhood of $(x_0,0)$ onto a neighborhood of $(y_0,0)$. Now note that, for x close to x_0 and y close to y_0,

$$\Psi^{-1}\circ\Phi(x,0)=\bigl(F(x),0\bigr),\quad \Phi^{-1}\circ\Psi(y,0)=\bigl(F^{-1}(y),0\bigr). \tag{5.4}$$

These identities imply that F and F^{-1} have the desired regularity.

Thus, when there are two such coordinate charts, $\varphi:\mathcal{O}\to U$, $\psi:\Omega\to U$, we have a C^k diffeomorphism $F:\mathcal{O}\to\Omega$ such that

$$\varphi=\psi\circ F. \tag{5.5}$$

By the chain rule,

$$D\varphi(x)=D\psi(y)\,DF(x),\quad y=F(x). \tag{5.6}$$

In particular this implies that $\operatorname{Range}D\varphi(x_0)=\operatorname{Range}D\psi(y_0)$, so T_pM in (5.1) is independent of the choice of coordinate chart. It is called the *tangent space* to M at p.

We next define an object called the *metric tensor* on M. Given a coordinate chart $\varphi:\mathcal{O}\to U$, there is associated an $m\times m$ matrix $G(x)=\bigl(g_{jk}(x)\bigr)$ of functions on $\mathcal{O}$, defined in terms of the inner product of vectors tangent to M :

$$g_{jk}(x)=D\varphi(x)e_j\cdot D\varphi(x)e_k=\frac{\partial\varphi}{\partial x_j}\cdot\frac{\partial\varphi}{\partial x_k}=\sum_{\ell=1}^{n}\frac{\partial\varphi_\ell}{\partial x_j}\frac{\partial\varphi_\ell}{\partial x_k}, \tag{5.7}$$

where $\{e_j:1\le j\le m\}$ is the standard orthonormal basis of $\mathbb{R}^m$. Equivalently,

$$G(x)=D\varphi(x)^t\,D\varphi(x). \tag{5.8}$$

We call (g_{jk}) the metric tensor of M, on U, with respect to the coordinate chart $\varphi:\mathcal{O}\to U$. Note that this matrix is positive-definite. From a coordinate-independent point of view, the metric tensor on M specifies inner products of vectors tangent to M, using the inner product of $\mathbb{R}^n$.

If we take another coordinate chart $\psi:\Omega\to U$, we want to compare (g_{jk}) with $H=(h_{jk})$, given by

$$h_{jk}(y)=D\psi(y)e_j\cdot D\psi(y)e_k,\quad \text{i.e.,}\ \ H(y)=D\psi(y)^t\ D\psi(y). \tag{5.9}$$

As seen above we have a diffeomorphism $F:\mathcal{O}\to\Omega$ such that (5.5)–(5.6) hold. Consequently,

$$G(x)=DF(x)^t\,H(y)\,DF(x), \tag{5.10}$$

or equivalently,

$$g_{jk}(x) = \sum_{i,\ell} \frac{\partial F_i}{\partial x_j} \frac{\partial F_\ell}{\partial x_k}\, h_{i\ell}(y). \tag{5.11}$$

We now define the notion of surface integral on M. If $f : M \to \mathbb{R}$ is a continuous function supported on U, we set

$$\int\limits_M f\, dS = \int\limits_{\mathcal{O}} f \circ \varphi(x)\, \sqrt{g(x)}\; dx, \tag{5.12}$$

where

$$g(x) = \det G(x). \tag{5.13}$$

We need to know that this is independent of the choice of coordinate chart $\varphi : \mathcal{O} \to U$. Thus, if we use $\psi : \Omega \to U$ instead, we want to show that (5.12) is equal to $\int_\Omega f \circ \psi(y)\, \sqrt{h(y)}\; dy$, where $h(y) = \det H(y)$. Indeed, since $f \circ \psi \circ F = f \circ \varphi$, we can apply the change of variable formula for multidimensional integrals, to get

$$\int\limits_\Omega f \circ \psi(y)\, \sqrt{h(y)}\; dy = \int\limits_{\mathcal{O}} f \circ \varphi(x)\, \sqrt{h(F(x))}\; |\det\, DF(x)|\; dx. \tag{5.14}$$

Now, (5.10) implies that

$$\sqrt{g(x)} = |\det\, DF(x)|\; \sqrt{h(y)}, \tag{5.15}$$

so the right side of (5.14) is seen to be equal to (5.12), and our surface integral is well defined, at least for f supported in a coordinate patch. More generally, if $f : M \to \mathbb{R}$ has compact support, write it as a finite sum of terms, each supported on a coordinate patch, and use (5.12) on each patch.

Let us pause to consider the special cases $m = 1$ and $m = 2$. For $m = 1$, we are considering a curve in $\mathbb{R}^n$, say $\varphi : [a, b] \to \mathbb{R}^n$. Then $G(x)$ is a 1×1 matrix, namely $G(x) = |\varphi'(x)|^2$. If we denote the curve in $\mathbb{R}^n$ by γ, rather than M, the formula (5.12) becomes

$$\int\limits_\gamma f\; ds = \int_a^b f \circ \varphi(x)\; |\varphi'(x)|\; dx. \tag{5.16}$$

In case $m = 2$, let us consider a surface $M \subset \mathbb{R}^3$, with a coordinate chart $\varphi : \mathcal{O} \to U \subset M$. For f supported in U, an alternative way to write the surface integral is

$$\int\limits_M f\; dS = \int\limits_{\mathcal{O}} f \circ \varphi(x)\; |\partial_1 \varphi \times \partial_2 \varphi|\; dx_1 dx_2, \tag{5.17}$$

where $u \times v$ is the cross product of vectors u and v in $\mathbb{R}^3$. To see this, we compare this integrand with the one in (5.12). In this case,

$$g = \det \begin{pmatrix} \partial_1\varphi \cdot \partial_1\varphi & \partial_1\varphi \cdot \partial_2\varphi \\ \partial_2\varphi \cdot \partial_1\varphi & \partial_2\varphi \cdot \partial_2\varphi \end{pmatrix} = |\partial_1\varphi|^2 |\partial_2\varphi|^2 - (\partial_1\varphi \cdot \partial_2\varphi)^2. \tag{5.18}$$

Recall from §1 that $|u \times v| = |u|\ |v|\ |\sin\theta|$, where θ is the angle between u and v. Equivalently, since $u \cdot v = |u|\ |v|\ \cos\theta$,

$$|u \times v|^2 = |u|^2|v|^2\big(1 - \cos^2\theta\big) = |u|^2|v|^2 - (u \cdot v)^2. \tag{5.19}$$

Thus we see that $|\partial_1\varphi \times \partial_2\varphi| = \sqrt{g}$, in this case, and (5.17) is equivalent to (5.12).

An important class of surfaces is the class of graphs of smooth functions. Let $u \in C^1(\Omega)$, for an open $\Omega \subset \mathbb{R}^{n-1}$, and let M be the graph of $z = u(x)$. The map $\varphi(x) = \big(x, u(u)\big)$ provides a natural coordinate system, in which the metric tensor is given by

$$g_{jk}(x) = \delta_{jk} + \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}. \tag{5.20}$$

If u is C^1, we see that g_{jk} is continuous. To calculate $g = \det(g_{jk})$, at a given point $p \in \Omega$, if $\nabla u(p) \neq 0$, rotate coordinates so that $\nabla u(p)$ is parallel to the x_1 axis. We see that

$$\sqrt{g} = \big(1 + |\nabla u|^2\big)^{1/2}. \tag{5.21}$$

In particular, the $(n-1)$-dimensional volume of the surface M is given by

$$V_{n-1}(M) = \int\limits_M dS = \int\limits_\Omega \big(1 + |\nabla u(x)|^2\big)^{1/2}\, dx. \tag{5.22}$$

Particularly important examples of surfaces are the unit spheres S^{n-1} in $\mathbb{R}^n$,

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}. \tag{5.23}$$

Spherical polar coordinates on $\mathbb{R}^n$ are defined in terms of a smooth diffeomorphism

$$R : (0,\infty) \times S^{n-1} \longrightarrow \mathbb{R}^n \setminus 0, \quad R(r,\omega) = r\omega. \tag{5.24}$$

Let $(h_{\ell m})$ denote the metric tensor on S^{n-1} (induced from its inclusion in $\mathbb{R}^n$) with respect to some coordinate chart $\varphi : \mathcal{O} \to U \subset S^{n-1}$. Then, with respect to the coordinate chart $\Phi : (0,\infty) \times \mathcal{O} \to \mathcal{U} \subset \mathbb{R}^n$ given by $\Phi(r,y) = r\varphi(y)$, the Euclidean metric tensor can be written

$$\big(e_{jk}\big) = \begin{pmatrix} 1 & \\ & r^2 h_{\ell m} \end{pmatrix}. \tag{5.25}$$

To see that the blank terms vanish, i.e., $\partial_r\Phi \cdot \partial_{x_j}\Phi = 0$, note that $\varphi(x)\cdot\varphi(x) = 1 \Rightarrow \partial_{x_j}\varphi(x)\cdot\varphi(x) = 0$. Now (5.25) yields

$$\sqrt{e} = r^{n-1}\sqrt{h}. \tag{5.26}$$

We therefore have the following result for integrating a function in spherical polar coordinates.

$$\int\limits_{\mathbb{R}^n} f(x)\,dx = \int\limits_{S^{n-1}} \Big[\int_0^\infty f(r\omega)r^{n-1}\,dr\Big]\,dS(\omega). \tag{5.27}$$

We next compute the $(n-1)$-dimensional area A_{n-1} of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, using (5.27) together with the computation

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{n/2}, \tag{5.28}$$

which can be reduced to the case $n = 2$ and done there in polar coordinates. First note that, whenever $f(x) = \varphi(|x|)$, (5.27) yields

$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\,dx = A_{n-1}\int_0^\infty \varphi(r)r^{n-1}\,dr. \tag{5.29}$$

In particular, taking $\varphi(r) = e^{-r^2}$ and using (5.28), we have

$$\pi^{n/2} = A_{n-1}\int_0^\infty e^{-r^2}r^{n-1}\,dr = \tfrac{1}{2}A_{n-1}\int_0^\infty e^{-s}s^{n/2-1}\,ds, \tag{5.30}$$

where we used the substitution $s = r^2$ to get the last identity. We hence have

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{5.31}$$

where $\Gamma(z)$ is Euler's Gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-s}s^{z-1}\,ds. \tag{5.32}$$

We need to complement (5.31) with some results on $\Gamma(z)$ allowing a computation of $\Gamma(n/2)$ in terms of more familiar quantities. Of course, setting $z = 1$ in (5.32), we immediately get

$$\Gamma(1) = 1. \tag{5.33}$$

Also, setting $n = 1$ in (5.30), we have

$$\pi^{1/2} = 2\int_0^\infty e^{-r^2}\,dr = \int_0^\infty e^{-s}s^{-1/2}\,ds,$$

or

$$\Gamma\Big(\frac{1}{2}\Big) = \pi^{1/2}. \tag{5.34}$$

We can proceed inductively from (5.33)–(5.34) to a formula for $\Gamma(n/2)$ for any $n \in \mathbb{Z}^+$, using the following.

Lemma 5.2. *For all $z > 0$,*

$$\Gamma(z+1) = z\Gamma(z). \tag{5.35}$$

Proof. We can write

$$\Gamma(z+1) = -\int_0^\infty \Bigl(\frac{d}{ds}e^{-s}\Bigr)s^z\,ds = \int_0^\infty e^{-s}\,\frac{d}{ds}\bigl(s^z\bigr)\,ds,$$

the last identity by integration by parts. The last expression here is seen to equal the right side of (5.35).

Consequently, for $k \in \mathbb{Z}^+$,

$$\Gamma(k) = (k-1)!,\quad \Gamma\Bigl(k+\frac{1}{2}\Bigr) = \Bigl(k-\frac{1}{2}\Bigr)\cdots\Bigl(\frac{1}{2}\Bigr)\pi^{1/2}. \tag{5.36}$$

Thus (5.31) can be rewritten

$$A_{2k-1} = \frac{2\pi^k}{(k-1)!},\quad A_{2k} = \frac{2\pi^k}{\bigl(k-\frac{1}{2}\bigr)\cdots\bigl(\frac{1}{2}\bigr)}. \tag{5.37}$$

We discuss another important example of a smooth surface, in the space $M(n) \approx \mathbb{R}^{n^2}$ of real $n \times n$ matrices, namely $SO(n)$, the set of matrices $T \in M(n)$ satisfying $T^tT = I$ and det $T > 0$ (hence det $T = 1$). The exponential map Exp: $M(n) \to M(n)$ defined by $\mathrm{Exp}(A) = e^A$ has the property

$$\mathrm{Exp} : \mathrm{Skew}(n) \longrightarrow SO(n), \tag{5.38}$$

where Skew(n) is the set of skew-symmetric matrices in $M(n)$. Also $D\ \mathrm{Exp}(0)A = A$; hence

$$D\ \mathrm{Exp}(0) = \iota : \mathrm{Skew}(n) \hookrightarrow M(n). \tag{5.39}$$

It follows from the Inverse Function Theorem that there is a neighborhood $\mathcal{O}$ of 0 in Skew(n) which is mapped by Exp diffeomorphically onto a smooth surface $U \subset M(n)$, of dimension $m = n(n-1)/2$. Furthermore, U is a neighborhood of I in $SO(n)$. For general $T \in SO(n)$, we can define maps

$$\varphi_T : \mathcal{O} \longrightarrow SO(n),\quad \varphi_T(A) = T\ \mathrm{Exp}(A), \tag{5.40}$$

and obtain coordinate charts in $SO(n)$, which is consequently a smooth surface of dimension $n(n-1)/2$ in $M(n)$. Note that $SO(n)$ is a closed bounded subset of $M(n)$; hence it is compact.

We use the inner product on $M(n)$ computed componentwise; equivalently,

$$\langle A, B\rangle = \mathrm{Tr}\,(B^tA) = \mathrm{Tr}\,(BA^t). \tag{5.41}$$

This produces a metric tensor on $SO(n)$. The surface integral over $SO(n)$ has the following important invariance property.

Proposition 5.3. *Given $f \in C\big(SO(n)\big)$, if we set*

$$\rho_T f(X) = f(XT), \quad \lambda_T f(X) = f(TX), \tag{5.42}$$

for $T, X \in SO(n)$, we have

$$\int\limits_{SO(n)} \rho_T f \ dS = \int\limits_{SO(n)} \lambda_T f \ dS = \int\limits_{SO(n)} f \ dS. \tag{5.43}$$

Proof. Given $T \in SO(n)$, the maps $R_T, L_T : M(n) \to M(n)$ defined by $R_T(X) = XT,\ L_T(X) = TX$ are easily seen from (5.41) to be isometries. Thus they yield maps of $SO(n)$ to itself which preserve the metric tensor, proving (5.43).

Since $SO(n)$ is compact, its total volume $V\big(SO(n)\big) = \int_{SO(n)} 1 \ dS$ is finite. We define the integral with respect to "Haar measure"

$$\int\limits_{SO(n)} f(g)\, dg = \frac{1}{V\big(SO(n)\big)} \int\limits_{SO(n)} f\, dS. \tag{5.44}$$

This is used in many arguments involving "averaging over rotations."

Exercises

1. Define $\varphi : [0, \theta] \to \mathbb{R}^2$ to be $\varphi(t) = (\cos t, \sin t)$. Show that, if $0 < \theta \le 2\pi$, the image of $[0, \theta]$ under φ is an arc of the unit circle, of length θ. Deduce that the unit circle in $\mathbb{R}^2$ has total length 2π. This result follows also from (5.37).
Remark. Use the definition of π given in the auxiliary problem set after §3.
This length formula provided the original definition of π, in ancient Greek geometry.

2. Compute the volume of the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$.
Hint. Apply (5.29) with $\varphi = \chi_{[0,1]}$.

3. Taking the upper half of the sphere S^n to be the graph of $x_{n+1} = (1 - |x|^2)^{1/2}$, for $x \in B^n$, the unit ball in $\mathbb{R}^n$, deduce from (5.22) and (5.29) that

$$A_n = 2A_{n-1} \int_0^1 \frac{r^{n-1}}{\sqrt{1 - r^2}}\, dr = 2A_{n-1} \int_0^{\pi/2} (\sin\, \theta)^{n-1}\, d\theta.$$

Use this to get an alternative derivation of the formula (5.37) for A_n.
Hint. Rewrite this formula as

$$A_n = A_{n-1} b_{n-1}, \quad b_k = \int_0^\pi \sin^k \theta \, d\theta.$$

To analyze b_k, you can write, on the one hand,

$$b_{k+2} = b_k - \int_0^\pi \sin^k \theta \, \cos^2 \theta \, d\theta,$$

and on the other, using integration by parts,

$$b_{k+2} = \int_0^\pi \cos \theta \, \frac{d}{d\theta} \sin^{k+1} \theta \, d\theta.$$

Deduce that

$$b_{k+2} = \frac{k+1}{k+2} \, b_k.$$

4. Suppose M is a surface in $\mathbb{R}^n$ of dimension 2, and $\varphi : \mathcal{O} \to U \subset M$ is a coordinate chart, with $\mathcal{O} \subset \mathbb{R}^2$. Set $\varphi_{jk}(x) = \big(\varphi_j(x), \varphi_k(x)\big)$, so $\varphi_{jk} : \mathcal{O} \to \mathbb{R}^2$. Show that the formula (5.12) for the surface integral is equivalent to

$$\int\limits_M f \, dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{\sum_{j<k} \Big(\det \, D\varphi_{jk}(x)\Big)^2} \, dx.$$

Hint. Show that the quantity under $\sqrt{}$ is equal to (5.18).

5. If M is an m-dimensional surface, $\varphi : \mathcal{O} \to M \subset M$ a coordinate chart, for $J = (j_1, \dots, j_m)$ set

$$\varphi_J(x) = \big(\varphi_{j_1}(x), \dots, \varphi_{j_m}(x)\big), \quad \varphi_J : \mathcal{O} \to \mathbb{R}^m.$$

Show that the formula (5.12) is equivalent to

$$\int\limits_M f \, dS = \int\limits_{\mathcal{O}} f \circ \varphi(x) \sqrt{\sum_{j_1 < \cdots < j_m} \Big(\det \, D\varphi_J(x)\Big)^2} \, dx.$$

Hint. Reduce to the following. For fixed $x_0 \in \mathcal{O}$, the quantity under $\sqrt{}$ is equal to $g(x)$ at $x = x_0$, in the case $D\varphi(x_0) = \big(D\varphi_1(x_0), \dots, D\varphi_m(x_0), 0, \dots, 0\big)$.

6. Let M be the graph in $\mathbb{R}^{n+1}$ of $x_{n+1} = u(x)$, $x \in \mathcal{O} \subset \mathbb{R}^n$. Show that, for $p = (x, u(x)) \in M$, T_pM (given as in (5.1)) has a 1-dimensional orthogonal complement N_pM, spanned by $(-\nabla u(x), 1)$. We set $N = \big(1 + |\nabla u|^2\big)^{-1/2}(-\nabla u, 1)$, and call it the (upward-pointing) unit normal to M.

7. Let M be as in Exercise 6, and define N as done there. Show that, for a continuous function $f : M \to \mathbb{R}^{n+1}$,

$$\int_M f \cdot N \, dS = \int_{\mathcal{O}} f\big(x, u(x)\big) \cdot \big(-\nabla u(x), 1\big) \, dx.$$

The left side is often denoted $\int_M f \cdot d\mathbf{S}$.

8. Let M be a 2-dimensional surface in $\mathbb{R}^3$, covered by a single coordinate chart, $\varphi : \mathcal{O} \to M$. Suppose $f : M \to \mathbb{R}^3$ is continuous. Show that, if $\int_M f \cdot d\mathbf{S}$ is defined as in Exercise 7, then

$$\int_M f \cdot d\mathbf{S} = \int_{\mathcal{O}} f\big(\varphi(x)\big) \cdot (\partial_1 \varphi \times \partial_2 \varphi) \, dx,$$

assuming N is a positive multiple of $\partial_1 \varphi \times \partial_2 \varphi$.

9. Consider a symmetric $n \times n$ matrix $A = (a_{jk})$ of the form $a_{jk} = v_j v_k$. Show that the range of A is the one-dimensional space spanned by $v = (v_1, \dots, v_n)$ (if this is nonzero). Deduce that A has exactly one nonzero eigenvalue, namely $\lambda = |v|^2$. Use this to give another derivation of (5.21) from (5.20).

10. Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be a C^k map. Fix $c \in \mathbb{R}$ and consider

$$S = \{x \in \Omega : u(x) = c\}.$$

Assume $S \neq \emptyset$ and that $\nabla u(x) \neq 0$ for all $x \in S$. Show that S is a C^k surface of dimension $n - 1$. Show that, for each $p \in S$, T_pS has a 1-dimensional orthogonal complement N_pS spanned by $\nabla u(p)$.
Hint. Use the Implicit Function Theorem.

11. Let S be as in Exercise 10. Assume moreover that there is a C^k map $\varphi : \mathcal{O} \to \mathbb{R}$, with $\mathcal{O} \subset \mathbb{R}^{n-1}$ open, such that $u(x', \varphi(x')) = c$, and that $x' \mapsto (x', \varphi(x'))$ parametrizes S. Show that

$$\int\limits_S f\, dS = \int\limits_{\mathcal{O}} f\, \frac{|\nabla u|}{|\partial_n u|}\, dx',$$

where the functions in the integrand on the right are evaluated at $(x', \varphi(x'))$.
Hint. Compare the formula in Exercise 6 for N with the fact that $\pm N = \nabla u/|\nabla u|$, and keep in mind the formula (5.22).

6. Vector fields on surfaces

Let M be an m-dimensional surface in $\mathbb{R}^n$, as in §5. Say $U \subset M$ and $\varphi : \mathcal{O} \to U$ is a coordinate chart, with $\mathcal{O}$ open in $\mathbb{R}^m$. A function $W : M \to \mathbb{R}^k$ is said to be smooth on U provided $W \circ \varphi : \mathcal{O} \to \mathbb{R}^k$ is smooth. By Lemma 5.1, this concept of smoothness is independent of the choice of a coordinate chart.

If $V : M \to \mathbb{R}^n$ is smooth and $V(p) \in T_pM$ for each $p \in M$, we say V is a smooth tangent vector field on M. It is easily established that, for any coordinate chart $\varphi : \mathcal{O} \to U$, there exist unique smooth $a_j : \mathcal{O} \to \mathbb{R}$ such that

$$V \circ \varphi(x) = \sum a_j(x) \frac{\partial \varphi}{\partial x_j}(x). \tag{6.1}$$

In fact, $A_x = D\varphi(x) : \mathbb{R}^m \to T_{\varphi(x)}M$ is an isomorphism, and

$$(a_1, \dots, a_m)^t = A_x^{-1} V(\varphi(x)).$$

Alternatively, with notation as in (5.2),

$$(a_1, \dots, a_m, 0, \dots, 0)^t = D\Phi(x, 0)^{-1} V\bigl(\varphi(x)\bigr).$$

If V is a tangent vector field and $W : M \to \mathbb{R}^k$ is smooth, we want to define $\nabla_V W : M \to \mathbb{R}^k$. We want the following property to hold. Given $p \in M$, let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be a smooth curve, contained in M, such that $\gamma(0) = p$ and $\gamma'(0) = V(p)$. Then we want

$$\nabla_V W(p) = \frac{d}{dt} W\bigl(\gamma(t)\bigr)\big|_{t=0}. \tag{6.2}$$

To show this is independent of the choice of γ and obtain other desirable properties, such as linearity in V, we produce an alternative formula, as follows.

Given such γ, there is a smooth curve $\sigma : (-\varepsilon, \varepsilon) \to \mathcal{O}$ such that $\sigma(0) = q$ (where $\varphi(q) = p$) and $\gamma(t) = \varphi(\sigma(t))$; in fact with Φ as in (5.2), $\Phi^{-1}(\gamma(t)) = (\sigma(t), 0)$. Let us set

$$\mathcal{W} = W \circ \varphi, \quad \mathcal{W} : \mathcal{O} \to \mathbb{R}^k.$$

Then $W \circ \gamma = \mathcal{W} \circ \sigma$, so

$$\frac{d}{dt} W\bigl(\gamma(t)\bigr) = D\mathcal{W}\bigl(\sigma(t)\bigr)\, \sigma'(t),$$

and hence (6.2) gives

$$\nabla_V W(p) = D\mathcal{W}(q) A_p^{-1} V(p). \tag{6.3}$$

The right side of (6.3) is independent of the choice of γ. It is also linear in V. Using (6.2) and (6.3) together, one readily verifies the following identities:

$$\nabla_{f_1V_1+f_2V_2}W = f_1\nabla_{V_1}W + f_2\nabla_{V_2}W, \tag{6.4}$$

$$\nabla_V(W_1+W_2) = \nabla_V W_1 + \nabla_V W_2, \tag{6.5}$$

$$\nabla_V(fW) = f\nabla_V W + (\nabla_V f)W, \tag{6.6}$$

$$\nabla_V(W_1\cdot W_2) = (\nabla_V W_1)\cdot W_2 + W_1\cdot(\nabla_V W_2), \tag{6.7}$$

where the last identity involves the dot product of two maps $W_j : M \to \mathbb{R}^k$, and it is assumed that V, V_j are tangent vector fields, f, f_j real-valued functions.

The following remark is also frequently useful. Suppose Ω is an open neighborhood of M in $\mathbb{R}^n$ and $W : M \to \mathbb{R}^k$ has a smooth extension $\widetilde{W} : \Omega \to \mathbb{R}^k$. Then if V is a tangent vector field to M,

$$\nabla_V W(p) = D\widetilde{W}(p)V(p), \quad \forall\, p \in M. \tag{6.8}$$

Exercise

1. Show that, if M is a smooth surface in $\mathbb{R}^n$ and W is a smooth vector field on M, then W has a smooth extension $\widetilde{W}$ to a neighborhood of M in $\mathbb{R}^n$.
Hint. Use the map Φ given in (5.2), which was used in the proof of Lemma 5.1.

7. Shape operators and curvature

We now restrict attention to two-dimensional surfaces $M \subset \mathbb{R}^3$. If $X : \mathcal{O} \to U$ is a coordinate chart on M, then, for $p = X(u,v)$, T_pM is spanned by X_u and X_v, and a unit normal field N is defined by

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}. \tag{7.1}$$

We define the shape operator (also known as the Weingarten map)

$$S : T_pM \longrightarrow \mathbb{R}^3 \tag{7.2}$$

by

$$S(V) = -\nabla_V N. \tag{7.3}$$

Proposition 7.1. *We have*

$$S : T_pM \longrightarrow T_pM. \tag{7.4}$$

Proof. It suffices to show that, for $V \in T_pM$, $S(V) \cdot N = 0$. Indeed, applying ∇_V to $N \cdot N = 1$, using (6.7), we have

$$0 = \nabla_V(N \cdot N) = 2(\nabla_V N) \cdot N.$$

Using (7.4), we define the Gauss curvature of M at p:

$$K(p) = \det S(p), \tag{7.5}$$

and the mean curvature of M at p:

$$H(p) = \frac{1}{2} \operatorname{Tr} S(p). \tag{7.6}$$

The metric tensor on M, in the X-coordinate system, is given by

$$g_{jk} = X_j \cdot X_k, \quad X_1 = X_u, \; X_2 = X_v, \tag{7.7}$$

or

$$\begin{gathered} \mathcal{G} = (g_{jk}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \\ E = X_u \cdot X_u, \quad F = X_u \cdot X_v, \quad G = X_v \cdot X_v. \end{gathered} \tag{7.8}$$

To make computations involving the shape operator, we also consider

$$h_{jk} = S(X_j) \cdot X_k, \tag{7.9}$$

or

$$\begin{gathered} \mathcal{H} = (h_{jk}) = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}, \\ \ell = S(X_u) \cdot X_u, \quad m = S(X_u) \cdot X_v, \quad n = S(X_v) \cdot X_v. \end{gathered} \tag{7.10}$$

Implicit in (7.10) is the identity

$$S(X_u) \cdot X_v = S(X_v) \cdot X_u, \tag{7.11}$$

which is equivalent to the assertion that S in (7.4) is symmetric:

$$S = S^t. \tag{7.12}$$

This will be established in the proof of the next result.

Proposition 7.2. *We have*

$$\ell = N \cdot X_{uu}, \quad m = N \cdot X_{uv} = N \cdot X_{vu}, \quad n = N \cdot X_{vv}. \tag{7.13}$$

Proof. To begin, we have

$$S(X_u) = -N_u, \quad S(X_v) = -N_v, \tag{7.14}$$

by (6.3). Hence $\ell = -N_u \cdot X_u$. Now applying $\partial/\partial u$ to the identity $N \cdot X_u = 0$ we have

$$0 = \frac{\partial}{\partial u}(N \cdot X_u) = N_u \cdot X_u + N \cdot X_{uu}.$$

This proves the identity for ℓ in (7.13). The other parts of (7.13) follow similarly, and the identity (7.11) follows from $X_{uv} = X_{vu}$.

Let us denote by $\mathcal{S} = (s_{jk})$ the matrix representation of S with respect to the basis $(X_1, X_2) = (X_u, X_v)$ of T_pM:

$$S(X_j) = \sum_\ell s_{j\ell} \, X_\ell. \tag{7.15}$$

Taking the dot product of both sides with X_k, we have

$$h_{jk} = \sum_\ell s_{j\ell} \, g_{\ell k}, \tag{7.16}$$

i.e.,

$$\mathcal{H} = \mathcal{S}\mathcal{G}, \quad \text{or} \;\; \mathcal{S} = \mathcal{H}\mathcal{G}^{-1}. \tag{7.17}$$

Taking determinants, we see that the Gauss curvature is given by

$$K = \frac{\ell n - m^2}{EG - F^2}. \tag{7.18}$$

Noting that

$$\mathcal{G}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}, \tag{7.19}$$

one readily computes that

$$H = \frac{1}{2} \frac{G\ell - 2Fm + En}{EG - F^2}. \tag{7.20}$$

The following furnishes other tools to compute the Gauss curvature and mean curvature.

Proposition 7.3. *Suppose V and W span T_pM. Then*

$$S(V) \times S(W) = K(p)(V \times W), \tag{7.21}$$

and

$$S(V) \times W + V \times S(W) = 2H(p)(V \times W). \tag{7.22}$$

Proof. Writing $S(V) = aV + bW$, $S(W) = cV + dW$, we have

$$S(V) \times S(W) = (aV + bW) \times (cV + dW) = (ad - bc)(V \times W),$$

which gives (7.21). The proof of (7.22) is similar.

In particular, for $p = X(u, v)$, using (7.14) plus (7.21) we have

$$N_u \times N_v = K(p)(X_u \times X_v). \tag{7.23}$$

The formula for N might be sufficiently more complicated than that for some other normal field to M (not of unit length), such as perhaps $X_u \times X_v$, that calculations become cumbersome. It is useful to produce formulas for curvature involving more general normal fields, not assumed to have unit length. Say Z is such a normal field; assume it is a positive multiple of N, and write

$$N = \varphi Z, \quad \varphi = |Z|^{-1}. \tag{7.24}$$

We have $N_u = \varphi_u Z + \varphi Z_u$, and hence

$$N_u \times N_v = \varphi^2 Z_u \times Z_v + \big\{\varphi\varphi_u Z \times Z_v + \varphi\varphi_v Z_u \times Z\big\}.$$

The vector field in brackets is tangent to M, while the left side is normal to M. Hence

$$Z \cdot N_u \times N_v = \varphi^2 Z \cdot Z_u \times Z_v.$$

If we take the dot product of both sides of (7.23) with Z we deduce that

$$K(p) = \frac{Z \cdot Z_u \times Z_v}{|Z|^3\,|X_u \times X_v|}. \tag{7.25}$$

Let us apply formula (7.25) when M is the graph of a function:

$$z = f(x, y). \tag{7.26}$$

Then a coordinate patch is given by $X(u, v) = (u, v, f(u, v))$, and we can take

$$Z = X_u \times X_v = (-f_u, -f_v, 1). \tag{7.27}$$

We compute

$$Z_u \times Z_v = (f_{uu} f_{vv} - f_{uv}^2) k,$$

and hence

$$Z \cdot Z_u \times Z_v = f_{uu} f_{vv} - f_{uv}^2. \tag{7.28}$$

Then (7.25) gives for the Gauss curvature

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}. \tag{7.29}$$

We can extend the scope of (7.25) to a setting not involving a particular coordinate system, as follows.

Proposition 7.4. *Suppose V and W span T_pM, and N is a positive multiple both of $V \times W$ and of a normal vector field Z. Then the Gauss curvature is given by*

$$K(p) = \frac{Z \cdot \nabla_V Z \times \nabla_W Z}{|Z|^3\,|V \times W|}. \tag{7.30}$$

Proof. Say $N = \varphi Z$. Start with

$$S(V) = -\nabla_V N = -\varphi \nabla_V Z - (\nabla_V \varphi) Z,$$

and a similar formula for $S(W)$. Hence

$$S(V) \times S(W) = \varphi^2 \nabla_V Z \times \nabla_W Z + R,$$

where R is seen to be tangent to M. Thus, by (7.21),

$$K(p)\, Z \cdot V \times W = \varphi^2\, Z \cdot \nabla_V Z \times \nabla_W Z.$$

Now $Z \cdot V \times W = |Z| \cdot |V \times W|$ and $\varphi = |Z|^{-1}$, so (7.30) follows.

We apply (7.30) to compute the Gauss curvature of a family of quadratic surfaces of the following form. Let A be an invertible symmetric 3×3 matrix, set

$$u(x) = \frac{1}{2} x \cdot Ax, \tag{7.31}$$

and consider

$$M_c = \{x \in \mathbb{R}^3 : u(x) = c\}. \tag{7.32}$$

Let us take

$$Z(x) = \nabla u(x) = Ax. \tag{7.33}$$

Then

$$\nabla_V Z = AV, \quad \nabla_W Z = AW, \tag{7.34}$$

and the numerator in (7.30) becomes

$$Z \cdot \nabla_V Z \times \nabla_W Z = Ax \cdot AV \times AW = (\det\, A) x \cdot V \times W. \tag{7.35}$$

Hence (7.30) gives, for $x \in M_c$,

$$K(x) = \frac{\det\, A}{|Ax|^3} \frac{x \cdot V \times W}{|V \times W|} = \frac{\det\, A}{|Ax|^3}\, x \cdot N, \tag{7.36}$$

with $N = (V \times W)/|V \times W|$. Also $N = Z/|Z| = Ax/|Ax|$, and since $x \cdot Ax = 2c$ for $x \in M_c$, we obtain

$$K(x) = 2c \frac{\det\, A}{|Ax|^4}, \quad x \in M_c. \tag{7.37}$$

Note that no specific formulas for V and W were required in this computation.

We have an analogue of Proposition 7.4 for mean curvature.

Proposition 7.5. *With notation as in Proposition 7.4, the mean curvature of M is given by*

$$H(p) = -\frac{1}{2}\,\frac{Z\cdot(\nabla_V Z\times W + V\times \nabla_W Z)}{|Z|^2\,|V\times W|}. \tag{7.38}$$

Proof. This is deduced from (7.22) in a fashion parallel to the proof of Proposition 7.4.

In the particular case when we have a coordinate chart $X:\mathcal{O}\to U\subset M$ and take $V=X_u, W=X_v$, we have the following complement to (7.25):

$$H(p) = -\frac{1}{2}\,\frac{Z\cdot(Z_u\times X_v + X_u\cdot Z_v)}{|Z|^2\,|X_u\times X_v|}. \tag{7.39}$$

If we consider the surface that is the graph of $z=f(x,y)$, as in (7.26), and take Z as in (7.27), we compute

$$\begin{aligned} Z_u\times X_v &= (-f_{uv}f_v, f_{uu}f_v, -f_{uu}),\\ X_u\times Z_v &= (f_{vv}f_u, -f_{uv}f_u, -f_{vv}), \end{aligned} \tag{7.40}$$

and hence (7.39) yields

$$H = \frac{1}{2}\bigl\{(1+f_v^2)f_{uu} - 2f_uf_vf_{uv} + (1+f_u^2)f_{vv}\bigr\}(1+f_u^2+f_v^2)^{-3/2}. \tag{7.41}$$

Next, we compute the mean curvature of a quadratic surface of the form (7.31)–(7.31), with $Z, \nabla_V Z$ and $\nabla_W Z$ given by (7.33)–(7.34). Thus we have

$$H(x) = -\frac{1}{2}\,\frac{Ax\cdot(AV\times W + V\times AW)}{|Ax|^2\,|V\times W|}. \tag{7.42}$$

Treating this seems to require more computation than (7.35)–(7.36). Let us assume for now that A is a diagonal matrix:

$$A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}. \tag{7.43}$$

Then one can grind out from

$$Ax\cdot AV\times W = \det\begin{pmatrix} a_1x_1 & a_2x_2 & a_3x_3\\ a_1V_1 & a_2V_2 & a_3V_3\\ W_1 & W_2 & W_3 \end{pmatrix}$$

and

$$Ax \cdot V \times AW = \det \begin{pmatrix} a_1 x_1 & a_2 x_2 & a_3 x_3 \\ V_1 & V_2 & V_3 \\ a_1 W_1 & a_2 W_2 & a_3 W_3 \end{pmatrix}$$

that

$$\begin{aligned} Ax \cdot (AV \times W + V \times AW) = a_1(a_2 + a_3)(V_2 W_3 - V_3 W_2) x_1 \\ + a_2(a_1 + a_3)(V_3 W_1 - V_1 W_3) x_2 \\ + a_3(a_1 + a_2)(V_1 W_2 - V_2 W_1) x_3. \end{aligned}$$

Recalling what are the components of $V \times W$, we see that this yields

$$Ax \cdot (AV \times W + V \times AW) = Y \cdot V \times W, \tag{7.44}$$

with

$$Y = \bigl(a_1(a_2 + a_3)x_1, a_2(a_1 + a_3)x_2, a_3(a_1 + a_2)x_3\bigr), \tag{7.45}$$

or equivalently

$$Y = \widetilde{A}Ax, \quad \widetilde{A} = (\operatorname{Tr} A)I - A. \tag{7.46}$$

Then, since $V \times W/|V \times W| = N = Ax/|Ax|$, we have

$$H(x) = -\frac{1}{2} \frac{1}{|Ax|^2} Y \cdot \frac{V \times W}{|V \times W|} = -\frac{1}{2} \frac{Y \cdot Ax}{|Ax|^3},$$

or equivalently

$$H(x) = -\frac{1}{2} \frac{Ax \cdot \widetilde{A}Ax}{|Ax|^3}, \quad x \in M_c. \tag{7.47}$$

Since one can always arrange (7.43) via a rotation, it follows that (7.47) holds for all invertible symmetric A.

Exercises

1. Let $M \subset \mathbb{R}^3$ be a *surface of revolution*, with coordinate chart

$$X(u, v) = \bigl(g(u), h(u)\cos v, h(u)\sin v\bigr). \tag{7.48}$$

This surface is obtained by taking the curve

$$\gamma(u) = \bigl(g(u), h(u), 0\bigr)$$

in the xy-plane and rotating it about the x-axis in $\mathbb{R}^3$. Show that

$$\begin{aligned}
E &= g'(u)^2 + h'(u)^2 = |\gamma'(u)|^2, \\
F &= 0, \\
G &= h(u)^2,
\end{aligned}$$

and that

$$\begin{aligned}
\ell &= \frac{h'(u)g''(u) - g'(u)h''(u)}{|\gamma'(u)|}, \\
m &= 0, \\
n &= \frac{h(u)g'(u)}{|\gamma'(u)|}.
\end{aligned}$$

Show that, with respect to the X_u, X_v-basis of T_pM, the matrix representation of S is

$$\mathcal{S} = \begin{pmatrix} \ell/E & 0 \\ 0 & n/G \end{pmatrix},$$

and that

$$K = \frac{g'(h'g'' - g'h'')}{h((g')^2 + (h')^2)^2}.$$

2. Consider the *torus* in $\mathbb{R}^3$ given by (7.48) with

$$\gamma(u) = (a\, \cos u, a\, \sin u + b, 0),$$

with $0 < a < b$. Compute explicitly the quantities mentioned in Exercise 1. Here $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Compute

$$\text{Area}\, M = \int\limits_M 1\, dS$$

and

$$\int\limits_M K\, dS.$$

3. Consider the *catenoid* in $\mathbb{R}^3$, given by (7.48) with

$$\gamma(u) = (u, c\cosh(u/c), 0),$$

a catenary. Compute explicitly the quantities mentioned in Exercise 1. Show that

$$H = 0.$$

4. Let $M \subset \mathbb{R}^3$ be a *helicoid*, with coordinate chart

$$X(u,v) = (u\,\cos v, u\,\sin v, bv),$$

where $b \neq 0$. Show that

$$K = -\frac{b^2}{(b^2+u^2)^2}, \quad H = 0.$$

5. Suppose $\gamma(t)$ is a unit-speed curve in $\mathbb{R}^3$, with curvature $\kappa(t) > 0$. Fix $\varepsilon > 0$ small and consider the boundary of the ε-tube about γ, parametrized by

$$X(u,v) = \gamma(u) + \varepsilon(\cos v)N(u) + \varepsilon(\sin v)B(u),$$

where (T, N, B) is the Frenet frame of γ. Show that

$$\frac{X_u \times X_v}{|X_u \times X_v|} = -(\cos v)N(u) - (\sin v)B(u).$$

Show that

$$K(u,v) = -\frac{\kappa(u)\,\cos v}{\varepsilon(1 - \varepsilon\kappa(u)\cos v)}.$$

6. For a smooth surface $M \subset \mathbb{R}^3$ with shape operator $S : T_pM \to T_pM$, we define the *principal curvatures* of M at p to be the eigenvalues of S. Show that the principal curvatures are the roots of

$$\lambda^2 - 2H(p)\lambda + K(p) = 0.$$

7. Show there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that $Se_j = k_j e_j$, where k_1, k_2 are the principal curvatures of M at p. We say e_1 and e_2 are *principal directions* (or directions of principal curvature) at $p \in M$.

8. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be symmetric, $b \in \mathbb{R}^3$, and consider

$$u(x) = \frac{1}{2}x \cdot Ax + b \cdot x, \quad M_c = \{x \in \mathbb{R}^3 : u(x) = c\}.$$

Using $Z(x) = \nabla u(x) = Ax + b$, so $\nabla_V Z = AV, \nabla_W Z = AW$ for V, W tangent to M_c, and using Proposition 7.4, show that

$$K(x) = (\det A)\frac{x \cdot (Ax+b)}{|Ax+b|^4} + \frac{b \cdot AV \times AW}{|Ax+b|^3\,|V \times W|}.$$

Here $V \times W/|V \times W| = (Ax+b)/|Ax+b|$. Show that, if A is invertible,

$$\frac{b \cdot AV \times AW}{|V \times W|} = (\det A)\,\frac{A^{-1}b \cdot (Ax+b)}{|Ax+b|}.$$

Then show that, whether or not A is invertible,

$$\frac{b \cdot AV \times AW}{|V \times W|} = \frac{(\operatorname{cof} A)b \cdot (Ax+b)}{|Ax+b|}.$$

Deduce that

$$K = \frac{(Ax+b) \cdot (\operatorname{cof} A)(Ax+b)}{|Ax+b|^4}.$$

Check this against (7.29) for the graph of $z = x^2 + y^2$ and the graph of $z = x^2 - y^2$.

9. If $U \in T_pM$ is a unit vector, we define the *normal curvature* of M (at p) in the direction U to be

$$k_U = U \cdot S(U).$$

Suppose γ is a unit speed curve in M with $\gamma(0) = p, \gamma'(0) = U$. Show that

$$k_U = \gamma''(0) \cdot N(p).$$

Hint. Take $U(t) = \gamma'(t)$ and differentiate $U(t) \cdot N(t) = 0$.

10. Given a surface $M \subset \mathbb{R}^3$, $U \in T_pM$ a unit vector, let Π denote the plane through p with tangent space generated by U and $N(p)$, the normal to M at p. Show that, near p, Π intersects M along a curve γ, whose *signed curvature* at p (with respect to the normal $N(p)$) is equal to k_U.

11. Suppose e_1, e_2 form an orthonormal basis of T_pM such that $S(p)e_j = k_j e_j$. Set

$$U(\theta) = (\cos\theta)e_1 + (\sin\theta)e_2.$$

Show that the normal curvature of M in the direction $U(\theta)$ is given by

$$k_{U(\theta)} = (\cos^2\theta)k_1 + (\sin^2\theta)k_2.$$

If $k_1 > k_2$, deduce that $k_{U(\theta)}$ is maximal at $\theta = 0, \pi$ and minimal at $\theta = \pm\pi/2$.

12. We say a point $p \in M$ is an *umbilic* provided the principal curvatures of M are equal at p, i.e., $k_1(p) = k_2(p)$. Show that the condition that p is an umbilic is equivalent to each of the following:

a) $S(p) = \lambda I$ for some $\lambda \in \mathbb{R}$ (I = identity on T_pM).
b) There exists $\lambda \in \mathbb{R}$ such that $\ell = \lambda E, m = \lambda F, n = \lambda G$ at p.
c) $K(p) = H(p)^2$.

Show that always $K(p) \leq H(p)^2$.

13. Consider the ellipsoid M:

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 1. \tag{7.49}$$

Assume $a_1 > a_2 > a_3 > 0$. Consider the point $p = (x_1, 0, x_3)$, with

$$x_1^2 = b(a_1 - a_2)a_3^2, \quad x_3^2 = b a_1^2 (a_2 - a_3),$$

and with b chosen so (7.49) holds. Show that p is an umbilic. Find a total of four umbilics on M.
Hint. Use (7.37) and (7.47) to compute $K(p)$ and $H(p)$.

14. Consider the torus described in Exercise 2. Show that this surface has no umbilics.

15. For a surface $M \subset \mathbb{R}^3$, with unit normal N, define the *Gauss map*

$$G : M \longrightarrow S^2, \quad G(p) = N(p). \tag{7.50}$$

Show that, for $p \in M$,

$$DG(p) : T_p M \longrightarrow T_{G(p)} S^2 = T_p M, \tag{7.51}$$

with equality as linear subspaces of $\mathbb{R}^3$. Show that the Gauss curvature is given by

$$K(p) = \det\, DG(p). \tag{7.52}$$

16. For the graph of $z = f(x, y)$, parametrized by $X(u, v) = (u, v, f(u, v))$, verify that

$$E = 1 + f_u^2, \quad F = f_u f_v, \quad G = 1 + f_v^2$$

and

$$\ell = \frac{f_{uu}}{\sqrt{1 + |\nabla f|^2}}, \quad m = \frac{f_{uv}}{\sqrt{1 + |\nabla f|^2}}, \quad n = \frac{f_{vv}}{\sqrt{1 + |\nabla f|^2}},$$

and re-derive the formulas (7.29) and (7.41) for K and H.

8. The covariant derivative on a surface and Gauss' Theorema Egregium

Let $M \subset \mathbb{R}^3$ be a smooth surface. Let $P(x) : \mathbb{R}^3 \to T_x M$ be the orthogonal projection of $\mathbb{R}^3$ onto $T_x M$. If V and W are smooth vector fields tangent to M, we define the covariant derivative on M by

$$\nabla^M_V W = P\,\nabla_V W. \tag{8.1}$$

Note that, for any $X \in \mathbb{R}^3$,

$$PX = X - (X \cdot N)N, \tag{8.2}$$

where N is a unit normal field to M. Hence

$$\nabla^M_V W = \nabla_V W - (N \cdot \nabla_V W)N. \tag{8.3}$$

Also note that, for a vector field W tangent to M,

$$N \cdot W = 0 \Longrightarrow N \cdot \nabla_V W = -(\nabla_V N) \cdot W = SV \cdot W, \tag{8.4}$$

where S is the shape operator for M. Thus we can write

$$\nabla^M_V W = \nabla_V W - (SV \cdot W)N. \tag{8.5}$$

This is called Weingarten's formula.

It is an easy consequence of (6.4)–(6.7) that ∇^M has the following properties:

$$\nabla^M_{f_1 V_1 + f_2 V_2} W = f_1 \nabla^M_{V_1} W + f_2 \nabla^M_{V_2} W, \tag{8.6}$$

$$\nabla^M_V (W_1 + W_2) = \nabla^M_V W_1 + \nabla^M_V W_2, \tag{8.7}$$

$$\nabla^M_V (fW) = f\nabla^M_V W + (\nabla_V f)W, \tag{8.8}$$

$$\nabla_V (W_1 \cdot W_2) = (\nabla^M_V W_1) \cdot W_2 + W_1 \cdot (\nabla^M_V W_2). \tag{8.9}$$

We next establish an important identity for $\nabla^M_V W - \nabla^M_W V$. Using (8.5), we have

$$\begin{aligned} \nabla^M_V W - \nabla^M_W V &= \nabla_V W - \nabla_W V - \big[(SV \cdot W) - (SW \cdot V)\big]N \\ &= \nabla_V W - \nabla_W V, \end{aligned} \tag{8.10}$$

since $S = S^t$ on $T_p M$. To consider this further, let us assume V and W are extended to smooth vector fields on an open neighborhood $\mathcal{O}$ of M in $\mathbb{R}^3$. This can always be done. Hence V and W are smooth vector fields on $\mathcal{O}$ that are tangent to M.

Generally, if $V = \sum V^j\, e_j, W = \sum W^j\, e_j$, $\{e_j\}$ the standard basis of $\mathbb{R}^3$, we have $\nabla_V W = \sum V^k\, \partial_k W^j\, e_j$, hence

$$\nabla_V W - \nabla_W V = \sum_{j,k} \Big(V^k \frac{\partial W^j}{\partial x_k} - W^k \frac{\partial V^j}{\partial x_k} \Big) e_j. \tag{8.11}$$

There is another way to look at (8.11). We think of a vector field V as defining a differential operator on real-valued functions, via

$$Vf = \nabla_V f = \sum V^j \frac{\partial f}{\partial x_j}. \tag{8.12}$$

Then VW and WV are both second order differential operators:

$$\begin{aligned} VWf &= \sum_{j,k} V^j \frac{\partial}{\partial x_j} \Big(W^k \frac{\partial f}{\partial x_k} \Big) \\ &= \sum_{j,k} V^j W^k \frac{\partial^2 f}{\partial x_j \partial x_k} + \sum_{j,k} V^j \frac{\partial W^k}{\partial x_j} \frac{\partial f}{\partial x_k}. \end{aligned} \tag{8.13}$$

Hence, if we define the *commutator*

$$[V, W]f = VWf - WVf, \tag{8.14}$$

we see that $[V, W]$ is also a vector field (called the *Lie bracket* of V and W):

$$[V, W]f = \sum_{j,k} \Big(V^k \frac{\partial W^j}{\partial x_k} - W^k \frac{\partial V^j}{\partial x_k} \Big) \frac{\partial f}{\partial x_j}. \tag{8.15}$$

Comparing this with (8.11) we have (for all vector fields on an open set in $\mathbb{R}^3$)

$$[V, W] = \nabla_V W - \nabla_W V. \tag{8.16}$$

Hence, from (8.10) we have (for V and W tangent to M)

$$\nabla^M_V W - \nabla^M_W V = [V, W]. \tag{8.17}$$

Not only is this a useful identity for $\nabla^M_V - \nabla^M_W V$, but also, since the left side of (8.17) is clearly tangent to M, we have:

Proposition 8.1. *If V and W are smooth vector fields on an open set $\mathcal{O} \subset \mathbb{R}^3$ and both are tangent to M, so is $[V, W]$.*

It is also useful to examine commutators of ∇_V and ∇_W, and of ∇^M_V and ∇^M_W. First, a routine calculation (parallel to (8.13)–(8.15)) gives

$$\nabla_V \nabla_W X - \nabla_W \nabla_V X = \nabla_{[V,W]} X, \tag{8.18}$$

for any smooth vector fields on an open set in $\mathbb{R}^3$. Now assume V, W, and X are tangent to M. To examine $\nabla^M_V \nabla^M_W X - \nabla^M_W \nabla^M_V X$, we repeatedly use (8.5). To begin,

$$\begin{aligned} \nabla^M_V (\nabla^M_W X) &= \nabla_V (\nabla^M_W X) - (SV \cdot \nabla^M_W X) N \\ &= \nabla_V \nabla_W X - \nabla_V \big((SW \cdot X)N\big) - (SV \cdot \nabla_W X)N \\ &= \nabla_V \nabla_W X + (SW \cdot X) SV \quad \operatorname{mod}(N), \end{aligned} \tag{8.19}$$

where by "mod(N)" we mean a scalar multiple of N has been omitted. Hence we have

$$\begin{aligned} \nabla^M_V \nabla^M_W X - \nabla^M_W \nabla^M_V X = \nabla_V \nabla_W X - \nabla_W \nabla_V X& \\ + (SW \cdot X) SV - (SV \cdot X) SW,& \quad \operatorname{mod}(N), \end{aligned} \tag{8.20}$$

and, bringing in (8.18), we obtain

$$\nabla^M_V \nabla^M_W X - \nabla^M_W \nabla^M_V X - \nabla^M_{[V,W]} X = (SW \cdot X) SV - (SV \cdot X) SW. \tag{8.21}$$

This time there is no "mod(N)" remainder, since both sides of (8.21) are tangent to M. (In fact, the vanishing of the remainder has some significance; see the exercises.) The left side of (8.21) is typically denoted $R(V, W)X$, i.e., by definition

$$R(V, W)X = \nabla^M_V \nabla^M_W X - \nabla^M_W \nabla^M_V X - \nabla^M_{[V,W]} X. \tag{8.22}$$

Hence the content of (8.21) is the identity

$$R(V, W)X = (SW \cdot X) SV - (SV \cdot X) SW. \tag{8.23}$$

R is called the Riemann tensor. We can relate this to Gauss curvature:

Proposition 8.2. *If $\{u_1, u_2\}$ is any orthonormal basis of T_pM, then*

$$u_1 \cdot R(u_1, u_2) u_2 = K(p). \tag{8.24}$$

Proof. Suppose $Su_1 = a_{11} u_1 + a_{21} u_2,\ Su_2 = a_{12} u_1 + a_{22} u_2$. (The symmetry of S implies $a_{21} = a_{12}$.) Now, by (8.23) we have

$$u_1 \cdot R(u_1, u_2) u_2 = (Su_2 \cdot u_2)(Su_1 \cdot u_1) - (Su_1 \cdot u_2)(Su_2 \cdot u_1) = a_{11} a_{22} - a_{12}^2.$$

Since the right side is $\det S$, we have (8.24).

The next result implies that ∇^M is determined by the intrinsic geometry of M.

Propoisition 8.3. *There is only one operator ∇^M satisfying (8.6)–(8.9) and (8.17). It satisfies the following identity, for all smooth vector fields X, Y, Z tangent to M:*

$$\begin{aligned} 2Z \cdot \nabla^M_X Y = {} & \nabla_X (Y \cdot Z) + \nabla_Y (X \cdot Z) - \nabla_Z (X \cdot Y) \\ & + [X, Y] \cdot Z - [X, Z] \cdot Y - [Y, Z] \cdot X. \end{aligned} \tag{8.25}$$

Proof. To obtain the formula (8.25), cyclically permute X, Y, and Z in the identity

$$\nabla_X (Y \cdot Z) = (\nabla^M_X Y) \cdot Z + Y \cdot (\nabla^M_X Z),$$

and take the appropriate alternating sum, using (8.17) to cancel out all terms involving ∇^M but two copies of $Z \cdot \nabla^M_X Y$.

Suppose you have a coordinate chart $X : \mathcal{O} \to U \subset M$, and pick the basis $D_1 = X_u, D_2 = X_v$ of $T_p M$. Then

$$[D_1, D_2] = 0. \tag{8.26}$$

In fact, if $f : U \to \mathbb{R}$ and $\tilde{f} = f \circ X : \mathcal{O} \to \mathbb{R}$, we have

$$(D_1 f) \circ X = \frac{\partial \tilde{f}}{\partial u}, \quad (D_2 f) \circ X = \frac{\partial \tilde{f}}{\partial v},$$

(or, in notation as we commonly abuse it, $D_1 f = \partial f / \partial u,\ D_2 f = \partial f / \partial v$), so

$$\frac{\partial}{\partial u} \frac{\partial}{\partial v} = \frac{\partial}{\partial v} \frac{\partial}{\partial u} \Longrightarrow D_1 D_2 = D_2 D_1.$$

Since $D_j \cdot D_k = g_{jk}$ are the coefficients of the metric tensor, as in (7.7), we deduce from (8.25) that

$$D_\ell \cdot \nabla^M_{D_j} D_k = \frac{1}{2} \big(\partial_j \, g_{k\ell} + \partial_k \, g_{j\ell} - \partial_\ell \, g_{jk} \big), \tag{8.27}$$

where $\partial_1 = \partial / \partial u, \partial_2 = \partial / \partial v$. Equivalently, we have

$$\nabla^M_{D_j} D_k = \sum_i \Gamma^i{}_{kj} \, D_i, \tag{8.28}$$

where the coefficients $\Gamma^i{}_{kj}$, known as connection coefficients, are given by

$$\Gamma^i{}_{kj} = \frac{1}{2} \sum_\ell g^{i\ell} (\partial_j \, g_{k\ell} + \partial_k \, g_{j\ell} - \partial_\ell \, g_{jk}), \tag{8.29}$$

where (g^{jk}) denotes the matrix inverse to (g_{jk}), i.e., as in (7.19),

$$(g^{jk}) = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}. \tag{8.30}$$

It is a consequence of (8.7)–(8.8) that

$$V = \sum_k v^k\, D_k \Longrightarrow \nabla^M_{D_j} V = \sum_k \Bigl(\partial_j v^k + \sum_\ell v^\ell \Gamma^k{}_{\ell j} \Bigr) D_k. \tag{8.31}$$

Another common notation is

$$\nabla^M_{D_j} V = \sum_k v^k{}_{;j}\, D_k, \quad v^k{}_{;j} = \partial_j v^k + \sum_\ell \Gamma^k{}_{\ell j}\, v^\ell. \tag{8.32}$$

We now make some computations that will result in expressing the Gauss curvature of M in terms of the connection coefficients. Define $R^\ell{}_{mjk}$ by

$$R(D_j, D_k) D_m = \sum_\ell R^\ell{}_{mjk}\, D_\ell. \tag{8.33}$$

Applying the definition (8.22), the fact that $[D_j, D_k] = 0$, and the formula (8.32), we obtain

$$R^\ell{}_{mjk} = \partial_j \Gamma^\ell{}_{mk} - \partial_k \Gamma^\ell{}_{mj} + \sum_i (\Gamma^\ell{}_{ij} \Gamma^i{}_{mk} - \Gamma^\ell{}_{ik} \Gamma^i{}_{mj}). \tag{8.34}$$

Now $\{D_1, D_2\}$ is typically not an orthonormal basis of $T_p M$, but we can expand D_j in terms of an orthonormal basis:

$$D_1 = a u_1 + b u_2, \quad D_2 = c u_1 + d u_2. \tag{8.35}$$

Then we can express $D_1 \cdot R(D_1, D_2) D_2$ in terms of $u_1 \cdot R(u_1, u_2) u_2$, by using a couple of antisymmetries which we now state.

Proposition 8.4. *If $V, W, X,$ and Y are smooth vector fields tangent to M, we have*

$$R(V, W)X = -R(W, V)X \tag{8.36}$$

and

$$Y \cdot R(V, W)X = -X \cdot R(V, W)Y. \tag{8.37}$$

Proof. The identity (8.36) follows directly from the definition (8.22). To prove (8.37), one can repeatedly apply (8.9), to get

$$0 = (VW - WV - [V, W])(X \cdot Y) = R(V, W)X \cdot Y + X \cdot R(V, W)Y. \tag{8.38}$$

Using (8.36), we obtain

$$R(D_1, D_2)D_2 = (ad - bc)R(u_1, u_2)D_2, \tag{8.39}$$

and then using (8.37) we get

$$D_1 \cdot R(D_1, D_2)D_2 = (ad - bc)^2\, u_1 \cdot R(u_1, u_2)u_2. \tag{8.40}$$

Note that the metric tensor satisfies

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}. \tag{8.41}$$

Hence

$$EG - F^2 = (ad - bc)^2. \tag{8.42}$$

Thus the formula (8.24) for the Gauss curvature yields

$$K = \frac{1}{EG - F^2}\, D_1 \cdot R(D_1, D_2)D_2. \tag{8.43}$$

Using (8.33) we have

$$D_i \cdot R(D_j, D_k)D_m = \sum_\ell g_{i\ell}\, R^\ell{}_{mjk}. \tag{8.44}$$

This establishes the following:

Proposition 8.5. *The Gauss curvature is given by*

$$K = \frac{1}{EG - F^2} \sum_\ell g_{1\ell}\, R^\ell{}_{212}. \tag{8.45}$$

For an alternative formulation, set

$$R_{imjk} = \sum_\ell g_{i\ell}\, R^\ell{}_{mjk}, \tag{8.46}$$

so

$$R_{imjk} = D_i \cdot R(D_j, D_k)D_m. \tag{8.47}$$

Then (8.45) is equivalent to the identity

$$K = \frac{R_{1212}}{EG - F^2}. \tag{8.48}$$

A derivation of (8.48) that avoids the computations in (8.35)–(8.42) can be given as follows. Directly from (8.23) we have

$$D_1 \cdot R(D_1, D_2)D_2 = (SD_2 \cdot D_2)(SD_1 \cdot D_1) - (SD_1 \cdot D_2)^2 = \ell n - m^2, \tag{8.49}$$

using the formula (7.10) for ℓ, m, n. Then (8.48) follows from (7.18).

Now formulas (8.29) and (8.34) show that the right side of (8.45) (or of (8.48)) is given by the metric tensor alone (and its derivatives of order ≤ 2). The shape operator does not play a role in this formula, as it did in the definition (7.5) or the formula (7.18). This result is known as Gauss' *Theorema Egregium.*

Exercises

1. Show directly from the definition (8.22) (using (8.6)–(8.8)) that, for smooth $f : M \to \mathbb{R}$,

$$R(fV, W)X = R(V, fW)X = R(V, W)(fX) = f\, R(V, W)X.$$

2. Define 2×2 matrices

$$\Gamma_j = (\Gamma^a{}_{bj}), \quad \mathfrak{R}_{jk} = (R^a{}_{bjk}).$$

Show that the formula (8.34) for $R^\ell{}_{mjk}$ is equivalent to

$$\mathfrak{R}_{jk} = \partial_j \Gamma_k - \partial_k \Gamma_j + [\Gamma_j, \Gamma_k], \tag{8.50}$$

where we use the matrix commutator: $[A, B] = AB - BA$.

3. Show that the "mod(N)" term missing from the right side of (8.19) is equal to

$$-\big(\nabla_V (SW \cdot X)\big)N - (SV \cdot \nabla^M_W X)N.$$

Compute the "mod(N)" term missing from the right sides of (8.20) and (8.21). Show that the vanishing of the one in (8.21) is equivalent to the identity

$$\nabla_V (SW \cdot X) + SV \cdot \nabla^M_W X = \nabla_W (SV \cdot X) + SW \cdot \nabla^M_V X + S([V, W]) \cdot X,$$

for vector fields V, W, X tangent to M. Show that this in turn is equivalent to the identity

$$\nabla^M_V (SW) - \nabla^M_W (SV) = S([V, W]). \tag{8.51}$$

This is a form of the *Codazzi equation.*

4. Let $M \subset \mathbb{R}^3$ be a connected surface. Assume every $p \in M$ is an umbilic, so $S(p) = \lambda(p)I$ for all $p \in M$. Show that $\lambda(p)$ is *constant.*
Hint. Apply the Codazzi equation (8.51). Using a coordinate chart $X : \mathcal{O} \to U \subset M$ and taking $V = X_u, W = X_v$, show that $\partial\lambda/\partial u = \partial\lambda/\partial v = 0$.

5. Retain the hypotheses of Exercise 4. Show that M lies in either a plane or a sphere (of radius $1/\lambda$).
Hint. If $S = \lambda I$ and $\lambda \neq 0$, define $\psi : M \to \mathbb{R}^3$ by $\psi(x) = x + (1/\lambda)N$. For V tangent to M, compute $\nabla_V \psi$.

6. Suppose you have a coordinate system in which the metric tensor is

$$\mathcal{G} = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}. \tag{8.52}$$

Show that

$$\begin{aligned} \Gamma_1 = \left(\Gamma^a{}_{b1}\right) &= \frac{1}{2}\begin{pmatrix} E_u/E & E_v/E \\ -E_v/G & G_u/G \end{pmatrix}, \\ \Gamma_2 = \left(\Gamma^a{}_{b2}\right) &= \frac{1}{2}\begin{pmatrix} E_v/E & -G_u/E \\ G_u/G & G_v/G \end{pmatrix}. \end{aligned} \tag{8.53}$$

Compute $\mathfrak{R}_{12}$ and show that the Gauss curvature is given by

$$K = -\frac{1}{2\sqrt{EG}}\Big(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\Big). \tag{8.54}$$

7. One says M has a *Clairaut parametrization* if there is a coordinate patch in which the metric tensor has the form (8.52), with $E = E(u)$ and $G = G(u)$. (Only exceptional surfaces have Clairaut parametrizations, but this class does include some interesting examples.) Show that, in a Clairaut parametrization, you have

$$\begin{aligned} \Gamma_1 &= \frac{1}{2}\begin{pmatrix} E_u/E & 0 \\ 0 & G_u/G \end{pmatrix}, \\ \Gamma_2 &= \frac{1}{2}\begin{pmatrix} 0 & -G_u/E \\ G_u/G & 0 \end{pmatrix}, \end{aligned} \tag{8.55}$$

and

$$K = -\frac{1}{2\sqrt{EG}}\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}}. \tag{8.56}$$

8. Suppose M is a surface of revolution, with coordinate chart

$$X(u,v) = \big(g(u), h(u)\cos v, h(u)\sin v\big).$$

Show this is a Clairaut parametrization. Work out the specific formulas for the quantities in (8.55)–(8.56) in this case. Compare the result for K with that obtained in Exercise 1 of §7.

9. Geodesics on surfaces

Suppose M is a smooth $(n-1)$-dimensional surface in $\mathbb{R}^n$. If $\gamma(t) = \big(x_1(t), \dots, x_n(t)\big),\ a \le t \le b$, is a smooth curve in M, its length is

$$L = \int_a^b \|\gamma'(t)\|\, dt. \tag{9.1}$$

where

$$\|\gamma'(t)\|^2 = \sum_{j=1}^n x_j'(t)^2. \tag{9.2}$$

A curve γ is said to be a geodesic if, for $|t_1 - t_2|$ sufficiently small, $t_j \in [a,b]$, the curve $\gamma(t),\ t_1 \le t \le t_2$ has the shortest length of all smooth curves in M from $\gamma(t_1)$ to $\gamma(t_2)$.

Our first goal is to derive an equation for geodesics. So let $\gamma_0(t)$ be a smooth curve in M $(a \le t \le b)$, joining p and q. Suppose $\gamma_s(t)$ is a smooth family of such curves. We look for a condition guaranteeing that $\gamma_0(t)$ has minimum length. Since the length of a curve is independent of its parametrization, we may as well suppose

$$\|\gamma_0'(t)\| = c_0,\ \text{ constant, for } a \le t \le b. \tag{9.3}$$

Let N denote a field of normal vectors to M. Note that, with $\partial_s \gamma_s(t) = (\partial/\partial s)\gamma_s(t)$,

$$V = \partial_s \gamma_s(t) \perp N. \tag{9.4}$$

Also, any vector field $V \perp N$ over the image of γ_0 can be obtained by some variation γ_s of γ_0, provided $V = 0$ at p and q. Recall we are assuming $\gamma_s(a) = p,\ \gamma_s(b) = q$. If $L(s)$ denotes the length of γ_s, we have

$$L(s) = \int_a^b \|\gamma_s'(t)\|\, dt, \tag{9.5}$$

and hence

$$\begin{aligned} L'(s) &= \frac{1}{2}\int_a^b \|\gamma_s'(t)\|^{-1} \partial_s \big(\gamma_s'(t), \gamma_s'(t)\big)\, dt \\ &= \frac{1}{c_0}\int_a^b \big(\partial_s \gamma_s'(t), \gamma_s'(t)\big)\, dt,\ \text{ at } s = 0. \end{aligned} \tag{9.6}$$

Using the identity

$$\frac{d}{dt}\big(\partial_s \gamma_s(t), \gamma_s'(t)\big) = \big(\partial_s \gamma_s'(t), \gamma_s'(t)\big) + \big(\partial_s \gamma_s(t), \gamma_s''(t)\big), \tag{9.7}$$

together with the fundamental theorem of calculus, in view of the fact that

$$\partial_s \gamma_s(t) = 0 \text{ at } t = a \text{ and } b, \tag{9.8}$$

we have

$$L'(s) = -\frac{1}{c_0} \int_a^b \big(V(t), \gamma_s''(t)\big)\, dt, \text{ at } s = 0. \tag{9.9}$$

Now, if γ_0 were a geodesic, we would have

$$L'(0) = 0, \tag{9.10}$$

for all such variations. In other words, we must have $\gamma_0''(t) \perp V$ for all vector fields V tangent to M (and vanishing at p and q), and hence

$$\gamma_0''(t) \| N. \tag{9.11}$$

This vanishing of the tangential curvature of γ_0 is the geodesic equation for an $(n-1)$-dimensional surface in $\mathbb{R}^n$.

For a unit speed curve γ_0 in M, the quantity

$$g(t) = \gamma_0''(t) \cdot U, \quad U = N \times T,$$

is called the *geodesic curvature* of γ_0. Note that $\gamma_0'' = T' \perp T$, so γ_0'' must be a linear combination of N and U. The condition for γ_0 to be a geodesic is hence the vanishing of its geodesic curvature.

We proceed to derive from (9.11) an ODE in standard form. Suppose M is defined locally by $u(x) = C$, $\nabla u \neq 0$. Then (9.11) is equivalent to

$$\gamma_0''(t) = K \nabla u(\gamma_0(t)) \tag{9.12}$$

for a scalar K which remains to be determined. But the condition that $u(\gamma_0(t)) = C$ implies

$$\gamma_0'(t) \cdot \nabla u(\gamma_0(t)) = 0,$$

and differentiating this gives

$$\gamma_0''(t) \cdot \nabla u(\gamma_0(t)) = -\gamma_0'(t) \cdot D^2 u(\gamma_0(t)) \gamma_0'(t) \tag{9.13}$$

where $D^2 u$ is the matrix of second order partial derivatives of u. Comparing (9.12) and (9.13) gives K, and we obtain the ODE

$$\gamma_0''(t) = -\frac{\gamma_0'(t) \cdot D^2 u\big(\gamma_0(t)\big) \gamma_0'(t)}{\big|\nabla u\big(\gamma_0(t)\big)\big|^2} \nabla u\big(\gamma_0(t)\big) \tag{9.14}$$

for a geodesic γ_0 lying in M.

The fact that any solution to the geodesic equation does have the length-minimizing property (at least locally) will be established in §11.

Exercises

1. Show that any curve γ_0 on a surface M satisfying (9.11) must have constant speed, i.e.,

$$\frac{d}{dt}|\gamma_0'(t)|^2 = 0.$$

Show that any solution to (9.14) on an interval containing t_0 must lie on a level set of u, *provided* $\gamma_0'(t_0) \perp \nabla u(\gamma(t_0))$.

2. Let $M \subset \mathbb{R}^3$ be a surface of revolution about the z-axis, given by

$$x^2 + y^2 = f(z),$$

where $f(z)$ is a smooth positive function of z. Write out the geodesic equation, making use of (9.14).

3. If $\gamma(t) = (x(t), y(t), z(t))$ is a geodesic on such a surface of revolution as considered in Exercise 2, show that

$$\frac{d}{dt}\big(x(t)y'(t) - y(t)x'(t)\big) = 0. \tag{9.15}$$

Hence, for each such geodesic, there is a constant a_γ such that

$$x(t)y'(t) - y(t)x'(t) = a_\gamma. \tag{9.16}$$

9B. Other special curves

A curve γ in $M \subset \mathbb{R}^3$ is called a *principal curve* if $\gamma'(t)$ is an eigenvector of the shape operator S at $\gamma(t)$ for all t (equivalently, $\gamma'(t)$ points in a principal direction).

A nonzero vector $V \in T_pM$ is said to be *asymptotic* provided $SV \cdot V = 0$. There exist asymptotic vectors in T_pM if and only if $K(p) \leq 0$. A curve γ in M is said to be an asymptotic curve provided $\gamma'(t)$ is an asymptotic vector, for each t.

In the remainder of this section, let us denote the unit normal field to M by Z rather than by N (to avoid confusing (9.16) below with ingredients in the Frenet-Serret formulas).

1. Show that a curve γ in M is principal if and only if $\gamma'(t)$ and $Z'(t)$ are parallel for all t. (Here, $Z(t)$ stands for $Z(\gamma(t))$.)

2. Show that a curve γ in M is asymptotic if and only if $\gamma''(t)$ is tangent to M for all t. (Note that this is the *opposite* to the condition (9.11) for a curve to be a geodesic.)

3. Given a unit-speed curve γ in $M \subset \mathbb{R}^3$, instead of the Frenet frame field on γ studied in §3, consider the following frame, called the *Darboux frame*. Take $T(t) = \gamma'(t)$, $Z(t)$ (the unit normal to M at $\gamma(t)$), and

$$U = Z \times T.$$

Show that

$$\begin{aligned} T' &= \quad gU + kZ \\ U' &= -gT \qquad\quad + bZ \\ Z' &= -kT - bU \end{aligned} \tag{9B.1}$$

where

$$g = U \cdot \gamma''$$

is the geodesic curvature of γ,

$$k = S(T) \cdot T$$

is the normal curvature of M in the direction T, and

$$b = S(T) \cdot U.$$

4. In the setting of Exercise 3, show that γ is

$$\begin{aligned} \text{a geodesic} &\iff g = 0, \\ \text{asymptotic} &\iff k = 0, \\ \text{principal} &\iff b = 0. \end{aligned}$$

10. Geodesic equation and covariant derivative

We want to re-do the derivation of the ODE for a geodesic on a surface M. As before, let $\gamma_s(t)$ be a one parameter family of curves satisfying $\gamma_s(a) = p,\ \gamma_s(b) = q$, and (9.3). Then

$$V = \partial_s \gamma_s(t)|_{s=0} \tag{10.1}$$

is a vector field defined on the curve $\gamma_0(t)$, vanishing at p and q, and a general vector field of this sort could be obtained by a variation $\gamma_s(t)$. Let

$$T = \gamma_s'(t). \tag{10.2}$$

With the notation of (9.1), we have, parallel to (9.6),

$$\begin{aligned} L'(s) &= \int_a^b V\langle T,T\rangle^{1/2}\,dt \\ &= \frac{1}{2c_0}\int_a^b V\langle T,T\rangle\,dt, \quad \text{at } s=0, \end{aligned} \tag{10.3}$$

assuming γ_0 has constant speed c_0, as in (9.3). Now we need a generalization of $(\partial/\partial s)\gamma_s'(t)$, and of the formula (9.7).

One natural approach involves the notion of a covariant derivative, as defined in §8. Using (8.9), we deduce from (10.3) that

$$L'(s) = \frac{1}{c_0}\int_a^b \langle \nabla_V^M T, T\rangle\,dt, \quad \text{at } s=0. \tag{10.4}$$

Since $\partial/\partial s$ and $\partial/\partial t$ commute, we have $[V,T]=0$ on γ_0, and (8.17) implies

$$L'(s) = \frac{1}{c_0}\int_a^b \langle \nabla_T^M V, T\rangle\,dt \ \ \text{at } s=0. \tag{10.5}$$

The replacement for (9.7) is

$$T\langle V,T\rangle = \langle \nabla_T^M V, T\rangle + \langle V, \nabla_T^M T\rangle, \tag{10.6}$$

so, by the fundamental theorem of calculus,

$$L'(0) = -\frac{1}{c_0}\int_a^b \langle V, \nabla_T^M T\rangle\,dt. \tag{10.7}$$

If this is to vanish for all smooth vector fields over γ_0, vanishing at p and q, we must have

$$\nabla^M_T T = 0. \tag{10.8}$$

This is the geodesic equation, in an alternative formulation.

If the metric tensor on M takes the form $g_{jk}(x)$, in a coordinate chart, the connection coefficients $\Gamma^k{}_{ij}$ are defined by (8.28), i.e.,

$$\nabla^M_{D_j} D_i = \sum_k \Gamma^k{}_{ij}\, D_k, \tag{10.9}$$

As seen in (8.29),

$$\Gamma^\ell{}_{ij} = \frac{1}{2} \sum_k g^{k\ell} \Big(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \Big). \tag{10.10}$$

We can rewrite the geodesic equation (10.8) for $\gamma_0(t)$ as follows. Say $X : \mathcal{O} \to U \subset M$ is a coordinate chart, and $\gamma_0(t) = X(x(t))$. As before, set $D_\ell = \partial X/\partial x_\ell$. With $x = (x_1, \dots, x_n)$, we have $T = \sum \dot{x}_\ell D_\ell$, and (10.8) yields

$$0 = \sum_\ell \nabla^M_T \big(\dot{x}_\ell D_\ell\big) = \sum_\ell \big[\ddot{x}_\ell D_\ell + \dot{x}_\ell \nabla^M_T D_\ell\big]. \tag{10.11}$$

In view of (10.9), this becomes

$$\ddot{x}_\ell + \sum_{j,k} \dot{x}_j\, \dot{x}_k\, \Gamma^\ell{}_{jk} = 0 \tag{10.12}$$

The standard existence and uniqueness theory applies to this system of second order ODE. We will call any smooth curve satisfying the equation (10.8), or equivalently (10.12), a geodesic. In the next section we will verify that such a curve is indeed locally length minimizing. Note that if $T = \gamma'(t)$, then $T\langle T, T\rangle = 2\langle \nabla^M_T T, T\rangle$, so if (10.8) holds $\gamma(t)$ automatically has constant speed.

We take a closer look at the geodesic equations in the special case when M is a 2-dimensional surface with a Clairaut parametrization, as in Exercise 7 of §8. (Recall that this applies to surfaces of revolution.) Then the geodesic equation (10.12) can be written out using the formulas in (8.55) for $\Gamma^\ell{}_{jk}$. It is convenient to present this information in the following fashion:

$$\begin{aligned} \big(\Gamma^1{}_{jk}\big) &= \frac{1}{2} \begin{pmatrix} E'/E & 0 \\ 0 & -G'/E \end{pmatrix}, \\ \big(\Gamma^2{}_{jk}\big) &= \frac{1}{2} \begin{pmatrix} 0 & G'/G \\ G'/G & 0 \end{pmatrix}. \end{aligned} \tag{10.13}$$

Recall that $E = E(u), F = 0$, and $G = G(u)$ in this case, with $(u, v) = (x_1, x_2)$. Hence the system (10.12) takes the form

$$\begin{aligned} \ddot{u} + \frac{1}{2}\frac{E'}{E}\,\dot{u}^2 - \frac{1}{2}\frac{G'}{E}\,\dot{v}^2 &= 0, \\ \ddot{v} + \frac{G'}{G}\,\dot{u}\,\dot{v} &= 0. \end{aligned} \tag{10.14}$$

The second equation in (10.14) is equivalent to $G\ddot{v} + G'\dot{u}\dot{v} = 0$, or

$$\frac{d}{dt}\big(G(u)\dot{v}\big) = 0. \tag{10.15}$$

That is to say, along each geodesic $\gamma(t) = X(u(t), v(t))$, we have a constant a_γ such that

$$G\big(u(t)\big)\,\dot{v}(t) = a_\gamma. \tag{10.16}$$

Since we have

$$\gamma'(t)\cdot X_v\big(\gamma(t)\big) = (\dot{u}, \dot{v})\begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = G(u)\dot{v},$$

the identity (10.16) is equivalent to

$$\gamma'\cdot X_v(\gamma) = a_\gamma, \tag{10.17}$$

for each geodesic γ on a surface with a Clairaut parametrization. Compare this with Exercise 3 of §9.

Using (10.16) we could eliminate $\dot{v}$ from the first equation in (10.14). But in fact it is more efficient to recall that a solution to the geodesic equations has constant speed (say c) and eliminate $\dot{v}$ from

$$E(u)\dot{u}^2 + G(u)\dot{v}^2 = c^2, \tag{10.18}$$

obtaining

$$E(u)\dot{u}^2 + \frac{a_\gamma^2}{G(u)} = c^2, \tag{10.19}$$

or

$$\frac{du}{dt} = \pm\sqrt{\frac{1}{E(u)}\Big(c^2 - \frac{a_\gamma^2}{G(u)}\Big)}, \tag{10.20}$$

an ODE amenable to separation of variables. Note that (10.19) forces $c^2 \geq a_\gamma^2/G(u)$, or equivalently

$$G(u) \geq \frac{a_\gamma^2}{c^2}, \quad \text{on} \ \ \gamma. \tag{10.21}$$

Exercises

1. Parametrize the surface of revolution $x^2 + y^2 = f(z)$ by

$$X(u,v) = \big(h(u) \cos v, h(u) \sin v, u\big), \tag{10.22}$$

where $h(u) = \sqrt{f(u)}$. Write out the geodesic equation in these coordinates, in the form (10.14). Note that the metric tensor is given by

$$\mathcal{G}(u,v) = \begin{pmatrix} 1 + h'(u)^2 & 0 \\ 0 & h(u)^2 \end{pmatrix}.$$

2. Consider a surface of revolution of the form (10.22) with $h(u) = e^{-u}$. Given $a \in \mathbb{R}$, consider the geodesic $\gamma_a(t)$ on this surface satisfying

$$\gamma_a(0) = (1,0,0), \quad \gamma_a'(0) = (-1,a,1).$$

(Check that $\gamma_a'(0)$ is tangent to the surface.) Show that, for some $t_a \in (0,\infty)$, $k \cdot \gamma_a(t_a)$ is maximal. Compute this maximum.
Hint. Use (10.21).

3. For a surface of revolution of the form (10.22), show that (10.17) and (9.16) are equivalent.

4. Suppose $(u(t), v(t))$ gives a unit-speed geodesic in a Clairaut patch, for t in an interval I containing t_0. Show that

$$\dot{v}(t_0) = 0 \Longrightarrow v(t) \text{ const. } \forall \ t \in I.$$

Show that

$$\dot{u}(t_0) = 0, \ G'(u(t_0)) = 0 \Longrightarrow u(t) \text{ const., and } \ \dot{v}(t) \text{ const. } \forall \ t \in I.$$

Interpret these results for a surface of revolution, of the form (10.22).

10B. Geodesics on an inner tube

To produce a unit speed geodesic on a surface of revolution, or more generally in a Clairaut patch, it is convenient to integrate numerically the system

$$\ddot{u} + \frac{1}{2}\frac{E'}{E}\,\dot{u}^2 - \frac{1}{2}\frac{G'}{E}\,\frac{a_\gamma^2}{G^2} = 0, \quad G(u)\dot{v} = a_\gamma, \tag{10B.1}$$

which results from (10.14) and (10.16). If one can guarantee that $\dot{u} \neq 0$ on a given geodesic γ, it would be easier to integrate the pair (10.16), (10.20), but this gives trouble when $\dot{u} = 0$. As noted in (10.21), we have $G(u) \geq a_\gamma^2$ on γ. If one picks an initial point $(u(0), v(0)) = (u_0, v_0)$, then a_γ can range over

$$0 \leq a_\gamma \leq \sqrt{G(u_0)}. \tag{10B.2}$$

One can use (10.20) to specify $\dot{u}(0)$:

$$\dot{u}(0) = \sqrt{\frac{1}{E(u_0)}\Bigl(1 - \frac{a_\gamma^2}{G(u_0)^2}\Bigr)}. \tag{10B.3}$$

We work this out for the torus in $\mathbb{R}^3$ obtained by taking the circle of radius a in the yz-plane, centered at $y = b > a,\ z = 0$, and rotating it about the z-axis. One has the pametrization

$$X(u, v) = \bigl((a \cos u + b) \cos v, (a \cos u + b) \sin v, a \sin u\bigr). \tag{10B.4}$$

One computes that

$$\begin{aligned} E &= X_u \cdot X_u = a^2, \\ F &= X_u \cdot X_v = 0, \\ G &= X_v \cdot X_v = (a \cos u + b)^2. \end{aligned} \tag{10B.5}$$

Thus the system (10B.1) becomes (with $w = \dot{u}$)

$$\begin{aligned} \dot{u} &= w, \\ \dot{v} &= \frac{a_\gamma}{G(u)}, \\ \dot{w} &= \frac{1}{2}\frac{G'(u)}{G(u)^2}\,\frac{a_\gamma^2}{a^2}, \end{aligned} \tag{10B.6}$$

with

$$G(u) = (a \cos u + b)^2. \tag{10B.7}$$

Note that

$$G_{\max} = G(0) = (a+b)^2, \quad G_{\min} = G(\pi) = (b-a)^2. \tag{10B.8}$$

We pick initial data

$$u_0 = 0, \quad v_0 = 0, \quad w_0 = \frac{1}{a}\sqrt{1 - \Bigl(\frac{a_\gamma}{a+b}\Bigr)^2}, \tag{10B.9}$$

the formula for w_0 arising from (10B.3), and we let a_γ range over $[0, a+b]$, producing a variety of geodesics on this torus, passing through the point $X(0,0) = (a+b, 0, 0)$. The pictures we show take the case $a = 1,\ b = 2$.

The system (10B.6) is integrated numerically using a Runge-Kutta difference scheme, which is described as follows for a general autonomous first order system of ODEs

$$X' = F(X), \quad X(0) = X_0. \tag{10B.10}$$

Choose a small time step h. Then the approximation X_n to $X(nh)$ is given recursively by

$$X_{n+1} = X_n + \frac{h}{6}\bigl(K_{n1} + 2\,K_{n2} + 2\,K_{n3} + K_{n4}\bigr), \tag{10B.11}$$

where

$$\begin{gathered} K_{n1} = F(X_n), \quad K_{n2} = F(X_n + \tfrac{1}{2}hK_{n1}), \\ K_{n3} = F(X_n + \tfrac{1}{2}hK_{n2}), \quad K_{n4} = F(X_n + hK_{n3}). \end{gathered} \tag{10B.12}$$

The pictures that follow include both graphs of the geodesics in (x,y,z)-space and graphs in (u,v)-space (with u the vertical coordinate). Each picture is labeled with the value of a_γ (denoted "`ag`" in these pictures), ranging over $(0,3]$. The geodesic through $(3,0,0)$ for which $a_\gamma = 1$ spirals in towards the closed geodesic given by $u = \pi$, never quite reaching it. A number of pictures show geodesics for which a_γ is close to 1. The geodesic through $(3,0,0)$ for which $a_\gamma = 3$ is the closed geodesic given by $u = 0$.

11. The exponential map

For a given $p \in M$, the exponential map

$$\mathrm{Exp}_p : U \longrightarrow M, \tag{11.1}$$

is defined on a neighborhood U of $0 \in T_pM$ by

$$\mathrm{Exp}_p(v) = \gamma_v(1) \tag{11.2}$$

where $\gamma_v(t)$ is the unique constant speed geodesic satisfying

$$\gamma_v(0) = p, \quad \gamma_v'(0) = v. \tag{11.3}$$

Note that $\mathrm{Exp}_p(tv) = \gamma_v(t)$. It is clear that Exp_p is well defined and C^∞ on a sufficiently small neighborhood U of $0 \in T_pM$, and its derivative at 0 is the identity. Thus, perhaps shrinking U, we have that Exp_p is a diffeomorphism of U onto a neighborhood $\mathcal{O}$ of p in M. This provides what is called an exponential coordinate system, or a normal coordinate system. Clearly the geodesics through p are the lines through the origin in this coordinate system. We claim that, in this coordinate system,

$$\Gamma^\ell{}_{jk}(p) = 0. \tag{11.4}$$

Indeed, since the line through the origin in any direction $aD_j + bD_k$ is a geodesic, we have

$$\nabla_{(aD_j+bD_k)}(aD_j + bD_k) = 0 \ \text{ at } p, \tag{11.5}$$

for all $a, b \in \mathbb{R}$, and all j, k. This implies

$$\nabla_{D_j} D_k = 0 \text{ at } p \text{ for all } j, k, \tag{11.6}$$

which implies (11.4). We note that (11.4) implies $\partial g_{jk}/\partial x_\ell = 0$ at p, in this exponential coordinate system. In fact, a simple manipulation of (10.10) gives

$$\frac{\partial g_{jk}}{\partial x_\ell} = \sum_m \Bigl(g_{mk}\Gamma^m{}_{j\ell} + g_{mj}\Gamma^m{}_{k\ell} \Bigr). \tag{11.7}$$

As a consequence, a number of calculations in differential geometry can be simplified by working in exponential coordinate systems.

We now establish a result, known as the *Gauss Lemma*, which implies that a geodesic is locally length minimizing. For a small, let $\Sigma_a = \{v \in \mathbb{R}^n : \|v\| = a\}$, and let $S_a = \mathrm{Exp}_p(\Sigma_a)$.

Proposition 11.1. *Any unit speed geodesic through p hitting S_a at $t = a$ is orthogonal to S_a.*

Proof. If $\gamma_0(t)$ is a unit speed geodesic, $\gamma_0(0) = p$, $\gamma_0(a) = q \in S_a$, and $V \in T_qM$ is tangent to S_a, there is a smooth family of unit speed geodesics, $\gamma_s(t)$, such that $\gamma_s(0) = p$ and $(\partial/\partial s)\gamma_s(a)\big|_{s=0} = V$. Using (10.5)–(10.6) for this family, with $0 \le t \le a$, since $L(s)$ is constant, we have

$$0 = \int_0^a T\langle V, T\rangle \, dt = \langle V, \gamma_0'(a)\rangle, \tag{11.8}$$

which proves the proposition.

Corollary 11.2. *Suppose $Exp_p : B_a \to M$ is a diffeomorphism of $B_a = \{v \in T_pM : |v| \le a\}$ onto its image $\mathcal{B}$. Then, for each $q \in \mathcal{B}$, $q = Exp_p(w)$, the curve $\gamma(t) = Exp_p(tw)$, $0 \le t \le 1$, is the unique shortest path from p to q.*

Proof. We can assume $|w| = a$. Let $\sigma : [0,1] \to M$ be another constant speed path from p to q, say $|\sigma'(t)| = b$. We can assume $\sigma(t) \in \mathcal{B}$ for all $t \in [0,1]$; otherwise restrict σ to $[0,\beta]$ where $\beta = \inf\{t : \sigma(t) \in \partial\mathcal{B}\}$ and the argument below will show this segment has length $\ge a$.

For all t such that $\sigma(t) \in \mathcal{B} \setminus p$, we can write $\sigma(t) = \mathrm{Exp}_p(r(t)\omega(t))$, for uniquely determined $\omega(t)$ in the unit sphere of T_pM, and $r(t) \in (0,a]$ If we pull the metric tensor of M back to B_a, we have

$$|\sigma'(t)|^2 = r'(t)^2 + r(t)^2|\omega'(t)|^2,$$

by the Gauss lemma. Hence

$$\begin{aligned} b = \ell(\sigma) &= \int_0^1 |\sigma'(t)| \, dt \\ &= \frac{1}{b}\int_0^1 |\sigma'(t)|^2 \, dt \\ &\ge \frac{1}{b}\int_0^1 r'(t)^2 \, dt. \end{aligned} \tag{11.9}$$

Cauchy's inequality yields

$$\int_0^1 |r'(t)| \, dt \le \Big(\int_0^1 r'(t)^2 \, dt\Big)^{1/2},$$

so the last quantity in (11.9) is $\ge a^2/b$. This implies $b \ge a$, with equality only if $|\omega'(t)| = 0$ for all t, so the corollary is proven.

The following is a useful converse.

Proposition 11.3. *Let $\gamma : [0,1] \to M$ be a constant speed Lipschitz curve from p to q that is absolutely length minimizing. Then γ is a smooth curve satisfying the geodesic equation.*

Proof. We make use of the following fact, which will be established below. Namely, there exists $a > 0$ such that, for each point $x \in \gamma$, $\mathrm{Exp}_x : B_a \to M$ is a diffeomorphism of B_a onto its image (and a is *independent* of $x \in \gamma$).

So choose $t_0 \in [0,1]$ and consider $x_0 = \gamma(t_0)$. The hypothesis implies that γ must be a length minimizing curve from x_0 to $\gamma(t)$, for all $t \in [0,1]$ By Corollary 11.2, $\gamma(t)$ coincides with a geodesic for $t \in [t_0, t_0 + \alpha]$ and for $t \in [t_0 - \beta, t_0]$, where $t_0 + \alpha = \min(t_0 + a, 1)$ and $t_0 - \beta = \max(t_0 - a, 0)$. We need only show that, if $t_0 \in (0,1)$, these two geodesic segments fit together smoothly, i.e., that γ is smooth in a neighborhood of t_0.

To see this, pick $\varepsilon > 0$ such that $\varepsilon < \min(t_0, a)$, and consider $t_1 = t_0 - \varepsilon$. The same argument as above applied to this case shows that γ coincides with a smooth geodesic on a segment including t_0 in its interior, so we are done.

The asserted lower bound on a follows from compactness plus the following observation. Given $p \in M$, there is a neighborhood $\mathcal{O}$ of $(p, 0)$ in TM on which

$$\mathcal{E} : \mathcal{O} \longrightarrow M, \quad \mathcal{E}(x, v) = \mathrm{Exp}_x(v), \quad (v \in T_x M) \tag{11.10}$$

is defined. Let us set

$$\mathcal{F}(x, v) = \big(x, \mathrm{Exp}_x(v)\big), \quad \mathcal{F} : \mathcal{O} \longrightarrow M \times M. \tag{11.11}$$

We readily compute that

$$D\mathcal{F}(p, 0) = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

as a map on $T_pM \oplus T_pM$, where we use Exp_p to identify $T_{(p,0)}T_pM \approx T_pM \oplus T_pM \approx T_{(p,p)}(M \times M)$. Hence the inverse function theorem implies that $\mathcal{F}$ is a diffeomorphism from a neighborhood of $(p, 0)$ in TM onto a neighborhood of (p, p) in $M \times M$.

Let us remark that, though a geodesic is locally length minimizing, it need not be globally length minimizing. There are many simple examples of this, which we leave to the reader to produce.

In addition to length, another quantity associated with a smooth curve $\gamma : [a, b] \to M$ is *energy*:

$$E = \frac{1}{2} \int_a^b \langle T, T \rangle \, dt, \tag{11.12}$$

which differs from the arclength integral in that the integrand here is $\langle T, T \rangle$, rather than the square root of this quantity. If one has a family γ_s of curves, with variation (10.1) and with fixed endpoints, then

$$E'(0) = \frac{1}{2} \int_a^b V \langle T, T \rangle \, dt. \tag{11.13}$$

This is just like the formula (10.3) for $L'(0)$, except for the factor $1/2c_0$ in (10.3), but to get (11.13) we do *not* need to assume that the curve γ_0 has constant speed. Now the computations (10.4)–(10.7) have a parallel here:

$$E'(0) = -\int_a^b \langle V, \nabla_T^M T\rangle \, dt. \tag{11.14}$$

Hence the stationary condition for the energy functional (11.12) is $\nabla_T^M T = 0$, which coincides with the geodesic equation (10.8).

12. Frame fields, connection coefficients, and connection forms

In §8 we did calculations of covariant derivatives in a local coordinate chart $X : \mathcal{O} \to U \subset M$ in terms of the connection coefficients ${\Gamma^\ell}_{ij}$, defined by

$$\nabla^M_{D_j} D_i = \sum_\ell {\Gamma^\ell}_{ij}\, D_\ell, \tag{12.1}$$

where $\{D_1, D_2\}$ is the frame field given by $D_1 = X_u, D_2 = X_v$, assuming $X = X(u, v)$.

Here we consider a more general frame field $\{E_1, E_2\}$, spanning T_pM for each $p \in U \subset M$. We also denote $E_\alpha \circ X$ by $E_\alpha(u, v)$. Given such a frame field, we define the connection coefficients ${}^E{\Gamma^\alpha}_{\beta j}$ by

$$\nabla^M_{D_j} E_\beta = \sum_\alpha {}^E{\Gamma^\alpha}_{\beta j}\, E_\alpha. \tag{12.2}$$

Note that both the coordinate frame field $\{D_1, D_2\}$ and the frame field $\{E_1, E_2\}$ play a role in (12.2). Parallel to (8.31) we have

$$V = \sum_\alpha v^\alpha\, E_\alpha \Longrightarrow \nabla^M_{D_j} V = \sum_\alpha \Bigl(\partial_j v^\alpha + \sum_\beta v^\beta\, {}^E{\Gamma^\alpha}_{\beta j}\Bigr) E_\alpha, \tag{12.3}$$

where, as before, $\partial_1 = \partial/\partial u,\ \partial_2 = \partial/\partial v$.

The Riemann tensor was defined in (8.22) as

$$R(V, W)X = \nabla^M_V \nabla^M_W X - \nabla^M_W \nabla^M_V X - \nabla^M_{[V,W]} X, \tag{12.4}$$

for V, W, X tangent to M. Parallel to (8.33), we define ${}^E{R^\alpha}_{\beta jk}$ by

$$R(D_j, D_k)E_\beta = \sum_\alpha {}^E{R^\alpha}_{\beta jk}\, E_\alpha. \tag{12.5}$$

Then, parallel to (8.34), we have

$${}^E{R^\alpha}_{\beta jk} = \partial_j {}^E{\Gamma^\alpha}_{\beta k} - \partial_k {}^E{\Gamma^\alpha}_{\beta j} + \sum_\gamma \bigl({}^E{\Gamma^\alpha}_{\gamma j}\, {}^E{\Gamma^\gamma}_{\beta k} - {}^E{\Gamma^\alpha}_{\gamma k}\, {}^E{\Gamma^\gamma}_{\beta j}\bigr). \tag{12.6}$$

As in (8.50) we can express this in a shorter form, by defining 2×2 matrices

$${}^E\Gamma_j = ({}^E{\Gamma^\alpha}_{\beta j}), \quad {}^E\mathfrak{R}_{jk} = ({}^E{R^\alpha}_{\beta jk}). \tag{12.7}$$

Then (12.6) becomes

$$ {}^E\mathfrak{R}_{jk} = \partial_j {}^E\Gamma_k - \partial_k {}^E\Gamma_k + [{}^E\Gamma_j, {}^E\Gamma_k]. \tag{12.8} $$

For V tangent to M, let us define ${}^E\Gamma^\alpha{}_\beta(V)$ by

$$ \nabla^M_V E_\beta = \sum_\alpha {}^E\Gamma^\alpha{}_\beta(V)\, E_\alpha, \tag{12.9} $$

so ${}^E\Gamma^\alpha{}_{\beta j} = {}^E\Gamma^\alpha{}_\beta(D_j)$ and, by (12.2),

$$ v = \sum_j v^j\, D_j \Longrightarrow {}^E\Gamma^\alpha{}_\beta(V) = \sum_j {}^E\Gamma^\alpha{}_{\beta j}\, v^j. \tag{12.10} $$

We also define

$$ {}^E\omega_{\alpha\beta}(V) = \nabla^M_V E_\alpha \cdot E_\beta, \tag{12.11} $$

so

$$ {}^E\omega_{\alpha\beta}(V) = \sum_\gamma e_{\beta\gamma}\, {}^E\Gamma^\gamma{}_\alpha(V), \quad e_{\beta\gamma} = E_\beta \cdot E_\gamma. \tag{12.12} $$

Throughout the rest of this section, we assume that $\{E_1, E_2\}$ forms an *orthonormal* basis of T_pM for each $p \in U$, i.e.,

$$ E_\alpha \cdot E_\beta = \delta_{\alpha\beta}, \tag{12.13} $$

where $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$, 0 for $\alpha \neq \beta$. We drop the superscript E from ${}^E\omega_{\alpha\beta}(V)$, simply writing $\omega_{\alpha\beta}(V)$. Differentiating the identity (12.13), we have $\nabla^M_V E_\alpha \cdot E_\beta + E_\alpha \cdot \nabla^M_V E_\beta = 0$, or

$$ \omega_{\alpha\beta}(V) = -\omega_{\beta\alpha}(V). \tag{12.14} $$

Hence, in this case,

$$ \big({}^E\Gamma^\alpha{}_\beta(V)\big) = {}^E\Gamma^1{}_2(V) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \omega_{12}(V) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12.15} $$

Thus, in the computation (12.8), the matrix commutator is zero, and we have

$$ \begin{aligned} {}^E\mathfrak{R}_{12} &= \partial_1 {}^E\Gamma_2 - \partial_2 {}^E\Gamma_1 \\ &= \big\{\partial_1\omega_{12}(D_2) - \partial_2\omega_{12}(D_1)\big\} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{12.16} $$

We now bring in the language of differential forms. We have

$$\omega_{12} = \omega_{12}(D_1)\,du + \omega_{12}(D_2)\,dv. \tag{12.17}$$

The exterior derivative of this 1-form is a 2-form, given by

$$d\omega_{12} = \big\{\partial_1\omega_{12}(D_2) - \partial_2\omega_{12}(D_1)\big\}du\wedge dv, \tag{12.18}$$

so

$$\partial_1\omega_{12}(D_2) - \partial_2\omega_{12}(D_1) = d\omega_{12}(D_1, D_2), \tag{12.19}$$

and we can write (12.16) as

$$^E\mathfrak{R}_{12} = d\omega_{12}(D_1, D_2)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12.20}$$

Equivalently,

$$R(D_1, D_2)(a_1E_1 + a_2E_2) = d\omega_{12}(D_1, D_2)(-a_2E_1 + a_1E_2). \tag{12.21}$$

Hence, for all V, W tangent to M,

$$R(V, W)(a_1E_1 + a_2E_2) = d\omega_{12}(V, W)(-a_2E_1 + a_1E_2). \tag{12.22}$$

In particular,

$$R(E_1, E_2)E_2 = -d\omega_{12}(E_1, E_2)E_1, \tag{12.23}$$

so

$$E_1\cdot R(E_1, E_2)E_2 = -d\omega_{12}(E_1, E_2). \tag{12.24}$$

Since we are assuming $\{E_1, E_2\}$ is an orthonormal basis of T_pM, we see from (8.24) that the left side of (12.24) is equal to the Gauss curvature K of M. Regarding the right side, suppose

$$D_j = \sum_\beta a_{j\beta}\,E_\beta, \quad A = (a_{j\beta}). \tag{12.25}$$

Then the metric tensor $\mathcal{G} = (g_{jk})$ is given by $g_{jk} = D_j\cdot D_k = \sum_\beta a_{j\beta}a_{k\beta}$, i.e., $\mathcal{G} = AA^t$. We have

$$\begin{aligned} d\omega_{12}(D_1, D_2) &= \sum_{\beta,\gamma} a_{1\beta}a_{2\gamma}\,d\omega_{12}(E_\beta, E_\gamma) \\ &= (a_{11}a_{22} - a_{12}a_{21})\,d\omega_{12}(E_1, E_2). \end{aligned} \tag{12.26}$$

Now $a_{11}a_{22} - a_{12}a_{21} = \det A$ and $g = \det \mathcal{G} = (\det A)^2$. We say that $\{E_1, E_2\}$ is positively oriented with respect to $\{D_1, D_2\}$ if $\det A > 0$. In such a case, we have

$$d\omega_{12}(D_1, D_2) = \sqrt{EG - F^2}\, d\omega_{12}(E_1, E_2) = -K\sqrt{EG - F^2}, \tag{12.27}$$

the last identity by the comment after (12.24). Equivalently,

$$d\omega_{12} = -K\, dA, \tag{12.28}$$

where dA denotes the area 2-form on M.

Exercises

1. Suppose $\{E_1, E_2\}$ and $\{F_1, F_2\}$ are two frame fields defined over $U \subset M$, with a fixed coordinate chart. Say they are related by

$$F_\beta = \sum_\alpha a^\alpha{}_\beta\, E_\alpha.$$

Show that ${}^E\Gamma^\alpha{}_\beta(V)$ and ${}^F\Gamma^\alpha{}_\beta(V)$ are related by the following identity:

$$\sum_\gamma a^\alpha{}_\gamma\, {}^F\Gamma^\gamma{}_\beta(V) = \sum_\gamma {}^E\Gamma^\alpha{}_\gamma(V)\, a^\gamma{}_\beta + \nabla_V a^\alpha{}_\beta,$$

for all α, β. Forming the 2×2 matrices

$$A = (a^\alpha{}_\beta), \quad {}^E\Gamma(V) = \big({}^E\Gamma^\alpha{}_\beta(V)\big), \quad {}^F\Gamma(V) = \big({}^F\Gamma^\alpha{}_\beta(V)\big),$$

show this is equivalent to

$${}^F\Gamma(V) = A^{-1}\, {}^E\Gamma(V)\, A + A^{-1}\nabla_V A,$$

where $\nabla_V A$ is computed componentwise.

13. The Gauss-Bonnet formula

Let M be a smooth, oriented surface in $\mathbb{R}^3$. Let $\overline{\Omega} \subset M$ be a compact subset, with boundary $\partial\Omega$, consisting of piecewise smooth curves, perhaps with corners. The Gauss-Bonnet formula relates $\int_\Omega K\,dA$ to geometrical data on $\partial\Omega$, where K is the Gauss curvature of M.

To begin the evaluation of $\int_\Omega K\,dA$, let us first assume that there is a smooth, positively oriented, orthonormal frame $\{E_1, E_2\}$, spanning T_pM for each $p \in \overline{\Omega}$. Take $N = E_1 \times E_2$. We use the connection form ω_{12} given by (12.11), i.e.,

$$\omega_{12}(V) = \nabla^M_V E_1 \cdot E_2 = -\omega_{21}(V), \tag{13.1}$$

and the identity (12.28), i.e.,

$$d\omega_{12} = -K\,dA, \tag{13.2}$$

to write

$$\int\limits_\Omega K\,dA = -\int\limits_\Omega d\omega_{12} = -\int\limits_{\partial\Omega} \omega_{12}, \tag{13.3}$$

the last identity following by Stokes' formula. The next result provides the crucial link with the geometry of $\partial\Omega$.

Assume that a part of $\partial\Omega$ is given by a smooth, unit speed curve $\gamma : (a,b) \to \partial\Omega$, $T(t) = \gamma'(t)$. Define a smooth function $\varphi : (a,b) \to \mathbb{R}$ such that

$$T(t) = \cos\varphi(t)\ E_1 + \sin\varphi(t)\ E_2. \tag{13.4}$$

Lemma 13.1. *If $k_g(t)$ denotes the geodesic curvature of $\gamma(t)$, then*

$$k_g(t) = \varphi'(t) + \omega_{12}(T). \tag{13.5}$$

Proof. Recall that

$$k_g(t) = T' \cdot U, \quad U = N \times T. \tag{13.6}$$

Differentiating (13.4) gives

$$\begin{aligned} T' = &-\varphi'\ \sin\varphi\ E_1 + \cos\varphi\,\nabla_T E_1 \\ &+\varphi'\ \cos\varphi\ E_2 + \sin\varphi\,\nabla_T E_2. \end{aligned} \tag{13.7}$$

Now $\nabla_T E_j \cdot U = \nabla^M_T E_j \cdot U$ and

$$\nabla^M_T E_1 = \sum_\alpha (\nabla^M_T E_1 \cdot E_\alpha) E_\alpha = \omega_{11}(T)E_1 + \omega_{12}(T)E_2 = \omega_{12}(T)E_2.$$

Making the same calculation for $\nabla^M_T E_2$, we obtain

$$\nabla^M_T E_1 = \omega_{12}(T)E_2, \quad \nabla^M_T E_2 = \omega_{21}(T)E_1, \tag{13.8}$$

so (13.7) becomes

$$T' = \sin\varphi\big(-\varphi' + \omega_{21}(T)\big)E_1 + \cos\varphi\big(\varphi' + \omega_{12}(T)\big)E_2, \quad \bmod (N). \tag{13.9}$$

To compute the dot product with U, we note that

$$\begin{aligned} U = N \times T &= \cos\varphi\,(N \times E_1) + \sin\varphi\,(N \times E_2) \\ &= -\sin\varphi\; E_1 + \cos\varphi\; E_2, \end{aligned} \tag{13.10}$$

so (13.6) gives

$$k_g(t) = -\sin^2\varphi\,\big(-\varphi' - \omega_{12}(T)\big) + \cos^2\varphi\,\big(\varphi' + \omega_{12}(T)\big) = \varphi' + \omega_{12}(T),$$

as desired.

We have the following consequence of (13.3) and (13.5).

Proposition 13.2. *Assume $\overline{\Omega}$ is contractible and $\partial\Omega$ is smooth. Then*

$$\int\limits_{\Omega} K\,dA = 2\pi - \int\limits_{\partial\Omega} k_g(s)\,ds. \tag{13.11}$$

Proof. We have

$$\int\limits_{\Omega} K\,dA = -\int\limits_{\partial\Omega} \omega_{12} = -\int\limits_{\partial\Omega} k_g(s)\,ds + \int\limits_{\partial\Omega} \varphi'(s)\,ds. \tag{13.12}$$

By the fundamental theorem of calculus, $\int_{\partial\Omega} \varphi'(s)\,ds = \varphi(\ell) - \varphi(0)$, if $\partial\Omega$ has length ℓ and is given by a unit speed curve $\gamma : [0,\ell] \to M$. Assuming $\partial\Omega$ is smooth, we have $T(\ell) = T(0)$, so $\varphi(\ell) - \varphi(0)$ must be an integral multiple of 2π. In fact we claim that

$$\varphi(\ell) - \varphi(0) = 2\pi. \tag{13.13}$$

This is a consequence of the contractibility of $\overline{\Omega}$. This contractibility implies that there is a continuous family Ω_σ, defined for $0 < \sigma \le 1$, with $\Omega_1 = \Omega$, such that $\partial\Omega_\sigma$ has length $\sigma\ell$ and, for small σ, Ω_σ is a disk (in some coordinate system), small enough that E_1 and E_2 have only small variation over Ω_σ. We have corresponding functions $\varphi_\sigma(t)$ defined, for $0 \le t \le \sigma\ell$. Now $\delta(\sigma) = \varphi_\sigma(\sigma\ell) - \varphi_\sigma(0)$ takes values in $2\pi\mathbb{Z}$ and is continuous in $\sigma \in (0,1]$. On the other hand, it follows from Euclidean geometry that $\delta(\sigma) \to 2\pi$ as $\sigma \to 0$. Hence we have (13.13), and this proves (13.11).

It is of interest to allow $\partial\Omega$ to have corners; say

$$\partial\Omega = \gamma_1 \cup \cdots \cup \gamma_k \tag{13.14}$$

where each γ_j is a smooth, unit speed curve, $\gamma_j : [\ell_j, \ell_{j+1}] \to \partial\Omega \subset M$, with $\ell_1 = 0$, $\ell_{k+1} = \ell =$ length of $\partial\Omega$, and $\gamma_j(\ell_{j+1}) = \gamma_{j+1}(\ell_{j+1})$, $\gamma_k(\ell) = \gamma_1(0)$. Replace (13.4) by

$$\gamma_j'(t) = \cos\varphi_j(t)\ E_1 + \sin\varphi_j(t)\ E_2, \quad \ell_j \le t \le \ell_{j+1}. \tag{13.15}$$

Then (13.12) is replaced by

$$\int_\Omega K\,dA = -\int_{\partial\Omega} k_g(s)\,ds + \sum_{j=1}^k \int_{\gamma_j} \varphi_j'(s)\,ds. \tag{13.16}$$

This time the fundamental theorem of calculus gives

$$\sum_{j=1}^k \int_{\gamma_j} \varphi_j'(s)\,ds = \sum_{j=1}^k \bigl[\varphi_j(\ell_{j+1}) - \varphi_j(\ell_j)\bigr]. \tag{13.17}$$

We can rewrite the right side of (13.16) as

$$\bigl[\varphi_1(\ell_2) - \varphi_2(\ell_2)\bigr] + \cdots + \bigl[\varphi_{k-1}(\ell_k) - \varphi_k(\ell_k)\bigr] + \bigl[\varphi_k(\ell) - \varphi_1(0)\bigr]. \tag{13.18}$$

Note that each $\varphi_j(t)$ is well defined up to an additive constant of the form $2\pi m_j$, $m_j \in \mathbb{Z}$. These ambiguities cancel out in each term in the sum on the right side of (13.17). They might not cancel out in each term in brackets in (13.18), though of course they all cancel out in the total sum. To make a definite choice, let us define the *turning angle* θ_j and the *interior angle* α_j of $\partial\Omega$ at the corner $p_j = \gamma_j(\ell_{j+1}) = \gamma_{j+1}(\ell_{j+1})$ by

$$\varphi_{j+1}(\ell_{j+1}) - \varphi_j(\ell_{j+1}) = \theta_j, \quad \alpha_j = \pi - \theta_j. \tag{13.19}$$

See Fig. 13.1. Then we demand that

$$-\pi < \varphi_1(0) \le \pi, \quad 0 < \alpha_j < 2\pi, \quad 1 \le j \le k-1. \tag{13.20}$$

(We could also consider limiting cases of cusps, with $\alpha_j = 0$ or $\alpha_j = 2\pi$, but we will not dwell on this.) Now the quantity in (13.18) is equal to

$$\varphi_k(\ell) - \varphi_1(0) - \sum_{j=1}^{k-1} \theta_j. \tag{13.21}$$

Let us suppose that $\gamma_1(0) = \gamma_k(\ell)$ is a regular point of $\partial\Omega$, so $\gamma_1'(0) = \gamma_k'(\ell)$, and $\varphi_k(\ell) - \varphi_1(0)$ is an integral multiple of 2π. If $\overline{\Omega}$ is contractible, a deformation argument such as used above gives

$$\varphi_k(\ell) - \varphi_1(0) = 2\pi. \tag{13.22}$$

We have the following result, known as the Gauss-Bonnet formula.

Theorem 13.3. *Assume $\overline{\Omega}$ is contractible and that $\partial\Omega$ is piecewise smooth, with corners $p_1, \dots, p_{k-1}$ having turning angles $\theta_1, \dots, \theta_{k-1}$. then*

$$\int_{\Omega} K\, dA = 2\pi - \sum_{j=1}^{k-1} \theta_j - \int_{\partial\Omega} k_g(s)\, ds. \tag{13.23}$$

For the next result, suppose M is a compact, oriented surface in $\mathbb{R}^3$, without boundary. Suppose we have M partitioned into F contractible pieces (faces), each of which has piecewise smooth boundary, with finitely many corners. We assume M has V vertices, at which these corners are located, and we assume M has E edges. Each edge is a smooth path connecting two vertices, making up part of the common boundary of two faces.

Each face Ω_ν $(1 \le \nu \le F)$ is a contractible set with piecewise smooth boundary, with corners at those vertices that belong to $\overline{\Omega}_\nu$; say they have turning angles $\theta_{\nu j}$. We can apply (13.23) to each Ω_ν, and then sum over ν. For each edge, there are two contributions that involve a line integral of the geodesic curvature, and these cancel each other. We get

$$\int_M K\, dA = 2\pi F - \sum_{\nu,j} \theta_{\nu j}. \tag{13.24}$$

Note that, if we sum the interior angles $\alpha_{\nu j} = \pi - \theta_{\nu j}$, we get 2π for each vertex, so

$$\sum_{\nu,j} (\pi - \theta_{\nu j}) = 2\pi V. \tag{13.25}$$

Furthermore, there are two terms in the sum on the left side of (13.25) for each edge, so this identity is equivalent to $2\pi E - \sum_{\nu,j} \theta_{\nu,j} = 2\pi V$, and we have

$$\int_M K\, dA = 2\pi(F - E + V). \tag{13.26}$$

The quantity

$$\chi(M) = F - E + V \tag{13.27}$$

is called the Euler characteristic of M. Since the left side of (13.26) is independent of the particular division of M into faces, so is the right side, so (13.27) is an intrinsic quantity associated to M. We record this result, as the Gauss-Bonnet formula for a compact, oriented surface in $\mathbb{R}^3$.

Proposition 13.4. *If $M \subset \mathbb{R}^3$ is a compact, oriented surface, without boundary, then*

$$\int_M K\, dA = 2\pi\chi(M). \tag{13.28}$$

14. Cartan structure equations and Gauss-Codazzi equations

Let $M \subset \mathbb{R}^3$ be an oriented surface, with unit normal field N. Let $U \subset M$ be an open subset over which there is a smooth, positively oriented, orthonormal frame field $\{E_1, E_2\}$, spanning T_pM for each $p \in U$. We complete this to an orthonormal basis of $\mathbb{R}^3$ at each p by throwing in $E_3 = N$. It will also be convenient to extend $E_\alpha(x)$ to a smooth orthonormal frame on an open neighborhood $\mathcal{O}$ of U in $\mathbb{R}^3$.

We define $\omega_{\alpha\beta}$ for $1 \le \alpha, \beta \le 3$ by

$$\omega_{\alpha\beta}(V) = \nabla_V E_\alpha \cdot E_\beta. \tag{14.1}$$

If $\alpha, \beta \in \{1, 2\}$ and V is tangent to M, this coincides with the definition of $\omega_{\alpha\beta}$ in (12.11); in particular

$$V \ \text{ tangent to } \ M \Longrightarrow \omega_{12}(V) = \nabla^M_V E_1 \cdot E_2. \tag{14.2}$$

In the present context, the shape operator is also captured:

$$V \ \text{ tangent to } \ M \Longrightarrow S(V) = -\sum_{\beta=1}^{2} \omega_{3\beta}(V) E_\beta. \tag{14.3}$$

As before, the orthonormality of $\{E_1, E_2, E_3\}$ yields $\nabla_V (E_\alpha \cdot E_\beta) = 0$, and hence

$$\omega_{\alpha\beta}(V) = -\omega_{\beta\alpha}(V). \tag{14.4}$$

Given the orthonormal frame field $\{E_1, E_2, E_3\}$ on $\mathcal{O} \subset \mathbb{R}^3$, we define the dual frame field $\{\theta_1, \theta_2, \theta_3\}$ of 1-forms on $\mathcal{O}$ by

$$\theta_\alpha(E_\beta) = \delta_{\alpha\beta}. \tag{14.5}$$

If $\{e_1, e_2, e_3\}$ is the standard orthonormal basis of $\mathbb{R}^3$, consisting of unit vectors e_j pointing in the x_j-direction, we can write

$$E_\alpha = \sum_j A_{\alpha j}\, e_j, \quad A_{\alpha j} = E_\alpha \cdot e_j. \tag{14.6}$$

Then

$$\theta_\alpha = \sum_j A_{\alpha j}\, dx_j. \tag{14.7}$$

It is convenient to use the 3×3 matrices

$$A = (A_{\alpha j}), \quad \Omega = (\omega_{\alpha\beta}). \tag{14.8}$$

If we write E_1, E_2, E_3 as *column* vectors, then $A = (E_1, E_2, E_3)$. Note that this is an orthogonal matrix. The following is a computationally convenient formula for $\omega_{\alpha\beta}$.

Proposition 14.1. *We have*

$$\omega_{\alpha\beta} = \sum_{k=1}^{3} dA_{\alpha k}\, A_{\beta k}. \tag{14.9}$$

Proof. If V is a vector field on $\mathcal{O}$, then

$$dA_{\alpha k}(V) = \nabla_V A_{\alpha k} = \nabla_V E_\alpha \cdot e_k, \tag{14.10}$$

since $\nabla_V e_k = 0$. Hence

$$\sum_{k=1}^{3} dA_{\alpha k}(V)\, A_{\beta k} = \sum_{k=1}^{3} (\nabla_V E_\alpha \cdot e_k)(E_\beta \cdot e_k), \tag{14.11}$$

which is clearly equal to the right side of (14.1).

In matrix notation, (14.9) says

$$\Omega = dA\, A^t = dA\, A^{-1}, \tag{14.12}$$

the last identity because A is orthogonal.

We now obtain identities known as Cartan's structural equations.

Proposition 14.2. *We have*

$$d\theta_\alpha = \sum_{\beta=1}^{3} \omega_{\alpha\beta} \wedge \theta_\beta, \qquad \textit{(first structural equation)}, \tag{14.13}$$

$$d\omega_{\alpha\beta} = \sum_{\gamma=1}^{3} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}, \quad \textit{(second structural equation)}. \tag{14.14}$$

Proof. By (14.7),

$$d\theta_\alpha = \sum_j dA_{\alpha j} \wedge dx_j. \tag{14.15}$$

Now (14.12) gives $dA = \Omega A$, or

$$dA_{\alpha j} = \sum_\beta \omega_{\alpha\beta}\, A_{\beta j}, \tag{14.16}$$

and plugging this into (14.15) gives (14.13). Next, since $d^2 = 0$, (14.12) gives

$$\begin{aligned} d\Omega &= -dA \wedge (dA)^t \\ &= -dA\, A^t \wedge A\,(dA)^t \\ &= -(dA\, A^t) \wedge (dA\, A)^t \\ &= \Omega \wedge \Omega, \end{aligned} \tag{14.17}$$

the second identity by the orthogonality of A. This gives (14.14).

In view of the antisymmetry $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, (14.14) contains three identities. One is

$$d\omega_{12} = \omega_{13} \wedge \omega_{32}. \qquad \text{(Gauss equation)} \tag{14.18}$$

In other words

$$d\omega_{12}(V, W) = \omega_{13}(V)\omega_{32}(W) - \omega_{13}(W)\omega_{32}(V). \tag{14.19}$$

Using (14.3) we see that, if V and W are tangent to M,

$$\omega_{\beta 3}(V) = S(V) \cdot E_\beta, \quad \omega_{\beta 3}(W) = S(W) \cdot E_\beta, \tag{14.20}$$

so

$$d\omega_{12}(V, W) = -(SV \cdot E_1)(SW \cdot E_2) + (SW \cdot E_1)(SV \cdot E_2), \tag{14.21}$$

and in particular

$$d\omega_{12}(E_1, E_2) = -\det S = -K. \tag{14.22}$$

This is a second derivation of (12.24), hence of (12.28), so (14.18) is a variant of the Gauss equation.

The other two identities contained in (14.14) are

$$\begin{aligned} d\omega_{13} &= \omega_{12} \wedge \omega_{23} \qquad \text{(Codazzi equation)} \\ d\omega_{23} &= \omega_{21} \wedge \omega_{13}. \end{aligned} \tag{14.23}$$

We use this to give a second proof that

$$\nabla^M_V(SW) - \nabla^M_W(SV) - S([V, W]) = 0, \tag{14.24}$$

for V, W tangent to M, which is the version of the Codazzi equation stated in §8. To begin, we have from (14.3) that

$$\nabla^M_V SW - \nabla^M_W SV = \sum_{\alpha=1}^{2} \big\{ \nabla^M_V \big(\omega_{\alpha 3}(W) E_\alpha\big) - \nabla^M_W \big(\omega_{\alpha 3}(V) E_\alpha\big) \big\}. \tag{14.25}$$

If we use the derivation identity and the formula

$$d\omega_{\alpha 3}(V,W) = V\omega_{\alpha 3}(W) - W\omega_{\alpha 3}(V) - \omega_{\alpha 3}([V,W]), \tag{14.26}$$

we see that the left side of (14.24) is equal to

$$\sum_{\alpha=1}^{2} \{\omega_{\alpha 3}(W)\nabla^M_V E_\alpha - \omega_{\alpha 3}(V)\nabla^M_W E_\alpha\} + \sum_{\alpha=1}^{2} d\omega_{\alpha 3}(V,W)\, E_\alpha. \tag{14.27}$$

Now, for V tangent to M, $\nabla^M_V E_\alpha = \sum_{\beta<3} \omega_{\alpha\beta}(V) E_\beta$ (if $1 \le \alpha \le 2$). If we plug this into (14.27) and then apply (14.23), we obtain the identity (14.24).

Suppose you have an oriented coordinate patch $X : \mathcal{O} \to U \subset M$ for which the metric tensor takes the form

$$\mathcal{G} = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}. \tag{14.28}$$

Then an orthonormal frame field for TU is given by

$$E_1 = E^{-1/2} X_u, \quad E_2 = G^{-1/2} X_v, \tag{14.29}$$

with dual frame field (pulled back to $\mathcal{O}$)

$$\theta_1 = E^{1/2}\, du, \quad \theta_2 = G^{1/2}\, dv. \tag{30}$$

Note that, if $\kappa : M \hookrightarrow \mathbb{R}^3$ is the inclusion, then, since θ_3 annihilates vectors tangent to M, we have

$$\kappa^* \theta_3 = 0. \tag{14.31}$$

Hence Cartan's first structure equations, when pulled back to M (or to $\mathcal{O}$) shorten to

$$\begin{aligned} d\theta_1 &= \omega_{12} \wedge \theta_2, \\ d\theta_2 &= \omega_{21} \wedge \theta_1. \end{aligned} \tag{14.32}$$

It is notable that these identities uniquely determine ω_{12} on M, given θ_1 and θ_2 on M. In the present case, suppose ω_{12} (pulled back to $\mathcal{O}$) is given by

$$\omega_{12} = \varphi\, du + \psi\, dv. \tag{14.33}$$

To solve for φ and ψ we compute the left sides of (14.32):

$$d\theta_1 = -(\partial_v E^{1/2})\, du \wedge dv, \quad d\theta_2 = (\partial_u G^{1/2})\, du \wedge dv, \tag{14.34}$$

and the right sides of (14.32):

$$\omega_{12} \wedge \theta_2 = \varphi G^{1/2}\, du \wedge dv, \quad \omega_{21} \wedge \theta_1 = \psi E^{1/2}\, du \wedge dv, \tag{14.35}$$

and compare, to get

$$\omega_{12} = -\frac{1}{2} \frac{E_v}{\sqrt{EG}}\, du + \frac{1}{2} \frac{G_u}{\sqrt{EG}}\, dv. \tag{14.36}$$

We can use (14.36) together with the equation

$$d\omega_{12} = -K\, dA \tag{14.37}$$

to compute the Gauss curvature. Recall that the area element is given by

$$dA = \sqrt{EG}\, du \wedge dv. \tag{14.38}$$

From (14.36) we have

$$d\omega_{12} = \frac{1}{2} \Big\{ \frac{\partial}{\partial u} \frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{E_v}{\sqrt{EG}} \Big\}\, du \wedge dv. \tag{14.39}$$

Hence

$$K = -\frac{1}{2\sqrt{EG}} \Big\{ \frac{\partial}{\partial u} \frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{E_v}{\sqrt{EG}} \Big\}. \tag{14.40}$$

This recovers the formula (8.54).

15. Minkowski space and hyperbolic space

Minkowski space $\mathbb{R}^{n,1}$ is the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ endowed with the inner product

$$\langle V, W\rangle = v_1 w_1 + \cdots + v_n w_n - v_{n+1} w_{n+1}, \tag{15.1}$$

for vectors $V = (v_1, \dots, v_{n+1}), W = (w_1, \dots, w_{n+1})$. Here we will concentrate on the 3-dimensional case $\mathbb{R}^{2,1}$, with coordinates (x, y, z). A natural replacement for the sphere in this case is the set

$$H^2 = \{(x, y, z) : x^2 + y^2 - z^2 = -1,\ z > 0\}, \tag{15.2}$$

the upper half of a 2-sheeted hyperboloid. This is parametrized by

$$X(u, \theta) = (\sinh u\ \cos\theta, \sinh u\ \sin\theta, \cosh u). \tag{15.3}$$

These coordinates are singular at $u = 0$, corresponding to $(x, y, z) = (0, 0, 1)$, but we need not worry about that. A computation gives

$$\begin{aligned} \langle X_u, X_u\rangle &= 1, \\ \langle X_u, X_\theta\rangle &= 0, \\ \langle X_\theta, X_\theta\rangle &= \sinh^2 u, \end{aligned} \tag{15.4}$$

using the inner product (15.1). In particular, we see that, for each $p \in H^2$, the inner product (15.1) restricted to T_pH^2 is *positive definite*. A surface $M \subset \mathbb{R}^{2,1}$ with this property is said to be *spacelike*.

Using the metric tensor on H^2 given by (15.4), we can define ∇^{H^2} via (8.25), then define the Riemann tensor via (8.22) and use (8.24) to define the Gauss curvature, i.e.,

$$K(p) = \langle u_1, R(u_1, u_2)u_2\rangle, \tag{15.5}$$

where $\{u_1, u_2\}$ is an orthonormal basis of T_pH^2. (It is readily verified that this is independent of the choice of orthonormal basis of T_pH^2.) Note that (15.4) implies the coordinates (u, θ) on H^2 are orthogonal. Hence the formula (8.54) for K holds. We have $G_u = 2\cosh u\ \sinh u$, $\sqrt{EG} = \sinh u$, $E_\theta = 0$, and hence

$$K = -\frac{1}{2}\frac{1}{\sinh u}\frac{\partial}{\partial u}(2\ \cosh u),$$

i.e.,

$$K = -1. \tag{15.6}$$

The space H^2 is called *hyperbolic space*.

Suppose $M \subset \mathbb{R}^{2,1}$ is a spacelike surface. One can verify that a nonzero vector $Z(p) \in \mathbb{R}^{2,1}$ orthogonal (with respect to (15.1)) to T_pM must be *timelike*, i.e., $\langle Z(p), Z(p)\rangle < 0$. We pick $N(p)$ orthogonal to T_pM with z-coordinate positive, such that $\langle N(p), N(p)\rangle = -1$, and call this our unit timelike normal to M.

If V and W are vector fields tangent to M, we can set

$$\nabla^M_V W = P\,\nabla_V W, \tag{15.7}$$

parallel to (8.1). In this case, we have

$$\nabla^M_V W = \nabla_V W + \langle N, \nabla_V W\rangle N. \tag{15.8}$$

Note the sign change, compared to (8.3), due to the fact that here $\langle N, N\rangle = -1$. We can write

$$\nabla^M_V W = \nabla_V W - \langle SV, W\rangle N, \tag{15.9}$$

parallel to (8.5), where, this time, we set

$$SV = \nabla_V N. \tag{15.10}$$

Note again a sign change compared to (7.3). Again we have

$$S(p) : T_pM \longrightarrow T_pM. \tag{15.11}$$

We leave it as an exercise to the reader to show that we still have

$$\langle SV, W\rangle = \langle V, SW\rangle, \quad V, W \in T_pM. \tag{15.12}$$

Then we again have the identities (8.6)–(8.10), (8.17), and (8.25). Regarding the analysis of the Riemann tensor on M, we have, for X, Y, Z tangent to M,

$$\begin{aligned} \nabla^M_V(\nabla^M_W X) &= \nabla_V\nabla_W X - \nabla_V(\langle SW, X\rangle N) \quad \mod N \\ &= \nabla_V\nabla_W X - \langle SW, X\rangle SV \qquad\quad \mod N. \end{aligned} \tag{15.13}$$

Note the sign change in the last term, compared to (8.19). With $R(V, W)X$ defined as in (8.22), this gives

$$R(V, W)X = -\langle SW, X\rangle SV + \langle SV, X\rangle SW, \tag{15.14}$$

the right side having the opposite sign from the right side of (8.23). Hence, defining the Gauss curvature of M by (15.5), for an orthonormal basis $\{u_1, u_2\}$ of T_pM, we have

$$K = -\langle Su_1, u_1\rangle\langle Su_2, u_2\rangle + \langle Su_1, u_2\rangle\langle Su_2, u_1\rangle. \tag{15.15}$$

If we define the Gauss map

$$G : M \longrightarrow H^2, \quad G(p) = N(p), \tag{15.16}$$

we see that

$$DG(p) : T_pM \longrightarrow T_{G(p)}H^2 = T_pM, \tag{15.17}$$

and (15.15) yields

$$K(p) = -\det DG(p). \tag{15.18}$$

Note once more a sign change, compared to (7.52). In case $M = H^2$, G is the identity map, and (15.18) is consistent with (15.6).

Exercises

1. Show that if you set

$$r = \tanh\frac{u}{2}, \quad 0 \le r < 1, \tag{15.19}$$

the metric tensor on H^2 in the (r, θ) coordinate system becomes

$$(g_{jk}) = \frac{4}{(1-r^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{15.20}$$

If you switch to Cartesian coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, the metric tensor becomes

$$(g_{jk}) = \frac{4}{(1-x^2-y^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{15.21}$$

on the disk $D = \{(x, y) : x^2 + y^2 < 1\}$. The disk D with this metric tensor is called the *Poincaré disk.*

2. Identify $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x+iy$. Show that the linear fractional transformations

$$\varphi_{a,b}(z) = \frac{az+b}{\overline{b}z+\overline{a}}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1, \tag{15.22}$$

map the disk D onto itself and preserve the metric tensor (15.21).

A. Exponentiation of matrices

Let A be an $n \times n$ matrix, real or complex. We consider the linear ODE

$$\frac{dy}{dt} = Ay; \quad y(0) = y_0. \tag{A.1}$$

We can produce the solution in the form

$$y(t) = e^{tA} y_0, \tag{A.2}$$

where we define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \tag{A.3}$$

We will establish estimates implying the convergence of this infinite series for all real t, indeed for all complex t. Then term by term differentiation is valid, and gives (A.1). To discuss convergence of (A.3), we need the notion of the *norm* of a matrix.

If $u = (u_1, \dots, u_n)$ belongs to $\mathbb{R}^n$ or to $\mathbb{C}^n$, set

$$\|u\| = \bigl(|u_1|^2 + \cdots + |u_n|^2\bigr)^{1/2}. \tag{A.4}$$

Then, if A is an $n \times n$ matrix, set

$$\|A\| = \sup\{\|Au\| : \|u\| \le 1\}. \tag{A.5}$$

The norm (A.4) possesses the following properties:

$$\|u\| \ge 0, \quad \|u\| = 0 \Longleftrightarrow u = 0, \tag{A.6}$$

$$\|cu\| = |c|\ \|u\|, \ \text{for real or complex } c, \tag{A.7}$$

$$\|u + v\| \le \|u\| + \|v\|. \tag{A.8}$$

The last, known as the triangle inequality, follows from Cauchy's inequality:

$$|(u, v)| \le \|u\| \cdot \|v\|, \tag{A.9}$$

where the inner product is $(u, v) = u_1 \overline{v}_1 + \cdots + u_n \overline{v}_n$. To deduce (A.8) from (A.9), just square both sides of (A.8). To prove (A.9), use $(u - v, u - v) \ge 0$ to get

$$2 \ \text{Re}\ (u, v) \le \|u\|^2 + \|v\|^2.$$

Then replace u by $e^{i\theta}u$ to deduce

$$2|(u,v)| \le \|u\|^2 + \|v\|^2.$$

Next, replace u by tu and v by $t^{-1}v$, to get

$$2|(u,v)| \le t^2\|u\|^2 + t^{-2}\|v\|^2,$$

for any $t > 0$. Picking t so that $t^2 = \|v\|/\|u\|$, we have Cauchy's inequality (A.9).

Granted (A.6)–(A.8), we easily get

$$\begin{aligned} \|A\| &\ge 0, \\ \|cA\| &= |c|\ \|A\|, \\ \|A+B\| &\le \|A\| + \|B\|. \end{aligned} \tag{A.10}$$

Also, $\|A\| = 0$ if and only if $A = 0$. The fact that $\|A\|$ is the smallest constant K such that $\|Au\| \le K\|u\|$ gives

$$\|AB\| \le \|A\| \cdot \|B\|. \tag{A.11}$$

In particular,

$$\|A^k\| \le \|A\|^k. \tag{A.12}$$

This makes it easy to check convergence of the power series (A.3).

Power series manipulations can be used to establish the identity

$$e^{sA}e^{tA} = e^{(s+t)A}. \tag{A.13}$$

Another way to prove this is the following. Regard t as fixed; denote the left side of (A.13) as $X(s)$ and the right side as $Y(s)$. Then differentiation with respect to s gives, respectively

$$\begin{aligned} X'(s) &= AX(s), \quad X(0) = e^{tA}, \\ Y'(s) &= AY(s), \quad Y(0) = e^{tA}, \end{aligned} \tag{A.14}$$

so uniqueness of solutions to the ODE implies $X(s) = Y(s)$ for all s. We note that (A.13) is a special case of the following.

Proposition A.1. *$e^{t(A+B)} = e^{tA}e^{tB}$ for all t, if and only if A and B commute.*

Proof. Let

$$Y(t) = e^{t(A+B)}, \quad Z(t) = e^{tA}e^{tB}. \tag{A.15}$$

Note that $Y(0) = Z(0) = I$, so it suffices to show that $Y(t)$ and $Z(t)$ satisfy the same ODE, to deduce that they coincide. Clearly

$$Y'(t) = (A+B)Y(t). \tag{A.16}$$

Meanwhile

$$Z'(t) = Ae^{tA}e^{tB} + e^{tA}Be^{tB}. \tag{A.17}$$

Thus we get the equation (A.16) for $Z(t)$ provided we know that

$$e^{tA}B = Be^{tA} \ \text{ if } \ AB = BA. \tag{A.18}$$

This follows from the power series expansion for e^{tA}, together with the fact that

$$A^kB = BA^k \text{ for all } k \geq 0, \text{ if } AB = BA. \tag{A.19}$$

For the converse, if $Y(t) = Z(t)$ for all t, then $e^{tA}B = Be^{tA}$, by (A.17), and hence, taking the t-derivative, $e^{tA}AB = BAe^{tA}$; setting $t = 0$ gives $AB = BA$.

If A is in diagonal form

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \tag{A.20}$$

then clearly

$$e^{tA} = \begin{pmatrix} e^{ta_1} & & \\ & \ddots & \\ & & e^{ta_n} \end{pmatrix} \tag{A.21}$$

The following result makes it useful to diagonalize A in order to compute e^{tA}.

Proposition A.2. *If K is an invertible matrix and $B = KAK^{-1}$, then*

$$e^{tB} = K\, e^{tA}\, K^{-1}. \tag{A.22}$$

Proof. This follows from the power series expansion (A.3), given the observation that

$$B^k = K\, A^k\, K^{-1}. \tag{A.23}$$

There are a number of ways to show that

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Longrightarrow e^{tJ} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{A.24}$$

We invite the reader to find one or more demonstrations of this.

B. A formula for $d\alpha$

We recall from §7 of [T] that, if α is a k-form on an open set $\mathcal{O} \subset \mathbb{R}^n$, given by

$$\alpha = \sum_j a_j(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k}, \tag{B.1}$$

where the sum is over k-multi-indices $j = (j_1, \dots, j_k),\ 1 \le j_\nu \le n$, and we have the anti-commutation rule $dx_\alpha \wedge dx_\beta = -dx_\beta \wedge dx_\alpha$, then

$$d\alpha = \sum_{j,\ell} \frac{\partial a_j}{\partial x_\ell}\, dx_\ell \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{B.2}$$

Also, in §6 of [T] such a k-form is interpreted as a k-linear functional on vector fields. In particular, as noted there, if β is a 2-form,

$$\beta = \sum_{j,k} b_{jk}(x)\, dx_j \wedge dx_k, \tag{B.3}$$

and we have vector fields

$$U = \sum u^j \frac{\partial}{\partial x_j}, \quad V = \sum v^k \frac{\partial}{\partial x_k}, \tag{B.4}$$

then

$$\beta(U, V) = \sum_{j,k} (b_{jk} - b_{kj}) u^j v^k. \tag{B.5}$$

We look at the exterior derivative of a 1-form

$$\alpha = \sum_k a_k(x)\, dx_k, \tag{B.6}$$

given by

$$d\alpha = \sum_{j,k} \frac{\partial a_k}{\partial x_j}\, dx_j \wedge dx_k. \tag{B.7}$$

We aim to establish the following identity, which is useful in §14 (cf. (14.26)).

Proposition B.1. *If α is a 1-form and U, V vector fields on $\mathcal{O} \subset \mathbb{R}^n$, then*

$$d\alpha(U,V) = U\alpha(V) - V\alpha(U) - \alpha([U,V]). \tag{B.8}$$

Proof. From (B.5) and (B.7) we have

$$d\alpha(U,V) = \sum_{j,k} \Bigl(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k}\Bigr) u^j v^k. \tag{B.9}$$

Since $\alpha(U) = \sum a_j u^j$ and $\alpha(V) = \sum a_k v^k$, we calculate that

$$\begin{aligned} U\alpha(V) &= \sum_{j,k} \Bigl(u^j v^k \frac{\partial a_k}{\partial x_j} + u^j \frac{\partial v^k}{\partial x_j} a_k\Bigr), \\ V\alpha(U) &= \sum_{j,k} \Bigl(u^j v^k \frac{\partial a_j}{\partial x_k} + v^j \frac{\partial u^k}{\partial x_j} a_k\Bigr). \end{aligned} \tag{B.10}$$

Meanwhile, using the formula (8.15) for $[U,V]$, we have

$$\alpha([U,V]) = \sum_{j,k} \Bigl(u^j \frac{\partial v^k}{\partial x_j} - v^j \frac{\partial u^k}{\partial x_j}\Bigr) a_k. \tag{B.11}$$

Combining (B.9)–(B.11), we have (B.8).

C. Sard's theorem

Let $F : \Omega \to \mathbb{R}^n$ be a C^1 map, with Ω open in $\mathbb{R}^n$. If $p \in \Omega$ and $DF(p) : \mathbb{R}^n \to \mathbb{R}^n$ is not surjective, then p is said to be a *critical point*, and $F(p)$ a *critical value*. The set C of critical points can be a large subset of Ω, even all of it, but the set of critical values $F(C)$ must be small in $\mathbb{R}^n$. This is part of Sard's Theorem.

Theorem C.1. *If $F : \Omega \to \mathbb{R}^n$ is a C^1 map, then the set of critical values of F has measure 0 in $\mathbb{R}^n$.*

Proof. If $K \subset \Omega$ is compact, cover $K \cap C$ with m-dimensional cubes Q_j, with disjoint interiors, of side δ_j. Pick $p_j \in C \cap Q_j$, so $L_j = DF(p_j)$ has rank $\le n-1$. Then, for $x \in Q_j$,

$$F(p_j + x) = F(p_j) + L_j x + R_j(x), \quad \|R_j(x)\| \le \rho_j = \eta_j \delta_j, \tag{C.1}$$

where $\eta_j \to 0$ as $\delta_j \to 0$. Now $L_j(Q_j)$ is certainly contained in an $(n-1)$-dimensional cube of side $C_0\delta_j$, where C_0 is an upper bound for $\sqrt{m}\|DF\|$ on K. Since all points of $F(Q_j)$ are a distance $\le \rho_j$ from (a translate of) $L_j(Q_j)$, this implies

$$\text{meas } F(Q_j) \le 2\rho_j (C_0\delta_j + 2\rho_j)^{n-1} \le C_1 \eta_j \delta_j^n, \tag{C.2}$$

provided δ_j is sufficiently small that $\rho_j \le \delta_j$. Now $\sum_j \delta_j^n$ is the volume of the cover of $K \cap C$. For fixed K this can be assumed to be bounded. Hence

$$\text{meas } F(C \cap K) \le C_K \, \eta, \tag{C.3}$$

where $\eta = \max \{\eta_j\}$. Picking a cover by small cubes, we make η arbitrarily small, so meas $F(C \cap K) = 0$. Letting $K_j \nearrow \Omega$, we complete the proof.

Sard's theorem also treats the more difficult case when Ω is open in $\mathbb{R}^m, m > n$. Then a more elaborate argument is needed, and one requires more differentiability, namely that F is class C^k, with $k = m - n + 1$. A proof can be found in [St].

The main application of Sard's theorem in this course is to the proof of the change of variable theorem we present in the next section. Here, we give another application of Sard's theorem, to the existence of lots of Morse functions. This application gives the typical flavor of how one uses Sard's theorem. We begin with a special case:

Proposition C.2. *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in C^\infty(\Omega)$. For $a \in \mathbb{R}^n$, set $f_a(x) = f(x) - a \cdot x$. Then, for almost every $a \in \mathbb{R}^n$, f_a is a Morse function, i.e., it has only nondegenerate critical points.*

Proof. Consider $F(x) = \nabla f(x); F : \Omega \to \mathbb{R}^n$. A point $x \in \Omega$ is a critical point of f_a if and only if $F(x) = a$, and this critical point is degenerate only if, in addition, a is a critical value of F. Hence the desired conclusion holds for all $a \in \mathbb{R}^n$ that are not critical values of F.

Now for the result on manifolds:

Proposition C.3. *Let M be an n-dimensional manifold, imbedded in $\mathbb{R}^K$. Let $f \in C^\infty(M)$, and, for $a \in \mathbb{R}^K$, let $f_a(x) = f(x) - a \cdot x$, for $x \in M \subset \mathbb{R}^K$. Then, for almost all $a \in \mathbb{R}^K$, f_a is a Morse function.*

Proof. Each $p \in M$ has a neighborhood Ω_p such that some n of the coordinates x_ν on $\mathbb{R}^K$ produce coordinates on Ω_p. Let's say $x_1, \dots, x_n$ do it. Let $(a_{n+1}, \dots, a_K)$ be fixed, but arbitrary. Then, by Proposition C.2, for almost every $(a_1, \dots, a_n) \in \mathbb{R}^n$, f_a has only nondegenerate critical points on Ω_p. By Fubini's theorem, we deduce that, for almost every $a \in \mathbb{R}^K$, f_a has only nondegenerate critical points on Ω_p. (The set of bad $a \in \mathbb{R}^K$ is readily seen to be a countable union of closed sets, hence measurable.) Covering M by a countable family of such sets Ω_p, we finish the proof.

D. A Change of Variable Theorem

Theorem D.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $F : \Omega \to \mathbb{R}^n$ be a C^1 map. For $x \in \mathbb{R}^n$ set $n(x) = card F^{-1}(x)$. Then n is measurable and for any measurable $u \geq 0$ on $\mathbb{R}^n$,*

$$\int_{\Omega} u\big(F(x)\big) |\det\, DF(x)|\, dx = \int_{\mathbb{R}^n} u(x)\, n(x)\, dx. \tag{D.1}$$

This is an extension of the standard change of variable formula, in which one assumes that F is a diffeomorphism of Ω onto its image. (Then $n(x)$ is the characteristic function of $F(\Omega)$.) We will make use of this standard result in the proof of the theorem.

We make a sequence of reductions. Let

$$K = \{x \in \Omega : \det\, DF(x) = 0\}, \quad \widetilde{\Omega} = \Omega \setminus K, \quad \widetilde{F} = F\big|_{\widetilde{\Omega}}, \quad \widetilde{n}(x) = \operatorname{card} \widetilde{F}^{-1}(x).$$

Sard's theorem implies $F(K)$ has measure zero. Hence $n(x) = \widetilde{n}(x)$ a.e. on $\mathbb{R}^n$, so if we can show that $\widetilde{n}$ is measurable and (D.1) holds with Ω, F, n replaced by $\widetilde{\Omega}, \widetilde{F}, \widetilde{n}$, we will have the desired result. Thus we will henceforth assume that $\det\, DF(x) \neq 0$ for all $x \in \Omega$.

Next, for $k \in \mathbb{Z}^+$, let

$$D_k = \{x \in \Omega : |x| \leq k,\ \operatorname{dist}(x, \partial\Omega) \geq 1/k\}.$$

Each D_k is a compact subset of Ω. Furthermore, $D_k \subset D_{k+1}$ and $\Omega = \cup D_k$. Let $F_k = F\big|_{D_k}$ and $n_k(x) = \operatorname{card} F_k^{-1}(x)$. Then, for each $x \in \mathbb{R}^n$, $n_k(x) \nearrow n(x)$ as $k \to \infty$. Suppose we can show that, for each k, n_k is measurable and

$$\int_{D_k} u\big(F(x)\big)\ |\det\, DF(x)|\, dx = \int_{\mathbb{R}^n} u(x)\, n_k(x)\, dx. \tag{D.2}$$

Now the monotone convergence theorem implies that, as $k \to \infty$ the left side of (D.2) tends to the left side of (D.1) and the right side of (D.2) tends to the right side of (D.1). Hence (D.1) follows from (D.2), so it remains to prove (D.2) (and the measurability of n_k).

For notational simplicity, drop the index k. We have a compact set $D \subset \mathbb{R}^n$, F is C^1 on a neighborhood Ω of D, and $\det\, DF(x)$ is nowhere vanishing. We set $n_D(x) = \operatorname{card} D \cap F^{-1}(x)$ and desire to prove that n_D is measurable and

$$\int_{D} u\big(F(x)\big)\, |\det\, DF(x)|\, dx = \int_{\mathbb{R}^n} u(x)\, n_D(x)\, dx. \tag{D.3}$$

By the inverse function theorem, each $x \in \Omega$ has a neighborhood $\mathcal{O}_x$ such that F is a diffeomorphism of $\mathcal{O}_x$ onto its image. Since D is compact, we can cover D with a finite number of open sets $\mathcal{O}_j$ on each of which F is a diffeomorphism. Then there exists $\delta > 0$ such that any subset of D of diameter $\le \delta$ is contained in one of these sets $\mathcal{O}_j$. Tile $\mathbb{R}^n$ with closed cubes of edge $\delta/\sqrt{n}$ (so they intersect only along their faces). Let $\{Q_k : 1 \le k \le N\}$ denote those cubes that intersect D, and let $E_k = D \cap Q_k$. Then E_k is compact and the standard change of variables theorem gives

$$\int\limits_{E_k} u\bigl(F(x)\bigr)\,|\det DF(x)|\,dx = \int\limits_{F(E_k)} u(x)\,dx. \tag{D.4}$$

If we sum the left side of (D.4) over k we get the left side of (D.3). Since the faces of each cube, and also their images under F, all have measure zero, we also have

$$n_D(x) = \operatorname{card}\{k : x \in F(E_k)\} = \sum_{k=1}^{N} \chi_{F(E_k)}(x) \quad \text{a.e. on } \mathbb{R}^n.$$

Since $F(E_k)$ is compact, this proves that n_D is measurable, and also implies that if we sum the right side of (D.4) over k we get the right side of (D.3). The theorem is hence proven.

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