Notes on Compact Riemann Surfaces

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Contents

1. Introduction

These notes began as a companion to §§30–34 of [T2], which develops the basic theory of elliptic functions and elliptic integrals. Material there starts with the Weierstrass $\wp$ function, defined as a double series, shows how this and related functions can be represented in terms of theta functions, and shows how elliptic integrals can be represented in terms of inverses of $\wp$ functions. This last result is known as the solution to the Abel inversion problem, and the route taken to do this in [T2] involves constructing the Riemann surface of $\sqrt{q(\zeta)}$, whenever $q(\zeta)$ is a polynomial of degree 3 or 4, with distinct roots.

We continue this theme in §2 of these notes, constructing the Riemann surface M of $\sqrt{q(\zeta)}$ for a polynomial $q(\zeta)$ of degree ≥ 5, with distinct roots, and investigating certain properties, such as its topology (it has genus g if $q(\zeta)$ has order $2g+1$ or $2g+2$), and a basis of holomorphic 1-forms on M. Such surfaces constitute the family of hyperelliptic surfaces. We record the well known fact that each compact Riemann surface of genus 2 is hyperelliptic. (The surfaces dealt with in [T2] have genus 1 and are called elliptic.) This result uses the Riemann-Roch theorem, discussed in Appendix A, which is of fundamental use in subsequent sections. At this point, these notes acquire the role of also being a companion to §9 of [T1].

In §3 we consider general compact Riemann surfaces. The analogue of the Abel inversion problem in this context (which for genus 1 was to present an isomorphism of M with a torus $\mathbb{C}/\Lambda$) is to map M into a complex g-dimensional torus $\mathbb{C}^g/\Lambda$, called the Jacobi variety and denoted $J(M)$, establish that this is an embedding, and examine the image, both of M and of k-fold products of M (especially for $k = g-1$ and $k = g$). This mapping $\Phi : M \to J(M)$ is constructed in §3 and shown there to be an immersion. The presentation in §3 and subsequent sections draws heavily from two books on Riemann surfaces, [FK] and, particularly, [N].

In §4 we prove Abel's theorem, characterizing which divisors on M are divisors of a meromorphic function, and use this to show that the map $\Phi : M \to J(M)$, shown to be a immersion in §3, is also one-to-one, and hence an embedding.

In §5 we define the Riemann theta function ϑ and related theta functions. These are functions on $\mathbb{C}^g$ with special properties under translation by elements of Λ that allow one to regard them as holomorphic sections of certain line bundles over $\mathbb{C}^g/\Lambda = J(M)$. In §6 we present basic results on the divisor of ϑ, and in §7 we present Riemann's factorization theorem, expressing a general meromorphic function on M as a product of factors of the form

$$\frac{\vartheta(\Phi(z) - \Phi(p_k) - \xi)}{\vartheta(\Phi(z) - \Phi(q_k) - \xi)}$$

times the exponential of $\int_{p_0}^{z} \zeta$, for some holomorphic 1-form on M, times a constant.

In addition to Appendix A, mentioned above, there are several other appendices, containing either background or complementary results. Appendix B discusses special divisors, which is useful for §6. Appendix C has further commentary on the factor $\exp \int_{p_0}^{z} \zeta$, arising in §4 and in §7, and relates the material there to the question of examining certain flat complex line bundles over M and determining whether they are holomorphically trivial. Appendices D and E draw parallels between $\overline{\partial}$ and $d \oplus \delta$ and discuss connections with Hodge theory. Appendix F discusses proofs of the Uniformization Theorem for compact Riemann surfaces, and gives two proofs when $g = 0$ and when $g = 1$, one set of proofs using the Riemann-Roch theorem (standard) and the other using basic linear PDE (the treatment for $g = 0$ being perhaps not well known).

Appendix G discusses holomorphic maps $\varphi_L : M \to \mathbb{CP}^{k-1}$ defined for certain holomorphic line bundles $L \to M$, with emphasis on the canonical map φ_κ, which one gets when $L = \kappa$ is the canonical bundle and M has genus $g \geq 2$. It is shown that the canonical map is an embedding when M is not hyperelliptic. Appendix H produces a basis for the canonical ring $\oplus_{n \geq 0} \mathcal{O}(\kappa^n)$ in case M has genus 2; this is a simple counterpoint to the more elaborate study of the canonical ring for non-hyperelliptic M, which can be found in [ACGH].

Appendices I and J deal with holomorphic vector bundles over M, including the extension of the Riemann-Roch theorem to this setting.

2. Hyperelliptic Riemann surfaces

A hyperelliptic Riemann surface is the Riemann surface M of $\sqrt{q(\zeta)}$, where $q(\zeta)$ has the form

$$q(\zeta) = (\zeta - e_1) \cdots (\zeta - e_m), \tag{2.1}$$

with $e_j \in \mathbb{C}$ distinct and $m \geq 5$. The cases $m = 2, 3$, and 4 were discussed in §34 of [T2], where the resulting Riemann surfaces M were seen to be $\approx S^2$ for $m = 2$ and tori $\approx \mathbb{C}/\Lambda$ for $m = 3$ or 4. What was seen in §34 of [T2], in case $m \geq 5$, was the following construction of M.

Work on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, and picture $e_j \in S^2$. If $m = 2g + 1$, one has slits from e_{2j-1} to e_{2j}, for $j = 1, \dots, g$, and a slit from e_{2g+1} to ∞, which we denote e_{2g+2}. If $m = 2g + 2$, one has slits from e_{2j-1} to e_{2j} for $j = 1, \dots, g + 1$. Say the slits are geodesics on S^2. (Reorder $\{e_j\}$ if necessary to arrange that these various geodesic segments are disjoint.) Such a slit version of S^2 is a manifold X with $g + 1$ boundary components. Cf. Fig. 1. Now

$$M = X_1 \cup X_2 / \sim, \tag{2.2}$$

where X_1 and X_2 are two copies of X, and the equivalence relation $\sim$ identifies the upper boundary of X_1 along the slit γ_j with the lower boundary of X_2 along this slit and vice-versa, for each j. The identification amounts to gluing together $g + 1$ pairs of pipes, and one gets a g-holed surface, i.e., a surface of genus g, and a natural map

$$\varphi : M \longrightarrow S^2, \tag{2.3}$$

which is two-to-one except for the $2g + 2$ points $p_j = \varphi^{-1}(e_j)$. The space M has a unique Riemann surface structure for which φ is holomorphic. If $e_j \neq \infty$, a coordinate taking a neighborhood of p_j in M bijectively onto a neighborhood of the origin in $\mathbb{C}$ is given by $\varphi_j(x) = (\varphi(x) - e_j)^{1/2}$. If $e_j = \infty$, a coordinate taking a neighborhood of p_j in M bijectively onto a neighborhood of the origin in $\mathbb{C}$ is given by $\varphi_j(x) = \varphi(x)^{-1/2}$.

As noted in §34 of [T2], the pull-back by φ of the double valued form $d\zeta/\sqrt{q(\zeta)}$ gives a single valued holomorphic 1-form α on M, whenever $m \geq 3$. If $m = 3$ or 4, this form is nowhere vanishing on M. These results are consequences of the facts that

$$w^2 = \zeta \Longrightarrow \frac{d\zeta}{\sqrt{\zeta}} = 2\, dw, \tag{2.4}$$

and

$$w^2 = \frac{1}{\zeta} \Longrightarrow \frac{d\zeta}{\sqrt{\zeta^3}} = -2\,dw. \tag{2.5}$$

If $m \geq 5$, one sees that α does vanish on $\varphi^{-1}(\infty)$, though it is nonzero elsewhere on M. To see the order of vanishing at points in $\varphi^{-1}(\infty)$, one can compute as follows. If $m = 2g+1,\ g \geq 1$, use

$$w^2 = \frac{1}{\zeta} \Longrightarrow \frac{d\zeta}{\zeta^{g+1/2}} = -2w^{2g-2}\,dw, \tag{2.6}$$

and if $m = 2g+2,\ g \geq 1$, use

$$w = \frac{1}{\zeta} \Longrightarrow \frac{d\zeta}{\zeta^{g+1}} = -w^{g-1}\,dw. \tag{2.7}$$

One can obtain other holomorphic 1-forms on M (i.e., elements of $\mathcal{O}(\kappa)$) as follows. If $m = 2g+1$, we see from (2.6) that

$$\zeta^j\,\frac{d\zeta}{\zeta^{g+1/2}} = -2w^{2g-2-2j}\,dw, \tag{2.8}$$

and if $m = 2g+2$, we see from (2.7) that

$$\zeta^j\,\frac{d\zeta}{\zeta^{g+1}} = -w^{g-1-j}\,dw. \tag{2.9}$$

Hence, in both cases,

$$\varphi(z)^j\,\alpha \in \mathcal{O}(\kappa), \quad 0 \leq j \leq g-1. \tag{2.10}$$

Such a form vanishes on $\varphi^{-1}(\infty)$ to order $2g-2-2j$ if $m = 2g+2$, and to order $g-1-j$ if $m = 2g+1$. In either case, $\varphi(z)^{g-1}\,\alpha$ is nonvanishing at $\varphi^{-1}(\infty)$, though of course it vanishes on $\varphi^{-1}(0)$.

It is clear from (2.8)–(2.9) that the set

$$\{\varphi(z)^j\,\alpha : 0 \leq j \leq g-1\} \subset \mathcal{O}(\kappa) \tag{2.11}$$

is linearly independent. Here, following standard usage, we denote by $\mathcal{O}(\kappa)$ the space of holomorphic 1-forms on M. More generally, if $L \to M$ is a complex (holomorphic) line bundle over M, $\mathcal{O}(L)$ denotes the space of holomorphic sections of L.

It is known that for a general compact Riemann surface M,

$$\text{Genus } M = g \Longrightarrow \dim \mathcal{O}(\kappa) = g. \tag{2.12}$$

See (A.13) and (D.16) for two different proofs. Thus for hyperelliptic M we see that (2.11) gives a basis of $\mathcal{O}(\kappa)$.

An initially more general looking definition of hyperelliptic Riemann surfaces is the class of compact Riemann surfaces M that have a holomorphic map $\varphi : M \to S^2$ that is a two-to-one branched cover. Actually, this is the same class as described above. To see this, suppose the branch points in M are $\{p_j : 1 \le j \le V\}$, branching over $\{e_j : 1 \le j \le V\}$. Connecting various of the points e_j by geodesic segments on S^2, one can divide S^2 into polygonal regions (say F of them), with E edges and V vertices, namely $\{e_j : 1 \le j \le V\}$. Then the preimages under φ partition M into polygonal regions, with $2F$ faces, $2E$ edges, and V vertices, namely $\{p_j : 1 \le j \le V\}$. Euler's formula gives

$$\begin{aligned} V - E + F &= \chi(S^2) = 2, \\ V - 2E + 2F &= \chi(M) = 2 - 2g, \end{aligned} \tag{2.13}$$

where g denotes the genus of M. Hence $E - F = 2g$ and

$$V = 2g + 2. \tag{2.14}$$

Thus we are in the situation described earlier, and M is biholomorphically diffeomorphic to the Riemann surface constructed by slitting S^2 and forming (2.2).

There are lots of Riemann surfaces that are not hyperelliptic. However, the following result is of interest.

Proposition 2.1. *Each compact Riemann surface of genus 2 is hyperelliptic.*

Proof. If M has genus > 0, to show M is hyperelliptic it suffices to show that there is a nonconstant meromorphic function on M with poles contained in $\{p, q\}$, for some $p, q \in M$, more precisely, a nonconstant $u \in \mathcal{M}(-p-q)$. Such u must have at least one pole, by the maximum principle, and it cannot have just one simple pole since $g \neq 0$ ([T1], Proposition 9.8), so such $u : M \to S^2$ must be a 2-to-1 branched cover. We have

$$\mathcal{M}(-p-q) \approx \mathcal{O}(E_{-p-q}), \tag{2.15}$$

where E_{-p-q} is the bundle of a divisor ([T1], (9.41)), so the goal becomes to show that

$$\dim \mathcal{O}(E_{-p-q}) \ge 2. \tag{2.16}$$

To get this, we apply the Riemann-Roch theorem, which yields

$$\dim \mathcal{O}(E_{-p-q}) - \dim \mathcal{O}(E_{p+q} \otimes \kappa) = c_1(E_{-p-q}) - \frac{1}{2} c_1(\kappa); \tag{2.17}$$

cf. [T1], (9.17). We also have ([T1], (9.19), (9.39))

$$c_1(\kappa) = 2g - 2, \quad c_1(E_{-p-q}) = 2. \tag{2.18}$$

Hence

$$\dim \mathcal{O}(E_{-p-q}) = \dim \mathcal{O}(\kappa \otimes E_{p+q}) + 3 - g. \tag{2.19}$$

We now investigate

$$\mathcal{O}(E_{p+q} \otimes \kappa) \approx \mathcal{M}(\kappa, p+q), \tag{2.20}$$

the space of holomorphic 1-forms with zeros at p and q. This is a linear subspace of $\mathcal{O}(\kappa)$, a space of dimension $g = 2 > 0$. Take a nonzero $u \in \mathcal{O}(\kappa)$. If $\nu_p(u)$ denotes the order of vanishing of u at p, one has

$$\sum_p \nu_p(u) = c_1(\kappa) = 2. \tag{2.21}$$

Cf. [T1], (9.28). Say u vanishes at p and q; one writes $\vartheta(u) = p + q$. If $p = q$, this means u vanishes to second order at p. We see that, for $g = 2$,

$$\exists\, p, q \in M \quad \text{such that} \dim \mathcal{O}(E_{p+q} \otimes \kappa) \geq 1. \tag{2.22}$$

This combined with (2.19) gives (2.16) and finishes the proof of Proposition 2.1.

Remark. If M is a compact Riemann surface for which dim $\mathcal{M}(-p-q) > 2$, then M must be equivalent to S^2 and dim $\mathcal{M}(-p-q) = 3$. Consequently,

$$g \geq 1 \Longrightarrow \dim \mathcal{M}(-p-q) \leq 2, \quad \forall\, p, q \in M. \tag{2.23}$$

Part of the proof of Proposition 2.1 shows that (for $g \geq 2$)

$$M \ \text{ is hyperelliptic} \iff \text{dim } \mathcal{M}(-p-q) = 2, \ \text{for some } p, q \in M. \tag{2.24}$$

The rest of the proof of Proposition 2.1 is the demonstration that

$$g = 2 \Longrightarrow \dim \mathcal{M}(-p-q) = 2, \ \text{for some } p, q \in M. \tag{2.25}$$

3. Holomorphic differentials, period lattice, and Jacobi variety

Results of §2 illustrate, for hyperelliptic Riemann surfaces, the following, valid for a general compact Riemann surface M of genus $g \geq 1$. Recall that $\mathcal{O}(\kappa)$ denotes the space of holomorphic 1-forms on M. This has dimension g.

Proposition 3.1. *Given $p \in M$, there exists $\omega \in \mathcal{O}(\kappa)$ such that $\omega(p) \neq 0$.*

We give a proof using the Riemann-Roch formula. We consider

$$\mathcal{M}(\kappa, p) = \{u \in \mathcal{O}(\kappa) : u(p) = 0\} \approx \mathcal{O}(\kappa \otimes E_p), \tag{3.1}$$

where E_p is a point bundle (cf. [T1], (9.41)). The Riemann-Roch formula ([T1], (9.17); see also Appendix A) gives

$$\dim \mathcal{O}(\kappa \otimes E_p) - \dim \mathcal{O}(\kappa^{-1} \otimes E_p^{-1} \otimes \kappa) = c_1(\kappa \otimes E_p) - \frac{1}{2} c_1(\kappa). \tag{3.2}$$

We also have ([T1], (9.19), (9.39))

$$c_1(\kappa) = 2g - 2, \quad c_1(E_p) = -1, \quad c_1(\kappa \otimes E_p) = c_1(\kappa) + c_1(E_p). \tag{3.3}$$

Also $\mathcal{O}(\kappa^{-1} \otimes E_p^{-1} \otimes \kappa) \approx \mathcal{O}(E_p^{-1}) = \mathcal{M}(-p)$ ([T1], (9.41)) and

$$\begin{aligned} \dim \mathcal{M}(-p) = 2 \quad &\text{if} \quad g = 0 \\ 1 \quad &\text{if} \quad g \geq 1. \end{aligned} \tag{3.4}$$

(Cf. [T1], Proposition 9.9.) Consequently,

$$g \geq 1 \Longrightarrow \dim \mathcal{M}(\kappa, p) = g - 1, \tag{3.5}$$

so $\mathcal{M}(\kappa, p)$ is a proper linear subspace of $\mathcal{O}(\kappa)$. This proves Proposition 3.1.

We comment further on results behind (2.12). Let $\{\zeta_j : 1 \leq j \leq g\}$ be a basis of $\mathcal{O}(\kappa)$. Then $\{\overline{\zeta}_j : 1 \leq j \leq g\}$ is a basis of $\overline{\mathcal{O}}(\overline{\kappa})$, and the space of complex valued harmonic 1-forms $\mathcal{H}_1^{\mathbb{C}}(M)$ has the direct sum decomposition

$$\mathcal{H}_1^{\mathbb{C}}(M) = \mathcal{O}(\kappa) \oplus \overline{\mathcal{O}}(\overline{\kappa}). \tag{3.6}$$

Equivalently, with $u_j = \operatorname{Re} \zeta_j,\ v_j = \operatorname{Im} \zeta_j$, $\{u_j, v_j : 1 \leq j \leq g\}$ is a basis of $\mathcal{H}_1(M)$. Using this, we can prove the following. Let $\{\gamma_j : 1 \leq j \leq 2g\}$ denote a set of closed curves giving a basis of $H_1(M, \mathbb{Z})$, and form

$$V_j = \Bigl(\int_{\gamma_j} \zeta_1, \dots, \int_{\gamma_j} \zeta_g \Bigr) \in \mathbb{C}^g, \quad 1 \leq j \leq 2g. \tag{3.7}$$

Proposition 3.2. *The set $\{V_j : 1 \le j \le 2g\}$ is linearly independent over $\mathbb{R}$.*

Proof. An equivalent formulation is that

$$W_j = \Bigl(\int_{\gamma_j} u_1, \int_{\gamma_j} v_1, \dots, \int_{\gamma_j} u_g, \int_{\gamma_j} v_g\Bigr) \in \mathbb{R}^{2g}, \quad 1 \le j \le 2g \tag{3.8}$$

is linearly independent (over $\mathbb{R}$). This in turn is a special case of deRham's theorem.

Let

$$\Lambda = \Bigl\{\sum_{j=1}^{2g} n_j V_j : n_j \in \mathbb{Z}\Bigr\}. \tag{3.9}$$

This is the period lattice associated to the basis $\{\zeta_1, \dots, \zeta_g\}$ of $\mathcal{O}(\kappa)$. Proposition 2.2 implies that $\mathbb{C}^g/\Lambda$ is compact. We have a holomorphic mapping

$$\Phi : M \longrightarrow \mathbb{C}^g/\Lambda, \tag{3.10}$$

given by picking $z_0 \in M$ and setting

$$\Phi(z) = \Bigl(\int_{z_0}^{z} \zeta_1, \dots, \int_{z_0}^{z} \zeta_g\Bigr). \tag{3.11}$$

The g-dimensional torus $\mathbb{C}^g/\Lambda$ is called the Jacobi variety associated to M, and sometimes denoted $J(M)$. The following is a central result.

Theorem 3.3. *If $g \ge 1$, the map Φ in (3.10)–(3.11) is a holomorphic embedding of M into $\mathbb{C}^g/\Lambda$.*

In case $g = 1$, the map Φ is a holomorphic diffeomorphism. This was demonstrated in §34 of [T2], given that M is the Riemann surface of $\sqrt{q(\zeta)}$ with $q(\zeta)$ as in (2.1), with $m = 3$ or 4. In fact, given Proposition 3.1 here, the proof of Proposition 34.1 in [T2] works under the general hypothesis that $g = 1$.

At this point we have for $g \ge 1$ that Φ is well defined and holomorphic. Also Proposition 3.1 guarantees that $\Phi'(z) \ne 0$ for each $z \in M$, so Φ is an immersion. It remains to show that Φ is one-to-one. This is a special case of Abel's theorem, which we will discuss in §4.

Here is a more invariant formulation of the holomorphic map (3.10). We can define a homomorphism of additive groups

$$\iota : H_1(M, \mathbb{Z}) \to \mathcal{O}(\kappa)', \quad \iota(\gamma)(\zeta) = \int_{\gamma} \zeta. \tag{3.12}$$

Then Proposition 3.2 can be formulated as saying that ι is injective, and produces a lattice $\Lambda = \iota H_1(M,\mathbb{Z})$, for which $\mathcal{O}(\kappa)'/\Lambda$ is compact, and we have a holomorphic mapping

$$\Phi : M \longrightarrow \mathcal{O}(\kappa)'/\Lambda, \tag{3.13}$$

given by picking $z_0 \in M$ and setting

$$\Phi(z)(\zeta) = \int_{z_0}^{z} \zeta, \quad \zeta \in \mathcal{O}(\kappa). \tag{3.14}$$

Having this invariant formulation, we see we are free to pick a convenient basis of $\mathcal{O}(\kappa)$ and of $H_1(M,\mathbb{Z})$ in order to establish Theorem 3.3.

A convenient choice of basis of $H_1(M,\mathbb{Z})$ for a Riemann surface M of genus g has the form $\{a_j, b_j : 1 \le j \le g\}$. This is pictured in Fig. 2 in case $g = 2$. If φ is a smooth 1-form on M, we set

$$A_k(\varphi) = \int\limits_{a_k} \varphi, \quad B_k(\varphi) = \int\limits_{b_k} \varphi. \tag{3.15}$$

The following is an important result of Riemann.

Proposition 3.4. *If $\zeta \in \mathcal{O}(\kappa)$, $\zeta \ne 0$, and if M has genus $g > 0$, then*

$$Im \sum_{k=1}^{g} A_k(\zeta)\overline{B_k(\zeta)} < 0. \tag{3.16}$$

See [N], p. 80, for a proof. Note the following:

Corollary 3.5. *The map*

$$\mathcal{A} : \mathcal{O}(\kappa) \longrightarrow \mathbb{C}^g, \quad \mathcal{A}(\zeta) = \Big(\int\limits_{a_1} \zeta, \dots, \int\limits_{a_g} \zeta\Big) \tag{3.17}$$

is injective, hence an isomorphism.

It follows that there is a basis $\{\zeta_j : 1 \le j \le g\}$ of $\mathcal{O}(\kappa)$, called a *normalized basis*, such that

$$\int\limits_{a_k} \zeta_j = \delta_{jk}, \quad 1 \le j,k \le g. \tag{3.18}$$

From here on we work with such a normalized basis. The following result constitutes Riemann's bilinear relations.

Proposition 3.6. *If (3.18) holds, then if*

$$B_{jk} = \int_{b_j} \zeta_k, \quad 1 \leq j, k \leq g, \tag{3.19}$$

the matrix $B = (B_{jk})$ is symmetric, and its imaginary part is positive definite.

For a proof, see [N], p. 81.

With the choices of $\gamma_j = a_j$ for $1 \leq j \leq g$, b_{j-g} for $g+1 \leq j \leq 2g$, and ζ_j as in (3.18), we see that V_j in (3.7) are given by

$$V_j = E_j, \quad V_{g+j} = B_j, \quad 1 \leq j \leq g. \tag{3.20}$$

where $\{E_j : 1 \leq j \leq g\}$ is the standard basis of $\mathbb{R}^g$ and $B_j = (B_{j1}, \dots, B_{jg})$.

Another ingredient in the proof of Theorem 3.3 is the following result, of intrinsic interest, concerning the conditions of existence of meromorphic 1-forms on M.

Proposition 3.7. *Given distinct $p_1, \dots, p_K \in M$ with disjoint neighborhoods $\mathcal{O}_j$ and meromorphic 1-forms u_j on $\mathcal{O}_j$, with poles at p_j, there exists a meromorphic 1-form u on M (we write $u \in \mathcal{M}(\kappa)$), such that $u - u_j$ is a holomorphic 1-form on $\mathcal{O}_j$, if and only if*

$$\sum_{j=1}^{K} Res_{p_j}(u_j) = 0. \tag{3.21}$$

Proof. We use the fact that the elliptic differential operator

$$\overline{\partial}_\kappa : H^s(M, \kappa) \longrightarrow H^{s-1}(M, \kappa \otimes \overline{\kappa}) \approx H^{s-1}(M) \tag{3.22}$$

is Fredholm, with closed range, of finite codimension, whose orthogonal complement is the null space of

$$\overline{\partial}_\kappa^t = -\overline{\partial} : H^{1-s}(M) \to H^{-s}(M, \overline{\kappa}). \tag{3.23}$$

Clearly this null space is the space of holomorphic functions on M, i.e., constants. Thus the range of $\overline{\partial}_\kappa$ in (3.22) has codimension 1. Thus a family of distributions with support on $\{p_1, \dots, p_K\}$ in $H^{-s}(M, \overline{\kappa})$ (for s large enough) belongs to the range of $\overline{\partial}_\kappa$ in (3.22) provided one linear relation is satisfied. It is relatively easily verified that for such a linear combination of distributions, (3.21) defines that linear relation. In fact, one can take a smoothly bounded open set $\mathcal{O}$ containing $\{p_j\}$, diffeomorphic to a disk, and deform $\partial\mathcal{O}$ to $a_1 \cup b_1 \cup a_1^{-1} \cup b_1^{-1} \cup \cdots \cup a_g \cup b_g \cup a_g^{-1} \cup b_g^{-1}$.

Corollary 3.8. *Given distinct $p, q \in M$, there exists $u \in \mathcal{M}(\kappa)$, regular on $M \setminus \{p, q\}$, with first order poles at p and q, such that*

$$\text{Res}_p\, u = 1, \quad \text{Res}_q\, u = -1. \tag{3.24}$$

Using Corollary 3.5, we see we can take such u and add an element of $\mathcal{O}(\kappa)$ to obtain an element we denote $\tau_{pq} \in \mathcal{M}(\kappa)$, holomorphic on $M \setminus \{p, q\}$, with first order poles at p and q, satisfying (3.24) and

$$\int_{a_j} \tau_{pq} = 0, \quad 1 \le j, k \le g. \tag{3.25}$$

Here is another proof of Corollary 3.8, using the Riemann-Roch formula. Consider the space of meromorphic sections of κ, with poles in $\{p, q\}$, of order ≤ 1,

$$\mathcal{M}(\kappa, -p - q) \approx \mathcal{O}(\kappa \otimes E_{-p-q}). \tag{3.26}$$

The Riemann-Roch formula gives

$$\begin{aligned} \dim \mathcal{O}(\kappa \otimes E_{-p-q}) - \dim \mathcal{O}(\kappa^{-1} \otimes E_{p+q} \otimes \kappa) &= c_1(\kappa \otimes E_{-p-q}) - \frac{1}{2} c_1(\kappa) \\ &= g - 1 + c_1(E_{-p-q}) \\ &= g + 1, \end{aligned} \tag{3.27}$$

since $c_1(E_{-p-q}) = 2$ ([T1], (9.39)). Also $\mathcal{O}(E_{p+q}) = 0$, so

$$\dim \mathcal{O}(\kappa \otimes E_{-p-q}) = g + 1, \tag{3.28}$$

or equivalently

$$\dim \mathcal{M}(\kappa, -p - q) = \dim \mathcal{O}(\kappa) + 1. \tag{3.29}$$

Thus there exists $u \in \mathcal{M}(\kappa, -p - q)$, $u \notin \mathcal{O}(\kappa)$. The final argument in the proof of Proposition 3.7 shows the residues sum to 0, so multiplication by a constant achieves (3.24).

4. Abel's Theorem and the Jacobi inversion problem

We prepare to state Abel's theorem, which was mentioned after Theorem 3.3 as a central ingredient in the proof of that imbedding result. The theorem involves the notion of a divisor.

By a divisor on M we mean a formal sum

$$\vartheta = \sum_{p_j} n_j p_j, \tag{4.1}$$

where $n_j \in \mathbb{Z}$ and $p_j \in M$, with equivalence relations $p_1 + p_2 = p_2 + p_1$ and $mp_j + np_j = (m+n)p_j$. Given a meromorphic function $u \in \mathcal{M}$, we say its divisor $\vartheta(u)$ is as in (4.1) (with p_j distinct) where if $n_j > 0$, u has a zero of order n_j at p_j and if $n_j < 0$, u has a pole of order $-n_j$ at p_j. An easy degree theory argument shows that $\sum_j n_j = 0$ whenever ϑ is the divisor of $u \in \mathcal{M}(M)$. More generally, if u is a meromorphic section of a complex line bundle L, $\vartheta(u)$ is defined similarly, and one has $\sum_j n_j = c_1(L)$ ([T1], (9.28)).

Theorem 4.1. *(Abel's Theorem) Given a divisor $\vartheta = \sum_j n_j p_j$, there exists $u \in \mathcal{M}(M)$ such that $\vartheta(u) = \vartheta$ if and only if $\sum n_j = 0$ and*

$$\sum n_j \Phi(p_j) = 0 \quad (mod\ \Lambda), \tag{4.2}$$

where Φ is as in (3.10)–(3.11).

Before discussing the proof, we show how Theorem 4.1 implies injectivity of $\Phi : M \to \mathbb{C}^g/\Lambda$ for $g \geq 1$. Indeed, if $p \neq q$ and $\Phi(p) = \Phi(q)$, Theorem 4.1 implies that there exists $u \in \mathcal{M}(M)$ such that $\vartheta(u) = p - q$, so u has a single simple pole. This implies that $u : M \to S^2$ is a holomorphic diffeomorphism ([T1], Proposition 9.9), so $g = 0$. Thus Theorem 3.3 is proven (modulo the proof of Theorem 4.1).

To show that (4.2) holds whenever $\vartheta = \vartheta(u)$ for some $u \in \mathcal{M}(M)$, we use an argument from [FK]. Namely, consider

$$\psi : S^2 \longrightarrow \mathbb{C}^g/\Lambda, \quad \psi(\zeta) = \sum_{p \in u^{-1}(\zeta)} \Phi(p), \tag{4.3}$$

where $\{p \in u^{-1}(\zeta)\}$ is counted with multiplicity. One sees that ψ is holomorphic. Since S^2 is simply connected, ψ lifts to a holomorphic map $\tilde{\psi} : S^2 \to \mathbb{C}^g$, which (by the maximum principle) must be constant. Then (4.2) records the fact that $\psi(0) = \psi(\infty)$.

We next present the converse implication in Theorem 4.1 (which is deeper and more significant). It is convenient to write

$$\vartheta = p_1 + \cdots + p_r - q_1 - \cdots - q_r, \tag{4.4}$$

with perhaps repetition among p_js and among q_js, but with $p_j \neq q_k$. The claim is that if

$$\sum_{j=1}^{r} \Phi(p_j) = \sum_{j=1}^{r} \Phi(q_j) \pmod{\Lambda}, \tag{4.5}$$

then $\vartheta = \vartheta(u)$ for some $u \in \mathcal{M}(M)$. One produces such u in the form

$$u = e^{\int_{z_0}^{z} (\tau_{p_1 q_1} + \cdots + \tau_{p_r q_r} + \zeta)}, \tag{4.6}$$

where τ_{pq} is defined as in Corollary 3.8 and (3.24)–(3.25), and $\zeta \in \mathcal{O}(\kappa)$ is selected to guarantee that (4.6) is single valued on M. (The hypothesis (4.5) is used in producing such ζ.) The condition for such single valuedness of u is that

$$\int_{\gamma} (\tau_{p_1 q_1} + \cdots + \tau_{p_r q_r} + \zeta) \in 2\pi i \mathbb{Z}, \tag{4.7}$$

for each closed curve γ in M. It suffices to arrange this for $\gamma = a_j$ and for $\gamma = b_j$, $1 \leq j \leq g$. We set

$$\zeta = 2\pi i (c_1 \zeta_1 + \cdots + c_g \zeta_g), \tag{4.8}$$

with $\zeta_j \in \mathcal{O}(\kappa)$ as in (3.18), and the task is to produce constants c_j such that (4.7) holds.

Let us set $\theta = \tau_{p_1 q_1} + \cdots + \tau_{p_r q_r} + \zeta$. Note that, by (3.18) and (3.25),

$$\int_{a_j} \theta = 2\pi i c_j. \tag{4.9}$$

To evaluate $\int_{b_j} \theta$, we use the following "reciprocity formula":

Lemma 4.2. *We have*

$$\int_{b_j} \tau_{pq} = 2\pi i \int_q^p \zeta_j, \tag{4.10}$$

the integral taken along a curve joining q to p in $M \setminus \cup a_j \setminus \cup b_j$.

For a proof of Lemma 4.2, see [N], pp. 82–83. Given this, we have

$$\int_{b_j} \theta = 2\pi i \sum_{\ell=1}^{r} \int_{q_\ell}^{p_\ell} \zeta_j + 2\pi i \sum_{k=1}^{g} c_k B_{jk}, \tag{4.11}$$

with B_{jk} as in (3.19). Now the hypothesis (4.5) says

$$\sum_{\ell=1}^{r}\int_{q_\ell}^{p_\ell}(\zeta_1,\dots,\zeta_g)\in\Lambda, \tag{4.12}$$

while (using the symmetry $B_{jk}=B_{kj}$) we have

$$\sum_{k=1}^{g}c_k(B_{1k},\dots,B_{gk})=\sum_{k=1}^{g}c_kB_k, \tag{4.13}$$

where $\{B_k : 1\le k\le g\}$ is half the set of generators of Λ given in (3.20). If we write (4.12) as

$$\sum_{\ell=1}^{r}\int_{q_\ell}^{p_\ell}(\zeta_1,\dots,\zeta_g)=\sum_{j=1}^{g}\nu_jE_j+\sum_{j=1}^{g}\mu_jB_j, \tag{4.14}$$

with $\nu_j,\mu_j\in\mathbb{Z}$, we have

$$\Bigl(\int_{b_1}\theta,\dots,\int_{b_g}\theta\Bigr)=2\pi i\sum_{j=1}^{g}\nu_jE_j+2\pi i\sum_{j=1}^{g}(\mu_j+c_j)B_j. \tag{4.15}$$

We see that if we take

$$c_j=-\mu_j \tag{4.16}$$

in (4.8), then (4.7) is achieved. This proves Theorem 4.1.

Here is an application of Abel's theorem. Given a holomorphic line bundle $L\to M$, we can define

$$\Psi(L)\in J(M)=\mathbb{C}^g/\Lambda \tag{4.17}$$

as follows. There is a nontrivial meromorphic section $u\in\mathcal{M}(L)$ ([T1], Proposition 9.2). Given such a section, set

$$\Psi(L)=\sum n_j\Phi(p_j)\pmod{\Lambda},\quad \vartheta(u)=\sum n_jp_j. \tag{4.18}$$

Note that if we pick another nontrivial $v\in\mathcal{M}(L)$, with $\vartheta(v)=\sum m_kq_k$, then u/v is a meromorphic function on M, with divisor $\sum n_jp_j-\sum m_kq_k$, so the easy half of Abel's theorem implies $\sum m_kq_k=\sum n_jp_j \bmod \Lambda$. Hence Ψ in (4.17)–(4.18) is well defined. Using the deeper half of Abel's theorem, we have the following.

Proposition 4.3. *If $L \to M$ is a holomorphic line bundle satisfying*

$$c_1(L) = 0, \quad \Psi(L) = 0 \pmod{\Lambda}, \tag{4.19}$$

then L is holomorphically trivial.

Proof. Pick a nontrivial $u \in \mathcal{M}(L)$; say $\vartheta(u) = \sum n_j p_j$. If (4.19) holds, then there is a meromorphic function f on M such that $\vartheta(f) = \sum n_j p_j$, by Abel's theorem. Then $v = u/f$ is a holomorphic section of L, with no zeros, so it provides a trivialization of L.

Corollary 4.4. *Holomorphic line bundles $L_j \to M$ ($j = 1, 2$) are holomorphically equivalent if and only if*

$$c_1(L_1) = c_1(L_2) \ \text{ and } \ \Psi(L_1) = \Psi(L_2) \pmod{\Lambda}. \tag{4.20}$$

Proof. First, L_1 and L_2 are holomorphically equivalent if and only if $L_1 \otimes L_2^{-1}$ is holomorphically trivial. Note that

$$\Psi(L_1 \otimes L_2^{-1}) = \Psi(L_1) - \Psi(L_2), \tag{4.21}$$

and also $c_1(L_1 \otimes L_2^{-1}) = c_1(L_1) - c_1(L_2)$.

Remark. If E_ϑ is the bundle of a divisor ϑ and $u \in \mathcal{M}(E_\vartheta)$ the "natural" section, we have

$$\vartheta(u) = -\vartheta. \tag{4.22}$$

Hence

$$\Psi(E_\vartheta) = -\Phi(\vartheta), \tag{4.23}$$

where we set

$$\Phi(\vartheta) = \sum m_j \Phi(p_j) \ \text{ for } \ \vartheta = \sum m_j p_j. \tag{4.24}$$

Here is another perspective. The set $\text{Div}(M)$ of divisors on M forms a group. The set $\text{Div}_0(M)$ of divisors of degree zero is a subgroup. Now (4.24) defines

$$\Phi : \text{Div}(M) \longrightarrow J(M) = \mathbb{C}^g/\Lambda. \tag{4.25}$$

This depends a priori on the base point used to define Φ in (3.11). Note however that its restriction to $\text{Div}_0(M)$,

$$\Phi : \text{Div}_0(M) \longrightarrow J(M), \tag{4.26}$$

is independent of the choice of base point, since

$$\int_{z_0}^{p} \zeta_j - \int_{z_0}^{q} \zeta_j = \int_{q}^{p} \zeta_j, \quad (\mathrm{mod}\ \Lambda), \tag{4.27}$$

so

$$\vartheta = p_1 + \cdots + p_k - q_1 - \cdots - q_k \Longrightarrow \Phi(\vartheta) = \sum_{j=1}^{k} \int_{q_j}^{p_j} (\zeta_1, \ldots, \zeta_g). \tag{4.28}$$

The set $\mathrm{Div}_P(M)$ of divisors of degree zero such that $\Phi(\vartheta) = 0$ is a subgroup of $\mathrm{Div}_0(M)$, called the group of principal divisors on M. (Abel's theorem says this is precisely the set of divisors of meromorphic functions on M.) The quotient

$$\mathrm{Pic}(M) = \mathrm{Div}_0(M) / \mathrm{Div}_P(M) \tag{4.29}$$

is called the Picard variety of M. Since $\Phi = 0$ on $\mathrm{Div}_P(M)$, we have an induced map

$$\Phi^b : \mathrm{Pic}(M) \longrightarrow J(M), \tag{4.30}$$

with the following fundamental property:

Proposition 4.5. *The map Φ^b in (4.30) is a group isomorphism.*

Proof. It is clear that Φ^b is a group homomorphism and that it is one-to-one, since in passing from (4.26) to (4.30) we just divided out the kernel. It remains to show that Φ^b is onto in (4.30), or equivalently that Φ is onto in (4.26). The task of proving this, to which we now turn, is known as the Jacobi inversion problem.

To begin, we use Proposition B.3 to produce g distinct points $p_1, \ldots, p_g$ with the property that

$$\omega \in \mathcal{O}(\kappa),\ \omega(p_j) = 0\ \forall\, j \in \{1, \ldots, g\} \Rightarrow \omega \equiv 0. \tag{4.31}$$

Let U_j be coordinate neighborhoods of p_j (diffeomorphic to the disk), let $\{\zeta_1, \ldots, \zeta_g\}$ be a basis of $\mathcal{O}(\kappa)$, and say

$$\zeta_j = \varphi_{jk}(z_k)\, dz_k \quad \text{on} \quad U_k. \tag{4.32}$$

It follows from (4.32) that the $g \times g$ matrix

$$\big(\varphi_{jk}(p_k)\big) \quad \text{is invertible.} \tag{4.33}$$

Now define

$$\widetilde{\Phi} : U_1 \times \cdots \times U_g \longrightarrow \mathbb{C}^g \tag{4.34}$$

by

$$(4.35)\qquad \begin{aligned}\widetilde{\Phi}(z_1,\dots,z_k) &= \Phi(z_1-p_1+\cdots+z_g-p_g)\\ &= \int_{p_1}^{z_1}(\zeta_1,\dots,\zeta_g)+\cdots+\int_{p_g}^{z_g}(\zeta_1,\dots,\zeta_g).\end{aligned}$$

We require the paths from p_j to z_j lie in U_j, so we actually have a map to $\mathbb{C}^g$ rather then merely to $\mathbb{C}^g/\Lambda$. Say $\widetilde{\Phi} = (\widetilde{\Phi}_1,\dots,\widetilde{\Phi}_g)$. Note that

$$(4.36)\qquad \frac{\partial \widetilde{\Phi}_j}{\partial z_k}(p_1,\dots,p_g) = \varphi_{jk}(p_k).$$

Thus the inverse function theorem implies

$$(4.37)\qquad \widetilde{\Phi} \ \text{ maps } \ U_1\times\cdots\times U_g \ \text{ onto a neighborhood } \mathcal{O} \text{ of } \ 0\in J(M)=\mathbb{C}^g/\Lambda.$$

From here the surjectivity of Φ^b in (4.30) follows easily. Take $z\in J(M)=\mathbb{C}^g/\Lambda$. Let $\tilde{z}\in\mathbb{C}^g$ denote a preimage. Then there exists $M\in\mathbb{N}$ such that $\tilde{z}/M\in\mathcal{O}$, given in (4.37). Thus there exist $q_j\in U_j,\ j=1,\dots,g$, such that

$$(4.38)\qquad \widetilde{\Phi}(q_1,\dots,q_g) = M^{-1}\tilde{z},$$

and hence

$$(4.39)\qquad \Phi(M(q_1-p_1+\cdots+q_g-p_g)) = z \pmod{\Lambda}.$$

The proof of Proposition 4.5 is complete.

Noting that $(p_1,\dots,p_g)$ in (4.36) is *fixed*, we have the following corollary. Let $\mathrm{Div}_k^+(M)$ denote the set of positive divisors of degree k, i.e., divisors of the form $p_1+\cdots+p_k$.

Corollary 4.6. *If $g\ge 2$,*

$$(4.40)\qquad \Phi:\mathrm{Div}_g^+(M)\longrightarrow J(M)$$

is surjective.

The following is a useful complement to Corollary 4.6. For a proof, see [N], pp. 87–88.

Proposition 4.7. *In the setting of Corollary 4.6, there exists an analytic set $Y\subset \mathrm{Div}_g^+(M)$, of dimension $<g$, such that*

$$(4.41)\qquad \Phi:\mathrm{Div}_g^+(M)\setminus Y\longrightarrow J(M)\setminus\Phi(Y) \quad \textit{is bijective.}$$

In fact, Y consists precisely of the class of special divisors (of degree g). Furthermore, given $D\in\mathrm{Div}_g^+(M)$, the rank of Φ at D is equal to

$$(4.42)\qquad g+1-\mathit{dim}\,\mathcal{M}(-D),$$

so $Y'=\Phi(Y)$ is an analytic set of dimension $\le g-2$.

5. Theta functions

For the torus $\mathbb{C}/\Lambda$ where Λ is the lattice generated by 1 and τ (with Im $\tau > 0$), there is the Jacobi theta function

$$\vartheta_3(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}, \tag{5.1}$$

discussed in §32 of [T2], whose relation to $\wp(z;\Lambda)$ arises from the formulas

$$\begin{aligned} \vartheta_3(z+1,\tau) &= \vartheta_3(z,\tau), \\ \vartheta_3(z+\tau,\tau) &= e^{-\pi i(\tau+2z)}\vartheta_3(z,\tau), \end{aligned} \tag{5.2}$$

and variants for $\vartheta_j(z,\tau)$, $j = 1,2,4$, given in (32.8)–(32.10) of [T2].

More generally, let $\Lambda \subset \mathbb{C}^g$ be a lattice, with generators as in (3.20), i.e.,

$$\begin{gathered} \Lambda = \Big\{ \sum_{j=1}^{2g} n_j V_j : n_j \in \mathbb{Z} \Big\}, \\ V_j = E_j, \quad V_{g+j} = B_j, \quad 1 \le j \le g, \end{gathered} \tag{5.3}$$

where $E_1, \dots, E_g$ form the standard basis of $\mathbb{R}^g$ and

$$B_j = (B_{j1}, \dots, B_{jg}), \tag{5.4}$$

with B_{jk} as in (3.19), or more generally $B = (B_{jk}) \in M(g,\mathbb{C})$ satisfying

$$B = B^t, \quad \operatorname{Im} B \ \text{ positive definite.} \tag{5.5}$$

Then, generalizing (5.1), we have the Riemann theta function

$$\vartheta(z) = \vartheta(z,B) = \sum_{n\in\mathbb{Z}^g} e^{\pi i n B n^t + 2\pi i n z^t}. \tag{5.6}$$

Here $n, z \in \mathbb{C}^g$ are row vectors, whose transposes n^t, z^t are column vectors. The function $\vartheta(z,B)$ is seen to be holomorphic in $z \in \mathbb{C}^g$ (and in B satisfying (5.5)). Parallel to (5.2), a computation gives

$$\begin{aligned} \vartheta(z+E_k) &= \vartheta(z), \\ \vartheta(z+B_k) &= e^{-\pi i(B_{kk}+2z_k)}\vartheta(z). \end{aligned} \tag{5.7}$$

Another way of writing this is

$$\vartheta(z+m+nB)=e^{-\pi i(nBn^t+2zn^t)}\vartheta(z),\quad m,n\in\mathbb{Z}^g. \tag{5.8}$$

Generalizations of the other Jacobi theta functions $\vartheta_j(z,\tau)$ are defined as follows. Pick $\varepsilon,\varepsilon'\in\mathbb{R}^g$ and set

$$\vartheta_{\varepsilon,\varepsilon'}(z)=\vartheta_{\varepsilon,\varepsilon'}(z,B)=\sum_{n\in\mathbb{Z}^g}e^{\pi i(n+\varepsilon/2)B(n+\varepsilon/2)^t+2\pi i(n+\varepsilon/2)(z+\varepsilon'/2)^t}. \tag{5.9}$$

These functions are holomorphic in $z\in\mathbb{C}^g$, and in B satisfying (5.5), and one computes that

$$\vartheta_{\varepsilon,\varepsilon'}(z+m+nB)=e^{-\pi i(nBn^t+2zn^t-\varepsilon m^t+\varepsilon' n^t)}\vartheta_{\varepsilon,\varepsilon'}(z), \tag{5.10}$$

for $m,n\in\mathbb{Z}^g$. Note that

$$\varepsilon,\varepsilon'\in\mathbb{Z}^g\Longrightarrow e^{-\pi i(-\varepsilon m^t+\varepsilon' n^t)}=(-1)^{\varepsilon m^t-\varepsilon' n^t}. \tag{5.11}$$

When $\varepsilon,\varepsilon'\in\mathbb{Z}^g$, one says $\vartheta_{\varepsilon,\varepsilon'}(z)$ is a first order theta function with integer characteristic. Another calculation gives

$$\varepsilon,\varepsilon',\nu,\nu'\in\mathbb{Z}^g\Longrightarrow\vartheta_{\varepsilon+2\nu,\varepsilon'+2\nu'}(z)=e^{\pi i\nu'\varepsilon^t}\vartheta_{\varepsilon,\varepsilon'}(z). \tag{5.12}$$

Hence, up to sign, there are 2^{2g} distinct first order theta functions with integer characteristic. Let us also note that

$$\varepsilon,\varepsilon'\in\mathbb{Z}^g\Longrightarrow\vartheta_{\varepsilon,\varepsilon'}(-z)=(-1)^{\varepsilon'\varepsilon^t}\vartheta_{\varepsilon,\varepsilon'}(z). \tag{5.13}$$

When $g=1$, a comparison with (5.1) and with (32.8)–(32.10) of [T2] gives (for $B=\tau$)

$$\begin{aligned}\vartheta_{0,0}(z)=\vartheta_3(z),\quad \vartheta_{0,1}(z)=\vartheta_4(z),\\ \vartheta_{1,0}(z)=\vartheta_2(z),\quad \vartheta_{1,1}(z)=-\vartheta_1(z).\end{aligned} \tag{5.14}$$

Another family of theta functions is given by

$$\theta_{r,s}(z)=\theta_{r,s}(z,B)=\sum_{n\in\mathbb{Z}^g}e^{\pi i(n+s/r)B(rn+s)^t+2\pi i(rn+s)z^t}, \tag{5.15}$$

with

$$r\in\mathbb{N},\quad s=(s_1,\dots,s_g)\in\mathbb{Z}^g,\quad 0\le s_j<r. \tag{5.16}$$

These functions are also holomorphic in $z \in \mathbb{C}^g$ and in B satisfying (5.5). This time one has

$$\theta_{r,s}(z+m+nB) = e^{-\pi i r(nBn^t+2zn^t)}\,\theta_{r,s}(z), \quad m,n \in \mathbb{Z}^g. \tag{5.17}$$

We note that

$$\theta_{1,0}(z) = \vartheta_{0,0}(z) = \vartheta(z), \tag{5.18}$$

and

$$\vartheta_{\varepsilon,\varepsilon'}(z,B) = \theta_{2,\varepsilon}\Bigl(\frac{z}{2}+\frac{\varepsilon'}{4},\frac{B}{2}\Bigr), \tag{5.19}$$

for $\varepsilon,\varepsilon' \in \{0,1\}^g$.

The identity (5.8) allows us to regard $\vartheta = \vartheta_{0,0} = \theta_{1,0}$ as a holomorphic section of a holomorphic line bundle

$$L \longrightarrow \mathbb{C}^g/\Lambda, \tag{5.20}$$

with Λ given by (5.3)–(5.5). (Cf. [N], p. 91.) More generally, (5.17) exhibits $\theta_{r,s}$ as sections of $\otimes^r L$. It is the case that for each $r \in \mathbb{N}$,

$$\dim \mathcal{O}(\otimes^r L) = r^g, \tag{5.21}$$

and a basis is given by $\theta_{r,s}$, as s runs over the set described in (5.16); cf. [N], p. 93. Let us note that

$$a \in \mathbb{C}^g \Longrightarrow \vartheta(z+a)\vartheta(z-a) \in \mathcal{O}(\otimes^2 L), \tag{5.22}$$

and more generally

$$a_j \in \mathbb{C}^g,\ a_1+\cdots+a_k = 0 \Longrightarrow \vartheta(z+a_1)\cdots\vartheta(z+a_k) \in \mathcal{O}(\otimes^k L). \tag{5.23}$$

6. The theta divisor

By (5.7)–(5.8) it is clear that the zero set of the theta function ϑ in $\mathbb{C}^g$ is invariant under translation by Λ, so it defines an analytic variety in $\mathbb{C}^g/\Lambda$ of complex dimension $g-1$, which we denote Θ. Here we explore what Θ is and how it and its translates intersect $\Phi(M)$, when $\mathbb{C}^g/\Lambda = J(M)$.

It is clear that for most $\xi \in J(M)$ the image $\Phi(M)$ is not contained in $\Theta_\xi = \Theta + \xi$. For such generic ξ we have a nontrivial element

$$F_\xi \in \mathcal{O}(L_\xi), \quad L_\xi = \Phi_\xi^* L, \tag{6.1}$$

where $L_\xi \to M$ is the pull-back of $L \to J(M)$ by Φ_ξ, with $\Phi_\xi(p) = \Phi(p) - \xi$, and

$$F_\xi(p) = \vartheta(\Phi(p) - \xi). \tag{6.2}$$

Note that the divisor $\vartheta(F_\xi)$ of F_ξ (on M, identified with its image $\Phi(M)$) is

$$\Phi(M) \cap \Theta_\xi = \vartheta(F_\xi), \tag{6.3}$$

provided we count the points in $\Phi(M) \cap \Theta_\xi$ with multiplicity. The following is a key result.

Proposition 6.1. *If $\Phi(M)$ is not a subset of Θ_ξ, then $\vartheta(F_\xi)$ has degree g.*

Proof. To begin, it is useful to examine F_ξ on Ω, obtained by slitting M along the curves $a_j, b_j,\ 1 \le j \le g$, as indicated in Figure 2. (Arrange that these curves avoid the zeros of F_ξ.) Thus $\partial\Omega$ consists of copies of a_j, b_j and of the reverse curves a_j', b_j'. Each point $p \in a_j$ (resp., $q \in b_j$) has two preimages in $\partial\Omega$, which we denote p, p' (resp., q, q'), as indicated in Figure 3.

Note that we can define Φ as a map from $\overline{\Omega}$ to $\mathbb{C}^g$, by taking a path in $\overline{\Omega}$ in (3.11), and hence we can define $F_\xi : \overline{\Omega} \to \mathbb{C}$. Then we have

$$\begin{aligned} p \in a_j &\Longrightarrow \Phi(p) - \Phi(p') = \int_{p'}^{p} (\zeta_1, \dots, \zeta_g) = -\int_{b_j} (\zeta_1, \dots, \zeta_g) = -B_j, \\ q \in b_j &\Longrightarrow \Phi(q) - \Phi(q') = \int_{q'}^{q} (\zeta_1, \dots, \zeta_g) = \int_{a_j} (\zeta_1, \dots, \zeta_g) = E_j, \end{aligned} \tag{6.4}$$

and hence

$$\begin{aligned} p \in a_j &\Longrightarrow F_\xi(p') = e^{-\pi i B_{jj}}\, e^{-2\pi i (\Phi_j(p) - \xi_j)}\, F_\xi(p), \\ q \in b_j &\Longrightarrow F_\xi(q') = F_\xi(q). \end{aligned} \tag{6.5}$$

Now deg $\vartheta(F_\xi)$ is equal to the number of zeros of F_ξ in Ω, counted with multiplicity, which in turn is equal to

$$\frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{F_\xi'(z)}{F_\xi(z)}\,dz=\frac{1}{2\pi i}\sum_{j=1}^{g}\Bigl(\int\limits_{a_j}+\int\limits_{b_j}\Bigr)d\,\log\frac{F_\xi^+(z)}{F_\xi^-(z)}, \tag{6.6}$$

where F_ξ^+ is F_ξ restricted to $a_j\cup b_j\subset\partial\Omega$ and F_ξ^- is F_ξ restricted to $a_j'\cup b_j'$. By (6.5) we have

$$\begin{aligned} z\in a_j \Longrightarrow \frac{F_\xi^+(z)}{F_\xi^-(z)} &= e^{\pi i B_{jj}}\,e^{2\pi i(\Phi_j(z)-\xi)} \\ \Longrightarrow d\,\log\frac{F_\xi^+(z)}{F_\xi^-(z)} &= 2\pi i\,d\Phi_j(z), \end{aligned} \tag{6.7}$$

and

$$z\in b_j\Longrightarrow\frac{F_\xi^+(z)}{F_\xi^-(z)}=1\Longrightarrow d\,\log\frac{F_\xi^+(z)}{F_\xi^-(z)}=0, \tag{6.8}$$

so the right side of (6.6) is equal to

$$\sum_{j=1}^{g}\int\limits_{a_j}d\Phi_j(z)=\sum_{j=1}^{g}1=g. \tag{6.9}$$

This completes the proof.

Remark. A result equivalent to Proposition 6.1 is that

$$c_1(L_\xi)=g, \tag{6.10}$$

for each $\xi\in J(M)$ such that $\Phi(M)$ is not a subset of Θ_ξ, hence, by continuity, for all $\xi\in J(M)$.

Here is a useful complement to Proposition 6.1:

Proposition 6.2. *In the setting of Proposition 6.1, we have*

$$\Phi(\vartheta(F_\xi))=\xi-\alpha_0, \tag{6.11}$$

for some $\alpha_0\in J(M)$*, independent of* ξ*.*

Here we are using $\Phi : \mathrm{Div}(M) \to J(M)$, defined by (4.24)–(4.25). Note that the left side of (6.11) is $\sum \Phi(p_j(\xi))$, where $p_1(\xi), \dots, p_g(\xi)$ are the zeros of F_ξ, counted with muitiplicity. The proof of Proposition 6.2 starts with

$$\sum_{j=1}^{g} \Phi(p_j(\xi)) = \frac{1}{2\pi i} \int_{\partial\Omega} \Phi(z)\, d\log F_\xi(z), \tag{6.12}$$

and makes use of (6.4)–(6.5). See [N], pp. 98–99, for details.

To state the next result about Θ, we use $W_k \subset J(M)$, defined as

$$W_k = \Phi(\mathrm{Div}_k^+(M)), \quad \Phi : \mathrm{Div}(M) \to J(M). \tag{6.13}$$

Proposition 6.3. *With α_0 as in (6.11), we have*

$$\Theta = W_{g-1} + \alpha_0. \tag{6.14}$$

We refer to [N], pp. 100–101, for a proof. We mention that part of the argument involves showing:

Lemma 6.4. *Suppose $D = p_1 + \cdots + p_g \in \mathrm{Div}_g^+(M) \setminus Y$, so (4.41) holds, and set $\xi = \Phi(D) + \alpha_0$. Then D is the divisor of F_ξ. In particular, the zeros of F_ξ are simple. Consequently, (6.14) is an identity of divisors, not just sets.*

We next want to specify α_0. To do this, it is helpful to have the following.

Lemma 6.5. *The divisor Θ is not invariant under any nonzero translation of $J(M)$.*

For a proof, see [N], pp. 94–95. In light of (6.14) it is equivalent to saying that W_{g-1} is not invariant under any nontrivial translation of $J(M)$. Here is the result on α_0.

Proposition 6.6. *With Ψ defined as in (4.17)–(4.18), and $\kappa \to M$ the canonical line bundle, we have*

$$\Psi(\kappa) = -2\alpha_0. \tag{6.15}$$

Proof. To start, pick an arbitrary $D \in \mathrm{Div}_{g-1}^+(M)$. There is a nontrivial $u \in \mathcal{O}(\kappa)$ such that $\vartheta(u) \ge D$, by the Riemann-Roch theorem (cf. (B.16)). Thus there exists $D' \in \mathrm{Div}_{g-1}^+(M)$ such that $\vartheta(u) = D + D'$, so $\Psi(\kappa) = \Phi(D) + \Phi(D')$. This holds for arbitrary $\Phi(D) \in W_{g-1}$. It follows that

$$\Psi(\kappa) - W_{g-1} = W_{g-1}. \tag{6.16}$$

Since $\vartheta(-z) = -\vartheta(z)$, we have

$$\begin{aligned} \Theta = W_{g-1} + \alpha_0 &= -W_{g-1} - \alpha_0 \\ &= W_{g-1} - \Psi(\kappa) - \alpha_0 \\ &= \Theta - \Psi(\kappa) - 2\alpha_0. \end{aligned} \tag{6.17}$$

Hence Lemma 6.5 implies $\Psi(\kappa) + 2\alpha_0 = 0$, giving (6.15).

We next specify when $\Phi(M) \subset \Theta_\xi$.

Proposition 6.7. *Let $\xi \in J(M)$. Then $\Phi(M) \subset \Theta_\xi$ if and only if $\xi - \alpha_0 = \Phi(D)$ where $D \in \mathrm{Div}_g^+(M)$ is a special divisor, i.e., if and only if $\xi - \alpha_0 \in \Phi(Y)$, with Y as in (4.41).*

Proof. To say that $\Phi(p) - \xi \in \Theta$ for each $p \in M$ is to say that $\xi - \Phi(p) \in W_{g-1} + \alpha_0$ for each $p \in M$, or equivalently that

$$\xi - \alpha_0 = \Phi(D) \;\text{ for some }\; D \in \mathrm{Div}_g^+(M) \;\text{ with }\; p \in \;\mathrm{supp}(D). \tag{6.18}$$

Consequently, if we pick some $D_0 \in \mathrm{Div}_g^+(M)$ such that $\Phi(D_0) = \xi - \alpha_0$, the condition is that for each $p \in M$ there is a divisor $D \in \mathrm{Div}_g^+(M)$ containing p such that (by Abel's theorem) $D - D_0$ is the divisor of a meromorphic function. This is equivalent to the assertion that dim $\mathcal{M}(-D_0) \geq 2$, i.e., to the condition that D_0 is special.

Bringing in Proposition 4.7, we have:

Corollary 6.8. *If $\xi \in J(M)$ and $\Phi(M)$ is not a subset of Θ_ξ, then there is a unique divisor $D \in \mathrm{Div}_g^+(M)$ such that $\Phi(D) + \alpha_0 = \xi$, namely the divisor of F_ξ.*

Also:

Corollary 6.9. *The set*

$$V_\xi = \{\xi \in J(M) : \Phi(M) \subset \Theta_\xi\} = \alpha_0 + \Phi(Y) \tag{6.19}$$

is an analytic set of complex dimension $\leq g - 2$.

7. Riemann's factorization theorem

Here we prove Riemann's result that each $u \in \mathcal{M}(M)$ can be written as a product of factors of the form

$$\frac{\vartheta(\Phi(z)-\Phi(p_k)-\xi)}{\vartheta(\Phi(z)-\Phi(q_k)-\xi)}, \tag{7.1}$$

times an exponential factor $\exp\{\int_{p_0}^z \zeta\}$ for some $\zeta \in \mathcal{O}(\kappa)$ (reminiscent of Abel's theorem), times a constant. Here ξ will be a suitably chosen point in $\Theta \subset J(M)$. Namely, we require

$$\xi \notin Y' + \alpha_0 - \Phi(p_k), \quad \text{and} \quad \xi \notin Y' + \alpha_0 - \Phi(q_k), \tag{7.2}$$

for each k, where $Y' = \Phi(Y)$, as in (4.41).

The first order of business is to record sufficient information on the zeros in M of

$$F_{\xi,p}(z) = \vartheta(\Phi(z) - \Phi(p) - \xi). \tag{7.3}$$

If the analogue of (7.2) holds, i.e., $\Phi(p) + \xi - \alpha \notin Y'$, then $F_{\xi,p}$ has exactly g zeros, $p_1, \dots, p_g$, and, by Proposition 4.7, $p_1 + \cdots + p_g$ is the only divisor in $\mathrm{Div}_g^+(M)$ satisfying $\Phi(D) = \Phi(p) + \xi - \alpha_0$. Given that we take $\xi \in \Theta$, we have $q_j^0 \in M$ such that

$$\xi = \Phi(q_1^0 + \cdots + q_{g-1}^0) + \alpha_0, \tag{7.4}$$

by Proposition 6.3. Hence

$$\sum_{j=1}^{g} \Phi(p_j) = \Phi(p) + \sum_{j=1}^{g-1} \Phi(q_j^0). \tag{7.5}$$

The uniqueness result mentioned above implies

$$p_1 + \cdots + p_g = p + q_1^0 + \cdots + q_{g-1}^0. \tag{7.6}$$

We have the following conclusion.

Proposition 7.1. *If $\xi \in \Theta$ and $\Phi(p) - \xi - \alpha_0 \notin Y'$, then the zeros of $F_{\xi,p}$ in (7.3) consist of*

$$p, q_1^0, \dots, q_{g-1}^0, \tag{7.7}$$

where $q_1^0, \dots, q_{g-1}^0$ depend only on ξ, not on p.

We now state Riemann's factorization theorem.

Theorem 7.2. *Let $u \in \mathcal{M}(M)$ be nonconstant, with divisor $\vartheta(u) = \sum_{k=1}^m (p_k - q_k)$. Take $\xi \in \Theta$, satisfying (7.2). Then there exist $\zeta \in \mathcal{O}(\kappa)$ and $c \in \mathbb{C}$ such that*

$$u(z) = c\, e^{\int_{p_0}^z \zeta} \prod_{k=1}^m \frac{\vartheta(\Phi(z) - \Phi(p_k) - \xi)}{\vartheta(\Phi(z) - \Phi(q_k) - \xi)}. \tag{7.8}$$

Proof. We can write ξ in the form (7.4), i.e., $\xi = \Phi(D_0) + \alpha_0$, $D_0 = q_1^0 + \cdots + q_{g-1}^0$. Now slit M along a_j, b_j, $1 \le j \le g$, obtaining the simply connected domain Ω with $\partial\Omega = \cup(a_j \cup b_j \cup a_j' \cup b_j')$, as indicated in Figure 2, and arrange a_j, b_j to avoid the points p_k, q_k, and q_j^0. Define

$$F(z) = \prod_{k=1}^m \frac{\vartheta(\Phi(z) - \Phi(p_k) - \xi)}{\vartheta(\Phi(z) - \Phi(q_k) - \xi)} \tag{7.9}$$

on Ω. Note that

$$\vartheta(F) = \Big(\sum p_k + mD_0\Big) - \Big(\sum q_k + mD_0\Big) = \sum (p_k - q_k),$$

which is the divisor of u.

We now seek $\zeta \in \mathcal{O}(\kappa)$ so that $\exp\{\int_{p_0}^z \zeta\}\, F(z) = G(z)$ defines a single valued function on M. For this, it suffices for the values of G on $\overline{\Omega}$ to match up at corresponding points $p \in a_j$ and $p' \in a_j'$, and also at corresponding points $q \in b_j$ and $q' \in b_j'$. (See Figure 3.) Recall the computations (6.4)–(6.5). For such corresponding q, q' as just mentioned, we have $\Phi(q') = \Phi(q) - E_j$, and hence $F(q) = F(q')$. Meanwhile, for their counterparts p, p', we have $\Phi(p') = \Phi(p) + B_j$, and hence

$$\begin{aligned} \frac{F(p')}{F(p)} &= \prod_{k=1}^m \frac{e^{-2\pi i(\Phi_j(p) - \Phi_j(p_k) - \xi_j)}}{e^{-2\pi i(\Phi_j(p) - \Phi_j(q_k) - \xi_j)}} \\ &= e^{2\pi i \sum_{k=1}^m (\Phi_j(p_k) - \Phi_j(q_k))}. \end{aligned} \tag{7.10}$$

Now by Abel's theorem (the easy half) we have $n, m \in \mathbb{Z}^g$ such that

$$\sum_{k=1}^m (\Phi(p_k) - \Phi(q_k)) = \sum_{\ell=1}^g n_\ell E_\ell + \sum_{\ell=1}^g m_\ell B_\ell. \tag{7.11}$$

We set

$$\zeta = -2\pi i \sum_{\ell=1}^g m_\ell \zeta_\ell, \tag{7.12}$$

where $\zeta_1, \dots, \zeta_g$ is the normalized basis of $\mathcal{O}(\kappa)$ satisfying (3.18), and then we set

$$\varphi(z) = \int_{p_0}^{z} \zeta, \tag{7.13}$$

with $z \in \overline{\Omega}$. We see that if $q \in b_j$ and $q \in b'_j$ are as above then $e^{\varphi(q)} = e^{\varphi(q')}$. Also, if $p \in a_j$ and $p' \in a'_j$ are as above, then

$$e^{\varphi(p')} = e^{\varphi(p)}\, e^{-2\pi i \sum_\ell m_\ell B_{\ell j}}. \tag{7.14}$$

Hence $z \mapsto e^{\varphi(z)} F(z)$ is single valued on M and defines an element of $\mathcal{M}(M)$ with the same divisor as u. This proves (7.8).

A. The Riemann-Roch formula

Given a holomorphic line bundle $L \to M$ over a compact Riemann surface M of genus g, it is of great interest to have information about the dimension of the space $\mathcal{O}(L)$ of holomorphic sections of L. The Riemann-Roch formula relates the dimension of this space and that of $\mathcal{O}(L^{-1} \otimes \kappa)$, where κ is the canonical line bundle, whose local sections over $U \subset M$ consist of holomorphic 1-forms on U. The result is

$$\dim \mathcal{O}(L) - \dim \mathcal{O}(L^{-1} \otimes \kappa) = c_1(L) - \frac{1}{2} c_1(\kappa). \tag{A.1}$$

Here $c_1(L) \in \mathbb{Z}$ is the first Chern class of L. In particular,

$$c_1(\kappa) = -\chi(M) = 2g - 2. \tag{A.2}$$

The original version of the Riemann-Roch formula has an alternative formulation, stated in (A.7) below. To get it, we bring in E_ϑ, the holomorphic line bundle associated to the divisor

$$\vartheta = \sum n_j p_j \tag{A.3}$$

on M. It is constructed to have a natural section u whose divisor is given by $\vartheta(u) = -\vartheta$. Cf. [T1], (9.36)–(9.38). Then if we set

$$\mathcal{M}(L, \vartheta) = \{u \in \mathcal{M}(L) : \vartheta(u) \geq \vartheta\}, \tag{A.4}$$

where $\mathcal{M}(L)$ denotes the space of meromorphic sections of L, we have

$$\mathcal{M}(L, \vartheta) \approx \mathcal{O}(L \otimes E_\vartheta). \tag{A.5}$$

Cf. [T1], (9.41). We also have ([T1], (9.39))

$$c_1(E_\vartheta) = -\sum n_j p_j = -\deg \vartheta. \tag{A.6}$$

Then (A.1), applied to $L = E_\vartheta$, yields

$$\dim \mathcal{M}(\vartheta) - \dim \mathcal{M}(\kappa, -\vartheta) = -\deg \vartheta + 1 - g. \tag{A.7}$$

Here $\mathcal{M}(\vartheta) = \mathcal{M}(1, \vartheta)$, where 1 denotes the trivial line bundle.

It is useful to know that

$$u \in \mathcal{M}(L) \Longrightarrow \deg \vartheta(u) = c_1(L). \tag{A.8}$$

Cf. [T1], (9.28). In particular, if $c_1(L) < 0$, then $\deg \vartheta(u) < 0$, so

(A.9) $$c_1(L) < 0 \Longrightarrow \mathcal{O}(L) = 0.$$

Since $c_1(L^{-1} \otimes \kappa) = 2g - 2 - c_1(L)$, we see from (A.1) that

(A.10) $$c_1(L) > 2g - 2 \Longrightarrow \dim \mathcal{O}(L) = c_1(L) - g + 1.$$

For $c_1(L) \leq 2g - 2$, (A.1) yields Riemann's inequality:

(A.11) $$\dim \mathcal{O}(L) \geq c_1(L) - g + 1,$$

which has content only if also $c_1(L) \geq g$. Note that

(A.12) $$c_1(L) = 2g - 2 \Longrightarrow \dim \mathcal{O}(L) \geq g - 1.$$

As mentioned, $c_1(\kappa) = 2g - 2$; in that case, as one sees by taking $L = \kappa$ in (A.1),

(A.13) $$L = \kappa \Longrightarrow \dim \mathcal{O}(L) = g.$$

The results given above, which are frequently used in these notes, are proven in a number of places, including [FK], [N], and [T1].

B. Special divisors and Weierstrass points

Let M be a compact Riemann surface of genus g. Consider a positive divisor $\vartheta = p_1 + \cdots + p_k$, and the space of meromorphic functions u with divisor $\vartheta(u) \geq -\vartheta$,

$$\mathcal{M}(-p_1 - \cdots - p_k) \approx \mathcal{O}(E_{-\vartheta}). \tag{B.1}$$

The Riemann-Roch formula gives

$$\dim \mathcal{M}(-p_1 - \cdots - p_k) = \dim \mathcal{O}(E_\vartheta \otimes \kappa) + c_1(E_{-\vartheta}) - \frac{1}{2}c_1(\kappa). \tag{B.2}$$

Note that

$$c_1(E_{-\vartheta}) = k, \quad c_1(E_\vartheta \otimes \kappa) = 2g - 2 - k, \tag{B.3}$$

so

$$\dim \mathcal{M}(-p_1 - \cdots - p_k) \geq k - g + 1, \tag{B.4}$$

and in particular

$$k \geq g + 1 \Longrightarrow \dim \mathcal{M}(-p_1 - \cdots - p_k) \geq 2, \tag{B.5}$$

so there is guaranteed to be a nonconstant meromorphic function on M with divisor $\geq -p_1 - \cdots - p_k$ as long as $k \geq g+1$. (Note parenthetically that $k > 2g-2$ implies equality in (B.4).)

If $k = g$ and there is such a meromorphic function, we say ϑ is a *special divisor*. If this holds for $\vartheta = gp$, we say p is a *Weierstrass point*, i.e.,

$$p \in M \ \text{ is a Weierstrass point } \Longleftrightarrow \dim \mathcal{M}(-gp) \geq 2. \tag{B.6}$$

Clearly when $g = 1$ there are no Weierstrass points. It turns out that whenever $g \geq 2$ the set of Weierstrass points is a nonempty, finite set. Clearly if $g \geq 2$ and M is hyperelliptic, so there is a 2-fold holomorphic branched covering $\varphi : M \to S^2$, then each of the $2g+2$ branch points of φ is a Weierstrass point of M. It turns out that these are all the Weierstrass points of such M. (See Proposition B.2 below.) Furthermore, a compact Riemann surface of genus g is hyperelliptic if and only if it has precisely $2g+2$ Weierstrass points ([FK], p. 95).

Here is a useful characterization of Weierstrass points, and more generally of special divisors:

Proposition B.1. *If $g \geq 2$, a positive divisor of degree g, $\vartheta = p_1 + \cdots + p_g$ is a special divisor if and only if*

$$\text{(B.7)} \qquad dim\,\mathcal{M}(\kappa, \vartheta) \geq 1.$$

In particular, a point $p \in M$ is a Weierstrass point if and only if $dim\,\mathcal{M}(\kappa, gp) \geq 1$, i.e., if and only if there is a holomorphic 1-form on M, vanishing to order g at p, but not identically zero.

Proof. Note that $\mathcal{M}(\kappa, \vartheta) \approx \mathcal{O}(E_\vartheta \otimes \kappa)$. Using (B.2) we have

$$\text{(B.8)} \qquad \dim \mathcal{M}(-\vartheta) = \dim \mathcal{O}(E_\vartheta \otimes \kappa) + 1,$$

so the criteria dim $\mathcal{M}(-\vartheta) \geq 2$ and (B.7) are equivalent.

Proposition B.1 can be exploited as follows. Take a basis $\zeta_1, \ldots, \zeta_g$ of $\mathcal{O}(\kappa)$. In a local coordinate chart, say $\zeta_j = u_j(z)\,dz$. Form the Wronskian

$$\text{(B.9)} \qquad W = \det \begin{pmatrix} u_1 & u_1' & \cdots & u_1^{(g-1)} \\ u_2 & u_2' & \cdots & u_2^{(g-1)} \\ \vdots & \vdots & & \vdots \\ u_g & u_g' & \cdots & u_g^{(g-1)} \end{pmatrix}.$$

Then a point in this coordinate chart is a Weierstrass point if and only if W vanishes there. Now pasting together various coordinate charts reveals W as a global section of $\otimes^M \kappa$, with $M = 1 + 2 + \cdots + g = g(g+1)/2$:

$$\text{(B.10)} \qquad W \in \mathcal{O}(\otimes^M \kappa), \quad M = \frac{1}{2} g(g+1).$$

Since we started with a basis, W is not identically zero. Hence

$$\text{(B.11)} \qquad \vartheta(W) = \sum \nu_p(W) p,$$

where $\nu_p(W) \geq 0$, $\nu_p(W) > 0$ if and only if p is a Weierstrass point, and

$$\text{(B.12)} \qquad \sum_p \nu_p(W) = c_1(\otimes^M \kappa) = M c_1(\kappa) = (g-1)g(g+1).$$

Thus the maximum number of Weierstrass points on M is $g^3 - g$. It turns out that the minimum number is $2g + 2$ ([FK], p. 85). This follows from the assertion that $\nu_p(W) \leq (1/2)g(g-1)$ for each p.

For notational convenience, denote (B.9) by $W(u_1, \ldots, u_g)$. Note that if f is holomorphic,

$$\text{(B.12)} \qquad W(fu_1, \ldots, fu_g) = f^g W(u_1, \ldots, u_g),$$

as can be seen by expanding the entries $(fu_j)^{(k)}$ in the analogue to (B.9) via the Leibniz formula and using the fact that matrices with two identical columns have determinant zero. With this in mind, we can compute W when M is hyperelliptic and we take the basis of $\mathcal{O}(\kappa)$ given by (2.11), using local ζ coordinates away from the branch points. (There is no loss in generality in assuming one of the branch points lies over $\infty \in S^2$.) We obtain

$$W = q(\zeta)^{-g/2}\, W(1, \zeta, \dots, \zeta^{g-1}), \tag{B.13}$$

which is clearly nonzero away from the branch points. This establishes the advertised result:

Proposition B.2. *If M is hyperelliptic (of genus $g \geq 2$), its Weierstrass points are precisely the branch points of the 2-fold branched cover $M \to S^2$.*

The identity (B.12) implies that most points are not Weierstrass points. Here is a complementary result, which is useful in the study of the Jacobi inversion problem.

Proposition B.3. *If $g \geq 2$, there is a set $\{p_1, \dots, p_g\}$ of g distinct points of M such that $\vartheta = p_1 + \cdots + p_g$ is not special.*

Proof. Via Proposition B.1, it suffices to find distinct $p_1, \dots, p_g$ such that

$$\mathcal{M}(\kappa, p_1 + \cdots + p_g) = \bigcap_{\ell=1}^{g} \mathcal{M}(\kappa, p_\ell) = 0. \tag{B.14}$$

Clearly $\cap_{p \in M} \mathcal{M}(\kappa, p) = 0$, and (B.2) with $k = 1$ implies

$$\dim \mathcal{M}(\kappa, p) = g - 1 = \dim \mathcal{O}(\kappa) - 1, \quad \forall\, p \in M, \tag{B.15}$$

since $\mathcal{M}(-p) \approx \mathbb{C}$ for $g \geq 2$. (See also (3.5).) Another way to put this is that V_p, the annihilator of $\mathcal{M}(\kappa, p)$ in $\mathcal{O}(\kappa)'$, has dimension 1 for each p, and $\oplus_p V_p = \mathcal{O}(\kappa)'$. We can hence choose $p_1, \dots, p_g$ such that elements $e_j \in V_{p_j}$ form a basis of $\mathcal{O}(\kappa)'$, and (B.14) is achieved.

Remark. One can avoid use of (B.2) in (B.15) by taking some nontrivial $u \in \mathcal{O}(\kappa)$, setting $\Omega = \{p \in M : u(p) \neq 0\}$, and noting that

$$\dim \mathcal{M}(\kappa, p) = g - 1, \quad \forall\, p \in \Omega,$$

and that $\cap_{p \in \Omega} \mathcal{M}(\kappa, p) = 0$. This sort of argument yields the following useful result (cf. [N], p. 70):

Proposition B.3′. *Let $L \to M$ be a holomorphic line bundle, $V \subset \mathcal{O}(L)$ a k-dimensional linear subspace. Then there is a set of k distinct points $p_1, \dots, p_k$ such that*

$$u \in V,\ u(p_1) = \cdots = u(p_k) = 0 \Longrightarrow u \equiv 0.$$

Here is a result complementary to (B.5).

Proposition B.4. *If $g \geq 2$, we have*

$$k \leq g - 1 \Longrightarrow dim\,\mathcal{M}(\kappa, p_1 + \cdots + p_k) \geq g - k. \tag{B.16}$$

Proof. Rewrite (B.2) as

$$\dim \mathcal{M}(\kappa, p_1 + \cdots + p_k) = \dim \mathcal{M}(-\vartheta) + g - 1 - k, \tag{B.17}$$

and use the fact that $\mathcal{M}(-\vartheta)$ has dimension ≥ 1, since it contains constants.

This leads to the following counterpoint to Proposition B.3.

Proposition B.5. *If $g \geq 2$ and $p_1, \dots, p_{g-1} \in M$ are given, there exists $p_g \in M$ such that $p_1 + \cdots + p_g$ is a special divisor.*

Proof. By (B.16), there exists a nontrivial $u \in \mathcal{M}(\kappa, p_1 + \cdots + p_{g-1})$. Now $\vartheta(u)$ is a positive divisor of degree $c_1(\kappa) = 2g - 2 > g - 1$, so one can take any point p_g such that $p_1 + \cdots + p_{g-1} + p_g \leq \vartheta(u)$.

Remark. We have defined special divisors $\vartheta = p_1 + \cdots + p_k$ only for $k = g$. In [ACGH] a divisor of this form, for any k, is called special provided $\dim \mathcal{O}(E_\vartheta \otimes \kappa) \geq 1$. (This does require $k \leq 2g - 2$.)

C. Flat line bundles and the factor $\exp \int_{p_0}^{z} \zeta$.

Let $\zeta = 2\pi i(c_1\zeta_1 + \cdots + c_g\zeta_g)$ be a holomorphic 1-form on M, where ζ_j form a basis of $\mathcal{O}(\kappa)$ satisfying (3.18)–(3.19). We consider

$$A(z) = \exp \int_{p_0}^{z} \zeta, \tag{C.1}$$

which appears as a factor in (4.6) and in (7.8). This is a single valued function on $\widetilde{M}$, the universal cover of M. We have

$$\widetilde{M} \longrightarrow M, \tag{C.2}$$

with covering group $\pi_1(M)$, which has a natural surjective homomorphism

$$\alpha : \pi_1(M) \longrightarrow H_1(M, \mathbb{Z}), \tag{C.3}$$

in which the homotopy classes of a_k and of b_k are mapped to their corresponding homology classes. If

$$z' = \pi(z), \quad \pi \in \pi_1(M), \quad \alpha(\pi) = \sum_{k=1}^{g} (\nu_k a_k + \mu_k b_k), \tag{C.4}$$

then

$$\begin{aligned} A(z') &= A(z) \exp 2\pi i \sum_{j=1}^{g} \sum_{k=1}^{g} c_j(\nu_k \delta_{jk} + \mu_k B_{jk}) \\ &= A(z) \exp 2\pi i \sum_{k=1}^{g} (\nu_k c_k + \mu_k d_k), \end{aligned} \tag{C.5}$$

where

$$d_k = \sum_{j=1}^{g} c_j B_{jk}. \tag{C.6}$$

This presents A as a holomorphic section of a flat line bundle

$$L_c \longrightarrow M, \tag{C.7}$$

produced from the principal $\pi_1(M)$-bundle $\widetilde{M} \to M$ via the representation R_c of $\pi_1(M)$ on $\mathbb{C}$ given by

$$R_c(\pi) = e^{2\pi i(c\nu^t + cB\mu^t)}, \quad \sum (\nu_k a_k + \mu_k b_k) = \alpha(\pi). \tag{C.8}$$

Here, c, ν, and μ are regarded as row vectors.

The bundle L_c is holomorphically trivial; the section $A \in \mathcal{O}(L_c)$ provides a trivialization.

A more general family of flat complex line bundles over M has the form

$$L_{c,d} \longrightarrow M, \tag{C.9}$$

produced from the principal $\pi_1(M)$-bundle $\widetilde{M} \to M$ via the representation $R_{c,d}$ of $\pi_1(M)$ on $\mathbb{C}$ given by

$$R_{c,d}(\pi) = e^{2\pi i(c\nu^t + d\mu^t)}, \quad \sum(\nu_k a_k + \mu_k b_k) = \alpha(\pi), \tag{C.10}$$

where $c, d \in \mathbb{C}^g$ are arbitrary row vectors, uniquely determined modulo $\mathbb{Z}^g$.

Proposition C.1. *The line bundle $L_{c,d}$ is holomorphically trivial if and only if there exist $c_0, d_0 \in \mathbb{Z}^g$ such that*

$$d - d_0 = (c - c_0)B, \tag{C.11}$$

or equivalently, if and only if

$$d - cB \in \Lambda, \tag{C.12}$$

where Λ is the lattice in $\mathbb{C}^g$ given by (3.9), (3.20).

Proof. One direction follows via the analysis above of (C.1). For the other direction, suppose A is a non-trivial holomorphic section of $L_{c,d}$ (necessarily nowhere vanishing since $c_1(L_{c,d}) = 0$). Then A defines a holomorphic function on $\widetilde{M}$, satisfying the analogue of (C.5):

$$z' = \pi z \Longrightarrow A(z') = A(z)\, e^{2\pi i(c\nu^t + d\mu^t)}. \tag{C.13}$$

Also A has a logarithm: $A(z) = e^{F(z)}$, $F : \widetilde{M} \to \mathbb{C}$, holomorphic, so

$$z' = \pi z \Longrightarrow F(z') = F(z) + \text{const}(\pi). \tag{C.14}$$

Hence $\zeta = dF$ is a well defined holomorphic 1-form on M, and A has the form (C.1), which gives (C.11)–(C.12).

One way of looking at part of the proofs of Theorem 4.1 and Theorem 7.2 is that

$$\exp \int_{z_0}^{z} (\tau_{p_1 q_1} + \cdots + \tau_{p_r q_r}) \tag{C.15}$$

in (4.6) and

$$\prod_{k=1}^{m} \frac{\vartheta(\Phi(z) - \Phi(p_k) - \xi)}{\vartheta(\Phi(z) - \Phi(q_k) - \xi)} \tag{C.16}$$

in (7.9) define sections of line bundles over M to which Proposition C.1 applies.

D. $\overline{\partial}$ versus $d \oplus \delta$

As always, M is a compact, connected Riemann surface. We assume M carries a Riemannian metric, consistent with the complex structure, so in local holomorphic coordinates it has the form $ds^2 = A(x,y)(dx^2 + dy^2)$. The operator

$$\overline{\partial} : C^\infty(M,\mathbb{C}) \longrightarrow C^\infty(M,\overline{\kappa}) \tag{D.1}$$

is given in local holomorphic coordinates by

$$\begin{aligned} \overline{\partial} f &= \frac{1}{2}\Bigl(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Bigr) d\overline{z} \\ &= \frac{1}{2}\Bigl[\Bigl(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Bigr) + i\Bigl(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Bigr)\Bigr]\, d\overline{z}, \end{aligned} \tag{D.2}$$

where

$$u = \operatorname{Re} f, \quad v = \operatorname{Im} f. \tag{D.3}$$

Compare this with

$$\overline{D} : C^\infty(M,\mathbb{R}^2) \longrightarrow C^\infty(M,\Lambda^1), \tag{D.4}$$

given by

$$\begin{aligned} \overline{D}\begin{pmatrix} u \\ v \end{pmatrix} &= du + {*dv} \\ &= \Bigl(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Bigr) dx + \Bigl(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Bigr) dy, \end{aligned} \tag{D.5}$$

where $*$ is the Hodge star operator, and we use local holomorphic coordinates. Related to this comparison is the bundle isomorphism

$$\overline{\kappa}_p \approx T_p^*, \quad (u+iv)\, d\overline{z} \leftrightarrow u\, dx + v\, dy. \tag{D.6}$$

Note that this correspondence intertwines multiplication by i on $\overline{\kappa}_p$ with $*$ on T_p^*. We also have the isomorphism

$$\mathbb{R}^2 \approx \mathbb{C}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow u + iv. \tag{D.7}$$

These bundle isomorphisms intertwine $2\overline{\partial}$ and $\overline{D}$.

Going further, we look at

$$d \oplus \delta : C^\infty(M, \Lambda^0 \oplus \Lambda^2) \longrightarrow C^\infty(M, \Lambda^1), \tag{D.8}$$

given by

$$(d \oplus \delta) \begin{pmatrix} u \\ \beta \end{pmatrix} = du + \delta\beta. \tag{D.9}$$

In this case we have the bundle isomorphism

$$\Lambda^0 \oplus \Lambda^2 \approx \mathbb{R}^2, \quad \begin{pmatrix} u \\ \beta \end{pmatrix} \leftrightarrow \begin{pmatrix} u \\ - * \beta \end{pmatrix}, \tag{D.10}$$

which intertwines $\overline{D}$ with $d \oplus \delta$, since $\delta = - * d *$ on 2-forms on the surface M. Consequently $2\overline{\partial}$ and $d \oplus \delta$ are intertwined.

Standard Hodge theory implies that $d \oplus \delta$ in (D.8) yields a Fredholm operator between Sobolov spaces:

$$d \oplus \delta : H^s(M, \Lambda^0 \oplus \Lambda^2) \longrightarrow H^{s-1}(M, \Lambda^1), \tag{D.11}$$

for all real s. In all cases the kernel is a 2-dimensional real linear subspace of $C^\infty(M, \Lambda^0 \oplus \Lambda^2)$, independent of s, and the annihilator of the range is a $2g$-dimensional real linear subspace of $C^\infty(M, \Lambda^1)$, also independent of s, namely the space of harmonic 1-forms. In view of the intertwining observed above, this is equivalent to the assertion that

$$\overline{\partial} : H^s(M, \mathbb{C}) \longrightarrow H^{s-1}(M, \overline{\kappa}) \tag{D.12}$$

is Fredholm for all real s. In all cases the kernel is a 1-dimensional complex linear subspace of $C^\infty(M, \mathbb{C})$, namely constants, and the annihilator of the range is a g-dimensional complex linear subspace of $C^\infty(M, \kappa)$, independent of s, namely $\mathcal{O}(\kappa)$.

One consequence of the reasoning above is the identification dim $\mathcal{O}(\kappa) = g$, without using the Riemann-Roch theorem. Here is another useful consequence:

Proposition D.1. *Let V be a complex linear subspace of $H^{s-1}(M, \overline{\kappa})$. If dim $V > g$, then there is a nonzero element of V in the range of $\overline{\partial}$ in (D.12).*

Note that if $s < 1/2$ we can take elements

$$f_j \in H^{s-1}(M, \overline{\kappa}), \quad \operatorname{supp} f_j = \{p_j\}, \quad 1 \le j \le g+1, \tag{D.13}$$

whose linear span has dimension $g+1$. We can arrange this for $s \in [-1/2, 1/2)$, with each f_j having a δ-function type singularity and all the p_j distinct. Alternatively, we can take $s < 1/2 - k$ and arrange f_j to have the singularity of a derivative of order $k_j \le k$ of a delta function, and various p_j could coincide. By Proposition D.1, some nonzero linear combination

$$f = a_1 f_1 + \cdots + a_{g+1} f_{g+1} \tag{D.14}$$

belongs to the range of $\overline{\partial}$:

$$\overline{\partial} u = f, \quad \text{for some } \ u \in H^s(M, \mathbb{C}). \tag{D.15}$$

Note that such u is meromorphic on M, with poles at p_j, of order k_j. This gives an alternative proof of (B.5) (once we throw in the constants).

Let us expand a little more on the isomorphism

$$\gamma : \mathcal{H}^1(M) \longrightarrow \mathcal{O}(\kappa), \tag{D.16}$$

where $\mathcal{H}^1(M)$ is the $2g$-dimensional real linear space of harmonic 1-forms on M, given the structure of a g-dimensional complex vector space by $*$. Namely, given a (real) harmonic 1-form α on M, set

$$\gamma(\alpha) = \alpha + i * \alpha. \tag{D.17}$$

Note that in a local holomorphic coordinate system we have

$$\alpha = u\,dx + v\,dy \Longrightarrow \gamma(\alpha) = (u - iv)\,dz, \tag{D.18}$$

with $dz = dx + i\,dy$. Note that

$$\gamma(*\alpha) = *\alpha - i\alpha = -i\gamma(\alpha). \tag{D.19}$$

The condition that α be harmonic is that $d\alpha = 0$ and $d * \alpha = 0$, i.e.,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \tag{D.20}$$

which is the condition that $u - iv$ be holomorphic.

Let us also look at a consequence of the fact that $* : \mathcal{H}^1(M) \to \mathcal{H}^1(M)$ satisfies $** = -I$ and is an isometry for the L^2 inner product, given for 1-forms by $(\alpha, \beta) = \int_M \alpha \wedge *\beta$. It follows that there is an orthonormal basis

$$\alpha_j, \ \beta_j \quad 1 \le j \le g \tag{D.21}$$

of $\mathcal{H}^1(M)$ such that

$$*\alpha_j = \beta_j, \quad *\beta_j = -\alpha_j. \tag{D.22}$$

Consequently,

$$\begin{aligned} (\alpha_j, \alpha_k) &= \int \alpha_j \wedge \beta_k = \delta_{jk}, \\ (\beta_j, \beta_k) &= \int \alpha_k \wedge \beta_j = \delta_{jk}, \\ (\alpha_j, \beta_k) &= \int \alpha_k \wedge \alpha_j = \int \beta_k \wedge \beta_j = 0. \end{aligned} \tag{D.23}$$

In such a case,

$$\gamma(\alpha_j) = \alpha_j + i\beta_j, \quad \gamma(\beta_j) = -i\gamma(\alpha_j). \tag{D.24}$$

E. Basic results on $H^1(M,\mathbb{R})$

We discuss some basic results on the deRham cohomology group $H^1(M,\mathbb{R})$, isomorphic via Hodge theory to $\mathcal{H}^1(M)$, the space of harmonic 1-forms on M. As stated before, this is a vector space of real dimension $2g$, if M has genus g. Our goal here is to make this argument, given that M is diffeomorphic to a g-holed torus.

A g-holed torus is depicted in Figure 4. There are closed curves $\tilde{a}_k$ and $\tilde{b}_k$, which, it is useful to know, generate the fundamental group $\pi_1(M)$. For each $k \in \{1,\dots,g\}$ there is a smooth map

$$F_k : M \longrightarrow \mathbb{T}^2 = S^1 \times S^1, \tag{E.1}$$

taking $\tilde{a}_k$ to $\tilde{a}$ and $\tilde{b}_k$ to $\tilde{b}$, and collapsing the other holes. Now $\mathbb{T}^2 = S^1 \times S^1$, with coordinates (θ_1,θ_2) mod $2\pi\mathbb{Z}$, has closed 1-forms $\alpha = d\theta_1,\ \beta = d\theta_2$, such that

$$\int\limits_{\tilde{a}} \alpha = \int\limits_{\tilde{b}} \beta = 1, \quad \int\limits_{\tilde{a}} \beta = \int\limits_{\tilde{b}} \alpha = 0. \tag{E.2}$$

Now if we set

$$\alpha_k = F_k^* \alpha, \quad \beta_k = F_k^* \beta, \tag{E.3}$$

we have

$$\int\limits_{\tilde{a}_k} \alpha_j = \int\limits_{\tilde{b}_k} \beta_j = \delta_{jk}, \quad \int\limits_{\tilde{a}_k} \beta_j = \int\limits_{\tilde{b}_k} \alpha_j = 0. \tag{E.4}$$

Consequently

$$\{\alpha_k,\ \beta_k : 1 \le k \le g\} \tag{E.5}$$

are closed 1-forms whose images in $H^1(M,\mathbb{R})$ are linearly independent, since a linear combination that is exact must integrate to 0 over each curve $\tilde{a}_k$ and $\tilde{b}_k$.

In fact, we have the following.

Proposition E.1. *The images of (E.5) in $H^1(M,\mathbb{R})$ form a basis of this deRham cohomology group.*

For the proof, it remains to show that if φ is a closed 1-form, then φ is cohomologous to $c_1\alpha_1 + \cdots + c_g\alpha_g + d_1\beta_1 + \cdots + d_g\beta_g$, where

$$\int\limits_{\tilde{\alpha}_k} \varphi = c_k, \quad \int\limits_{\tilde{b}_k} \varphi = d_k. \tag{E.6}$$

This is an immediate consequence of the following lemma, applied to $\psi = \varphi - (c_1\alpha_1 + \cdots + d_g\beta_g)$.

Lemma E.2. *If ψ is a closed 1-form on M such that*

$$\int\limits_{\tilde{a}_k} \psi = \int\limits_{\tilde{b}_k} \psi = 0, \quad \forall\, k, \tag{E.7}$$

then ψ is exact.

Proof. Fix a base point $p_0 \in M$ and consider

$$u(z) = \int_{p_0}^{z} \psi. \tag{E.8}$$

If we show that u is single valued on M, we have $\psi = du$. To show u is single valued, it suffices to show that

$$\int\limits_{\gamma} \psi = 0, \tag{E.9}$$

for each piecewise smooth closed curve γ on M. The topological fact that gives this is that $\{\tilde{a}_k, \tilde{b}_k : 1 \le k \le g\}$ generates $\pi_1(M)$, so (E.9) follows from (E.7).

F. Uniformization when $g = 0$ or 1

Here we discuss some proofs of the following results.

Proposition F.1. *If M is a Riemann surface homeomorphic to S^2, then M is conformally diffeomorphic to S^2, with its standard complex structure.*

Proposition F.2. *If M is a Riemann surface homeomorphic to a torus, then there exists a lattice $\Lambda \subset \mathbb{C}$ such that M is conformally diffeomorphic to $\mathbb{C}/\Lambda$.*

These results can be proven using the Riemann-Roch formula. For Proposition F.1, note that, given $p \in M$,

$$g = 0 \Longrightarrow \dim \mathcal{M}(-p) \geq 2, \tag{F.1}$$

so M has a nonconstant meromorphic function f with a simple pole at p. Such f produces a map $f : M \to \mathbb{C} \cup \{\infty\} \approx S^2$, which is easily seen to be a conformal diffeomorphism.

For Proposition F.2, note that

$$g = 1 \Longrightarrow \dim \mathcal{O}(\kappa) = 1, \quad c_1(\kappa) = 0, \tag{F.2}$$

so there exists a nonzero $\zeta \in \mathcal{O}(\kappa)$, and it is nowhere vanishing. Then $\int_p^z \zeta$ provides a holomorphic covering map $M \to \mathbb{C}/\Lambda'$, hence a conformal diffeomorphism $M \to \mathbb{C}/\Lambda$ (actually $\Lambda = \Lambda'$).

We now discuss alternative proofs of Propositions F.1–F.2, not using the Riemann-Roch theorem, but rather some basic PDE.

To start with Proposition F.1, following an argument presented in [MT], we pick a Riemannian metric on M, compatible with its conformal structure. Then pick $p \in M$, and pick $g \in \mathcal{D}'(M)$, supported at p, given in local coordinates as a first-order derivative of δ_p (plus perhaps a multiple of δ_p), such that $\langle 1, g\rangle = 0$. Hence there exists a solution $u \in \mathcal{D}'(M)$ to

$$\Delta u = g. \tag{F.3}$$

Then $u \in C^\infty(M \setminus p)$ and u is harmonic on $M \setminus p$ and has a $\operatorname{dist}(x,p)^{-1}$ type singularity. Now if M is homeomorphic to S^2, then $M \setminus p$ is simply connected, so u has a single-valued harmonic conjugate on $M \setminus p$, given by $v(x) = \int_q^x *du$, where we pick $q \in M \setminus p$. We see that v also has a $\operatorname{dist}(x,p)^{-1}$ type singularity. Then $f = u + iv$ is holomorphic on $M \setminus p$ and has a simple pole at p, so as argued above f provides a conformal diffeomorphism of M with the standard Riemann sphere.

Moving on to Proposition F.2, again we put a compatible Riemannian metric g on M. We claim there exists $u \in C^\infty(M)$ such that $g' = e^{2u} g$ is a flat Riemannian metric, i.e., has Gauss curvature zero. Given this, it is an elementary geometric fact that (M, g') is isometric to $\mathbb{R}^2/\Lambda$ for some lattice Λ, and Proposition F.2 follows. To get such u, we use the fact that if g has Gauss curvature $k \in C^\infty(M)$, then g' has Gauss curvature

(F.4) $$K = (-\Delta u + k)e^{-2u}.$$

Cf. [T], Appendix C, (3.46). To achieve $K \equiv 0$, we must solve

(F.5) $$\Delta u = k \quad \text{on} \quad (M, g).$$

This has a smooth solution as long as $\int_M k\, dS_g = 0$. Now the Gauss-Bonnet theorem gives

(F.6) $$\int\limits_M k\, dS_g = 2\pi\chi(M) = 0,$$

so (F.5) is solvable and hence such a proof of Proposition F.2 works.

We mention the rest of the Uniformization Theorem for compact Riemann Surfaces:

Proposition F.3. *If M is a compact Riemann surface of genus $g \ge 2$, then M has a compatible metric tensor g' with Gauss curvature $K \equiv -1$. Hence M is covered by the Poincaré disk.*

To prove this, one can start with some compatible metric tensor g and seek $g' = e^{2u}g$ so that (F.4) holds with $K \equiv -1$, i.e., so that u solves

(F.7) $$\Delta u = k + e^{-2u}.$$

A proof of such solvability, due to M. Berger, is presented in [T], Chapter 14, §2.

Remark. Different PDE proofs of Propositions F.1–F.3 can be found in [J].

Remark. For a different proof of Proposition F.1 (observed in [Don], pp. 112–114), note that if M is a compact Riemann surface of genus g, then, by the analysis of (D.12) leading to Proposition D.1 (see also (I.3)–(I.4) below),

(F.8) $$\overline{\partial} : \mathcal{D}'(M) \longrightarrow \mathcal{D}'(M, \overline{\kappa})$$

has range of codimension g. Hence, if $g = 0$, $\overline{\partial}$ is surjective. Thus we can solve

(F.9) $$\overline{\partial} f = \delta_p,$$

where δ_p is a delta function on M (adjusted to be a section of $\overline{\kappa}$), to get the desired meromorphic function on M with one simple pole.

G. Embeddings into complex projective spaces

If $L \to M$ is a holomorphic line bundle over a compact Riemann surface M and if $\dim \mathcal{O}(L) = k \geq 2$, then, under a condition to be specified below, we can define a holomorphic map

$$\varphi_L : M \longrightarrow \mathbb{CP}^{k-1}, \tag{G.1}$$

as follows. Pick a basis $u_1, \dots, u_k$ of $\mathcal{O}(L)$. Then

$$z \mapsto [u_1(z), \dots, u_k(z)] \tag{G.2}$$

defines (G.1). In more detail, if $U \subset M$ is an open set on which there is a nonvanishing holomorphic section v of L, we can define

$$\varphi_L : U \longrightarrow \mathbb{CP}^{k-1}, \quad \varphi_L(z) = \Big[\frac{u_1(z)}{v(z)}, \dots, \frac{u_k(z)}{v(z)}\Big], \tag{G.3}$$

where, for $z \in U$, $u_j(z)/v(z) \in \mathbb{C}$, and the right side is the class in $\mathbb{CP}^{k-1}$ of $(u_1(z)/v(z), \dots, u_k(z)/v(z)) \in \mathbb{C}^k$. For this to be well defined, we need the following condition on L:

$$\text{For each } p \in M, \text{ there exists } u \in \mathcal{O}(L) \text{ such that } u(p) \neq 0. \tag{G.4}$$

Then for each $z \in U$, $(u_1(z)/v(z), \dots, u_k(z)/v(z)) \in \mathbb{C}^k \setminus 0$, so (G.3) is well defined. It follows easily that (G.3) is independent of the choice of the nonvanishing section v of L over U. Hence, for open U_j to which (G.3) applies, the maps coincide on $U_1 \cap U_2$, so (G.1) is a well defined holomorphic map. The following is easy to verify.

Proposition G.1. *Assume $\dim \mathcal{O}(L) = k \geq 2$ and (G.4) holds. Then φ_L in (G.1) is one-to-one if and only if the following holds:*

(G.5) *Given $p, q \in M$, $p \neq q$, there exist $u \in \mathcal{O}(L)$ such that $u(p) = 0$, $u(q) \neq 0$.*

The map φ_L has injective derivative if and only if the following holds:
(G.6)
Given $p \in M$, there exists $u \in \mathcal{O}(L)$ such that u vanishes to just first order at p.

If both (G.5) and (G.6) hold, $\varphi_L : M \to \mathbb{CP}^{k-1}$ is an embedding.

Remark. Given a different choice of basis $\tilde{u}_1, \dots, \tilde{u}_k$ of $\mathcal{O}(L)$, one gets a map that agrees with (G.1) up to a holomorphic diffeomorphism $\mathbb{CP}^{k-1} \to \mathbb{CP}^{k-1}$. As a related matter, we can give a version of φ_L that avoids picking a basis of $\mathcal{O}(L)$, as follows. If V is a complex vector space, let $\mathbb{CP}(V)$ denote the manifold of one-dimensional complex subspaces of V, so $\mathbb{CP}(\mathbb{C}^k) = \mathbb{CP}^{k-1}$. Assume (G.4) holds. Then, for each $p \in M$,

$$\mathcal{L}_p = \{u \in \mathcal{O}(L) : u(p) = 0\} \subset \mathcal{O}(L)$$

is a codimension 1 linear subspace, whose annihilator

$$\mathcal{L}_p^{\perp} \subset \mathcal{O}(L)'$$

is a one-dimensional linear subspace. We have

$$\varphi_L : M \longrightarrow \mathbb{CP}(\mathcal{O}(L)'), \quad \varphi_L(p) = \mathcal{L}_p^{\perp}.$$

Let us specialize to the case $L = \kappa$, the canonical bundle, for which $\dim \mathcal{O}(\kappa) = g$. That (G.4) holds whenever $g \geq 2$ follows from Proposition 3.1. Hence, whenever M has genus $g \geq 2$, we have

$$\varphi_\kappa : M \longrightarrow \mathbb{CP}^{g-1}, \tag{G.7}$$

called the canonical map. In case $g = 2$, we have $\varphi_\kappa : M \to \mathbb{CP}^1 \approx S^1$. This is a branched cover, but obviously not an embedding. For $g \geq 3$, we have the following.

Proposition G.2. *If M is not hyperelliptic, the canonical map $\varphi_\kappa : M \to \mathbb{CP}^{g-1}$ is an embedding.*

In light of Proposition G.1, to prove Proposition G.2 we need only establish the following.

Proposition G.3. *Assume M is not hyperelliptic. Then the following hold.*
(a) Given $p, q \in M,\ p \neq q$, there exist $u \in \mathcal{O}(\kappa)$ such that $u(p) = 0,\ u(q) \neq 0$.
(b) Given $p \in M$, there exists $u \in \mathcal{O}(\kappa)$ with a simple zero at p.

Proof. The proof is somewhat parallel to that of Proposition 3.1. As there, we consider

$$\mathcal{M}(\kappa, p) = \{u \in \mathcal{O}(\kappa) : u(p) = 0\} \approx \mathcal{O}(\kappa \otimes E_p). \tag{G.8}$$

Now we also consider

$$\mathcal{M}(\kappa, p, q) = \{u \in \mathcal{O}(\kappa) : u(p) = u(q) = 0\} \approx \mathcal{O}(\kappa \otimes E_{p+q}). \tag{G.9}$$

The content of (a) is that

$$\dim \mathcal{M}(\kappa, p) > \dim \mathcal{M}(\kappa, p, q), \tag{G.10}$$

provided M is not hyperelliptic. As seen in (3.5),

$$g \geq 1 \Longrightarrow \dim \mathcal{M}(\kappa, p) = g - 1. \tag{G.11}$$

Meanwhile, the Riemann-Roch formula gives

$$\dim \mathcal{O}(\kappa \otimes E_{p+q}) - \dim \mathcal{O}(\kappa^{-1} \otimes E_{p+q}^{-1} \otimes \kappa) = c_1(\kappa \otimes E_{p+q}) - \frac{1}{2} c_1(\kappa). \tag{G.12}$$

Supplementing (3.3), we have

$$c_1(E_{p+q}) = -2, \quad c_1(\kappa \otimes E_{p+q}) = c_1(\kappa) + c_1(E_{p+q}). \tag{G.13}$$

Also $\mathcal{O}(\kappa^{-1} \otimes E_{p+q}^{-1} \otimes \kappa) \approx \mathcal{O}(E_{p+q}^{-1}) = \mathcal{M}(-p-q)$. Thus we have

$$\dim \mathcal{M}(\kappa, p, q) = \dim \mathcal{M}(-p-q) + g - 3, \tag{G.14}$$

and this equals (G.11) if and only if

$$\dim \mathcal{M}(-p-q) = 2. \tag{G.15}$$

As noted in (2.24), if there exist $p, q \in M$ such that (G.15) holds, then M must be hyperelliptic. This proves the first part of Proposition G.3.

For the second part of Proposition G.3, we consider

$$\mathcal{M}(\kappa, 2p) = \{u \in \mathcal{O}(\kappa) : u \text{ has a double zero at } p\} \approx \mathcal{O}(\kappa \otimes E_{2p}), \tag{G.16}$$

and note that the content of (b) is that, for each $p \in M$,

$$\dim \mathcal{M}(\kappa, p) > \dim \mathcal{M}(\kappa, 2p), \tag{G.17}$$

provided M is not hyperelliptic. This is established by the same sort of arguments used above to establish (G.11).

Similar arguments involving the Riemann-Roch formula can be used to prove the following.

Proposition G.4. *If $L \to M$ is a holomorphic line bundle such that $c_1(L) > 2g$, then the map φ_L in (G.1) is well defined and is an embedding.*

Remark. In such a case, (A.10) implies $\dim \mathcal{O}(L) = c_1(L) - g + 1 > g + 1$.

We turn attention to the case where M is hyperelliptic and see how Proposition G.3 fails. Such M yields a 2-fold branched covering $\varphi : M \to S^2$. If $p \in M$ is not a branch point, pick $q \neq p$ such that $\varphi(p) = \varphi(q)$. Then $\dim \mathcal{M}(-p-q) \geq 2$, and hence by (G.11) and (G.14) $\dim \mathcal{M}(\kappa, p) = \dim \mathcal{M}(\kappa, p, q)$, so $\varphi_\kappa(p) = \varphi_\kappa(q)$. On the other hand, if $\varphi(p) \neq \varphi(q)$, then any $f \in \mathcal{M}(-p-q)$ must be constant, so $\dim \mathcal{M}(-p-q) = 1$ and the arguments in Proposition G.3 show that $\varphi_\kappa(p) \neq \varphi_\kappa(q)$. The conclusion is:

Proposition G.5. *If M is hyperelliptic, then φ_κ maps M as a 2-to-1 branched cover of its image in $\mathbb{CP}^{g-1}$.*

Let's next look at

$$\varphi_{\kappa^2} : M \longrightarrow \mathbb{CP}^{3g-4}, \tag{G.18}$$

where $\kappa^2 = \kappa \otimes \kappa$. Note that

$$c_1(\kappa^2) = 4g - 4, \tag{G.19}$$

and $\mathcal{O}(\kappa^{-2} \otimes \kappa) = 0$ for $g \geq 2$, so, by (A.1)

$$g \geq 2 \Longrightarrow \dim \mathcal{O}(\kappa^2) = 3g - 3. \tag{G.20}$$

From Proposition G.4 we have:

Proposition G.6. *The map φ_{κ^2} in (G.18) is an embedding as long as $g \geq 3$, whether or not M is hyperelliptic.*

When $g = 2$, (G.19) gives $c_1(\kappa^2) = 4 = 2g$, and Proposition G.4 does not apply. In fact, the following holds.

Proposition G.7. *If M has genus 2, $\varphi_{\kappa^2} : M \to \mathbb{CP}^2$ is a 2-to-1 branched covering of its image.*

Proof. We check whether (G.5) and (G.6) hold, for $L = \kappa^2,\ g = 2$. Parallel to the proof of Proposition G.3, this amounts to checking whether

$$\dim \mathcal{M}(\kappa^2, p) > \dim \mathcal{M}(\kappa^2, p, q) \tag{G.21}$$

and

$$\dim \mathcal{M}(\kappa^2, p) > \dim \mathcal{M}(\kappa^2, 2p), \tag{G.22}$$

where

$$\mathcal{M}(\kappa^2, p) \approx \mathcal{O}(\kappa^2 \otimes E_p), \quad \mathcal{M}(\kappa^2, p, q) \approx \mathcal{O}(\kappa^2 \otimes E_{p+q}). \tag{G.23}$$

The Riemann-Roch theorem gives

$$\dim \mathcal{O}(\kappa^2 \otimes E_p) = c_1(\kappa^2 \otimes E_p) - \frac{1}{2} c_1(\kappa), \tag{G.24}$$

and

$$\dim \mathcal{O}(\kappa^2 \otimes E_{p+q}) = \dim \mathcal{O}(\kappa^{-1} \otimes E_{-p-q}) + c_1(\kappa^2 \otimes E_{p+q}) - \frac{1}{2} c_1(\kappa), \tag{G.25}$$

hence

$$\dim \mathcal{O}(\kappa^2 \otimes E_p) - \dim \mathcal{O}(\kappa^2 \otimes E_{p+q}) = 1 - \dim \mathcal{O}(\kappa^{-1} \otimes E_{-p-q}). \tag{G.26}$$

Given that $g = 2$, we have $c_1(\kappa^2 \otimes E_{-p-q}) = 0$, and

$$1 - \dim \mathcal{O}(\kappa^2 \otimes E_{-p-q}) = \begin{matrix} 0 & \text{if} \ \ \kappa \approx E_{-p-q} \\ 1 & \text{if} \ \ \kappa \text{ not } \approx E_{-p-q}. \end{matrix} \tag{G.27}$$

Furthermore, when $g = 2$,

$$\kappa \approx E_{-p-q} \Longleftrightarrow \mathcal{M}(\kappa, p, q) \neq 0. \tag{G.28}$$

Constructions of holomorphic 1-forms made in §1 and in §H readily show that when $g = 2$ and $\varphi : M \to S^2$ is a 2-to-1 branched covering, if $p \neq q \in M$ have the same image under φ, there is a nonzero $u \in \mathcal{O}(\kappa)$ vanishing at p and q. Consequently

$$\begin{aligned} p \neq q \in M, \ \varphi(p) = \varphi(q) &\Longrightarrow \dim \mathcal{M}(\kappa^2, p) = \dim \mathcal{M}(\kappa^2, p, q) \\ &\Longrightarrow \varphi_{\kappa^2}(p) = \varphi_{\kappa^2}(q), \end{aligned} \tag{G.29}$$

and Proposition G.7 is proven.

Note that

$$g = 2 \Longrightarrow c_1(\kappa^3) = 6g - 6 = 6, \quad \text{and} \ \dim \mathcal{O}(\kappa^3) = 5g - 5 = 5, \tag{G.30}$$

and, by Proposition G.4,

$$\varphi_{\kappa^3} : M \longrightarrow \mathbb{CP}^4 \tag{G.31}$$

is an embedding. A basis is $\mathcal{O}(\kappa^3)$ is described in §H. This follows a description of a basis of $\mathcal{O}(\kappa^2)$, which leads to an alternative proof of Proposition G.7.

H. The canonical ring of a genus 2 surface

Let M be a compact Riemann surface of genus 2. Thus M is hyperelliptic, and there exists a 2-to-1 holomorphic branched cover

$$\varphi : M \longrightarrow S^2 = \mathbb{C} \cup \{\infty\}, \tag{H.1}$$

branching over six points $a_j \in S^2,\ 1 \le j \le 6$. We can assume that $\infty \notin \{a_j\}$ and $0 \notin \{a_j\}$.

Denote by κ the canonical bundle over M and by $\mathcal{O}(\kappa)$ the space of its holomorphic sections, i.e., the holomorphic 1-forms on M. In this case, $\dim \mathcal{O}(\kappa) = 2$ and a basis of $\mathcal{O}(\kappa)$ is given by

$$\omega_1 = \varphi^* \frac{dz}{\sqrt{q(z)}} = \frac{d\varphi}{\sqrt{q(\varphi)}}, \quad \omega_2 = \varphi\omega_1, \tag{H.2}$$

where

$$q(z) = \prod_{j=1}^{6} (z - a_j). \tag{H.3}$$

We desire to produce bases of each space $\mathcal{O}(\kappa^n),\ n = 2, 3, \dots$. By the Riemann-Roch formula,

$$\dim \mathcal{O}(\kappa^n) = c_1(\kappa^n) - \frac{1}{2} c_1(\kappa) = 2n - 1, \tag{H.4}$$

when the genus $g = 2$. Here is a first result.

Lemma H.1. *A basis of $\mathcal{O}(\kappa^2)$ is*

$$\omega_1^2, \omega_1\omega_2, \omega_2^2. \tag{H.5}$$

Proof. The form ω_1 has exactly two simple zeros, at $\varphi^{-1}(\infty)$; similarly ω_2 has exactly two simple zeros, at $\varphi^{-1}(0)$. From this it is clear that the set (H.5) is linearly independent. Since $\dim \mathcal{O}(\kappa^2) = 3$, the result follows.

Note that

$$\dim \mathcal{O}(\kappa^3) = 5. \tag{H.6}$$

To form a basis of $\mathcal{O}(\kappa^3)$, we take

$$\omega_1^3, \omega_1^2\omega_2, \omega_1\omega_2^2, \omega_2^3, \tag{H.7}$$

and add a fifth element:

$$W = \omega_1^2\, d\varphi. \tag{H.8}$$

This is the "Wronskian" of ω_1 and ω_2. (Cf. (B.9).) The double zeros of ω_1^2 cancel the double poles of $d\varphi$, and W is holomorphic, with zeros at the Weierstrass points of M.

Lemma H.2. *A basis of $\mathcal{O}(\kappa^3)$ consists of (H.7)–(H.8).*

Proof. Given (H.6), it suffices to show that (H.7)–(H.8) is a linearly independent set. Suppose

$$\sum_{j=0}^{3} b_j \omega_2^j \omega_1^{3-j} + cW = 0. \tag{H.9}$$

We use the identities

$$\omega_2 = \varphi\omega_1, \quad \omega_1 = \frac{d\varphi}{\sqrt{q(\varphi)}}, \quad W = \omega_1^2\, d\varphi = \sqrt{q(\varphi)}\,\omega_1^3 \tag{H.10}$$

to deduce from (H.9) that

$$\sum_{j=0}^{3} b_j \varphi^j \omega_1^3 + c\sqrt{q(\varphi)}\,\omega_1^3 = 0, \tag{H.11}$$

hence

$$\sum_{j=0}^{3} b_j \varphi^j + c\sqrt{q(\varphi)} = 0, \tag{H.12}$$

which forces $b_0 = \cdots = b_3 = c = 0$.

The following is a natural extension.

Proposition H.3. *For $n \geq 3$, a basis of $\mathcal{O}(\kappa^n)$ consists of*

$$\omega_2^j \omega_1^{n-j},\ 0 \leq j \leq n, \quad \textit{and} \ \ \omega_2^k \omega_1^{n-3-k} W,\ 0 \leq k \leq n-3. \tag{H.13}$$

Proof. Given (H.4), it remains to show that the holomorphic sections of κ^n listed in (H.13) are linearly independent. Suppose

$$\sum_{j=0}^{n} b_j \omega_2^j \omega_1^{n-j} + \sum_{k=0}^{n-3} c_k \omega_2^k \omega_1^{n-3-k} W = 0. \tag{H.14}$$

Using (H.10) again, we obtain

$$\sum_{j=0}^{k} b_j \varphi^j \omega_1^n + \sum_{k=0}^{n-3} c_k \varphi^k \sqrt{q(\varphi)}\omega_1^n = 0, \tag{H.15}$$

hence

$$\sum_{j=0}^{n} b_j \varphi^j + \sum_{k=0}^{n-3} c_k \varphi^k \sqrt{q(\varphi)} = 0, \tag{H.16}$$

which forces $b_0 = \cdots = b_n = c_0 = \cdots = c_{n-3} = 0$.

Note that

$$W^2 = q(\varphi)\omega_1^6 = \tilde{q}(\omega_2, \omega_1), \tag{H.17}$$

where

$$\tilde{q}(z, w) = \prod_{j=1}^{6} (z - a_j w) = q(z/w) w^6. \tag{H.18}$$

We deduce that the canonical ring

$$\bigoplus_{n \geq 0} \mathcal{O}(\kappa^n) \tag{H.19}$$

is isomorphic to

$$\mathbb{C}[\omega_1, \omega_2, W]/(W^2 - \tilde{q}(\omega_2, \omega_1)), \tag{H.20}$$

when M has genus 2.

For results on the canonical ring when M is not hyperelliptic, see [ACGH], Chapter 3, §3.

We relate basic constructions done above to results on maps of M into complex projective spaces discussed in §G. We continue to assume M has genus 2. From (H.2) we see that

$$\varphi_\kappa = \varphi : M \longrightarrow \mathbb{CP}^1, \tag{H.21}$$

where φ is the 2-to-1 branched covering in (H.1). From Lemma H.1 we see that

$$\varphi_{\kappa^2} = \mathcal{V} \circ \varphi_\kappa, \tag{H.22}$$

where

$$\mathcal{V} : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2 \tag{H.23}$$

is the embedding given in homogeneous coordinates by

$$[u, v] = [u^2, uv, v^2]. \tag{H.24}$$

This observation gives another proof of Proposition G.7. From Lemma H.2 we see that

$$\varphi_{\kappa^3} : M \longrightarrow \mathbb{CP}^4 \tag{H.25}$$

is given by

$$\varphi_{\kappa^3}(p) = [1, \varphi(p), \varphi(p)^2, \varphi(p)^3, \sqrt{q(\varphi(p))}], \tag{H.26}$$

away from $\varphi^{-1}(\infty)$, and by

$$\varphi_{\kappa^3}(p) = \left[\varphi(p)^{-3}, \varphi(p)^{-2}, \varphi(p)^{-1}, 1, \varphi(p)^{-3}\sqrt{q(\varphi(p))}\right], \tag{H.27}$$

near $\varphi^{-1}(\infty)$, extended by continuity to $\varphi^{-1}(\infty)$. Two points $p_1 \neq p_2$ with the same image under φ (and hence under φ_κ and φ_{κ^2}) have different images under φ_{κ^3}, because

$$\sqrt{q(\varphi(p_1))} = -\sqrt{q(\varphi(p_2))}, \tag{H.28}$$

and $q(\varphi(p_j)) \neq 0$. If $p_1 \neq p_2 \in \varphi^{-1}(\infty)$,

$$\{\varphi_{\kappa^3}(p_j) : j = 1, 2\} = \{[0, 0, 0, 1, \varepsilon] : \varepsilon = \pm 1\}. \tag{H.29}$$

Note also that φ_{κ^3} descends to an embedding

$$\psi : M \longrightarrow \mathbb{CP}^2, \tag{H.30}$$

given by

$$\psi(p) = [1, \varphi(p), \sqrt{q(\varphi(p))}], \tag{H.31}$$

away from $\varphi^{-1}(\infty)$, and

$$\psi(p) = \left[\varphi(p)^{-3}, \varphi(p)^{-2}, \varphi(p)^{-3}\sqrt{q(\varphi(p))}\right], \tag{H.32}$$

near $\varphi^{-1}(\infty)$, again extended by continuity to $\varphi^{-1}(\infty)$.

I. Holomorphic vector bundles and Riemann-Roch

Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface M. The Riemann-Roch formula in this situation is

$$\dim \mathcal{O}(E) - \dim \mathcal{O}(E' \otimes \kappa) = c_1(E) - \frac{\text{rank } E}{2} c_1(\kappa). \tag{I.1}$$

Here E' is the dual bundle to E, rank E is the complex fiber dimension of E, and, as before, $c_1(\kappa) = -\chi(M) = 2g - 2$. Also,

$$c_1(E) = c_1(\det E), \quad \det E = \Lambda^k E, \quad k = \text{rank } E. \tag{I.2}$$

This can be deduced from the Atiyah-Singer index theorem in the same way (A.1) is established in [T1]. To begin, one has

$$\overline{\partial}_E : C^\infty(M, E) \longrightarrow C^\infty(M, E \otimes \overline{\kappa}), \tag{I.3}$$

with adjoint

$$\overline{\partial}_E^t : C^\infty(M, E' \otimes \kappa) \longrightarrow C^\infty(M, E \otimes \kappa \otimes \overline{\kappa}); \tag{I.4}$$

a Hermitian metric on the complex tangent space of M gives rise to a trivialization of $\kappa \otimes \overline{\kappa}$ and a duality of $L^2(M, E \otimes \overline{\kappa})$ with $L^2(M, E' \otimes \kappa)$. Hence the left side of (I.1) is equal to the index of $\overline{\partial}_E$. As in (9.14)–(9.15) of [T1], we have

$$\text{Index}\, \overline{\partial}_E = \Big\langle e^{-c_1(\kappa)/2}\, \text{Ch}(E) \widehat{A}(M), [M] \Big\rangle, \tag{I.5}$$

and, as there, $\widehat{A}(M) = 1$ since M has real dimension 2. Furthermore, as in (6.36) of [T1],

$$\text{Ch}(E) = \text{Tr}\, e^{-\Phi/2\pi i}, \tag{I.6}$$

where Φ is the End(E)-valued curvature 2-form of a connection on E. Hence

$$e^{-c_1(\kappa)/2}\, \text{Ch}(E) = c_1(E) - \frac{k}{2} c_1(\kappa), \tag{I.7}$$

where $k = \text{rank } E$, and we have (I.1).

In [HSW] there is a different proof of (I.1). This work starts with the Riemann-Roch theorem for line bundles, proven a la Serre (as presented in [Gun]), and then proves (I.1) by induction on the rank of E. To do this, the following is brought in.

Lemma I.1. *If $E \to M$ is a holomorphic vector bundle over a compact Riemann surface, then E has a holomorphic line subbundle L.*

From there the work applies homological algebra to the sequence

$$0 \longrightarrow \mathcal{O}(L) \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}(E/L) \longrightarrow 0, \tag{I.8}$$

which is exact on the level of sheaves, plus the fact that

$$\dim \mathcal{O}(E' \otimes \kappa) = \dim H^1(M, \mathcal{O}(E)), \tag{I.9}$$

which follows from Serre duality.

We sketch a proof of Lemma I.1, similar to that of [HSW], except we take as already known the identity (I.1), which implies

$$\dim \mathcal{O}(E) \geq c_1(E) - k(g-1). \tag{I.10}$$

For a line bundle $L \to M$ to be isomorphic to a subbundle of E, it is necessary and sufficient for there to be a nowhere vanishing holomorphic section of $L' \otimes E$. Now (I.10), with E replaced by $L' \otimes E$, implies

$$\dim \mathcal{O}(L' \otimes E) \geq c_1(L') + c_1(E) - k(g-1), \tag{I.11}$$

so

$$c_1(L') \geq 1 - c_1(E) + k(g-1) \Longrightarrow \dim \mathcal{O}(L' \otimes E) \geq 1. \tag{I.12}$$

As noted in (A.3)–(A.6), the bundle E_ϑ of a divisor ϑ satisfies

$$c_1(E_\vartheta) = -\deg \vartheta. \tag{I.13}$$

Sp we can pick $p \in M$ and take

$$L_0 = E_{mp}, \quad m \geq 1 - c_1(E) + k(g-1), \tag{I.14}$$

and then $c_1(L_0') = m$, so

$$\dim \mathcal{O}(L_0' \otimes E) \geq 1. \tag{I.15}$$

Take a section $u \in \mathcal{O}(L_0' \otimes E)$, not identically zero. If u is nowhere vanishing, we are done, with $L = L_0$. If not, u has a finite number of zeros, say

$$\vartheta(u) = \sum n_j p_j = \vartheta_u. \tag{I.16}$$

The bundle E_{ϑ_u} has a natural meromorphic section v with divisor $-\theta_u$, and then $v \otimes u$ is a nowhere vanishing holomorphic section of $L' \otimes E$, where

$$L' = E_{\vartheta_u} \otimes L_0', \quad \text{i.e.,} \;\; L = E_{-\vartheta_u} \otimes L_0. \tag{I.17}$$

This finishes the proof of Lemma I.1.

J. Stable vector vector bundles

Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface M. The identity I is a holomorphic section of $\operatorname{End} E$. For a class of bundles called *stable*, the only holomorphic sections of $\operatorname{End} E$ are constant multiples of I, so

$$E \text{ stable} \Longrightarrow \dim \mathcal{O}(\operatorname{End} E) = 1. \tag{J.1}$$

We can apply (I.1) with E replaced by

$$\operatorname{End} E = E' \otimes E. \tag{J.2}$$

Note that this bundle is isomorphic to its dual, and

$$c_1(\operatorname{End} E) = 0, \tag{J.3}$$

so (I.1) gives

$$\dim \mathcal{O}(\operatorname{End} E) - \dim \mathcal{O}(\kappa \otimes \operatorname{End} E) = k^2(1-g), \tag{J.4}$$

with $k = \text{rank } E$. Consequently, by (J.1),

$$E \text{ stable} \Longrightarrow \dim \mathcal{O}(\kappa \otimes \operatorname{End} E) = 1 + k^2(g-1). \tag{J.5}$$

Given a holomorphic section $A \in \mathcal{O}(\kappa \otimes \operatorname{End} E)$, one can form the characteristic polynomial

$$\det(\lambda I - A) = \lambda^k + \alpha_1 \lambda^{k-1} + \cdots + \alpha_k, \quad \alpha_j \in \mathcal{O}(\kappa^j). \tag{J.6}$$

Thus we can regard

$$\det(\lambda I - A) \in \mathcal{O}(\kappa \oplus \cdots \oplus \kappa^k). \tag{J.7}$$

It follows from (A.2) and (A.10)–(A.13) that

$$\dim \mathcal{O}(\kappa) = g, \quad \text{(cf. also (D.16))} \tag{J.8}$$

and

$$g > 1,\ j > 1 \Longrightarrow \dim \mathcal{O}(\kappa^j) = (2j-1)(g-1). \tag{J.9}$$

Hence, when $g > 1$,

$$\dim(\kappa \oplus \cdots \oplus \kappa^k) = g + \sum_{j=2}^{k} (2j-1)(g-1) = 1 + k^2(g-1), \tag{J.10}$$

which is equal to the dimension in (J.5).

As an alternative to (J.7), one can consider

$$\det(\alpha I - A) \in \mathcal{O}(\kappa^k), \quad \text{given} \ \ \alpha \in \mathcal{O}(\kappa). \tag{J.11}$$

One also looks at the *spectral curve* of $A \in \mathcal{O}(\kappa \otimes \operatorname{End} E)$, which consists of points α in the total space $\kappa \to M$ on which

$$\det(\alpha I - A) = 0. \tag{J.12}$$

I'm not sure why, but this seems to connect to the area of integrable systems.

Note that we can take any holomorphic line bundle $L \to M$ and consider

$$\alpha \in \mathcal{O}(L), \quad A \in \mathcal{O}(L \otimes \operatorname{End} E), \quad \det(\alpha I - A) \in \mathcal{O}(L^k). \tag{J.13}$$

The special relevance of (J.11) seems to be connected to the fact that $\mathcal{O}(\kappa \otimes \operatorname{End} E)$ is naturally isomorphic to the cotangent space at E of the moduli space $\mathcal{M}$ of stable rank k holomorphic vector bundles over M (assuming E is stable).

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