Bessel Functions and Hankel Transforms

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1. Bessel functions

Bessel functions arise as a natural generalization of harmonic analysis of radial functions. To see this, let $F(x)$ be a radial function on $\mathbb{R}^n$, $F(x) = f(|x|)$. Then

$$\widehat{F}(\xi) = (2\pi)^{-n/2} \int_0^\infty f(r)\psi_n(r|\xi|)r^{n-1}\,dr, \tag{1.1}$$

where

$$\psi_n(|\xi|) = \Psi_n(\xi) = \int\limits_{S^{n-1}} e^{i\xi\cdot\omega}\,dS(\omega). \tag{1.2}$$

For $n \geq 2$, we get

$$\psi_n(r) = A_{n-2} \int_{-1}^1 e^{irs}(1-s^2)^{(n-3)/2}\,ds. \tag{1.3}$$

Recall $A_{n-2} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$. It is standard to write

$$\psi_n(r) = (2\pi)^{n/2} r^{1-n/2} J_{n/2-1}(t), \tag{1.4}$$

where, for $\operatorname{Re}\nu > -1/2$, J_ν is the Bessel function, defined by

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^1 (1-t^2)^{\nu-1/2} e^{izt}\,dt. \tag{1.5}$$

(Shortly we will define J_ν for all $\nu \in \mathbb{C}$.) Then (1.1) becomes

$$\widehat{F}(\xi) = |\xi|^{1-n/2} \int_0^\infty f(r) J_{n/2-1}(r|\xi|) r^{n/2}\,dr. \tag{1.6}$$

This is a special case of a Hankel transform. In general, we define the (modified) Hankel transform $\widetilde{H}_\nu$ by

$$\begin{aligned} \widetilde{H}_\nu f(\lambda) &= \int_0^\infty f(r) \frac{J_\nu(\lambda r)}{(\lambda r)^\nu} r^{2\nu+1}\,dr \\ &= \lambda^{-\nu} \int_0^\infty f(r) J_\nu(\lambda r) r^{\nu+1}\,dr. \end{aligned} \tag{1.7}$$

From (1.6), we have

$$\widehat{F}(\xi) = \widetilde{H}_{n/2-1} f(|\xi|). \tag{1.8}$$

We can compare (1.8) with other Euclidean space results results, coming from

$$\begin{aligned} \widehat{F}(x) &= (2\pi)^{n/2} f(\sqrt{-\Delta})\delta(x) \\ &= (2\pi)^{(n-1)/2} \Phi_n(|x|), \end{aligned} \tag{1.9}$$

where

$$\Phi_{2k+1}(r) = \Bigl(-\frac{1}{2\pi r}\frac{d}{dr}\Bigr)^k \hat{f}(r), \tag{1.10}$$

and

$$\Phi_{2k}(r) = 2\int_r^\infty \Phi_{2k+1}(s) \frac{s}{\sqrt{s^2-r^2}}\, ds. \tag{1.11}$$

See [T].

One consequence of (1.9)–(1.11), coupled to (1.8), is that

$$-\frac{1}{r}\frac{d}{dr}\widetilde{H}_\nu f(r) = \widetilde{H}_{\nu+1} f(r), \tag{1.12}$$

at least for $\nu = n/2 - 1,\ n \geq 2$. This is equivalent to

$$\frac{1}{2}\frac{d}{dr}\frac{J_\nu(r)}{r^\nu} = -\frac{J_{\nu+1}(r)}{r^{\nu+1}}, \tag{1.13}$$

which in turn is equivalent to

$$\frac{d}{dz}J_\nu(z) - \frac{\nu}{z}J_\nu(z) = -J_{\nu+1}(z). \tag{1.14}$$

This can be verified for all ν with $\operatorname{Re}\nu > 1/2$, directly by differentiating the integral formula (1.5), applying an integration by parts, and using

$$\Gamma(s+1) = s\Gamma(s), \tag{1.15}$$

with $s = \nu + 1/2$. A similar argument also gives the identity

$$\frac{d}{dz}J_\nu(z) + \frac{\nu}{z}J_\nu(z) = J_{\nu-1}(z), \tag{1.16}$$

initially for $\operatorname{Re}\nu > 1/2$. Having this, since the left side of (1.16) is well defined for $\operatorname{Re}\nu > -1/2$, so is the right side, i.e., $J_\nu(z)$ is analytically continued to $\{\operatorname{Re}\nu >$

$-3/2\}$. We can iterate, and analytically continue $J_\nu(z)$ to all $\nu \in \mathbb{C}$. Then the identity (1.14) also holds for all $\nu \in \mathbb{C}$. Putting together (1.14) and (1.16), we see that $J_\nu(z)$ solves the following second order differential equation,

$$\Bigl[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \Bigl(1 - \frac{\nu^2}{z^2}\Bigr)\Bigr] J_\nu(z) = 0, \tag{1.17}$$

known as Bessel's equation.

From (1.5) it is elementary to see that

$$J_{1/2}(z) = \Bigl(\frac{2}{\pi z}\Bigr)^{1/2} \sin z. \tag{1.18}$$

Applying (1.16) to (1.18) gives

$$J_{-1/2}(z) = \Bigl(\frac{2}{\pi z}\Bigr)^{1/2} \cos z, \tag{1.19}$$

while applying (1.14) repeatedly to (1.18) gives

$$J_{k+1/2}(z) = (-1)^k \Bigl\{\prod_{j=1}^{k} \Bigl(\frac{d}{dz} - \frac{j-1/2}{z}\Bigr)\Bigr\} \frac{\sin z}{\sqrt{2\pi z}}. \tag{1.20}$$

Note that, while we needed $n \ge 2$ in (1.3)–(1.4), (1.19) also gives (1.6) for $n = 1$, and, via (1.14), the contact with (1.9)–(1.10) is complete, for all odd n.

To complete the contact of (1.6) with (1.11) for $n = 2k$, by (1.14) it is sufficient to treat the case $k = 1$, i.e., $n = 2$. Thus we are comparing the formula

$$\widehat{F}(x) = \int_0^\infty f(r) J_0(r|x|) r\, dr, \tag{1.21}$$

for radial $F \in \mathcal{S}(\mathbb{R}^2)$ with

$$\widehat{F}(x) = \sqrt{2\pi}\, \Phi_2(|x|) = 2\sqrt{2\pi} \int_{|x|}^\infty \Phi_3(t) \frac{t}{\sqrt{t^2 - |x|^2}}\, dt. \tag{1.22}$$

By (1.10),

$$\Phi_3(t) = -\frac{1}{2\pi t}\frac{d}{dt}\hat{f}(t), \tag{1.23}$$

so

$$\widehat{F}(x) = -\sqrt{\frac{2}{\pi}} \int_{|x|}^\infty \hat{f}'(t) \frac{1}{\sqrt{t^2 - |x|^2}}\, dt. \tag{1.24}$$

We next express $\hat{f}'(t)$ as an integral and change order of integration. In order to get absolutely convergent integrals, we throw in a convergence factor:

$$
\begin{aligned}
\widehat{F}(x) &= \lim_{\varepsilon\searrow 0} -\sqrt{\frac{2}{\pi}}\int_{|x|}^{\infty} \hat{f}'(t)\frac{e^{-\varepsilon t}}{\sqrt{t^2-|x|^2}}\,dt \\
&= \lim_{\varepsilon\searrow 0} \frac{2}{\pi}\int_{|x|}^{\infty}\int_0^{\infty} f(r)r\sin rt\,\frac{e^{-\varepsilon t}}{\sqrt{t^2-|x|^2}}\,dr\,dt \\
&= \lim_{\varepsilon\searrow 0} \frac{2}{\pi}\int_0^{\infty} f(r)\Big[\int_{|x|}^{\infty}\frac{e^{-\varepsilon t}\sin rt}{\sqrt{t^2-|x|^2}}\,dt\Big] r\,dr.
\end{aligned} \tag{1.25}
$$

Furthermore, the inner integral in (1.25) transforms under $t=|x|s$ to

$$
\int_1^{\infty}\frac{e^{-\varepsilon|x|s}\sin r|x|s}{\sqrt{s^2-1}}\,ds. \tag{1.26}
$$

Comparison with (1.21) gives

$$
\begin{aligned}
J_0(r) &= \lim_{\varepsilon\searrow 0}\frac{2}{\pi}\int_1^{\infty}\frac{e^{-\varepsilon s}\sin rs}{\sqrt{s^2-1}}\,ds \\
&= \frac{2}{\pi}\int_1^{\infty}\frac{\sin rs}{\sqrt{s^2-1}}\,ds,
\end{aligned} \tag{1.27}
$$

a known formula for $J_0(r)$. The last integal is not absolutely convergent, but it is equal to

$$
\lim_{T\to+\infty}\int_1^{T}\frac{\sin rs}{\sqrt{s^2-1}}\,ds, \tag{1.28}
$$

as can be seen by a standard alternating series comparison.

We move on to a power series expansion for $J_\nu(z)$. For this, we take (1.5), write e^{izt} as a power series in zt, and integrate term by term (the odd powers all yielding zero integrals). To carry this out, we need two facts about the gamma function, namely Euler's formula for the beta function, which implies

$$
\int_{-1}^{1} t^{2k}(1-t^2)^{\nu-1/2}\,dt = \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}, \tag{1.29}
$$

and Legendre's duplication formula, which implies

$$
\frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(2k+1)} = \frac{2^{-2k}}{\Gamma(k+1)}. \tag{1.30}
$$

So, from (1.5), we have

$$
\begin{aligned}
J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \sum_{k=0}^\infty \frac{1}{(2k)!} \int_{-1}^1 (izt)^{2k}(1-t^2)^{\nu-1/2}\,dt \\
&= \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \sum_{k=0}^\infty \frac{1}{(2k)!} (iz)^{2k} \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)},
\end{aligned}
\tag{1.31}
$$

hence

$$
J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Big(\frac{z}{2}\Big)^{2k}. \tag{1.32}
$$

The second identity in (1.31) uses (1.29) and passage to (1.32) uses (1.30). This argument works for $\operatorname{Re}\nu > -1/2$, but then each side of (1.32) is entire in ν, and the identity holds for all $\nu \in \mathbb{C}$. Taking $\nu = \pm 1/2$, one recovers (1.18)–(1.19).

We note the leading behavior as $z \to 0$,

$$
J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} + O(z^{\nu+1}), \tag{1.33}
$$

and

$$
J'_\nu(z) = \frac{z^{\nu-1}}{2^\nu \Gamma(\nu)} + O(z^\nu). \tag{1.34}
$$

The leading coefficients are nonzero as long as ν is not a negative integer (or 0, for (1.34)).

Note that both J_ν and $J_{-\nu}$ solve Bessel's equation (1.17). From the expression (1.32) it is clear that J_ν and $J_{-\nu}$ are linearly independent provided ν is not an integer. On the other hand, comparison of power series shows

$$
J_{-n}(z) = (-1)^n J_n(z), \quad n = 0, 1, 2, \dots. \tag{1.35}
$$

We want to construct a basis of solutions to Bessel's equation, uniformly good for all ν. This construction can be motivated by a calculation of the Wronskian.

Generally, for a pair of solutions u_1 and u_2 to a second-order differential equation

$$
a(z)u'' + b(z)u' + c(z)u = 0, \tag{1.36}
$$

u_1 and u_2 are linearly independent if and only if their Wronskian

$$
W(z) = W(u_1, u_2)(z) = u_1 u_2' - u_2 u_1' \tag{1.37}
$$

is nonvanishing. Note that the Wronskian satisfies the first-order differential equation

$$W'(z) = -\frac{b}{a} W(z). \tag{1.38}$$

In the case of Bessel's equation (1.17), this becomes

$$W'(z) = -\frac{W(z)}{z}, \tag{1.39}$$

so

$$W(z) = \frac{K}{z}, \tag{1.40}$$

for some K, independent of z, but perhaps depending on ν. If $u_1 = J_\nu$, $u_2 = J_{-\nu}$, we can compute K by considering the behavior of $W(z)$ as $z \to 0$. From (1.33) and (1.34), we get

$$\begin{aligned} W(J_\nu, J_{-\nu})(z) &= -\Big[\frac{1}{\Gamma(\nu)\Gamma(1-\nu)} - \frac{1}{\Gamma(\nu+1)\Gamma(-\nu)}\Big]\frac{1}{z} \\ &= -2\frac{\sin \pi\nu}{\pi z}, \end{aligned} \tag{1.41}$$

where we have used the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{1.42}$$

This recaptures the observation that J_ν and $J_{-\nu}$ are linearly independent, and consequently a basis of solutions to (1.17), if and only if ν is not an integer.

To construct a basis of solutions uniformly good for all ν, we set

$$Y_\nu(z) = \frac{J_\nu(z)\cos \pi\nu - J_{-\nu}(z)}{\sin \pi\nu} \tag{1.43}$$

when ν is not an integer, and define

$$Y_n(z) = \lim_{\nu \to n} Y_\nu(z) = \frac{1}{\pi}\Big[\frac{\partial J_\nu(z)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(z)}{\partial \nu}\Big]\Big|_{\nu=n}. \tag{1.44}$$

We have

$$W(J_\nu, Y_\nu)(z) = \frac{2}{\pi z}, \tag{1.45}$$

for all ν.

Another important pair of solutions to Bessel's equation is the pair of Hankel functions

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \quad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \tag{1.46}$$

For $H_\nu^{(1)}$, there is the integral formula

$$H_\nu^{(1)}(z) = \frac{2e^{-\pi i\nu}}{i\sqrt{\pi}\Gamma(\nu+1/2)}\Bigl(\frac{z}{2}\Bigr)^\nu \int_1^\infty e^{izt}(t^2-1)^{\nu-1/2}\,dt, \tag{1.47}$$

for $\mathrm{Re}\,\nu > -1/2, \mathrm{Im} z > 0$. To prove this, one can show that the right side of (1.47) satisfies the same recursion formulas as $J_\nu(z)$ and hence solves the Bessel equation; thus it is a linear combination of $J_\nu(z)$ and $Y_\nu(z)$. The coefficients can be found by examining the limiting behavior as $z \to 0$, to establish the asserted identity. We note that (1.27) follows from (1.47).

2. Hankel transforms

We recall the modified Hankel transform, defined in (1.7),

$$\begin{aligned} \widetilde{H}_\nu f(\lambda) &= \int_0^\infty f(r)\frac{J_\nu(\lambda r)}{(\lambda r)^\nu} r^{2\nu+1}\,dr \\ &= \lambda^{-\nu}\int_0^\infty f(r)J_\nu(\lambda r)r^{\nu+1}\,dr. \end{aligned} \tag{2.1}$$

We take $\lambda \in (0,\infty)$, but allow $\nu \in \mathbb{C}$. By (1.8) plus the Fourier inversion formula, we get $\widetilde{H}_\nu \widetilde{H}_\nu f = f$ for $\nu = n/2 - 1,\ n \in \mathbb{N}$. One of our goals here will be to obtain this "Hankel inversion formula" for arbitrary $\nu \in [-1/2,\infty)$. We begin with the following mapping property. Set

$$\mathcal{S}(\mathbb{R}^+) = \{f|_{\mathbb{R}^+} : f \in \mathcal{S}(\mathbb{R}) \text{ is even}\}. \tag{2.2}$$

Lemma 2.1. *If* $\nu \ge -1/2$, *then*

$$\widetilde{H}_\nu : \mathcal{S}(\mathbb{R}^+) \longrightarrow \mathcal{S}(\mathbb{R}^+). \tag{2.3}$$

Proof. By (1.2), $J_\nu(\lambda r)/(\lambda r)^\nu$ is a smooth function of λr. The formula (1.5) yields

$$\Bigl|\frac{J_\nu(\lambda r)}{(\lambda r)^\nu}\Bigr| \le C_\nu < \infty,$$

for $\lambda r \in [0,\infty),\ \nu > -1/2$, a result that, by (1.19), also holds for $\nu = -1/2$. This readily yields

$$\widetilde{H}_\nu : \mathcal{S}(\mathbb{R}^+) \longrightarrow L^\infty(\mathbb{R}^+), \tag{2.4}$$

whenever $\nu \geq -1/2$. Now consider the differential operator $\widetilde{L}_\nu$, given by

$$\begin{aligned}\widetilde{L}_\nu f(r) &= -r^{-2\nu-1}\frac{\partial}{\partial r}\Big(r^{2\nu+1}\frac{\partial f}{\partial r}\Big) \\ &= -\frac{\partial^2 f}{\partial r^2} - \frac{2\nu+1}{r}\frac{\partial f}{\partial r}.\end{aligned} \tag{2.5}$$

Using Bessel's equation (1.17), we have

$$\widetilde{L}_\nu\Big(\frac{J_\nu(\lambda r)}{(\lambda r)^\nu}\Big) = \lambda^2\frac{J_\nu(\lambda r)}{(\lambda r)^\nu}, \tag{2.6}$$

and, for $f \in \mathcal{S}(\mathbb{R}^+)$,

$$\begin{aligned}\widetilde{H}_\nu(\widetilde{L}_\nu f)(\lambda) &= \lambda^2\widetilde{H}_\nu f(\lambda), \\ \widetilde{H}_\nu(r^2 f)(\lambda) &= \widetilde{L}_\nu\widetilde{H}_\nu f(\lambda).\end{aligned} \tag{2.7}$$

Since $f \in L^\infty(\mathbb{R}^+)$ belongs to $\mathcal{S}(\mathbb{R}^+)$ if and only if arbitrary iterated applications of $\widetilde{L}_\nu$ and multiplication by r^2 to f yield elements of $L^\infty(\mathbb{R}^+)$, the result (2.3) follows. We also have that this map is continuous with respect to the natural Frechet space structure on $\mathcal{S}(\mathbb{R}^+)$.

Lemma 2.2. *Consider the elements $E_b \in \mathcal{S}(\mathbb{R}^+)$, given for $b > 0$ by*

$$E_b(r) = e^{-br^2}. \tag{2.8}$$

We have

$$\widetilde{H}_\nu E_{1/2}(\lambda) = E_{1/2}(\lambda), \tag{2.9}$$

and more generally

$$\widetilde{H}_\nu E_b(\lambda) = (2b)^{-\nu-1}E_{1/4b}(\lambda). \tag{2.10}$$

Proof. To establish (2.9), plug the power series (1.32) for $J_\nu(z)$ into (2.1) and integrate term by term, to get

$$\widetilde{H}_\nu E_{1/2}(\lambda) = \sum_{k=0}^\infty \frac{(-1)^k 2^{-\nu-2k}}{k!\Gamma(k+\nu+1)}\lambda^{2k}\int_0^\infty r^{2k+2\nu+1}e^{-r^2/2}\,dr. \tag{2.11}$$

This last integral is seen to equal $2^{k+\nu}\Gamma(k+\nu+1)$, so we have

$$\widetilde{H}_\nu E_{1/2}(\lambda) = \sum_{k=0}^\infty \frac{1}{k!}\Big(-\frac{\lambda^2}{2}\Big)^k = e^{-\lambda^2/2} = E_{1/2}(\lambda). \tag{2.12}$$

Having (2.9), we get (2.10) by an easy change of variable argument.

In more detail, set $r^2/2 = bs^2$, or $s = r/\sqrt{2b}$. Then set $\mu = \sqrt{2b}\lambda$, so $\lambda r = \mu s$. Then (2.12), which we can write as

$$\int_0^\infty e^{-r^2/2} J_\nu(\lambda r) r^{\nu+1}\, dr = \lambda^\nu e^{-\lambda^2/2}, \tag{2.13}$$

translates to

$$\int_0^\infty e^{-bs^2} J_\nu(\mu s)(2b)^{(\nu+1)/2} s^{\nu+1}(2b)^{1/2}\, ds = (2b)^{-\nu/2}\mu^\nu e^{-\mu^2/4b}, \tag{2.14}$$

or, changing notation back,

$$\int_0^\infty e^{-bs^2} J_\nu(\lambda s) s^{\nu+1}\, ds = (2b)^{-\nu-1}\lambda^\nu e^{-\lambda^2/4b}, \tag{2.15}$$

which gives (2.10).

From (2.10) we have, for each $b > 0$,

$$\widetilde{H}_\nu \widetilde{H}_\nu E_b = (2b)^{-\nu-1}\widetilde{H}_\nu E_{1/4b} = E_b, \tag{2.16}$$

which verifies our stated Hankel inversion formula for $f = E_b,\ b > 0$. To get the inversion formula for general $f \in \mathcal{S}(\mathbb{R}^+)$, it suffices to establish the following.

Lemma 2.3. *The space*

$$\mathcal{V} = Span\{E_b : b > 0\} \tag{2.17}$$

is dense in $\mathcal{S}(\mathbb{R}^+)$*.*

Proof. Let $\overline{\mathcal{V}}$ denote the closure of $\mathcal{V}$ in $\mathcal{S}(\mathbb{R}^+)$. From

$$\frac{1}{\varepsilon}\big(e^{-br^2} - e^{-(b+\varepsilon)r^2}\big) \to r^2 e^{-br^2}, \tag{2.18}$$

we deduce that $r^2 e^{-br^2} \in \overline{\mathcal{V}}$, and inductively, we get

$$r^{2j} e^{-br^2} \in \overline{\mathcal{V}}, \quad \forall\, j \in \mathbb{Z}^+. \tag{2.19}$$

From here, one has

$$(\cos \xi r) e^{-r^2} \in \overline{\mathcal{V}}, \quad \forall\, \xi \in \mathbb{R}. \tag{2.20}$$

Now each even $\omega \in \mathcal{S}'(\mathbb{R})$ annihilating (2.20) for all $\xi \in \mathbb{R}$ has the property that $e^{-r^2}\omega$ has Fourier transform zero, which implies $\omega = 0$. The assertion (2.17) then follows by the Hahn-Banach theorem.

Putting the results of Lemmas 2.1–2.3 together, we have

Proposition 2.4. *Given $\nu \geq -1/2$, we have*

$$\widetilde{H}_\nu \widetilde{H}_\nu f = f, \tag{2.21}$$

for all $f \in \mathcal{S}(\mathbb{R}^+)$.

We promote this to

Proposition 2.5. *If $\nu \geq -1/2$, we have a unique extension of $\widetilde{H}_\nu$ from $\mathcal{S}(\mathbb{R}^+)$ to*

$$\widetilde{H}_\nu : L^2(\mathbb{R}^+, r^{2\nu+1}\, dr) \longrightarrow L^2(\mathbb{R}^+, \lambda^{2\nu+1}\, d\lambda), \tag{2.22}$$

as a unitary operator, and (2.21) holds for all $f \in L^2(\mathbb{R}^+, r^{2\nu+1}\, dr)$.

Proof. Take $f, g \in \mathcal{S}(\mathbb{R}^+)$, and use the inner product

$$(f, g) = \int_0^\infty f(r)\overline{g(r)} r^{2\nu+1}\, dr. \tag{2.23}$$

Using Fubini's theorem and the fact that $J_\nu(\lambda r)/(\lambda r)^\nu$ is real valued and symmtric in (λ, r), we get the first identity in

$$(\widetilde{H}_\nu f, \widetilde{H}_\nu g) = (\widetilde{H}_\nu \widetilde{H}_\nu f, g) = (f, g), \tag{2.24}$$

the second identity following by Proposition 2.4. From here, given that the linear space $\mathcal{S}(\mathbb{R}^+) \subset L^2(\mathbb{R}^+, r^{2\nu+1}\, dr)$ is dense, the assertions of Proposition 2.5 are apparent.

We now introduce H_ν, called the Hankel transform:

$$H_\nu g(\lambda) = \int_0^\infty g(r) J_\nu(\lambda r) r\, dr. \tag{2.25}$$

Note that

$$H_\nu(r^\nu f)(\lambda) = \lambda^\nu \widetilde{H}_\nu f(\lambda), \tag{2.26}$$

and that $M_\nu f(r) = r^\nu f(r)$ has the property that

$$M_\nu : L^2(\mathbb{R}^+, r^{2\nu+1}\, dr) \longrightarrow L^2(\mathbb{R}^+, r\, dr) \ \text{ is unitary.} \tag{2.27}$$

Thus Proposition 2.5 yields

Proposition 2.6. *For $\nu \geq -1/2$, we have*

$$H_\nu : L^2(\mathbb{R}^+, r\, dr) \longrightarrow L^2(\mathbb{R}^+, \lambda\, d\lambda) \tag{2.28}$$

unitary, and

$$H_\nu H_\nu g = g, \quad \forall\, g \in L^2(\mathbb{R}^+, r\, dr). \tag{2.29}$$

Reference

[T] M. Taylor, Functions of $\sqrt{-\Delta}$ and the wave equation,
available at http://www.unc.edu/math/Faculty/met/funct.pdf