The Hopf Bracket

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Abstract

Given a smooth map $f : M \to N$ between smooth manifolds, we construct a hierarchy of bilinear forms on suitable closed differential forms on N. This construction, which we call the *Hopf bracket*, simultaneously generalizes both the classical Hopf invariant and the Abelian Chern-Simons functional. It is a special case of a more general construction of N. Steenrod, known as the functional cup product. We show that the Hopf bracket has a pleasant development via differential forms.

1 The Basic Construction

The classical Hopf invariant is a homotopy invariant of maps $f : S^{2n-1} \to S^n$, where n is even. Our objective here is to point out that this invariant is but one manifestation of an operation, which we will call the *Hopf bracket*, that is defined for any smooth map between manifolds.

Let M and N be smooth manifolds, of dimensions m and n, respectively. We will use $\mathcal{E}^p(M)$ to denote the smooth, real-valued p-forms on M,

$$\mathcal{Z}^p(M) = \ker \left[d : \mathcal{E}^p(M) \to \mathcal{E}^{p+1}(M)\right] \tag{1.1}$$

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to denote the closed p-forms on M, and

$$\mathcal{H}^p(M) = \mathcal{Z}^p(M)/d\mathcal{E}^{p-1} \tag{1.2}$$

to denote the p^{th} deRham cohomology of M. Let $f : M \to N$ be a smooth map. Set

$$\mathcal{H}^p_f(N) = \ker\left[f^* : \mathcal{H}^p(N) \to \mathcal{H}^p(M)\right], \tag{1.3}$$

and let

$$\mathcal{Z}^p_f(N) = \{\phi \in \mathcal{Z}^p(N) \mid [\phi] \in \mathcal{H}^p_f(N)\}. \tag{1.4}$$

We note in passing that $\mathcal{H}^*_f(N) \subset \mathcal{H}^*(N)$ and $\mathcal{Z}^*_f(N) \subset \mathcal{Z}^*(N)$ are ideals with respect to the cup and wedge products.

Now suppose that $\phi \in \mathcal{Z}^p_f(N)$ and $\varphi \in \mathcal{Z}^q_f(N)$, where $p+q > n = \dim N$. Since $f^*[\phi] = 0 \in \mathcal{H}^p(M)$, we have

$$f^*\phi = d\theta \tag{1.5}$$

for some $\theta \in \mathcal{E}^{p-1}(M)$. Set

$$\langle \phi \mid \varphi \rangle_f = [\theta \wedge f^*\varphi] \in \mathcal{H}^{p+q-1}(M). \tag{1.6}$$

We will call this expression the *Hopf bracket* induced by f. This Hopf bracket is a special case of Steenrod's functional cup product (cf. [10]), one amenable to development via the use of the calculus of differential forms.

Proposition 1.1 *For any smooth map $f : M^m \to N^n$ and for integers p, q with $p+q > n$, the Hopf bracket (1.6) is a well defined bilinear map*

$$\langle \cdot \mid \cdot \rangle_f : \mathcal{Z}^p_f(N) \times \mathcal{Z}^q_f(N) \longrightarrow \mathcal{H}^{p+q-1}(M). \tag{1.7}$$

Moreover,

$$\langle \phi \mid \varphi \rangle_f = (-1)^{pq} \langle \varphi \mid \phi \rangle_f. \tag{1.8}$$

Proof. We have $f^*\phi = d\theta$ and $f^*\varphi = d\vartheta$ for appropriate differential forms $\theta \in \mathcal{E}^{p-1}(M)$ and $\vartheta \in \mathcal{E}^{q-1}(M)$. Now notice that

$$d(\theta \wedge f^*\varphi) = f^*\phi \wedge f^*\varphi = f^*(\phi \wedge \varphi) = f^*0 = 0, \tag{1.9}$$

since, by assumption, the degree of $\phi \wedge \varphi$ exceeds the dimension of N. Thus it makes sense to speak, as in (1.6) , of the deRham class $[\theta \wedge f^*\varphi]$. Of course, the deRham class $[\vartheta \wedge f^*\phi]$ is also defined, for precisely the same reason.

Let us now compare these two deRham classes on M. Since

$$\begin{aligned} d(\theta \wedge \vartheta) &= d\theta \wedge \vartheta + (-1)^{p-1}\theta \wedge d\vartheta \\ &= (-1)^{p(q-1)}\vartheta \wedge d\theta + (-1)^{p-1}\theta \wedge d\vartheta \\ &= (-1)^{p}\left[(-1)^{pq}\vartheta \wedge f^*\phi - \theta \wedge f^*\varphi\right], \end{aligned}$$

it follows that

$$[\theta \wedge f^*\varphi] = (-1)^{pq}[\vartheta \wedge f^*\phi]. \tag{1.10}$$

Now the left-hand side is independent of the choice of ϑ, whereas the right-hand side is independent of the choice of θ. It follows that both sides depend only on ϕ and φ. Moreover, the left-hand side is manifestly linear in φ, whereas as the right-hand side is manifestly linear in ϕ. Thus the Hopf bracket (1.6) is well defined, bilinear, and satisfies (1.8). ∎

We now up the ante, and demand that $p + q$ be slightly larger.

Proposition 1.2 *If $p + q > n + 1$, the Hopf bracket descends to a bilinear map*

$$(\cdot \mid \cdot)_f : \mathcal{H}^p_f(N) \times \mathcal{H}^q_f(N) \longrightarrow \mathcal{H}^{p+q-1}(M) \tag{1.11}$$

defined by

$$([\phi] \mid [\varphi])_f = \langle \phi \mid \varphi \rangle_f. \tag{1.12}$$

Proof. By bilinearity and (1.8), it suffices to show that $\langle \phi \mid \varphi \rangle_f$ vanishes if ϕ is exact. But if $\phi = d\psi$, we can take $\theta = f^*\psi$. Thus

$$\langle d\psi \mid \varphi \rangle_f = [f^*\psi \wedge f^*\varphi] = [f^*(\psi \wedge \varphi)] = 0, \tag{1.13}$$

since, by assumption, the degree of $\psi \wedge \varphi$ exceeds the dimension of N. ∎

Evidently, one must have $m \geq n$ for the Hopf bracket to be non-zero. We now verify that the bracket is a non-trivial operation by considering a simple family of examples.

Example 1.1. Let N be a compact, connected $2k$-manifold that admits a symplectic structure. Since $H^2(N, \mathbb{Q}) \subset H^2(N, \mathbb{R})$ is dense, we may choose a symplectic form $\omega \in \mathcal{Z}^2(N)$ such that $[\omega]$ is in the image of $H^2(N, \mathbb{Z}) \to \mathcal{H}^2(N)$. By the Chern-Weil theorem, there is then a complex line bundle with Hermitian connection of curvature $F = -2\pi i\omega$; thus, in particular,

$c_1(L) \in H^2(N, \mathbb{Z})$ has image $[\omega]$ in deRham cohomology. Let $M \subset L$ be the set of unit vectors with respect to the inner product on L, so that $M \to N$ is a principal $U(1)$ connection, and let $f : M^{2k+1} \to N$ be the bundle projection.

Now consider the connection 1-form θ on M, with kernel equal to the horizontal subspace and $\theta(\xi) = 1$, where ξ is the vector field which generates the S^1-action on M. Then $d\theta = if^*F = -2\pi f^*\omega$. Hence

$$f^*\omega^j = d\Big(-\frac{1}{2\pi}\theta \wedge f^*\omega^{j-1}\Big), \tag{1.14}$$

so that

$$[\omega]^j \in H_f^{2j}(N), \quad \forall j. \tag{1.15}$$

Giving M the out-pointing orientation, remembering that θ has integral 2π on each fiber of f, and using integration as the standard isomorphism $\mathcal{H}^{2k-1}(M) \cong \mathbb{R}$, we then have

$$\begin{aligned} \int_M ([\omega]^j \mid [\omega]^{k-j+1})_f &= \int_M (-\frac{1}{2\pi}\theta \wedge f^*\omega^{j-1}) \wedge f^*\omega^{k-j+1} \\ &= -\int_N \omega^k \neq 0. \end{aligned}$$

Let us now focus on the special case of $N = \mathbb{CP}_k$, where we may take our symplectic form to satisfy $[\omega] = \ell[\alpha]$, where ℓ is any non-zero integer and $[\alpha]$ is the deRham class corresponding to the standard generator of $H^2(\mathbb{CP}_k, \mathbb{Z})$. Then

$$\int_M ([\omega]^j \mid [\omega]^{k-j+1})_f = -\ell^k \tag{1.16}$$

and hence

$$\int_M ([\alpha]^j \mid [\alpha]^{k-j+1})_f = -\frac{\ell^k}{\ell^{k+1}} = -\frac{1}{\ell}. \tag{1.17}$$

Notice that, while this is a rational number, it is usually not an integer. We will later see that this is absolutely typical. ♢

It is also worth observing that the Hopf bracket really *does not* descend to cohomology when $p + q = n + 1$.

Example 1.2. Let N be a compact oriented 3-manifold, let $M = N$, and let $f : M \to N$ be the identity map. Thus $\mathcal{Z}_f^2(N)$ is precisely the space $d\mathcal{E}^1(N)$

of exact 2-forms. Using integration on N to identify $\mathcal{H}^3(N)$ with $\mathbb{R}$, we thus have

$$\langle d\theta \mid d\theta\rangle_f = \int_N \theta \wedge d\theta. \tag{1.18}$$

For example, if θ is a contact form on N, $\theta \wedge d\theta$ is a volume form, and the above Hopf bracket is certainly non-zero, despite the fact that each argument represents 0 in cohomology.

Thinking of θ as representing a connection on the trivial principal $U(1)$-bundle over N, the expression $\langle d\theta, d\theta\rangle_{identity}$ may be recognized as the $U(1)$ version [8] of the Chern-Simons functional [2] for connections on the trivial circle bundle over N. Notice that the same expression would also arise if one considered the Hopf bracket for the projection map $S^1 \times N \to N$. Applying the same idea to any $U(1)$-bundle $f : P \to N$ allows one to equate the Chern-Simons functional on connections on P as the Hopf bracket of curvature with itself. ♢

We now come to some important general properties of the Hopf bracket.

Proposition 1.3 *Suppose that $f : M \to N$ is the composition $F \circ g$ of smooth maps $F : W \to N$ and $g : M \to W$, where W is some smooth manifold. If $p + q > n$, we then have*

$$\phi \in \mathcal{Z}^p_F,\ \varphi \in \mathcal{Z}^q_F \Longrightarrow \langle \phi \mid \varphi\rangle_f = g^*\langle \phi \mid \varphi\rangle_F. \tag{1.19}$$

In particular, for $p + q > n + 1$ we have

$$(\cdot \mid \cdot)_f|_{\mathcal{H}^p_F \times \mathcal{H}^q_F} = g^*(\cdot \mid \cdot)_F. \tag{1.20}$$

Proof. If $F^*\phi = d\psi$ for $\psi \in \mathcal{E}^{p-1}(W)$, we may take $\theta = g^*\psi$, so that

$$\langle \phi \mid \varphi\rangle_f = [\theta \wedge f^*\varphi] = [g^*\psi \wedge g^*F^*\varphi] = g^*[\psi \wedge F^*\varphi] = \langle \phi \mid \varphi\rangle_F. \tag{1.21}$$

Descent to cohomology classes then yields the analogous claim for $(\cdot \mid \cdot)$. ∎

In particular, the Hopf bracket depends only on the homotopy class of f.

Corollary 1.4 *Suppose that $f : M \to N$ and $\tilde{f} : M \to N$ are homotopic maps. Then $\langle \cdot \mid \cdot\rangle_f = \langle \cdot \mid \cdot\rangle_{\tilde{f}}$, and $(\cdot \mid \cdot)_f = (\cdot \mid \cdot)_{\tilde{f}}$.*

Proof. Because the smooth maps are homotopic, they are smoothly homotopic, so there is a smooth map $F : M \times \mathbb{R} \to N$ with $f = F \circ i_0$ and $\tilde{f} = F \circ i_1$, where $i_j : M \to M \times \mathbb{R}$ is $p \mapsto (p, j)$. In particular, $\mathcal{Z}^p_f(N) = \mathcal{Z}^p_F(N) = \mathcal{Z}^p_{\tilde{f}}(N)$. Now set $W = M \times \mathbb{R}$, and apply Proposition 1.3 twice, first to $g = i_0$, and then to $g = i_1$. ∎

2 The Hopf Invariant Revisited

We now specialize our discussion to a context much closer to that of the classical Hopf invariant. Namely, let M and N be smooth, compact, oriented manifolds without boundary, where

$$\dim N = n \geq 2, \quad \dim M = 2n - 1. \tag{2.1}$$

Also assume

$$N \text{ connected}, \quad \mathcal{H}^n(M) = 0. \tag{2.2}$$

Let $f : M \to N$ be any smooth map. Let μ be an n-form on N with $\int_N \mu = 1$. Since (2.2) guarantees that $f^* : \mathcal{H}^n(N) \to \mathcal{H}^n(M)$ is the zero map, we have $\mathcal{H}^n_f(N) = \mathcal{H}^n(N) = \mathbb{R}$, and so $[\mu] \in \mathcal{H}^n_f(N)$ is the generator. Similarly, mapping $\mathcal{H}^{2n-1}(M)$ to $\mathbb{R}$ by integration, it is natural to associate a real number with f by

$$\mathfrak{H}(f) = \int_M ([\mu] \mid [\mu])_f \in \mathbb{R}; \tag{2.3}$$

more concretely, that is to say that

$$\mathfrak{H}(f) = \int_M \theta \wedge d\theta \tag{2.4}$$

for any $(n-1)$-form θ on M with $d\theta = f^*\mu$. Of course, since $([\mu] \mid [\mu])_f = (-1)^n([\mu] \mid [\mu])_f$ by (1.8), this expression automatically vanishes if n is odd, so that in practice we will generally want to take n to be even. We will call $\mathfrak{H}(f)$ the Hopf invariant of f, since this agrees [1] with the usual definition when $M = S^{2n-1}$ and $N = S^n$. We can also apply Proposition 1.2 to write

$$\mathfrak{H}(f) = \int_M \theta' \wedge d\theta, \tag{2.5}$$

where also $\int_N \mu' = 1$ and $f^*\mu' = d\theta'$.

Corollary 1.4 tells us that $\mathfrak{H}(f)$ depends only on the homotopy class of the map f. More generally, Proposition 1.3 allows one to show that $\mathfrak{H}(f)$ satisfies a limited form of cobordism invariance:

Proposition 2.1 *Let M and N be as above, and suppose that $M = \partial W$, where W is a compact, oriented, $2n$-dimensional manifold-with-boundary such that*

$$\mathcal{H}^n(W) = 0. \tag{2.6}$$

If there is a smooth map $F : W \longrightarrow N$ with $f = F|_{\partial W}$, then $\mathfrak{H}(f) = 0$.

Proof. Let W_ϵ be the open manifold, diffeomorphic to to the interior of W, obtained by attaching a collar $M \times [0, \epsilon)$ to the boundary of W, and let $\hat{F}$ be any extension of F to W_ϵ. Let $g : M \to W_\epsilon$ be the inclusion map. Stokes' theorem asserts that the composition

$$\mathcal{H}^{2n-1}(W_\epsilon) \xrightarrow{g^*} \mathcal{H}^{2n-1}(M) \xrightarrow{\int_M} \mathbb{R} \tag{2.7}$$

vanishes. Moreover, (2.6) asserts that $\mathcal{H}^n_{\hat{F}}(N) = \mathcal{H}^n(N)$. Thus Proposition 1.3 tells us that

$$\mathfrak{H}(f) = \int_M ([\mu] \mid [\mu])_f = \int_M g^*([\mu] \mid [\mu])_{\hat{F}} = 0, \tag{2.8}$$

as claimed. ∎

We will see in the example (2.10) below that this result does not hold in the absence of hypothesis (2.6).

Proposition 1.3 also gives us a useful relationship between the Hopf invariant and mapping degree.

Proposition 2.2 *Let $f : M \to N$ be as above, and suppose that we have a smooth factorization*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \\ \tilde{M} \arrow[r, "\tilde{f}"] & \tilde{N} \arrow[u, "h"'] \end{tikzcd}$$

where $\tilde{M}$ and $\tilde{N}$ are connected compact oriented manifolds of dimensions $2n-1$ and n, respectively, and where $\mathcal{H}^n(\tilde{M}) = 0$. Then

$$\mathfrak{H}(f) = (\deg g)\ \mathfrak{H}(\tilde{f})\ (\deg h)^2. \tag{2.9}$$

Proof. Let $\tilde{\mu}$ be an n-form on $\tilde{N}$ with integral 1. Then, by Proposition 1.3,

$$\begin{aligned} ([\mu] \mid [\mu])_{h\circ\tilde{f}\circ g} &= (h^*[\mu] \mid h^*[\mu])_{\tilde{f}\circ g} \\ &= ((\deg h)\ [\tilde{\mu}] \mid (\deg h)\ [\tilde{\mu}])_{\tilde{f}\circ g} \\ &= (\deg h)^2\ ([\tilde{\mu}] \mid [\tilde{\mu}])_{\tilde{f}\circ g} \\ &= (\deg h)^2\ g^*([\tilde{\mu}] \mid [\tilde{\mu}])_{\tilde{f}} \end{aligned}$$

Hence

$$\begin{aligned} \mathfrak{H}(f) &= \int_M (\deg h)^2\ g^*([\tilde{\mu}] \mid [\tilde{\mu}])_{\tilde{f}} \\ &= (\deg h)^2 (\deg g) \int_{\tilde{M}} ([\tilde{\mu}] \mid [\tilde{\mu}])_{\tilde{f}} \\ &= (\deg g)\ \mathfrak{H}(\tilde{f})\ (\deg h)^2, \end{aligned}$$

as claimed. ∎

Example 2.1. Let $N = S^2$, and let M be the unit tangent bundle of S^2. The map $M \to N$ amounts to the natural map

$$f : SO(3) \longrightarrow S^2, \tag{2.10}$$

arising from the transitive action of $SO(3)$ by isometries. Now $L = TS^2$ has a natural structure of a complex line bundle, whose curvature is $F = -2\pi i\omega$, where ω is $1/2\pi$ times the standard area form on S^2, yielding $c_1(L)[S^2] = 2$. By Example 1.1, we therefore have

$$\mathfrak{H}(f) = -\frac{1}{2}. \tag{2.11}$$

Note that composing the covering homomorphism $SU(2) \to SO(3)$ with f and identifying $SU(2) \approx S^3$ gives the classical Hopf map

$$\hat{f} : S^3 \longrightarrow S^2, \tag{2.12}$$

and then Proposition 2.2 gives the classical formula

$$\mathfrak{H}(\hat{f}) = -1. \tag{2.13}$$

(The sign of the invariant would be changed, of course, if we gave S^3 the opposite orientation.) A different approach to this calculation will follow from results established in the Section 4. ♢

3 The Hopf Invariant is Rational

For $f : S^{2n-1} \to S^n$, it is well known [1] that $\mathfrak{H}(f)$ is an integer, and can be reinterpreted as a linking number. Here we will show that if M and N satisfy our standing hypotheses and $f : M \to N$, then $\mathfrak{H}(f)$ is a rational number. This will be a consequence of Proposition 3.1 below, which provides an extension of the classical linking number formula for the Hopf invariant.

To begin our derivation, let $p \in N$ be a regular value of f; by Sard's theorem, almost every p in N has this property. The fiber $Y = f^{-1}(p)$ is then a compact, oriented $(n-1)$-dimensional submanifold of M, and Y has a tubular neighborhood which is diffeomorphic to $Y \times U$, where U is a neighborhood of $p \in N$, in such a manner that the restriction of f to the neighborhood amounts to projection $(Y \times U) \to U$ to the second factor. Let δ_p be the delta distribution centered at p, considered as a generalized n-form or *current* [7] on N, and let μ_j be a sequence of smooth n-forms on N, all with integral 1, all supported in U, such that $\mu_j \to \delta_p$ as distributions on N. For any fixed $(n-1)$-form ψ on M, we then have

$$\lim_{j\to\infty} \int_M \psi \wedge f^*\mu_j = \int_Y \psi, \tag{3.1}$$

as can easily be seen by applying Fubini's theorem on our tubular neighborhood $Y \times U$.

Now let μ be any fixed n-form on N with integral 1, and let θ be an $(n-1)$-form on M with $d\theta = f^*\mu$. Then for every j we have, as in (2.5),

$$\mathfrak{H}(f) = \int_M ([\mu] \mid [\mu_j])_f = \int_M \theta \wedge f^*\mu_j. \tag{3.2}$$

By (3.1), we thus have

$$\mathfrak{H}(f) = \lim_{j\to\infty} \int_M \theta \wedge f^*\mu_j = \int_Y \theta. \tag{3.3}$$

Now since our compact, oriented manifold M is assumed to have $H^n(M, \mathbb{R}) = \mathcal{H}^n(M) = 0$, Poincaré duality gives us the vanishing statement $H_{n-1}(M, \mathbb{R}) = 0$ for homology with real coefficients. The universal coefficients theorem thus tells us that $H_{n-1}(M, \mathbb{Z})$ consists of elements of finite order; indeed, it is therefore a finite group, since the compactness of M guarantees that it is finitely generated. In particular, there is a nonzero $m \in \mathbb{Z}$ and a smooth integral chain X such that

$$mY = \partial X. \tag{3.4}$$

Using this and applying Stokes' theorem to (4.2), we have

$$\mathfrak{H}(f) = \frac{1}{m}\int_{\partial X} \theta = \frac{1}{m}\int_X d\theta = \frac{1}{m}\int_X f^*\mu. \tag{3.5}$$

Taking μ to be supported away from $\{p\}$, we thus deduce the following:

Proposition 3.1 *With notations as above, we have*

$$\mathfrak{H}(f) = \frac{1}{m}\, deg\left[f|_X : (X, Y) \to (N, p)\right]. \tag{3.6}$$

In particular, the Hopf invariant is always a rational number.

Remark. One may of course always take m to divide the order of the group $H_{n-1}(M, \mathbb{Z}) = H^n(M, \mathbb{Z})$ in (3.4) and (3.6); it follows that $\mathfrak{H}(f)$ times $|H^n(M, \mathbb{Z})|$ is always an integer. For example, $H^n(M, \mathbb{Z}) = 0 \Longrightarrow \mathfrak{H}(f) \in \mathbb{Z}$. In particular, the Hopf invariant is always an integer if $M = S^{2n-1}$. $\square$

The proof of the above result actually gives one a practical method for calculating Hopf invariants. In the next section, we will exploit this method in a systematic way.

4 Euler Classes and Hopf Brackets

Theorem 4.1 *Let N be a smooth, compact, oriented manifold of dimension n, and let $\varpi : E \to N$ be an oriented real vector bundle of rank k. Let*

$e \in \mathcal{H}^k(N)$ be the Euler class of E, and assume that $e \neq 0$. Let μ be a smooth n-form on N with integral 1. Finally, let $M \subset E$ be the set of unit vectors for some positive-definite inner product on E, and let $f : M \to N$ be the restriction of the bundle projection ϖ to M. Then $e \in \mathcal{H}^k_f(N)$, $[\mu] \in \mathcal{H}^n_f(N)$, and, with respect to the out-pointing *orientation of M,*

$$\int_M ([\mu] \,|\, e)_f = -1. \tag{4.1}$$

Proof. The Gysin exact sequence

$$H^0(N) \xrightarrow{\smile e} H^k(N) \xrightarrow{f^*} H^k(M) \to H^1(N) \to \cdots \tag{4.2}$$

tells us that $e \in \mathcal{H}^k_f(N)$. On the other hand, since we have assumed that $e \neq 0$, Poincaré duality implies that there is some $\alpha \in \mathcal{H}^{n-k}(M)$ such that $e \smile \alpha = [\mu]$. Thus $f^*([\mu]) = f^*(e) \smile f^*(\alpha) = 0$, and $[\mu] \in \mathcal{H}^n_f(N)$.

Let u be a section of E which is transverse to the zero section, and let Z be the zero locus of u. Thus $Z \subset N$ is a smooth oriented compact submanifold of dimension $n-k$, and $Y = f^{-1}(Z) \subset M$ is a smooth oriented submanifold of dimension $n-1$. Let U be a tubular neighborhood of Z, which we can identify with the unit disk bundle of $E|_Z$, so that we have a family of local diffeomorphisms $\Phi_t : U \to U$, $t \in \mathbb{R}^+$, corresponding to scalar multiplication in $E|_Z$ by e^{-t}. Let ϕ be a closed k-form on N, supported in U, which represents the Poincaré dual of Z, and consider the family of forms $\phi_t = (\Phi_t^*)^{-1}\phi$, defined to be identically zero outside of $\Phi_t(U)$. Then for any $(n-k)$-form ψ on N we have

$$\begin{aligned}
\lim_{t\to\infty} \int_N \psi \wedge \phi_t &= \lim_{t\to\infty} \int_U (\Phi_t^*\psi) \wedge \phi = \int_U (\lim_{t\to\infty} \Phi_t^*\psi) \wedge \phi \\
&= \int_U (\varpi^*(\psi|_Z)) \wedge \phi = \int_Z \psi,
\end{aligned}$$

so the closed forms ϕ_t may be thought of as approximations of the integration current $[Z]$. Morever, if ω is any $(n-1)$ form on M, we have

$$\lim_{t\to\infty} \int_M \omega \wedge f^*\phi_t = \int_Y \omega \tag{4.3}$$

by exactly the same calculation.

Consider the section $u/\|u\|$ of E_{N-Z}, and notice that its image is contained in M. Moreover, the closure in M of the image of this section is an n-dimensional manifold-with-boundary $X \subset M$, the boundary ∂X of which is precisely $\bar{Y}$, meaning Y with the reverse orientation. If θ is any $n-1$ form on M, we therefore have

$$\int_X d\theta = -\int_Y \theta \tag{4.4}$$

by Stokes' theorem.

Let μ be an n-form on N with integral 1 which is supported away from Z, and recall that $f^*\mu$ is exact. Thus $f^*\mu = d\theta$ for some $(n-1)$-from θ on M. Now, since $e = [\phi_t]$, we have

$$\int_M ([\mu] \,|\, e)_f = \int_M \theta \wedge f^*\phi_t \tag{4.5}$$

for every t. Taking the limit as $t \to \infty$, it thus follows that

$$\int_M ([\mu] \,|\, e)_f = -\int_Y \theta = -\int_X f^*\mu = -\int_N \mu = -1, \tag{4.6}$$

as claimed. ▮

Proposition 4.2 *Let N be a smooth, compact, simply connected manifold of even dimension $n = 2m$. Let $E \to N$ be an oriented rank-n real vector bundle over N, and let $M \subset E$ be the set of unit vectors of some positive-definite inner product on E. Give $M \subset E$ the outward-pointing orientation, and let $f : M \to N$ be the induced projection. Assume that the Euler number $\mathbf{e} = \int_N e(E)$ of E is non-zero. Then M satisfies* (2.2), *and*

$$\mathfrak{H}(f) = -\frac{1}{\mathbf{e}}. \tag{4.7}$$

Proof. Taking an n-form μ on N such that $\int_N \mu = 1$, we can assume $[e] = \mathbf{e}[\mu]$, and then (4.1) gives

$$\int_M ([\mu] \,|\, [\mu])_f = -\frac{1}{\mathbf{e}}. \tag{4.8}$$

This gives (4.7), once we observe that our hypotheses put us in the framework of (2.2), i.e., that $\mathcal{H}^n(M) = 0$. To see this we again use the Gysin exact sequence (4.2), but in integer cohomology. (Here $k = n$.) The hypotheses that $H^1(N, \mathbb{Z}) = 0$ and that $e \neq 0$ imply that we have an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\smile e} \mathbb{Z} \to H^n(M, \mathbb{Z}) \to 0, \tag{4.9}$$

so $H^n(M, \mathbb{Z})$ is a finite cyclic group. ∎

Corollary 4.3 *Let $n = 2m$ be any positive even integer, and let $r \in \mathbb{Q}$ be any rational number. Then there is an oriented $(n-2)$-connected rational homology sphere M^{2n-1}, and a smooth map $f : M \to S^n$, such that*

$$\mathfrak{H}(f) = r. \tag{4.10}$$

5 Further examples

Combining results from §1 with results from §4, we can analyze further cases of the Hopf bracket. We begin with the following general result.

Proposition 5.1 *Let M and N be smooth, compact, oriented manifolds satisfying (2.1)–(2.2), and assume $f : M \to N$ is smooth. Assume $\pi : P \to M$ is a fibration, with P oriented and connected. Consider*

$$g = f \circ \pi : P \longrightarrow N. \tag{5.1}$$

If μ is an n-form on N such that $\int_N \mu = 1$, then

$$([\mu] \mid [\mu])_g = \mathfrak{H}(f)\gamma, \tag{5.2}$$

where $\gamma \in \mathcal{H}^{2n-1}(P)$ is the Poincaré dual of a fiber $\Gamma = \pi^{-1}(m_0)$, as a cycle in P, with appropriate orientation.

Proof. By Proposition 1.3, $([\mu] \mid [\mu])_g = \pi^*([\mu] \mid [\mu])_f$, so $([\mu] \mid [\mu])_g = \pi^*[\omega]$, where $\langle [\omega], [M] \rangle = 1$. This is readily seen to coincide with (5.2). ∎

Example 5.1. The natural projection

$$g : SO(2k+1) \longrightarrow S^{2k} \tag{5.3}$$

factors through $S_1S^{2k} \to S^{2k}$, where S_1S^{2k} is the bundle of unit tangent vectors to S^{2k}. Proposition 4.2 applies to f, so we have $([\mu] \,|\, [\mu])_g = -(1/2)\gamma$, where γ is the Poincaré dual of the $(k-1)(2k-1)$-cycle $SO(2k-1) \subset SO(2k+1)$. ♢

A similar argument applying Proposition 1.3 to Example 1.1 gives the following.

Example 5.2. The natural projection

$$g : SU(k+1) \longrightarrow \mathbb{CP}_k \tag{5.4}$$

factors through the standard projection $f : S^{2k+1} \to \mathbb{CP}_k$, a circle bundle with Chern class 1. Thus, if $[\alpha] \in H^2(\mathbb{CP}_k, \mathbb{Z})$ is the standard generator, we have

$$([\alpha]^j \,|\, [\alpha]^{k-j+1})_g = -\gamma, \tag{5.5}$$

where γ is the Poincaré dual of the cycle $SU(k) \subset SU(k+1)$. ♢

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