CR Manifolds

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Contents

1. Introduction

Let M be a manifold of $\mathbb{R}$-dimension $n = 2k + \ell$, and let $H \subset TM$ be a smooth sub-bundle of fiber dimension $2k$. Assume $J \in C^\infty(M, \operatorname{End} H)$ gives each fiber of H a complex structure, i.e., $J^2 = -I$. We say (M, H, J) is a CR manifold (of CR-codimension ℓ) provided that if X_j are vector fields in $C^\infty(M, H)$ and

$$Z_j = X_j + iJX_j, \tag{1.1}$$

then $[Z_1, Z_2]$ is a complex vector field of the same sort, i.e.,

$$[Z_1, Z_2] = W + iJW, \quad W \in C^\infty(M, H). \tag{1.2}$$

From the computation

$$\begin{aligned} [X_1 + iJX_1, X_2 + iJX_2] = [X_1, X_2] &- [JX_1, JX_2] \\ &+ i\Big\{[JX_1, X_2] + [X_1, JX_2]\Big\}, \end{aligned} \tag{1.3}$$

we have the following.

Proposition 1.1. *Let TM have the sub-bundle H with complex structure J, as described above. Then (M, H, J) is a CR manifold if and only if, for each $X, Y \in C^\infty(M, H)$,*

$$[X, Y] - [JX, JY] \in C^\infty(M, H), \tag{1.4}$$

and

$$\mathcal{N}(X, Y) = 0, \tag{1.5}$$

where

$$\mathcal{N}(X,Y) = J\Big([X,Y] - [JX,JY]\Big) - \Big([JX,Y] + [X,JY]\Big). \tag{1.6}$$

One verifies that if $X, Y \in C^\infty(M,H)$ and (1.4) holds, and if $f, g \in C^\infty(M)$ are scalar, then

$$\mathcal{N}(fX, gY) = fg\mathcal{N}(X,Y). \tag{1.7}$$

Hence, if (1.4) holds, we have $\mathcal{N} \in C^\infty(M, \mathrm{Hom}(H \otimes H, TM))$. In fact, if (1.4) holds, we can apply it to X and JY, to get

$$[X, JY] + [JX, Y] \in C^\infty(M,H), \tag{1.8}$$

so actually

$$\mathcal{N} \in C^\infty(M, \mathrm{Hom}(H \otimes H, H)). \tag{1.9}$$

Definition. In the setting of Proposition 1.1, if we just assume (1.4) holds, we say (M,H,J) is an almost-CR manifold.

Remark. In case $n = 2k, \ell = 0$, (1.4) is automatic. One says M is an almost complex manifold. In this situation, if (1.5) holds, it is the content of the Newlander-Nirenberg theorem that M is a complex manifold.

Generally, if (M,H,J) is an almost-CR manifold, we have

$$\begin{aligned} \mathcal{N}(X,Y) &= -\mathcal{N}(Y,X), \qquad \text{and} \\ \mathcal{N}(JX,Y) &= -J\mathcal{N}(X,Y) = \mathcal{N}(X,JY). \end{aligned} \tag{1.10}$$

In particular,

$$\dim_{\mathbb{R}} H_p = 2 \Longrightarrow \mathcal{N} = 0. \tag{1.11}$$

This gives:

Proposition 1.2. *If (M,H,J) is an almost-CR manifold of dimension $2k+\ell$, CR-codimension ℓ, and $k=1$, then it is a CR manifold. In particular, if it is an almost-CR manifold of CR-codimension 1, and $\dim M = 3$, then it is a CR manifold.*

Beyond complex manifolds, the most basic examples of CR manifolds, are $2n-1$ dimensional smooth submanifolds of $\mathbb{C}^n$, followed by submanifolds of $\mathbb{C}^n$ of higher codimension, subject to a natural hypothesis (cf. (2.7)). We start out §2 by considering these examples and explaining how they are CR manifolds. Then we look at the problem of obtaining submanifolds of codimension-1 CR manifolds that in turn are CR manifolds.

In §3 we discuss the Levi form of a CR manifold (M, H, J),

$$\mathcal{L} \in C^\infty(M, \mathrm{Hom}(H \otimes H, TM/H)), \tag{1.12}$$

which simultaneously measures whether the subbundle H of TM is integrable and how the presence or absence of integrability interacts with the complex structure on the fibers of H. When M is a codimension 1 surface in $\mathbb{C}^n$, we give a formula for $\mathcal{L}$ in terms of the second fundamental form of M. We look further at particular classes of CR manifolds, such as Levi-flat CR manifolds, for which $\mathcal{L} \equiv 0$, which are seen to be foliated by complex submanifolds. We look more generally at cases where $\mathcal{L}$ hs constant rank. For this class, one extreme is that M is Levi-flat, and the other extreme is that $\mathcal{L}$ is nondegenerate.

Material in §§2–3 focuses strictly on subbundles of the real tangent bundle TM, but recall that the definition of a CR manifold given at the beginning of this introduction involved complex vector fields, as in (1.1)–(1.3). In §4 we bring in the complex vector bundle $\mathcal{A}$, a subbundle of $\mathbb{C}TM$, with fibers

$$\mathcal{A}_p = \{X + iJX : X \in H_p\}, \tag{1.13}$$

its dual bundle $\mathcal{A}' = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$, and the exterior powers $\Lambda^j \mathcal{A}'$, of use in subsequent sections.

The need for complex vector fields in this subject is simply expained. When M is a complex manifold, the Cauchy-Riemann equations, defining when a function $u : M \to \mathbb{C}$ is holomorphic, involve complex vector fields. There is a natural generalization of these equations to the more general case of CR manifolds (M, H, J). A function $u : M \to \mathbb{C}$ is a CR function provided

$$Zu = 0, \quad \forall\, Z \in C^\infty(M, \mathcal{A}). \tag{1.14}$$

This can be rewritten as

$$\overline{\partial}_M u = 0, \tag{1.15}$$

for a first-order differential operator

$$\overline{\partial}_M : C^\infty(M) \longrightarrow C^\infty(M, \mathcal{A}'), \tag{1.16}$$

constructed in §5. The operator (1.16) is the first part of a "$\overline{\partial}_M$-complex,"

$$\overline{\partial}_M : C^\infty(M, \Lambda^j \mathcal{A}') \longrightarrow C^\infty(M, \Lambda^{j+1} \mathcal{A}'), \tag{1.17}$$

which is constructed in §6.

2. CR submanifolds

We discuss the production of CR submanifolds of a CR manifold (M, H, J), in some basic cases.

We start with the case that originated the investigation of CR manifolds. Namely, let $n = 2k + \ell$ with $\ell = 0$, so $H = TM$ and M is a complex manifold. Let $S \subset M$ be a smooth hypersurface, of dimension $n - 1$. Then, for $p \in S$, set

$$H_p^S = \{X \in T_pS : JX \in T_pS\}. \tag{2.1}$$

It follows that

$$\dim_{\mathbb{R}} H_p^S = 2k - 2, \quad \forall\, p \in S. \tag{2.2}$$

Furthermore, by definition,

$$J : H_p^S \longrightarrow H_p^S, \quad \forall\, p \in S. \tag{2.3}$$

We have the following basic result.

Proposition 2.1. *In the setting described above, (S, H^S, J) is a CR manifold, of CR-codimension 1.*

Proof. Let $X, Y \in C^\infty(S, H^S)$, and extend these to smooth vector fields $\widetilde{X}$ and $\widetilde{Y}$ on M, tangent to S. By (2.3), $J\widetilde{X}$ and $J\widetilde{Y}$ are also tangent to S. Clearly

$$[\widetilde{X}, \widetilde{Y}] - [J\widetilde{X}, J\widetilde{Y}] \in C^\infty(M, H), \tag{2.4}$$

and also, of course, $\mathcal{N}(\widetilde{X}, \widetilde{Y}) = 0$, so

$$J\Big([\widetilde{X}, \widetilde{Y}] - [J\widetilde{X}, J\widetilde{Y}]\Big) = [J\widetilde{X}, \widetilde{Y}] + [\widetilde{X}, J\widetilde{Y}], \tag{2.5}$$

the latter being clearly tangent to S. This implies that

$$[X, Y] - [JX, JY] \in C^\infty(S, H^S). \tag{2.6}$$

The condition $\mathcal{N}(X, Y) = 0$ is automatically inherited from (M, H, J), so Proposition 2.1 follows.

Remark. The proof of Proposition 2.1 as written above is basically a variant of a direct verification of (1.2) (with M replaced by S) in this setting.

This construction generalizes to the higher codimension case, as follows. Assume $S \subset M$ is a smooth submanifold of dimension $m < n$, and, for $p \in S$, define H_p^S as in (2.1). Then make the assumption that

$$\dim_{\mathbb{R}} H_p^S = 2\kappa \ \text{ is independent of } \ p \in S, \tag{2.7}$$

which automatically holds, as in (2.2), if $m = n-1$, but not otherwise. Again we have (2.3), and the hypothesis (2.7) implies H^S is a smooth subbunde of TS. From here, it is straightforward to extend the proof of Proposition 2.1 and obtain that (S, H^S, J) is a CR manifold.

To proceed, let us now suppose that (M, H, J) is a CR manifold of dimension $n = 2k+1 \geq 5$, and of CR-codimension 1. Assume $S \subset M$ is a smooth submanifold of dimension $n-2 = 2\kappa+1$, with $\kappa = k-1$. Let us assume that for $p \in S$,

$$\begin{aligned} &H_p^S = H_p \cap T_p S \ \text{ has } \mathbb{R}\text{-dimension } \ 2k-2, \\ &\text{so } H^S \text{ is a smooth subbundle of } TS. \end{aligned} \tag{2.8}$$

We furthermore assume that

$$J_p : H_p^S \longrightarrow H_p^S, \quad \forall\, p \in S. \tag{2.9}$$

Proposition 2.2. *Under the hypotheses (2.8) and (2.9), (S, H^S, J) is a CR manifold of CR-codimension* 1.

Proof. Again, let $X, Y \in C^\infty(S, H^S)$, which this time is equal to $C^\infty(S, TS \cap H)$. Thus X and Y have extensions to elements $\widetilde{X}$ and $\widetilde{Y}$ of $C^\infty(M, H)$, characterized by the property of being tangent to S. Given (2.9), $J\widetilde{X}$ and $J\widetilde{Y}$ are tangent to S. Since M is a CR manifold, we have

$$[\widetilde{X}, \widetilde{Y}] - [J\widetilde{X}, J\widetilde{Y}] \in C^\infty(M, H). \tag{2.10}$$

Over S, this is tangent to S, so $[X, Y] - [JX, JY]$ is a section of $TS \cap H = H^S$, and hence (1.4) holds, with S in place of M. Similarly, (1.5) holds.

Proposition 2.2 has the weakness that the hypothesis (2.9) can be hard to verify. This problem arises when $\dim M = 5$ and one seeks 3-dimensional CR submanifolds S.

As Robert Bryant has noted, one can do the following when M is a 5-dimensional surface in $\mathbb{C}^3$, with its induced CR structure. Namely, given $p \in M$, let Σ be a complex surface in $\mathbb{C}^3$, of complex dimension 2, such that $T_p\Sigma$ meets T_pM transversally and

$$T_p\Sigma \cap H_p^M \ \text{ has complex dimension } \ 1. \tag{2.11}$$

Then $S = \Sigma \cap M$ is (near p) a 3-dimensional hypersurface in Σ that carries a CR structure.

If M is not CR-embeddable in $\mathbb{C}^3$, matters remain mysterious.

3. The Levi form

The Levi form associated to a CR (or almost-CR) manifold simultaneously measures whether the sub-bundle H of TM is integrable and, if not, how the lack of integrability interacts with the complex structure on the fibers of H.

To begin, we ignore the CR condition, and let H be an arbitrary smooth sub-bundle of TM, and set

$$\mathcal{F}(X,Y) = [X,Y] \mod H, \quad X,Y \in C^\infty(M,H). \tag{3.1}$$

Note that, if $f,g \in C^\infty(M)$, then

$$\begin{aligned} \mathcal{F}(fX,gY) &= [fX,gY] && \mod H \\ &= g(Yf)X + f[X,gY] && \mod H \\ &= f(Xg)Y + fg[X,Y] && \mod H \\ &= fg\mathcal{F}(X,Y), \end{aligned} \tag{3.2}$$

so $\mathcal{F}$ (which we might call the Frobenius form) satisfies

$$\mathcal{F} \in C^\infty(M, \mathrm{Hom}(H \otimes H, TM/H)). \tag{3.3}$$

Now let us assume (M,H,J) is an almost-CR manifold, so, given $X,Y \in C^\infty(M,H)$,

$$[X,Y] - [JX,JY] \in C^\infty(M,H). \tag{3.4}$$

We then define the Levi form by

$$\mathcal{L}(X,Y) = \mathcal{F}(X,JY) = [X,JY] \mod H. \tag{3.5}$$

By (3.3),

$$\mathcal{L} \in C^\infty(M, \mathrm{Hom}(H \otimes H, TM/H)). \tag{3.6}$$

While $\mathcal{F}$ is clearly anti-symmetric:

$$\mathcal{F}(X,Y) = -\mathcal{F}(Y,X), \tag{3.7}$$

we have

Proposition 3.1. *The Levi form is symmetric:*

$$\mathcal{L}(X,Y)=\mathcal{L}(Y,X). \tag{3.8}$$

Proof. In fact, given $X,Y\in C^\infty(M,H)$,

$$\begin{aligned}\mathcal{L}(X,Y)-\mathcal{L}(Y,X)&=[X,JY]-[Y,JX] \mod H\\ &=[X,JY]+[JX,Y] \mod H\\ &=0,\end{aligned} \tag{3.9}$$

the last identity by (3.4), with Y replaced by JY.

If (M,H,J) has CR-codimension 1, then TM/H is a real line bundle over M, and, if it is orientable, one can choose a smooth family of isomorphisms $\alpha_p : T_pM/H_p\to\mathbb{R}$, and write

$$\widetilde{\mathcal{L}}(X,Y)=\alpha\circ\mathcal{L}(X,Y), \tag{3.10}$$

defining $\widetilde{\mathcal{L}}$ as a real-valued symmetric bilinear form on H.

For example, suppose Ω is a smoothly bounded open subset of $\mathbb{C}^{k+1}=\mathbb{C}^m$, and let N be the unit outward normal to $\partial\Omega$. Then $M=\partial\Omega$ is a CR manifold of CR codimension 1, and we can set

$$\widetilde{\mathcal{L}}(X,Y)=\langle[X,JY],JN\rangle, \tag{3.11}$$

for $X,Y\in C^\infty(M,H)$, using the Euclidean inner product on $\mathbb{R}^{2m}=\mathbb{C}^m$. We can relate $\widetilde{\mathcal{L}}(X,Y)$ to the second fundamental form of $M\subset\mathbb{R}^{2m}$, which has the form

$$\mathrm{II}\in C^\infty(M,\mathrm{Hom}(TM\otimes TM,NM)),$$

where NM denotes the normal bundle to M. Here is the result.

Proposition 3.2. *If $M=\partial\Omega$ with Ω a smoothly bounded open set in $\mathbb{C}^m$, then, for $X\in C^\infty(M,H)$,*

$$\widetilde{\mathcal{L}}(X,X)=\langle\mathrm{II}(X,X)+\mathrm{II}(JX,JX),N\rangle. \tag{3.12}$$

The proof makes use of the following result.

Lemma 3.3. *If II is the second fundamental form of the hypersurface $M\subset\mathbb{R}^{2m}$, and if $X\in C^\infty(M,H)$, then*

$$\begin{aligned}\mathrm{II}(X,X)&=-P_NJ\nabla_X(JX)\\ &=-JP_{JN}\nabla_X(JX).\end{aligned} \tag{3.13}$$

Here, ∇ is the Levi-Civita connection on M, P_N is the orthogonal projection of $\mathbb{R}^{2m}$ onto the span of N, and P_{JN} the orthogonal projection onto the span of JN. For background, we refer to §4 in Appendix C (Connections and Curvature) of [T].

Proof of Lemma 3.3. We start with two formulas for $\mathrm{II}(X,Y)$, given X and Y tangent to M. One is

$$\mathrm{II}(X,Y) = P_N D_X Y, \tag{3.14}$$

where D_X is the standard flat connection on $\mathbb{R}^{2m}$. The other is

$$\mathrm{II}(X,Y) = D_X Y - \nabla_X Y. \tag{3.15}$$

Note that $D_X(JY) = JD_XY$, so

$$\mathrm{II}(JX,X) = \mathrm{II}(X,JX) = P_N D_X(JX) = P_N J(D_X X). \tag{3.16}$$

Hence

$$\mathrm{II}(JX,X) = P_N J\ \mathrm{II}(X,X) + P_N J \nabla_X X. \tag{3.17}$$

Similarly,

$$\begin{aligned} \mathrm{II}(JX,JX) &= P_N J D_{JX} X \\ &= P_N J\ \mathrm{II}(JX,X) + P_N J \nabla_{JX} X, \end{aligned} \tag{3.18}$$

and subtracting (3.17) yields

$$\mathrm{II}(JX,JX) = P_N J P_N J\ \mathrm{II}(X,X) + P_N J P_N J \nabla_X X + P_N J \nabla_{JX} X. \tag{3.19}$$

Now $JP_NJ = -P_{JN}$, which is orthogonal to P_N, so $P_N J P_N J = 0$, and we have

$$\begin{aligned} \mathrm{II}(JX,JX) &= P_N J \nabla_{JX} X \\ &= J P_{JN} \nabla_{JX} X. \end{aligned} \tag{3.20}$$

Replacing X by JX in (3.20) yields (3.13), proving Lemma 3.3.

Proof of Proposition 3.2. Having Lemma 3.2, we can add (3.13) and (3.20), obtaining

$$\begin{aligned} \mathrm{II}(X,X) + \mathrm{II}(JX,JX) &= P_N J\Big(\nabla_{JX} X - \nabla_X(JX)\Big) \\ &= P_N J[JX,X]. \end{aligned} \tag{3.21}$$

To get (3.12), apply (3.21) to (3.11) and use the fact that $J^t = -J$.

We make some complementary remarks. First, it is readily seen that

$$\begin{aligned}\mathcal{L}(X, JY) &= -\mathcal{L}(JX, Y), \quad \text{and} \\ \mathcal{L}(JX, JY) &= \mathcal{L}(X, Y).\end{aligned} \tag{3.22}$$

Next, using

$$e^{tJ}X = (\cos t)X + (\sin t)JX, \tag{3.23}$$

and expanding, we see that

$$\mathcal{L}(e^{tJ}X, e^{tJ}Y) = \mathcal{L}(X, Y), \quad \forall\, t \in \mathbb{R}. \tag{3.24}$$

Using this, we can deduce from (3.12) that, in the setting of Proposition 3.2,

$$\widetilde{\mathcal{L}}(X, X) = \frac{1}{\pi} \int_0^{2\pi} \langle \mathrm{II}(e^{tJ}X, e^{tJ}X), N\rangle \, dt. \tag{3.25}$$

We can also expand the scope of (3.11) as follows. Assume M is a smooth $(2k+1)$-dimensional submanifold of $\mathfrak{X}$, a complex manifold of complex dimension $k+1$, with a Hermitian metric. Then one can use (3.11) to define $\widetilde{\mathcal{L}}$, for $X, Y \in C^\infty(M, H)$, H as in (2.4) (with $M \subset \mathfrak{X}$ in place of $S \subset M$). Looking at Proposition 3.2 and its proof, we see that, if D denotes the Levi-Civita connection on $\mathfrak{X}$, then (3.14)–(3.15) hold. Also,

$$D_X(JY) = JD_XY, \tag{3.26}$$

provided $\mathfrak{X}$ is *Kahler*. Thus, if $\mathfrak{X}$ is Kahler, we have (3.16)–(3.20), and hence we have the formulas (3.12) and (3.25).

Levi-flat CR manifolds

We say a CR mainfold (M, H, J) is Levi-flat if $\mathcal{L} \equiv 0$. By (3.5), this is equivalent to $\mathcal{F} \equiv 0$. Thus, by Frobenius' theorem, M is smoothly foliated by leaves Σ such that, for each $p \in \Sigma \subset M$,

$$T_p\Sigma = H_p. \tag{3.27}$$

Thus J endows each leaf Σ with an almost complex structure, and the vanishing of $\mathcal{N}(X, Y)$ for $X, Y \in C^\infty(M, H)$ given by (1.5) implies the almost complex structure on each leaf is integrable. Hence the Newlander-Nirenberg theorem implies that local holomorphic coordinates exist on each leaf. Going further, [Ni] provided a Newlander-Nirenberg theorem with parameters, which yields the following.

Proposition 3.4. *If (M,H,J) is a Levi-flat CR manifold, then for each $p\in M$, there is a neighborhood U of p in M and a smooth diffeomorphism*

$$\Phi : U \longrightarrow \mathcal{O}\subset \mathbb{C}^k\times\mathbb{R}^{n-2k}, \tag{3.28}$$

such that for each leaf Σ that intersects U, there exists $y=y(\Sigma)\in\mathbb{R}^{n-2k}$ such that

$$\Phi\big|_{U\cap\Sigma} = (\varphi_\Sigma, y(\Sigma)), \tag{3.29}$$

where $\varphi_\Sigma : U\cap\Sigma\to\mathbb{C}^k$ is a holomorphic diffeomorphism onto an open subset of $\mathbb{C}^k$.

Here, $n=\dim M$ and $2k=\dim_{\mathbb{R}} H_p$. For more on this, applicable to CR manifolds with limited smoothness, one can see §5 of [HT].

CR manifolds with constant Levi rank

Given a CR manifold (M,H,J), $p\in M$, let us set

$$E_p=\{X\in H_p : \mathcal{L}(X,Y)=0,\ \forall\, Y\in H_p\}. \tag{3.30}$$

We say that (M,H,J) has constant Levi rank if $\dim E_p$ is independent of p. Then we have a smooth vector bundle $E\to M$, with fibers E_p. Note that the second identity in (3.12) implies

$$J : E_p \longrightarrow E_p, \tag{3.31}$$

so $\dim_{\mathbb{R}} E_p$ is even. Call it 2κ. Following [Fr], we have:

Lemma 3.5. *Let (M,H,J) be a constant Levi-rank CR manifold, and let $E\subset H$ be the smooth subbundle given by (3.30). Then*

$$\begin{aligned} X,Y\in C^\infty(M,E),\ & Z\in C^\infty(M,H) \\ \Longrightarrow & [X,Z]\in C^\infty(M,H), \quad \textit{and} \\ & [X,Y]\in C^\infty(M,E). \end{aligned} \tag{3.32}$$

Proof. For such X,Y, and Z, we have $\mathcal{L}(X,Z)=\mathcal{L}(Y,Z)=0$. Also, $JZ\in C^\infty(M,H)$, so $\mathcal{F}(X,Z)=\mathcal{F}(Y,Z)=0$. Hence

$$[X,Z],\ [Y,Z]\in C^\infty(M,H), \tag{3.33}$$

and we have the first implication in (3.32). Furthermore, we can reiterate this argument, with Z replaced by $[X,Z]$ and $[Y,Z]$, to get

$$[Y,[X,Z]],\ [X,[Y,Z]]\in C^\infty(M,H). \tag{3.34}$$

It then follows from Jacobi's identity that

$$[[X,Y],Z] \in C^\infty(M,H). \tag{3.35}$$

Since this holds for all $Z \in C^\infty(M,H)$ (and hence for JZ), we deduce that $\mathcal{L}([X,Y],Z)=0$, for all $Z \in C^\infty(M,H)$, giving the last implication in (3.32).

It follows that the triple (M,E,J) satisfies (1.4), with H replaced by E, and then it inherits (1.5) from (M,H,J). Thus (M,E,J) is a CR manifold, of CR-codimension $n-2\kappa$, and it is *subordinate* to (M,H,J), in the following sense.

Definition. Let (M,H,J) be a CR manifold, and let $E\subset H$ be a smooth subbundle, satisfying (3.31). If (M,E,J) is a CR manifold, which is the case provided

$$X,Y \in C^\infty(M,E) \Longrightarrow [X,Y]-[JX,JY] \in C^\infty(M,E), \tag{3.36}$$

we say that (M,E,J) is subordinate to (M,H,J).

If (M,E,J) arises from a CR manifold (M,H,J) with constant Levi rank, as in (3.30), then (M,E,J) is a Levi-flat CR manifold. As before, M is smoothly foliated by leaves Σ, with the property that, if $p\in\Sigma\subset M$,

$$T_p\Sigma = E_p. \tag{3.37}$$

Thus Proposition 3.4 applies, with H replaced by E (and k by κ). Furthermore, the first implication in (3.32) gives additional structure. Namely, given $X\in C^\infty(M,E)$, the local flow $\mathcal{F}_X^t$ on M it generates, which preserves each leaf of the foliation associated to E, also has the property that

$$\text{the derived action of } \mathcal{F}_X^t \text{ on } TM \text{ preserves } H. \tag{3.38}$$

If we form the (local) space $\widetilde{M}$ of leaves, by identifying each leaf with a point, then given $p\in M$ (which we identify with $p\in\widetilde{M}$), we have

$$T_p\widetilde{M} = T_pM/E_p, \tag{3.39}$$

and $T\widetilde{M}$ has a smooth subbundle $\widetilde{H}$, with

$$\widetilde{H}_p = H_p/E_p. \tag{3.40}$$

Furthermore, the action of J on H_p, preserving E_p, induces

$$\widetilde{J}:\widetilde{H}_p \longrightarrow \widetilde{H}_p, \quad \widetilde{J}^2=-I, \tag{3.41}$$

and we have a triple $(\widetilde{M},\widetilde{H},\widetilde{J})$. However, this is not necessarily a CR manifold.

CR manifolds with nondegenerate Levi form

If, in (3.30), $E_p = 0$, we say the Levi form of M is nondegenerate at p. In case M has CR-codimension 1, this is equivalent to the real-valued quadratic form $\widetilde{\mathcal{L}}$ in (3.10) being nondegenerate at p. If $\widetilde{\mathcal{L}}$ is positive definite (or negative definite) at p, we say M is strongly pseudoconvex at p. If $\Omega \subset \mathbb{C}^n$ is a smoothly bounded open set, with boundary $\partial\Omega = M$, then we can use (3.11) to define $\widetilde{\mathcal{L}}$. By (3.12), we see that if Ω is strongly convex in $\mathbb{C}^n$ (i.e., its second fundamental form is positive definite) then it is strongly pseudoconvex. The following is a partial converse.

Proposition 3.6. *Let $\Omega \subset \mathbb{C}^n$ be smoothly bounded, $p \in \partial\Omega$. Assume $\partial\Omega$ is strongly pseudoconvex at p. Then there is a neighborhood $\mathcal{O}$ of p in $\mathbb{C}^n$ and a biholomorphic diffeomorphism $f : \mathcal{O} \to \widetilde{\mathcal{O}} \subset \mathbb{C}^n$ such that $f(\partial\Omega \cap \mathcal{O})$ is strongly convex at $f(p)$.*

For a proof, see §3.2 of [Kr].

4. Associated complex vector bundles

Let (M, H, J) be a CR manifold. We define complex subbundles $\mathcal{H}$ and $\mathcal{A}$ of the complexified tangent bundle $\mathbb{C}TM$, with fibers

$$\mathcal{H}_p = \{X - iJX : X \in H_p\}, \quad \mathcal{A}_p = \{X + iJX : X \in H_p\}. \tag{4.1}$$

Note that $Z = X - iJX \in H_p \Rightarrow iZ = JX + iX = JX - iJ(JX) \in H_p$. Similarly, $\overline{Z} = X + iJX \in \mathcal{A}_p \Rightarrow i\overline{Z} \in \mathcal{A}_p$. Note also that

$$\mathcal{A}_p = \overline{\mathcal{H}}_p, \quad \mathcal{H}_p \cap \mathcal{A}_p = 0, \quad \mathcal{H}_p + \mathcal{A}_p = \mathbb{C}H_p. \tag{4.2}$$

Furthermore, given $Z \in \mathbb{C}H_p$,

$$Z \in \mathcal{H}_p \Longleftrightarrow JZ = iZ, \quad Z \in \mathcal{A}_p \Longleftrightarrow JZ = -iZ. \tag{4.2A}$$

If we take

$$\begin{aligned} Z = X - iJX, \quad W = Y - iJY \in C^\infty(M, \mathcal{H}), \\ \overline{Z} = X + iJX, \quad \overline{W} = Y + iJY \in C^\infty(M, \mathcal{A}), \end{aligned} \tag{4.3}$$

then, by (1.1)–(1.6), the defining condition that (M, H, J) is a CR manifold is

$$[\overline{Z}, \overline{W}] \in C^\infty(M, \mathcal{A}). \tag{4.4}$$

Similarly,

$$[Z, W] \in C^\infty(M, \mathcal{H}). \tag{4.5}$$

Thus $\mathcal{A}$ and $\mathcal{H}$ are involutive subbundles of $\mathbb{C}TM$. On the other hand,

$$\begin{aligned} [Z, \overline{W}] &= [X - iJX, Y + iJY] \\ &= [X, Y] + [JX, JY] \\ &\quad + i\big\{[X, JY] - [JX, Y]\big\}. \end{aligned} \tag{4.6}$$

By (1.4), the real part is equal to $2[X, Y]$ mod $C^\infty(M, H)$, and by (1.8) the imaginary part is equal to $2[X, JY]$ mod $C^\infty(M, H)$. Thus the imaginary part is equal to $2\mathcal{L}(X, Y)$ mod $C^\infty(M, H)$. In particular,

$$\begin{aligned} \frac{1}{2i}[Z, \overline{Z}] &= [X, JX] \\ &= \mathcal{L}(X, X) \mod C^\infty(M, H). \end{aligned} \tag{4.7}$$

Generally, if one has a vector bundle over M, constructions of linear algebra give rise to a string of other vector bundles. We will find it useful to consider the vector bundles

$$\mathcal{A}', \quad \text{with fibers} \ \ \mathcal{A}'_p = \operatorname{Hom}(\mathcal{A}_p, \mathbb{C}), \tag{4.8}$$

and their exterior powers

$$\Lambda^j \mathcal{A}', \quad \text{with fibers } \Lambda^j \mathcal{A}'_p. \tag{4.9}$$

We will recall some facts about such exterior powers, referring to §21 of [T2] for details.

To take a general setting, let V be a d-dimensional vector space over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), with dual V'. We define $\Lambda^j V'$ to consist of $\mathbb{F}$-multilinear maps

$$\beta : V \times \cdots \times V \longrightarrow \mathbb{F} \quad (j \text{ factors}), \tag{4.10}$$

that are *alternating*, i.e., $\beta(v_1, \dots, v_j)$ changes sign when any two of its arguments are switched. Equivalently, if $\sigma \in S_j$ is a permutation,

$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(j)}) = (\operatorname{sgn} \sigma) \beta(v_1, \dots, v_j). \tag{4.11}$$

We set $\Lambda^0 V' = \mathbb{F}$. Note that $\Lambda^1 V' = V'$. It is readily verified that

$$\dim_{\mathbb{F}} \Lambda^d V' = 1, \quad \text{and} \ \ j > d \Rightarrow \Lambda^j V' = 0. \tag{4.12}$$

The family of vector spaces $\Lambda^j V'$ comes equipped with a *wedge product*, an $\mathbb{F}$-bilinear map

$$w : \Lambda^k V' \times \Lambda^\ell V' \longrightarrow \Lambda^{k+\ell} V', \quad w(\alpha, \beta) = \alpha \wedge \beta, \tag{4.13}$$

which is associative:

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma, \tag{4.14}$$

and satisfies the anti-commutativity condition

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha, \quad \text{if} \ \ \alpha \in \Lambda^k V', \ \beta \in \Lambda^\ell V'. \tag{4.15}$$

It has the property that, if $\{e_1, \dots, e_d\}$ is a basis of V and $\{\varepsilon_1, \dots, \varepsilon_d\}$ the dual basis of V', then, if $\{j_1, \dots, j_k\}$ are distinct elements of $\{1, \dots, d\}$,

$$\begin{aligned} &(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k})(e_{j_1}, \dots, e_{j_k}) = 1, \quad \text{and} \\ &(\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k})(e_{\ell_1}, \dots, e_{\ell_k}) = 0, \quad \text{if} \ \ \{\ell_1, \dots, \ell_k\} \neq \{j_1, \dots, j_k\}. \end{aligned} \tag{4.16}$$

Basis-independent formulas for w are given as follows, for $\alpha \in \Lambda^k V'$, $\beta \in \Lambda^\ell V'$, $v_j \in V$. First, if $k = \ell = 1$,

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1). \tag{4.17}$$

Next, if $k = 1$ and ℓ is general,

$$(\alpha \wedge \beta)(v_1, \dots, v_{\ell+1}) = \sum_{j=1}^{\ell} (-1)^{j+1} \alpha(v_j)\beta(v_1, \dots, \widehat{v}_j, \dots, v_{\ell+1}). \tag{4.18}$$

Finally, for general k and ℓ,

$$\begin{aligned} &(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) \\ &= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}). \end{aligned} \tag{4.19}$$

We will only have explicit need for (4.17).

5. CR functions

If $\Omega \subset \mathbb{C}^n$ is open and $u \in C^1(\Omega)$ is complex valued, then u is holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial}{\partial \overline{z}_j} u = 0, \ 1 \le j \le n, \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big). \tag{5.1}$$

An equivalent formulation is that, for each smooth real vector field X on Ω,

$$Z = X + iJX \Longrightarrow Zu = 0, \tag{5.2}$$

where $J : \mathbb{C}^n \to \mathbb{C}^n$ defines the standard complex structure on $\mathbb{C}^n$. This leads to the following concept.

Let (M, H, J) be a CR manifold, $u \in C^1(M)$. We say u is a CR function on M if and only if

$$Zu = 0, \quad \forall\, Z \in C^\infty(M, \mathcal{A}), \tag{5.3}$$

where $\mathcal{A}$ is given by (4.1). Here is another convenient formulation. Define

$$\overline{\partial}_M : C^\infty(M) \longrightarrow C^\infty(M, \mathcal{A}') \tag{5.4}$$

by

$$\overline{\partial}_M u(Z_p) = Zu(p), \quad p \in M, \tag{5.5}$$

for $Z \in C^\infty(M, \mathcal{A})$. Here $\mathcal{A}' = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$. Then, given $u \in C^1(M)$,

$$u \text{ is a CR function on } M \Leftrightarrow \overline{\partial}_M u = 0. \tag{5.6}$$

One way CR functions arise is as follows. Let (M, H, J) be a CR submanifold of $\mathbb{C}^n$, and let $u = v|_M$, where $v \in C^1(\mathbb{C}^n)$. Then

$$\begin{aligned} &\overline{\partial} v = 0 \ \text{ on } \ M \\ &\Longleftrightarrow Zv = 0 \text{ on } M, \ \forall\, Z = X + iJX, X \text{ real vector field on } \mathbb{C}^n \\ &\Longrightarrow Zu = 0 \text{ on } M, \ \forall\, Z \in C^\infty(M, \mathcal{A}). \end{aligned} \tag{5.7}$$

Thus the restriction to M of any $v \in C^1(\mathbb{C}^n)$ that satisfies the Cauchy-Riemann equations on M is a CR function on M. In particular, the restriction to M of a function on $\mathbb{C}^n$ that is holomorphic on a neighborhood of M is a CR function on M.

Let us specialize to $M = \partial\Omega$, Ω a smoothly bounded open subset of $\mathbb{C}^n$. We see from the discussion above that if $v \in C^1(\overline{\Omega})$ and v is holomorphic on Ω, then $u = v|_{\partial\Omega}$ is a CR function on M. The following converse result was proved by S. Bochner.

Proposition 5.1. *Let Ω be a bounded open subset of $\mathbb{C}^n$ with C^∞, connected boundary. Assume $n \geq 2$. Take $u \in C^2(\partial\Omega)$. Then there exists a holomorphic v on Ω such that $v|_{\partial\Omega} = u$ if and only if u is a CR function on $\partial\Omega$.*

For a proof, see [Bog], Chapter 19, with complements in Chapter 24. The proof brings in an integral operator containing the "Bochner-Martinelli kernel." Note that if $u = v|_{\partial\Omega}$ and v is holomorphic, then v is harmonic on Ω, so also v is obtained from u via the Poisson integral. In view of this, many precise results relating the regularity of u on $\partial\Omega$ to that of v on $\overline{\Omega}$ are available (supplementing material in [Bog]).

There are also local extension results, which depend on the nature of the Levi form of M, when M is a codimension-1 surface in $\mathbb{C}^n$. To state the result, we define $\widetilde{\mathcal{L}}$ as in (3.11), where N is a smooth unit normal to M in $\mathbb{C}^n$. If $p \in M$ has a neighborhood $\mathcal{O}_1$, by $\mathcal{O}_1^+$ we mean the component of $\mathcal{O}_1 \setminus M$ out of which N points, and by $\mathcal{O}_1^-$ we mean the other component.

Proposition 5.2. *Let M be a codimension 1 surface in $\mathbb{C}^n$, and consider $\widetilde{\mathcal{L}}$, as described above. Take $p \in M$.*

(a) If $\widetilde{\mathcal{L}}_p$ has at least one positive eigenvalue, then there exist neighborhoods $\mathcal{O}_1 \subset \mathcal{O}$ of p in $\mathbb{C}^n$ such that each smooth CR function u on $\mathcal{O} \cap M$ is the boundary value of a (unique) function holomorphic in $\mathcal{O}_1^+$.

(b) If $\widetilde{\mathcal{L}}_p$ has at least one negative eigenvalue, the same sort of conclusion holds, with $\mathcal{O}_1^+$ replaced by $\mathcal{O}_1^-$.

(c) If $\widetilde{\mathcal{L}}_p$ has two eigenvalues of opposite signs, then such a conclusion holds with $\mathcal{O}_1^\pm$ replaced by $\mathcal{O}_1$.

Parts (b) and (c) follow from part (a), which was proved by H. Lewy in the case $n = 2$. For a proof of Proposition 5.2, see Chapters 14–16 of [Bog], or Chapter 5 of [Tai].

In case (a) of Proposition 5.2, there exists $X \in C^\infty(M, H)$ such that $J[X, JX]$ points into $\mathcal{O}_1^+$ at p, and in case (b) there exists $X \in C^\infty(M, H)$ such that $J[X, JX]$ points into $\mathcal{O}_1^-$ at p.

Generally, if $M \subset \mathbb{C}^n$ is an embedded CR manifold, we say a vector $\xi \in \mathbb{R}^{2n} = \mathbb{C}^n$ is in the range of the Levi form at $p \in M$ provided there exists $X \in C^\infty(M, H)$ such that

$$J[X, JX] = \xi \quad \text{at} \quad p. \tag{5.8}$$

The set $\mathcal{L}_p(M)$ of such vectors is a cone in $\mathbb{R}^{2n} = \mathbb{C}^n$. Denote its convex hull by $\Gamma_p(M)$, and let $\overset{\circ}{\Gamma}_p(M)$ denote the interior of $\Gamma_p(M)$.

The following CR extension result was proved in [BP] To state it, we define an embedded CR manifold $M \subset \mathbb{C}^n$ to be *generic* provided

$$M \text{ has CR-codimension } \ell \text{ and } \dim M + \ell = 2n. \tag{5.9}$$

Proposition 5.3. *Let $M \subset \mathbb{C}^n$ be a smooth, generic CR manifold. Take $p \in M$ and assume*

$$\overset{\circ}{\Gamma}_p(M) \neq \emptyset. \tag{5.10}$$

Let U be a neighborhood of p in M. There exist U_1, Ω, with the following properties.

First, U_1 is a neighborhood of p in M, and Ω is open in $\mathbb{C}^n$, and

$$p \in U_1 \subset \overline{\Omega} \cap M \subset U. \tag{5.11}$$

Next, if Γ is an open cone in $\mathbb{C}^n$ whose closure $\overline{\Gamma}$ has compact base and satisfies

$$\overline{\Gamma} \setminus 0 \subset \overset{\circ}{\Gamma}_p(M), \tag{5.12}$$

then there is a neighborhood $U_\Gamma \subset U$ of p and an $\varepsilon_\Gamma > 0$ such that

$$U_\Gamma + (\Gamma \cap B_{\varepsilon_\Gamma}(0)) \subset \Omega, \tag{5.13}$$

such that the following holds.

If f is a smooth CR function on U, then there is a unique holomorphic function v on Ω such that $v \in C(\overline{\Omega})$ and $v = f$ on U_1.

Further results on holomorphic extensions of CR functions, including cases where the Levi form vanishes at a point but some "higher Levi form" does not, can be found in Chapter 5 of [Tai] and in Part 3 of [Bog].

6. The $\overline{\partial}_M$-complex

Let (M, H, J) be a CR manifold. In §5 we defined

$$\overline{\partial}_M : C^\infty(M) \longrightarrow C^\infty(M, \mathcal{A}') \tag{6.1}$$

by

$$\overline{\partial}_M u(Z) = Zu, \quad \text{for} \ \ Z \in C^\infty(M, \mathcal{A}), \tag{6.2}$$

with $\mathcal{A}$ as in (4.1), $\mathcal{A}' = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$. Note that if also $f \in C^\infty(M)$, then, since $Z(fu) = (Zf)u + f(Zu)$, we have

$$\overline{\partial}_M(fu) = (\overline{\partial}_M f)u + f(\overline{\partial}_M u). \tag{6.2A}$$

Here we extend the domain of the operator $\overline{\partial}_M$, starting with defining

$$\overline{\partial}_M : C^\infty(M, \mathcal{A}') \longrightarrow C^\infty(M, \Lambda^2\mathcal{A}'). \tag{6.3}$$

Taking a cue from formulas for the exterior derivative, acting on differential forms, we set

$$\overline{\partial}_M \alpha(Z_1, Z_2) = Z_1\alpha(Z_2) - Z_2\alpha(Z_1) - \alpha([Z_1, Z_2]), \tag{6.4}$$

for

$$\alpha \in C^\infty(M, \mathcal{A}'), \quad Z_j \in C^\infty(M, \mathcal{A}). \tag{6.5}$$

Recall that the implication

$$Z_j \in C^\infty(M, \mathcal{A}) \Longrightarrow [Z_1, Z_2] \in C^\infty(M, \mathcal{A}) \tag{6.6}$$

holds when (M, H, J) is a CR manifold. It is readly verified that

$$\overline{\partial}_M \alpha(Z_2, Z_1) = -\overline{\partial}_M \alpha(Z_1, Z_2), \tag{6.7A}$$

and if $f_j \in C^\infty(M)$, then

$$\overline{\partial}_M \alpha(f_1 Z_1, f_2 Z_2) = f_1 f_2 \overline{\partial}_M \alpha(Z_1, Z_2), \tag{6.7B}$$

so (6.4) defines $\overline{\partial}_M \alpha \in C^\infty(M, \Lambda^2\mathcal{A}')$. Also, use of

$$Z_1(f\alpha)(Z_2) = (Z_1 f)\alpha(Z_2) + f Z_1\alpha(Z_2) \tag{6.8}$$

and its counterpart for $Z_2\alpha(Z_1)$ gives

$$\overline{\partial}_M(f\alpha)(Z_1, Z_2) = \Big\{(Z_1 f)\alpha(Z_2) - (Z_2 f)\alpha(Z_1)\Big\} + f\overline{\partial}_M \alpha(Z_1, Z_2), \tag{6.9}$$

or equivalently (cf. (4.17))

$$\overline{\partial}_M(f\alpha) = \overline{\partial}_M f \wedge \alpha + f\overline{\partial}_M \alpha. \tag{6.10}$$

The next result is the crucial ingredient in making $\overline{\partial}_M$ a “complex.”

Proposition 6.1. *If $u \in C^\infty(M)$, then*

$$\overline{\partial}_M^2 u = 0. \tag{6.11}$$

Proof. We have, for $Z_j \in C^\infty(M, \mathcal{A})$,

$$\begin{aligned} \overline{\partial}_M^2 u(Z_1, Z_2) &= \overline{\partial}_M(\overline{\partial}_M)(Z_1, Z_2) \\ &= Z_1(\overline{\partial}_M u)(Z_2) - Z_2(\overline{\partial}_M u)(Z_1) - (\overline{\partial}_M u)([Z_1, Z_2]) \\ &= Z_1(Z_2 u) - Z_2(Z_1 u) - [Z_1, Z_2]u \\ &= 0. \end{aligned} \tag{6.12}$$

We next produce a convenient formula for $\overline{\partial}_M \alpha$, defined by (6.4)–(6.5). Assume $\dim M = 2k + \ell$ and (M, H, J) has CR-codimension ℓ, so, for $p \in M$,

$$\dim_{\mathbb{R}} H_p = 2k, \quad \text{hence} \quad \dim_{\mathbb{C}} \mathcal{A}_p = k. \tag{6.13}$$

Fix $p \in M$ and pick $Z_1, \dots, Z_k \in C^\infty(M, \mathcal{A})$ such that $\{Z_1(p), \dots, Z_k(p)\}$ is a basis of $\mathcal{A}_p$, $Z_j = X_j + iJX_j$. Then pick smooth $v_j : M \to \mathbb{R}$ such that $X_i v_j(p) = \delta_{ij}$. It follows that

$$\{\overline{\partial}_M v_j(p) : 1 \le j \le k\} \text{ is a basis of } \mathcal{A}'_p. \tag{6.14}$$

Consequently, there is a neighborhood $\mathcal{O}$ of p in M such that

$$\{\overline{\partial}_M v_j(x) : 1 \le j \le k\} \text{ is a basis of } \mathcal{A}'_x, \quad \forall\, x \in \mathcal{O}. \tag{6.15}$$

This yields the following.

Proposition 6.2. *Given $p \in M$, pick a neighborhood $\mathcal{O}$ of p and $\{v_j : 1 \le j \le k\}$ as in (6.15). Then, given $\alpha \in C^\infty(\mathcal{O}, \mathcal{A}')$, there exist unique $f_j \in C^\infty(\mathcal{O})$ such that*

$$\alpha = \sum_{j=1}^k f_j \overline{\partial}_M v_j. \tag{6.16}$$

Now, applying (6.10)–(6.11) to (6.16) gives

$$\overline{\partial}_M \alpha = \sum_{j=1}^k (\overline{\partial}_M f_j) \wedge (\overline{\partial}_M v_j) \text{ on } \mathcal{O}. \tag{6.17}$$

We are ready for the main result of this section.

Proposition 6.3. *There is a unique first-order differential operator*

$$\overline{\partial}_M : C^\infty(M, \Lambda^j \mathcal{A}') \longrightarrow C^\infty(M, \Lambda^{j+1}\mathcal{A}'), \quad j \ge 0, \tag{6.18}$$

agreeing with (6.1)–(6.2) for $j = 0$ and with (6.3)–(6.4) for $j = 1$, and having the property that, for each $\beta \in C^\infty(M, \Lambda^j \mathcal{A}')$, $f \in C^\infty(M)$, and $\alpha \in C^\infty(M, \mathcal{A}')$,

$$\overline{\partial}_M(f\beta) = (\overline{\partial}_M f) \wedge f + f\overline{\partial}_M \beta, \tag{6.19}$$

and

$$\overline{\partial}_M(\alpha \wedge \beta) = (\overline{\partial}_M \alpha) \wedge \beta - \alpha \wedge (\overline{\partial}_M \beta). \tag{6.20}$$

Remark. Of course, $\Lambda^j \mathcal{A}' = 0$ for $j > k$, so the action of $\overline{\partial}_M$ in (6.18) is trivial for $j \ge k$.

To prove Proposition 6.3, it suffices to work locally. To get started, take as given $\mathcal{O} \subset M$ for which we have $\{v_j : 1 \le j \le k\}$ as in (6.15). Then, as long as $j \le k$, an arbitrary $\beta \in C^\infty(\mathcal{O}, \Lambda^j \mathcal{A}')$ can be written uniquely as

$$\beta = \sum_\alpha f_\alpha \, \overline{\partial}_M v_{\alpha_1} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}, \quad f_\alpha \in C^\infty(\mathcal{O}), \tag{6.21}$$

where $\alpha = (\alpha_1, \dots, \alpha_j)$ and $1 \le \alpha_1 < \cdots < \alpha_j \le k$. If (6.19) holds, then

$$\overline{\partial}_M \beta = \sum_\alpha (\overline{\partial}_M f_\alpha) \wedge (\overline{\partial}_M v_{\alpha_1}) \wedge \cdots \wedge (\overline{\partial}_M v_{\alpha_j}) + R(\beta), \tag{6.22}$$

with

$$R(\beta) = \sum_\alpha f_\alpha \, \overline{\partial}_M (\overline{\partial}_M v_{\alpha_1} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}). \tag{6.23}$$

We claim that $R(\beta) = 0$. In fact, we claim that

$$\overline{\partial}_M (\overline{\partial}_M v_{\alpha_1} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}) = 0. \tag{6.24}$$

Indeed, if (6.20) holds, then the left side of (6.24) is equal to

$$\begin{aligned} &\overline{\partial}_M^2 v_{\alpha_1} \wedge (\overline{\partial}_M v_{\alpha_1} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}) \\ &\qquad + \overline{\partial}_M v_{\alpha_1} \wedge \overline{\partial}_M (\overline{\partial}_M v_{\alpha_2} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}). \end{aligned} \tag{6.25}$$

The first term vanishes by (6.11). Then an inductive argument shows that the second term in (6.25) vanishes, so we have (6.24). Consequently, for β as in (6.21),

$$\overline{\partial}_M \beta = \sum_\alpha (\overline{\partial}_M f_\alpha) \wedge (\overline{\partial}_M v_{\alpha_1}) \wedge \cdots \wedge (\overline{\partial}_M v_{\alpha_j}). \tag{6.26}$$

This shows that if $\overline{\partial}_M$ exists, satisfying (6.18)–(6.20), it is unique.

For the existence part of Proposition 6.3, we use (6.21) and (6.26) to define $\overline{\partial}_M$. The remaining task is to show that $\overline{\partial}_M$, so defined, satisfies (6.19) and (6.20), for arbitrary $f \in C^\infty(\mathcal{O})$, $\alpha \in C^\infty(\mathcal{O}, \mathcal{A}')$. To start, we have

$$f\beta = \sum_\alpha f f_\alpha \, \overline{\partial}_M v_{\alpha_1} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}, \tag{6.27}$$

so applying the formula (6.26) to $f\beta$ yields

$$\overline{\partial}_M (f\beta) = \sum_\alpha (\overline{\partial}_M f f_\alpha) \wedge (\overline{\partial}_M v_{\alpha_1}) \wedge \cdots \wedge (\overline{\partial}_M v_{\alpha_j}). \tag{6.28}$$

Then (6.2A) gives

$$\overline{\partial}_M (f f_\alpha) = (\overline{\partial}_M f) f_\alpha + f \overline{\partial}_M f_\alpha, \tag{6.29}$$

and (6.19) readily follows.

To check (6.20), we can set

$$\alpha = \sum_i g_i \, \overline{\partial}_M v_i, \tag{6.30}$$

so

$$\alpha \wedge \beta = \sum_{i,\alpha} g_i f_\alpha \, (\overline{\partial}_M v_i) \wedge (\overline{\partial}_M v_{\alpha_1}) \wedge \cdots \wedge (\overline{\partial}_M v_{\alpha_j}). \tag{6.31}$$

While it is not necessarily the case that $1 \le i < \alpha_1 < \cdots < \alpha_j$, nevertheless one can reorder the factors (when the product is nonzero) and verify that

$$\overline{\partial}_M (\alpha \wedge \beta) = \sum_{i,\alpha} \overline{\partial}_M (g_i f_\alpha) \wedge (\overline{\partial}_M v_i) \wedge (\overline{\partial}_M v_{\alpha_1}) \wedge \cdots \wedge (\overline{\partial}_M v_{\alpha_j}). \tag{6.32}$$

As in (6.2A),

$$\overline{\partial}_M (g_i f_\alpha) = (\overline{\partial}_M g_i) f_\alpha + g_i (\overline{\partial}_M f_\alpha), \tag{6.33}$$

and from here (6.20) follows.

Having Proposition 6.3, we can extend the scope of Proposition 6.1.

Proposition 6.4. *The operator* $\overline{\partial}_M$ *in (6.18) satisfies*

$$\overline{\partial}_M^2 \beta = 0, \quad \forall\, \beta \in C^\infty(M, \Lambda^j \mathcal{A}'). \tag{6.34}$$

Proof. Locally, we have $\overline{\partial}_M \beta$ given by (6.26). Applying $\overline{\partial}_M$ to this gives

$$\begin{aligned} \overline{\partial}_M^2 \beta = & \sum_\alpha (\overline{\partial}_M^2 f_\alpha) \wedge (\overline{\partial}_M v_{\alpha_1}) \wedge \cdots \wedge (\overline{\partial}_M v_{\alpha_j}) \\ & + \sum_\alpha (\overline{\partial}_M f_\alpha) \wedge \overline{\partial}_M (\overline{\partial}_M v_{\alpha_1} \wedge \cdots \wedge \overline{\partial}_M v_{\alpha_j}). \end{aligned} \tag{6.35}$$

since, by Proposition 6.1, $\overline{\partial}_M^2 f_\alpha = 0$, the first sum on the right side of (6.35) vanishes. Meanwhile, an analysis parallel to that applied to (6.23) shows that the last sum there vanishes.

Here is an extension of (6.4), which is also parallel to a well known formula for the exterior derivative of a differential form.

Proposition 6.5. *If* $\beta \in C^\infty(M, \Lambda^j \mathcal{A}')$*, then, for* $Z_i \in C^\infty(M, \mathcal{A})$*,*

$$\begin{aligned} & \overline{\partial}_M \beta(Z_1, \dots, Z_{j+1}) \\ & \quad = \sum_{i=1}^{j+1} (-1)^{i+1} Z_i \beta(Z_1, \dots, \widehat{Z}_i, \dots, Z_{j+1}) \\ & \quad\quad + \sum_{1 \le h < i \le j+1} (-1)^{h+i} \beta([Z_h, Z_i], Z_1, \dots, \widehat{Z}_h, \dots, \widehat{Z}_i, \dots, Z_{j+1}). \end{aligned} \tag{6.36}$$

For a proof, see Chapter 2 of [Tai].

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