Musings on the Discrete Heisenberg Group

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1. Introduction

The 3D continuous Heisenberg group $\mathcal{H}^3$ consists of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & p & t \\ & 1 & q \\ & & 1 \end{pmatrix}, \tag{1.1}$$

with $t, q, p \in \mathbb{R}$. We denote this element by (t, q, p) for short. The discrete Heisenberg group $\mathcal{H}^3_d$ consists of matrices of the form (1.1) with $t, q, p \in \mathbb{Z}$. We want to construct irreducible unitary representations of $\mathcal{H}^3_d$. First we describe a complete set of irreducible unitary representations of $\mathcal{H}^3$. See [T], Chapter 1, for more.

There are one dimensional representations arising from the surjective homomorphism $\mathcal{H}^3 \to \mathbb{R}^2$ given by

$$(t, q, p) \mapsto (q, p). \tag{1.2}$$

There are also unitary representations on $L^2(\mathbb{R})$, given for $\lambda, \mu \in \mathbb{R} \setminus 0$ by

$$R_{\lambda,\mu}(0, q, p)f(x) = e^{i\lambda qx} f(x + \mu p). \tag{1.3}$$

In such a case,

$$R_{\lambda,\mu}(t, 0, 0) = e^{i\lambda\mu t} I. \tag{1.3A}$$

We have

$$R_{\lambda,\mu} \approx R_{\lambda',\mu'} \Longleftrightarrow \lambda\mu = \lambda'\mu', \tag{1.4}$$

so we could restrict attention to $R_{1,\mu},\ \mu \in \mathbb{R} \setminus 0$.

We restrict these representations to $\mathcal{H}^3_d$. The 1-dimensional representations come from the surjective homomorphism $\mathcal{H}^3_d \to \mathbb{Z}^2$, again given by (1.2). The restriction of $R_{\lambda,\mu}$ to $\mathcal{H}^3_d$ is not irreducible. In such a case, the group of unitary operators on $L^2(\mathbb{R})$ is generated by M_λ and T_μ, defined by

$$M_\lambda f(x) = e^{i\lambda x} f(x), \quad T_\mu f(x) = f(x + \mu). \tag{1.5}$$

If $\mathcal{A}_{\lambda,\mu}$ denotes the von Neumann algebra generated by these operators, then its commutant satisfies

$$\mathcal{A}'_{\lambda,\mu} \supset \mathcal{A}_{2\pi/\mu, 2\pi/\lambda}, \tag{1.6}$$

since M_λ and T_μ clearly commute with $M_{2\pi/\mu}$ and $T_{2\pi/\lambda}$.

Having this, we are motivated to define representations ρ_μ of $\mathcal{H}_d^3$ on $L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, as follows. For each $\mu \in \mathbb{T} \setminus 0$, set

$$\begin{aligned} Mf(x) &= \rho_\mu(0,1,0)f(x) = e^{ix}f(x), \\ T_\mu f(x) &= \rho_\mu(0,0,1)f(x) = f(x+\mu). \end{aligned} \tag{1.7}$$

In such a case,

$$\rho_\mu(t,q,p)f(x) = e^{it\mu}e^{iqx}f(x+\mu p), \quad t,q,p \in \mathbb{Z},\ x \in \mathbb{T}. \tag{1.7A}$$

Note that if $A \in \mathcal{L}(L^2(\mathbb{T}))$ commutes with M then it is a multiplication operator, $Af(x) = a(x)f(x)$. If also A commutes with T_μ, we must have

$$a(x+\mu) = a(x) \text{ on } \mathbb{T}. \tag{1.8}$$

Conclusion. If $\mu/2\pi$ is irrational, ρ_μ is irreducible.

Finally, we decompose ρ_μ as a direct integral of unitary representations of $\mathcal{H}_d^3$ on $\ell^2(\mathbb{Z}/n) \approx \mathbb{C}^n$, in case

$$\mu = 2\pi\frac{m}{n} \in (0, 2\pi), \tag{1.9}$$

with m and n relatively prime. We do this by using (1.7) with x restricted from $\mathbb{R}/(2\pi\mathbb{Z})$ to points of the form $2\pi j/n + \zeta$, where

$$0 \le \zeta < \frac{2\pi}{n}. \tag{1.10}$$

Thus, for $f \in \ell^2(\mathbb{Z}/n)$, we set

$$\begin{aligned} M_\zeta f(j) &= \sigma_{\mu,\zeta}(0,1,0)f(j) = e^{(2\pi j/n+\zeta)i}f(j), \\ T_m f(j) &= \sigma_{\mu,\zeta}(0,0,1)f(j) = f(j+m). \end{aligned} \tag{1.11}$$

An operator A on $\ell^2(\mathbb{Z}/n)$ commuting with M_ζ must be a multiplication operator, $Af(j) = a(j)f(j)$, and if A also commutes with T_m then $a(j+m) \equiv a(j)$, which implies a is constant (since m and n are relatively prime), so the representations $\sigma_{\mu,\zeta}$ of $\mathcal{H}_d^3$ are all irreducible.

Remark. These results extend to the $(2k+1)$-dimensional Heisenberg group $\mathcal{H}^{2k+1}$, consisting of matrices in $M(k+2,\mathbb{R})$ of the form

$$\begin{pmatrix} 1 & p^t & t \\ & I & q \\ & & 1 \end{pmatrix}$$

(with $t \in \mathbb{R},\ p,q \in \mathbb{R}^k,\ I \in M(k,\mathbb{R})$ the identity matrix), acting on $L^2(\mathbb{R}^k)$, and its discrete analogue, acting on $\text{Ł}^2(\mathbb{T}^k)$. Details left to the reader.

2. Inversion formula

Given $u \in \ell^1(\mathcal{H}_d^3)$, and a unitary representation ρ_μ, we form

$$\rho_\mu(u) : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}), \tag{2.1}$$

defined as

$$\begin{aligned} \rho_\mu(u)f(x) &= \sum_{(t,q,p)\in\mathcal{H}_d^3} u(t,q,p)\rho_\mu(t,q,p)f(x) \\ &= \sum_{(t,q,p)\in\mathcal{H}_d^3} u(t,q,p)e^{i\mu t}e^{iqx}f(x+\mu p). \end{aligned} \tag{2.2}$$

Here, to reiterate,

$$t,q,p \in \mathbb{Z}, \quad \mu \in \mathbb{T}\setminus 0, \quad x \in \mathbb{T}, \quad f \in L^2(\mathbb{T}). \tag{2.3}$$

Note that

$$\|\rho_\mu(u)f\|_{L^2(\mathbb{T})} \le \|u\|_{\ell^1(\mathcal{H}_d^3)}\|f\|_{L^2(\mathbb{T})}. \tag{2.3A}$$

The inversion problem is to recover u from the family of operators $\rho_\mu(u)$.

To start, we obtain from (2.2) that

$$\rho_\mu(u)f(x) = \sum_{p,q\in\mathbb{Z}} b_\mu(q,p)e^{iqx}f(x+\mu p), \tag{2.4}$$

with

$$b_\mu(q,p) = \sum_{t\in\mathbb{Z}} u(t,q,p)e^{i\mu t} = \hat{u}_1(\mu,q,p). \tag{2.5}$$

We next note that, with $e_k(x) = e^{ikx}$,

$$\rho_\mu(u)e_k(x) = \sum_{p,q\in\mathbb{Z}} b_\mu(q,p)e^{iqx}e^{ik\mu p}e_k(x), \tag{2.6}$$

so

$$\begin{aligned} e_{-k}(x)\,\rho_\mu(u)e_k(x) &= \sum_{p,q\in\mathbb{Z}} b_\mu(q,p)e^{iqx}e^{ik\mu p} \\ &= \hat{b}_\mu(x,k\mu). \end{aligned} \tag{2.7}$$

That is,

$$\hat{b}_\mu(x, k\mu) = \psi_u(\mu, k, x) = e_{-k}(x)\,\rho_\mu(u)e_k(x). \tag{2.8}$$

To proceed, note that, for $x, \xi \in \mathbb{T}$,

$$\hat{b}_\mu(x, \xi) = \sum_{p,q\in\mathbb{Z}} b_\mu(q, p)e^{iqx}e^{ip\xi}, \tag{2.9}$$

and

$$\begin{aligned} \hat{u}(\mu, x, \xi) &= \sum_{t,q,p\in\mathbb{Z}} u(t, q, p)e^{i\mu t}e^{iqx}e^{ip\xi} \\ &= \hat{b}_\mu(x, \xi). \end{aligned} \tag{2.10}$$

We see that

$$\begin{aligned} u \in \ell^1(\mathcal{H}^3_d) &\Longrightarrow \hat{u} \in C(\mathbb{T}^3) \\ &\Longrightarrow \hat{b}_\mu \in C(\mathbb{T}^2), \quad \forall\, \mu \in \mathbb{T}. \end{aligned} \tag{2.11}$$

Consequently

$$b_\mu(q, p) = \int_{\mathbb{T}^2} \hat{b}_\mu(x, \xi)e^{-ixq}e^{-i\xi p}\,dx\,d\xi \tag{2.12}$$

is a continuous function of $\mu \in \mathbb{T}$, and

$$u(t, q, p) = \int_{\mathbb{T}} b_\mu(q, p)e^{-it\mu}\,d\mu. \tag{2.13}$$

In light of this, the following result will lead to a solution to the inversion problem.

Proposition 2.1. *If $\mu \in \mathbb{T}$ and $\mu/2\pi$ is irrational, then*

$$b_\mu(q, p) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \int_{\mathbb{T}} \hat{b}_\mu(x, k\mu)e^{-iqx}e^{-ipk\mu}\,dx, \tag{2.14}$$

with $\hat{b}_\mu(x, k\mu)$ given by (2.8). Equivalently,

$$b_\mu(q, p) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} (\rho_\mu(u)e_k, e_{k+q})_{L^2(\mathbb{T})}\,e^{-ik\mu p}. \tag{2.15}$$

Here is the inversion formula.

Theorem 2.2. *Let $u \in \ell^1(\mathcal{H}_d^3)$. Then, for $t, q, p \in \mathbb{Z}$,*

$$u(t,q,p) = \lim_{N\to\infty} \frac{1}{N} \int_{\mathbb{T}} \sum_{k=0}^{N} (\rho_\mu(u)e_k, e_{k+q})_{L^2(\mathbb{T})}\, e^{-ik\mu p} e^{-it\mu}\, d\mu. \tag{2.16}$$

Proof. Set

$$\varphi_N(\mu,q,p) = \frac{1}{N} \sum_{k=0}^{N} (\rho_\mu(u)e_k, e_{k+q})_{L^2(\mathbb{T})}\, e^{-ik\mu p}, \tag{2.17}$$

for $\mu \in \mathbb{T} \setminus 0,\ q,p \in \mathbb{Z}$. Then (2.15) says

$$\mu \in \mathbb{T} \setminus \pi\mathbb{Q} \Longrightarrow \varphi_N(\mu,q,p) \to b_\mu(q,p), \tag{2.18}$$

for all $q, p \in \mathbb{Z}$, as $N \to \infty$. We also have

$$\mu \in \mathbb{T} \setminus 0 \Longrightarrow |\varphi_N(\mu,q,p)| \le \|\rho_\mu(u)\|_{\mathcal{L}(L^2(\mathbb{T}))} \le \|u\|_{\ell^1(\mathcal{H}_d^3)}. \tag{2.19}$$

Hence the Lebesgue dominated convergence theorem implies

$$\int_{\mathbb{T}} \varphi_N(\mu,q,p)e^{-it\mu}\, d\mu \longrightarrow \int_{\mathbb{T}} b_\mu(q,p)e^{-it\mu}\, d\mu = u(t,q,p), \tag{2.20}$$

and we have the theorem.

3. Plancherel formula

Given $u \in \ell^2(\mathcal{H}^3_d)$, we have, with $b_\mu(q,p)$ and $\hat{b}_\mu(x,\xi)$ as in (2.5) and (2.9),

$$\begin{aligned}\|u\|^2_{\ell^2} &= \sum |u(t,q,p)|^2 \\ &= \int_{\mathbb{T}} \sum_{q,p} |b_\mu(q,p)|^2 \, d\mu, \\ &= \int_{\mathbb{T}^3} |\hat{b}_\mu(x,\xi)|^2 \, dx \, d\xi \, d\mu.\end{aligned} \tag{3.1}$$

If we strengthen our hypothesis on u to $u \in \ell^1(\mathcal{H}^3_d)$, then $\hat{b}_\mu(x,\xi)$ is continuous, and for each $\mu \in \mathbb{T} \setminus \pi\mathbb{Q}$,

$$\begin{aligned}\int_{\mathbb{T}^2} |\hat{b}_\mu(x,\xi)|^2 \, dx \, d\xi &= \lim_{N\to\infty} \int_{\mathbb{T}} \frac{1}{N} \sum_{k=0}^{N} |\hat{b}_\mu(x,k\mu)|^2 \, dx \\ &= \lim_{N\to\infty} \int_{\mathbb{T}} \frac{1}{N} \sum_{k=0}^{N} |\rho_\mu(u) e_k(x)|^2 \, dx \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \|\rho_\mu(u) e_k\|^2_{L^2(\mathbb{T})},\end{aligned} \tag{3.2}$$

the second identity by (2.8). This leads to the following Plancherel formula.

Proposition 3.1. *Let $u \in \ell^1(\mathcal{H}^3_d)$. Then*

$$\|u\|^2_{\ell^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \int_{\mathbb{T}} \|\rho_\mu(u) e_k\|^2_{L^2(\mathbb{T})} \, d\mu. \tag{3.3}$$

Proof. Set

$$\psi_N(\mu) = \frac{1}{N} \sum_{k=0}^{N} \|\rho_\mu(u) e_k\|^2_{L^2(\mathbb{T})}, \tag{3.4}$$

for $\mu \in \mathbb{T} \setminus 0$. Then, by (3.2),

$$\mu \in \mathbb{T} \setminus \pi\mathbb{Q} \Longrightarrow \psi_N(\mu) \to \int_{\mathbb{T}^2} |\hat{b}_\mu(x,\xi)|^2 \, dx \, d\xi, \tag{3.5}$$

as $N \to \infty$. We also have

$$\mu \in \mathbb{T} \setminus 0 \Longrightarrow |\psi_N(\mu)| \le \|\rho_\mu(u)\|^2_{\mathcal{L}(L^2(\mathbb{T}))} \le \|u\|^2_{\ell^1(\mathcal{H}^3_d)}. \tag{3.6}$$

From (3.5)–(3.6), the Lebesgue dominated convergence theorem gives

$$\int\limits_{\mathbb{T}} \psi_N(\mu)\, d\mu \to \int\limits_{\mathbb{T}^3} |\hat{b}_\mu(x,\xi)|^2 \, d\xi \, dx \, d\mu = \|u\|^2_{\ell^2}, \tag{3.7}$$

proving the proposition.

Another presentation of (3.3) is

$$\|u\|^2_{\ell^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \int\limits_{\mathbb{T}} (\rho_\mu(u)^* \rho_\mu(u) e_k, e_k)_{L^2(\mathbb{T})} \, d\mu. \tag{3.8}$$

Polarization gives, for $u, v \in \ell^1(\mathcal{H}^3_d)$,

$$(u,v)_{\ell^2} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \int\limits_{\mathbb{T}} (\rho_\mu(v)^* \rho_\mu(u) e_k, e_k)_{L^2(\mathbb{T})} \, d\mu. \tag{3.9}$$

Taking $v = \delta_{(t,q,p)}$ leads back to the inversion forumula of §2.

Reference

[T] M. Taylor, Noncommutative Harmonic Analysis, Amer. Math. Soc., Providence RI, 1986.