The Pentagon, the Dodecahedron, and the Icosahedron

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Introduction

Here we produce formulas for the vertices of a regular pentagon in the plane, and for a regular dodecahedron and a regular icosahedron in 3D Euclidean space. Once the pentagon is constructed, the key to constructing the dodecahedron lies in realizing one knows the dot product of vectors representing any pair of intersecting edges. Carrying out the computations also involves algebraic identities crucial to the construction of the pentagon. The icosahedron is then produced from the dodecahedron by computing the center of each of its faces. The resulting formulas for the vertices of the icosahedron have a pleasingly simple form.

These formulas for the vertices make it easy to establish the duality between the dodecahedron and the icosahedron. We also use these formulas to analyze the group of symmetries of the dodecahedron and icosahedron, and show that the group of orientation-preserving symmetries is isomorphic to the alternating group $\mathcal{A}_5$.

Contents

1. The pentagon

Let us construct the regular pentagon in the complex plane, with vertices

$$z_\nu = e^{2\pi i\nu/5} = c_\nu + is_\nu, \tag{1.1}$$

where

$$c_\nu = \cos\frac{2\pi\nu}{5}, \quad s_\nu = \sin\frac{2\pi\nu}{5}. \tag{1.2}$$

Here $\nu = \{0,\dots,4\}$, or equivalently $\nu \in \mathbb{Z}/(5)$. Note that for $1 \le \nu \le 4$, z_ν solves

$$z^4 + z^3 + z^2 + z + 1 = 0, \tag{1.3}$$

or equivalently

$$z^2 + z + 1 + z^{-1} + z^{-2} = 0. \tag{1.4}$$

If we set

$$w = z + z^{-1}, \tag{1.5}$$

then $w^2 = z^2 + 2 + z^{-2}$, and (1.4) yields

$$w^2 + w - 1 = 0, \tag{1.6}$$

with solutions

$$w_\pm = \frac{-1 \pm \sqrt{5}}{2}. \tag{1.7}$$

Then (1.5) can be rewritten

$$z^2 - wz + 1 = 0, \tag{1.8}$$

with solutions

$$z = \frac{w}{2} \pm \frac{1}{2}\sqrt{w^2 - 4}. \tag{1.9}$$

Note in particular that $w^2 - 4 = -3 - w$, so

$$w_\pm^2 - 4 = -\frac{5 \pm \sqrt{5}}{2}. \tag{1.10}$$

These are negative real numbers, so we see that in (1.9)

$$\operatorname{Re} z = \frac{w}{2}. \tag{1.11}$$

In particular,

$$c_1 = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \tag{1.12}$$

and

$$c_2 = \cos\frac{4\pi}{5} = \frac{-\sqrt{5}-1}{4}. \tag{1.13}$$

Also of course $c_4 = c_1$ and $c_3 = c_2$. Note from (1.12)–(1.13) that

$$c_1 + c_2 = -\frac{1}{2}. \tag{1.14}$$

The vertices of the pentagon given in (1.1) all have distance 1 from the origin. Each edge of this pentagon has length L, satisfying

$$\begin{aligned} L^2 = (c_1-1)^2 + s_1^2 &= 2 - 2c_1 \\ &= \frac{5-\sqrt{5}}{2}. \end{aligned} \tag{1.15}$$

See Fig. 1.1 for a display of the information obtained in this section.

2. The dodecahedron

The dodecahedron has 20 vertices and twelve faces, each face being a regular pentagon. Here we will give formulas for the vertices of a dodecahedron $\mathcal{D}$ with center of mass on the z-axis and two faces parallel to the xy-plane. We take the bottom face of $\mathcal{D}$ to have vertices

$$p_\nu = c_\nu i + s_\nu j, \quad \nu \in \mathbb{Z}/(5), \tag{2.1}$$

where (i, j, k) is the standard orthonormal basis of $\mathbb{R}^3$. The coefficients c_ν and s_ν are given by (1.2), and formulas for c_1 and c_2 are given in (1.12)–(1.13).

Let us set

$$e_\nu = p_\nu - p_{\nu-1}, \quad \nu \in \mathbb{Z}/(5). \tag{2.2}$$

If we assume the center of $\mathcal{D}$ lies on the positive z-axis, five more vertices of $\mathcal{D}$ are given by

$$q_\nu = p_\nu + f_\nu, \tag{2.3}$$

where each vector f_ν is uniquely specified by the requirements

$$f_\nu \cdot e_{\nu+1} = -f_\nu \cdot e_\nu = -e_{\nu+1} \cdot e_\nu, \quad f_\nu \cdot k > 0. \tag{2.4}$$

See Fig. 2.1. Note that

$$e_{\nu+1} \cdot e_\nu = e_1 \cdot e_0, \quad \nu \in \mathbb{Z}/(5). \tag{2.5}$$

Since

$$e_0 = (1 - c_1)i + s_1 j, \quad e_1 = -(1 - c_1)i + s_1 j, \tag{2.6}$$

we have

$$e_{\nu+1} \cdot e_\nu = -(c_1 - 1)^2 + s_1^2 = c_1(2 - 2c_1). \tag{2.7}$$

If we compare the calculation (1.15) for the edge length, we get

$$e_{\nu+1} \cdot e_\nu = c_1 L^2. \tag{2.8}$$

Making use of the formula (1.12) for c_1, we also see that

$$e_{\nu+1} \cdot e_\nu = \frac{3\sqrt{5} - 5}{2}. \tag{2.9}$$

To continue the calculation of f_ν, we see that

$$f_\nu = a p_\nu + bk, \tag{2.10}$$

with a determined by the requirement $f_0 \cdot e_0 = c_1(2-2c_1)$, i.e., $a\, p_0 \cdot e_0 = 2c_1(1-c_1)$, hence, by (2.6),

$$a = 2c_1. \tag{2.11}$$

Then b is determined by the condition $b > 0$ together with $|f_\nu|^2 = L^2$, hence

$$L^2 = a^2 + b^2, \tag{2.12}$$

so, by (1.15) and (2.11),

$$b^2 = 2 - 2c_1 - 4c_1^2. \tag{2.13}$$

This implies the following striking conclusion:

$$b = 1. \tag{2.14}$$

In fact, (2.14) would be equivalent to $c = c_1$ satisfying

$$4c^2 + 2c - 1 = 0, \tag{2.15}$$

a quadratic equation whose solution is

$$c = -\frac{1}{4} \pm \frac{\sqrt{5}}{4}. \tag{2.16}$$

As seen in (1.12), c_1 is indeed one of these numbers.

To summarize, we have the formulas

$$f_\nu = 2c_1 p_\nu + k, \tag{2.17}$$

and hence the five additional vertices of $\mathcal{D}$:

$$q_\nu = (1 + 2c_1)p_\nu + k. \tag{2.18}$$

Next we locate the geometrical center of $\mathcal{D}$. This will be the point on the z-axis hit by the line through p_0 parallel to the vector

$$\xi = e_1 - e_0 + f_0 = (4c_1 - 2)i + k. \tag{2.19}$$

In fact, this line through p_0 is given by

$$\gamma(t) = p_0 + t\xi = [1 + (4c_1 - 2)t]i + tk. \tag{2.20}$$

This hits the z-axis at $t = 1/(2 - 4c_1)$, so the center of $\mathcal{D}$ is

$$\gamma_0 = \frac{1}{2 - 4c_1} k = (1 + c_1)k, \tag{2.21}$$

the latter identity, i.e., $1 = (2 - 4c_1)(1 + c_1)$, by (2.15).

Let us translate the center of mass to the origin, producing a new dodecahedron $\widetilde{\mathcal{D}}$, ten of whose vertices are given by

$$\begin{aligned} p_\nu^\flat &= c_\nu i + s_\nu j - (1 + c_1)k, \\ q_\nu^\flat &= (1 + 2c_1)(c_\nu i + s_\nu j) - c_1 k, \end{aligned} \tag{2.22}$$

with the other ten vertices given by

$$p_\nu^\# = -p_\nu^\flat, \quad q_\nu^\# = -q_\nu^\flat, \tag{2.23}$$

for $\nu \in \mathbb{Z}/(5)$.

Just a few additional calculations are required to justify the assertion that $\widetilde{\mathcal{D}}$ is indeed a regular dodecahedron. In fact, in light of various symmetry considerations, it will suffice to verify that

$$|q_3^{\#} - q_0^b|^2 = |p_1^b - p_0^b|^2, \tag{2.24}$$

and

$$(q_3^{\#} - q_0^b) \cdot (p_0^b - q_0^b) = (p_0^b - q_0^b) \cdot (p_0^b - p_1^b). \tag{2.25}$$

Look at Fig. 2.2. Note that the right side of (2.24) is equal to $L^2 = 2 - 2c_1$ and the right side of (2.25) is equal to $f_0 \cdot e_1 = -c_1(2 - 2c_1)$. Thus, since $q_3^{\#} = -q_3^b$, we need to check

$$\begin{aligned} |q_0^b + q_3^b|^2 &= 2 - 2c_1, \\ (q_0^b + q_3^b) \cdot f_0 &= -c_1(2 - 2c_1). \end{aligned} \tag{2.26}$$

From (2.22) we have

$$q_0^b + q_3^b = (2c_1 + 1)(c_3 + 1)i + (2c_1 + 1)s_3 j - 2c_1 k, \tag{2.27}$$

and hence

$$\begin{aligned} |q_0^b + q_3^b|^2 &= (2c_1 + 1)^2(c_3 + 1)^2 + (2c_1 + 1)^2 s_3^2 + 4c_1^2 \\ &= (2c_1 + 1)^2(2c_3 + 2) + 4c_1^2. \end{aligned} \tag{2.28}$$

By (1.14) and $c_3 = c_2$, we have $2c_3 + 2 = 1 - 2c_1$, so

$$\begin{aligned} |q_0^b + q_3^b|^2 &= (1 + 2c_1)^2(1 - 2c_1) + 4c_1^2 \\ &= 1 + 2c_1 - 8c_1^3 \\ &= 2 - 2c_1, \end{aligned} \tag{2.29}$$

the last identity by (2.15). This verifies the first identity in (2.26).

As for the last identity in (2.26), since $f_0 = 2c_1 i + k$, we have

$$\begin{aligned} (q_0^b + q_3^b) \cdot f_0 &= 2c_1(2c_1 + 1)(c_3 + 1) - 2c_1 \\ &= c_1(1 + 2c_1)(1 - 2c_1) - 2c_1, \end{aligned} \tag{2.30}$$

the latter identity by $c_3 = c_2$ and (1.14). This in turn is equal to

$$c_1(1 - 4c_1^2) - 2c_1 = 2c_1^2 - 2c_1, \tag{2.31}$$

by (2.15), verifying the last identity in (2.26). Hence $\widetilde{\mathcal{D}}$ is a regular dodecahedron. See Fig. 2.3 for a picture of $\widetilde{\mathcal{D}}$.

3. The icosahedron

We can form a regular icosahedron $\widetilde{\mathcal{I}}$ whose twelve vertices consist of the centers of the twelve faces of the dodecahedron $\widetilde{\mathcal{D}}$, constructed in §2. In particular, the bottom vertex is

$$W = -(1 + c_1)k. \tag{3.1}$$

The center of the face of $\widetilde{\mathcal{D}}$ whose intersection with the bottom face is the edge joining p_0^b to p_1^b is given by V, where

$$5V = p_0^b + p_1^b + q_0^b + q_1^b - q_3^b. \tag{3.2}$$

See Fig. 2.2. Formulas derived in §2 yield

$$5V = (2 + 2c_1)p_0 + (2 + 2c_1)p_1 - (1 + 2c_1)p_3 - (2 + 3c_1)k. \tag{3.3}$$

To simplify this, note that

$$p_0 + p_1 = -(1 + 2c_1)p_3, \tag{3.4}$$

which is equivalent to

$$z_2 + z_3 = -(1 + 2c_1), \tag{3.5}$$

itself a consequence of (1.14); see also Fig. 1.1. Then further calculation, using (2.15) and (1.12), gives

$$V = -\frac{2 + 3c_1}{5}(2p_3 + k) = -\frac{1 + c_1}{\sqrt{5}}(2p_3 + k). \tag{3.6}$$

Having (3.6), we set

$$V_\nu = -\frac{1 + c_1}{\sqrt{5}}(2p_\nu + k). \tag{3.7}$$

Then six of the twelve vertices of $\widetilde{\mathcal{I}}$ are given by W and V_ν, and the other six by $-W$ and $-V_\nu$.

Finally, it is natural to re-scale, producing an icosahedron $\mathcal{I}$, with six vertices given by

$$w = k, \quad v_\nu = \frac{1}{\sqrt{5}}(2p_\nu + k), \tag{3.8}$$

and the other six by

$$w^b = -k, \quad v_\nu^b = -\frac{1}{\sqrt{5}}(2p_\nu + k). \tag{3.9}$$

See Fig. 3.1.

There is an important *duality* between the dodecahedron $\widetilde{\mathcal{D}}$ and the icosahedron $\mathcal{I}$. Namely, if we take the center of each face of $\mathcal{I}$, this collection of points is the set of vertices of another dodecahedron, a dilated version of $\widetilde{\mathcal{D}}$. To be precise, the following identities hold:

$$\begin{aligned} k + v_\nu + v_{\nu+1} &= Ap^\#_{\nu+3}, \\ v_\nu + v_{\nu+1} - v_{\nu+3} &= Aq^\#_{\nu+3}, \end{aligned} \tag{3.10}$$

for $\nu \in \mathbb{Z}/(5)$, associating 10 of the faces of $\mathcal{I}$ with vertices of a dilate of $\widetilde{\mathcal{D}}$, and there are similar identities associating the other 10 faces of $\mathcal{I}$ and the other 10 vertices of this dilate of $\widetilde{\mathcal{D}}$. Here,

$$A = 1 + \frac{1}{\sqrt{5}}. \tag{3.11}$$

Verification of (3.10) is straightforward, given the formulas (3.8) for v_ν, (2.22)–(2.23) for $p^\#_\nu$ and $q^\#_\nu$, and (1.12)–(1.13) for c_1 and c_2. Checking this out involves using the identity

$$p_\nu + p_{\nu+1} = 2c_2 p_{\nu+3}, \quad \nu \in \mathbb{Z}/(5), \tag{3.12}$$

which follows from $p_2 + p_3 = 2c_2 i$ plus rotational symmetry of the pentagon. From here, it is an exercise to verify (3.10).

4. Symmetries of the icosahedron

The icosahedron $\mathcal{I}$ is of interest because it is a regular polyhedron. Here we verify this fact directly from the list of vertices (3.8)–(3.9) and look at the group of symmetries that arises. In view of the relationship between the vertices of $\mathcal{I}$ and the centers of the faces of $\widetilde{\mathcal{D}}$, and the dual relationship between the centers of the faces of $\mathcal{I}$ and the vertices of $\widetilde{\mathcal{D}}$ established in §3, it will be apparent that the group we describe is also the group of symmetries of $\widetilde{\mathcal{D}}$.

From the list of vertices (3.8)–(3.9) plus the formulas for p_ν discussed in previous sections, it is clear that a rotation through $2\pi/5$ about the k-axis produces a symmetry of $\mathcal{I}$. We have a unique $R \in SO(3)$ satisfying

$$Rk = k, \quad Rv_\nu = v_{\nu+1}, \quad \nu \in \mathbb{Z}/(5). \tag{4.1}$$

We claim there are rotations about each v_ν-axis that are symmetries of $\mathcal{I}$. For example, we claim there is a unique $S_0 \in SO(3)$ satisfying

$$S_0 v_0 = v_0, \quad S_0 v_1 = k, \tag{4.2}$$

and that such S_0 has the further properties:

$$S_0 k = v_4, \quad S_0 v_4 = -v_2, \quad S_0(-v_2) = -v_3, \quad S_0(-v_3) = v_1. \tag{4.3}$$

To show this, it is convenient to use the following elementary linear algebra result.

Lemma 4.1. *Let e_1, e_2, e_3 be linearly independent unit vectors in $\mathbb{R}^3$, and let f_1, f_2, f_3 be unit vectors in $\mathbb{R}^3$. Assume*

$$f_1 \cdot f_2 = e_1 \cdot e_2. \tag{4.4}$$

Then there exists a unique $T \in SO(3)$ such that

$$Te_j = f_j, \quad j = 1, 2. \tag{4.5}$$

If in addition we have

$$f_j \cdot f_3 = e_j \cdot e_3, \quad j = 1, 2, \tag{4.6}$$

and

$$\det(f_1, f_2, f_3) = \det(e_1, e_2, e_3), \tag{4.7}$$

i.e., both sides of (4.7) have the same sign, then

$$Te_3 = f_3. \tag{4.8}$$

We leave the proof of this lemma as an exercise for the reader and proceed. To obtain $S_0 \in SO(3)$ satisfying (4.2), we take $e_1 = v_0,\ e_2 = v_1,\ f_1 = v_0$, and $f_2 = k$. The formula (3.8) gives

$$v_0 \cdot v_1 = \frac{1}{5}(4c_1 + 1), \quad v_0 \cdot k = \frac{1}{\sqrt{5}}, \tag{4.9}$$

and the formula (1.12) for c_1 readily yields $v_0 \cdot v_1 = v_0 \cdot k$, so we have (4.2). To verify (4.3), we successively apply the lemma to the following cases:

$$\begin{array}{rl} e_1, e_2, e_3 & f_1, f_2, f_3 \\ v_0, v_1, k & v_0, k, v_4 \\ v_0, k, v_4 & v_0, v_4, -v_2 \\ v_0, v_4, -v_2 & v_0, -v_2, -v_3 \\ v_0, -v_2, -v_3 & v_0, -v_3, v_1. \end{array} \tag{4.10}$$

To verify (4.6) in each of these cases it remains to show that

$$\begin{aligned}v_0 \cdot k = v_1 \cdot k = v_0 \cdot v_4 = k \cdot v_4 = v_0 \cdot (-v_2) \\ = v_4 \cdot (-v_2) = v_0 \cdot (-v_3) = (-v_2) \cdot (-v_3) = (-v_3) \cdot v_1.\end{aligned} \tag{4.11}$$

Direct computation from (3.8) gives

$$v_0 \cdot k = v_1 \cdot k = k \cdot v_4 = \frac{1}{\sqrt{5}}, \tag{4.12}$$

$$v_0 \cdot v_4 = \frac{1}{5}(4c_1 + 1), \quad v_0 \cdot (-v_2) = -\frac{1}{5}(4c_2 + 1), \quad v_0 \cdot (-v_3) = -\frac{1}{5}(4c_3 + 1), \tag{4.13}$$

(4.14)

$$v_4 \cdot (-v_2) = -\frac{1}{5}(4c_2c_4 + 4s_2s_4 + 1), \quad (-v_2) \cdot (-v_3) = \frac{1}{5}(4c_2c_3 + 4s_2s_3 + 1),$$
$$(-v_3) \cdot v_1 = -\frac{1}{5}(4c_1c_3 + 4s_1s_3 + 1).$$

The quantities in (4.12) are all good. The analysis used for (4.9) also applies to the first quantity in (4.13), then the identity $c_2 = -1/2 - c_1$ (cf. (1.14)) treats the second quantity, and then the identity $c_3 = c_2$ treats the third.

It remains to treat (4.14). Note that

$$c_2c_4 + s_2s_4 = c_1c_2 - s_1s_2. \tag{4.15}$$

Furthermore (working with complex numbers, as in §1),

$$\begin{aligned}(c_1 + is_1)(c_2 + is_2) = c_3 + is_3 = c_2 - is_2 \\ \Longrightarrow c_1c_2 - s_1s_2 = c_2,\end{aligned} \tag{4.16}$$

which treats the first quantity in (4.14). As for the second, we have

$$c_2c_3 + s_2s_3 = c_2^2 - s_2^2, \tag{4.17}$$

and

$$\begin{aligned}(c_2 + is_2)(c_2 + is_2) = c_4 + is_4 = c_1 - is_1 \\ \Longrightarrow c_2^2 - s_2^2 = c_1.\end{aligned} \tag{4.18}$$

As for the third quantity in (4.14), since $c_1c_3 + s_1s_3 = c_1c_2 - s_1s_2$, (4.16) applies.

The agreement of signs of the two sides of (4.7) are also readily verified (all signs are positive). Thus we have $S_0 \in SO(3)$ satisfying (4.2)–(4.3), and these formulas guarantee that $S_0 : \mathcal{I} \to \mathcal{I}$.

More generally, we have unique $S_\nu \in SO(3)$ satisfying

$$S_\nu v_\nu = v_\nu, \quad S_\nu v_{\nu+1} = k, \quad \nu \in \mathbb{Z}/(5), \tag{4.19}$$

and S_ν preserves $\mathcal{I}$. In fact we see that

$$S_\nu = R S_{\nu-1} R^{-1}, \tag{4.20}$$

since both sides give elements of $SO(3)$ that satisfy (4.19). Hence

$$S_\nu = R^\nu S_0 R^{-\nu}. \tag{4.21}$$

Let $\mathcal{G}_+$ be the subgroup of $SO(3)$ generated by R and S_0. We then see that $S_\nu \in \mathcal{G}_+$ for each $\nu \in \mathbb{Z}/(5)$, and $\mathcal{G}_+$ is a group of symmetries of $\mathcal{I}$. We claim that

$$\mathcal{G}_+ \text{ acts transitively on the vertices of } \mathcal{I}. \tag{4.22}$$

To see this, look at the orbit $\mathcal{O}_k$ of k under this group action. Of course $k \in \mathcal{O}_k$, and we also have

$$v_4, -v_2, -v_3, v_1 \in \mathcal{O}_k,$$

by (4.3). Applying powers of R yields

$$v_\nu, -v_\nu \in \mathcal{O}_k, \quad \nu \in \mathbb{Z}/(5),$$

and then

$$S_0(-v_1) = -k \Longrightarrow -k \in \mathcal{O}_k.$$

This proves (4.22).

We now make some observations about the geometry of $\mathcal{I}$. It is clear from (3.8) that the vertex k has five nearest neighbors, namely v_ν, $\nu \in \mathbb{Z}/(5)$. Each v_ν is joined to k by a line segment, called an edge of $\mathcal{I}$. It follows from (4.22) that each vertex $x \in \mathcal{I}$ (e.g., $x = v_\nu$, or $-v_\nu$, or $-k$) has five nearest neighbors, each connected to x by an edge. Furthermore, a vertex y of $\mathcal{I}$ is a nearest neighbor of x if and only if $|x - y| = |k - v_0|$, and in such a case x is also a nearest neighbor of y.

If x and y are nearest neighbors, we consider the oriented line segment, going *from x to y*, and call it an *oriented edge.* Apparently each of the twelve vertices of $\mathcal{I}$ has five oriented edges issuing from it, so $\mathcal{I}$ has 60 oriented edges. (Each edge of $\mathcal{I}$ has two orientations, so $\mathcal{I}$ has 30 edges.)

Suppose that e is an oriented edge of $\mathcal{I}$, going from the vertex x to y. By (4.22) there exists $U \in \mathcal{G}_+$ such that $Ux = k$. Then Uy is a nearest neighbor of k, say $Uy = v_\mu$, and then $R^{-\mu} U y = v_0$. Setting $T_e = (R^{-\mu} U)^{-1}$ and invoking Lemma 4.1 once more, we have:

Proposition 4.2. *Given an oriented edge e of $\mathcal{I}$, going from the vertex x to y, there is a unique $T_e \in SO(3)$ such that*

$$T_e k = x, \quad T_e v_0 = y. \tag{4.23}$$

Furthermore,

$$T_e \in \mathcal{G}_+. \tag{4.24}$$

Hence we have a one-to-one correspondence between $\mathcal{G}_+$ and the set of oriented edges of $\mathcal{I}$.

Corollary 4.3. *Given any two oriented edges e_j of $\mathcal{I}$, going from vertices x_j to y_j, there is a unique $T_{e_1e_2} \in SO(3)$ such that*

$$T_{e_1e_2} x_1 = x_2, \quad T_{e_1e_2} y_1 = y_2. \tag{4.25}$$

Furthermore,

$$T_{e_1e_2} = T_{e_2} T_{e_1}^{-1} \in \mathcal{G}_+. \tag{4.26}$$

There are other useful descriptions of members of $\mathcal{G}_+$. Any element of $SO(3)$, other than the identity I, is a rotation about a uniquely determined axis. As we have seen, R is a rotation about the k-axis and S_ν is a rotation about the v_ν-axis. These rotations about axes through the vertices of $\mathcal{I}$ and their powers (including the identity) provide $1 + 4 \cdot 6 = 25$ elements of $\mathcal{G}_+$. We have rotations about other axes of symmetry of $\mathcal{I}$. There is an axis of symmetry through the center of each edge e (and its opposite edge $-e$). If we denote by e_1 and e_2 the two orientations of e, this is given by $J_e = T_{e_1e_2}$, the element of $\mathcal{G}_+$ described by Corollary 4.3. Note that $J_e^2 = I$. There are 15 edge pairs, so we get 15 additional symmetries this way. Also there is an axis of symmetry through each triangular face $\mathfrak{f}$ of $\mathcal{I}$ (and its opposite face $-\mathfrak{f}$). Say x_1, x_2, x_3 are the vertices of $\mathfrak{f}$, running counterclockwise, and joined by oriented edges e_{12}, e_{23}, e_{31}, with e_{12} going from x_1 to x_2, etc. Then such a rotation is given by $K_\mathfrak{f} = T_{e_{12}e_{23}}$. Note that $K_\mathfrak{f}^2 = T_{e_{12}e_{31}}$ and $K_\mathfrak{f}^3 = I$. There are 20 faces, hence 10 face pairs, so this provides 20 additional symmetries. Note that $25 + 15 + 20 = 60$, so this gives all the elements of $\mathcal{G}_+$ (as expected).

The transformation $-I$, which belongs to $O(3)$ but not to $SO(3)$, is also a symmetry of $\mathcal{I}$. More generally, if we set

$$\mathcal{G}_- = \{-T : T \in \mathcal{G}_+\}, \tag{4.27}$$

then

$$\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}_- \tag{4.28}$$

is a group of symmetries of $\mathcal{I}$, of order 120, containing both orientation-preserving symmetries (in $\mathcal{G}_+$) and orientation-reversing symmetries (in $\mathcal{G}_-$).

5. Five cubes inscribed in $\widetilde{\mathcal{D}}$

Here we describe some cubes that are inscribed in the dodecahedron $\widetilde{\mathcal{D}}$. We begin with a look at a frame of three vectors based at p_1^b, namely

$$E_1 = p_4^b - p_1^b, \quad E_2 = q_3^\# - p_1^b, \quad E_3 = q_2^b - p_1^b. \tag{5.1}$$

See Fig. 5.1. Clearly these three vectors have the same length, since a rotation through $2\pi/3$ about the axis through the origin and p_1^b is a symmetry of $\widetilde{\mathcal{D}}$ that permutes the vertices $p_4^b, q_3^\#$, and q_2^b. We claim that

$$E_1 \cdot E_2 = E_2 \cdot E_3 = E_1 \cdot E_3 = 0. \tag{5.2}$$

To see this, note that

$$E_1 = -2s_1 j, \tag{5.3}$$

while

$$\begin{aligned} E_2 &= -(1+2c_1)(c_3 i + s_3 j) + c_1 k - (c_1 i + s_1 j) + (1+2c_1)k \\ &= -[c_1 + (1+2c_1)c_3]i - [s_1 + (1+2c_1)s_3]j + (1+2c_1)k. \end{aligned} \tag{5.4}$$

The identity $E_1 \cdot E_2 = 0$ then follows from the identity $s_1 + (1+2c_1)s_3 = 0$, which is equivalent to

$$s_1 = (1+2c_1)s_2, \tag{5.5}$$

or, since (by (1.14)) $1 + 2c_1 = -2c_2$, to

$$s_1 = -2c_2 s_2. \tag{5.6}$$

To establish (5.6), we have, complementary to (4.18),

$$\begin{aligned} (c_2 + is_2)(c_2 + is_2) &= c_1 - is_1 \\ &\Longrightarrow 2c_2 s_2 = -s_1. \end{aligned} \tag{5.7}$$

This gives $E_1 \cdot E_2 = 0$. The rest of the identities in (5.2) follow from the symmetry of $\widetilde{\mathcal{D}}$ mentioned above.

The frame just constructed forms one corner of a cube, with vertices

$$p_1^b,\ p_4^b,\ q_3^\#,\ q_2^b, \ \text{ and }\ q_3^\# + E_1,\ q_3^\# + E_3,\ q_2^b + E_1,\ q_3^\# + E_3 + E_1. \tag{5.8}$$

In fact, each of these vertices is a vertex of $\widetilde{\mathcal{D}}$. This follows from the identities

$$
\begin{aligned}
q_3^{\#} + p_4^b - p_1^b &= q_2^{\#} \\
q_3^{\#} + q_2^b - p_1^b &= p_4^{\#} \\
p_4^b + q_2^b - p_1^b &= q_3^b \\
q_3^{\#} + q_2^b - p_1^b + p_4^b - p_1^b &= p_1^{\#}.
\end{aligned} \tag{5.9}
$$

These all follow from the formulas (2.22)–(2.23). For example, regarding the first identity in (5.9), we have

$$
\begin{aligned}
q_3^{\#} + p_4^b - p_1^b &= -(1+2c_1)p_3 + p_4 - p_1 + c_1 k, \\
q_2^{\#} &= -(1+2c_1)p_2 + c_1 k,
\end{aligned} \tag{5.10}
$$

so the desired identity is equivalent to

$$p_1 - p_4 = (1+2c_1)(p_2 - p_3), \tag{5.11}$$

hence to

$$s_1 = (1+2c_1)s_2, \tag{5.12}$$

which is (5.5) again. The next two identities in (5.9) have similar proofs. The last identity in (5.9) is then established by using the second identity to write the left side as $p_4^{\#} + p_4^b - p_1^b$ and recalling that $p_\nu^{\#} = -p_\nu^b$. Refer to Fig. 5.2 for a picture of the cube we have produced. Call this cube $\mathcal{Q}_0$.

Note that $\mathcal{Q}_0$ has twelve edges and $\widetilde{\mathcal{D}}$ has twelve faces. It is clear that no one face of $\widetilde{\mathcal{D}}$ can contain two different edges of $\mathcal{Q}_0$, so each face of $\widetilde{\mathcal{D}}$ has exactly one edge of $\mathcal{Q}_0$ lying in it, connecting two non-adjacent vertices of such a face.

We will call a line between two non-adjacent vertices of a regular pentagon a "diagonal" of the pentagon. Thus each pentagon has five distinct diagonals. We can say that the edges of $\mathcal{Q}_0$ consist of one diagonal in each face of $\widetilde{\mathcal{D}}$.

Consider the group of rotations through $2\pi/5$ about the k-axis, which is a group of symmetries of $\widetilde{\mathcal{D}}$, namely the subgroup of $\mathcal{G}_+$ generated by R. Each element R^ν takes $\mathcal{Q}_0$ to a cube, which we will denote $\mathcal{Q}_\nu$, and as ν runs over $\mathbb{Z}/(5)$, the edge of $\mathcal{Q}_\nu$ contained in the bottom face of $\widetilde{\mathcal{D}}$ runs over the five diagonals of this face.

At this point we make another observation about $\mathcal{Q}_0$. Namely, the diagonal along the bottom face, from p_1^b to p_4^b, parallel to E_1, uniquely determines the diagonals in the other two faces of $\widetilde{\mathcal{D}}$ with vertex p_1^b, which make the corners of $\mathcal{Q}_0$ with vertex p_1^b. Hence this one diagonal uniquely determines $\mathcal{Q}_0$.

It follows that as ν runs over $\mathbb{Z}/(5)$, the edges of $\mathcal{Q}_\nu$ contained in any fixed face of $\widetilde{\mathcal{D}}$ run once over all the diagonals in that face.

6. The isomorphism of $\mathcal{G}_+$ and $\mathcal{A}_5$

The material developed in §5 enables us to recognize the group $\mathcal{G}_+$ as being isomorphic to the group $\mathcal{A}_5$, the subgroup of the group $\mathcal{S}_5$ of permutations of a set of five objects, consisting of even permutations. Our treatment here is an amalgam of results presented on pp. 17–20 of [K] and on p. 36 of [G]. In these texts the results on $\mathcal{I}$ and $\widetilde{\mathcal{D}}$ that we have derived in §§2–5 were taken as "given."

Recall the five cubes $\mathcal{Q}_\nu,\ \nu \in \mathbb{Z}/(5)$ constructed in §5. Each $T \in \mathcal{G}_+$ takes each cube $\mathcal{Q}_\alpha$ to some cube $\mathcal{Q}_\beta$, and $\mathcal{Q}_\alpha \neq \mathcal{Q}_{\alpha'} \Rightarrow T(\mathcal{Q}_\alpha) \neq T(\mathcal{Q}_{\alpha'})$. Hence $\mathcal{G}_+$ acts as a group of permutations of $\Sigma = \{\mathcal{Q}_\nu : \nu \in \mathbb{Z}/(5)\}$. Thus we have a group homomorphism

$$\gamma : \mathcal{G}_+ \longrightarrow \mathcal{S}_5. \tag{6.1}$$

In fact, $\mathcal{G}$ acts on Σ, but $-I$ acts trivially, so $\mathcal{G}$ does not act effectively on this set. We claim that $\mathcal{G}_+$ does act effectively:

Lemma 6.1. *The homomorphism γ in (6.1) is injective.*

Proof. We can characterize how each $T \in \mathcal{G}_+$ acts on $\widetilde{\mathcal{D}}$, in a fashion parallel to our analysis of the action on $\mathcal{I}$ in §4, and use this to study $\pi(T)$. For example, if T is a rotation about the k-axis, hence a power of R, then as mentioned near the end of §5 we certainly have T moving about each element of Σ. The same will hold when T is a rotation about any other axis through the center of a face of $\widetilde{\mathcal{D}}$ (in which case T is conjugate to a power of R).

Next, suppose T is a rotation through $\pm 2\pi/3$ about an axis through a vertex x_0 of $\widetilde{\mathcal{D}}$. Let $\mathfrak{f}$ be a face of $\widetilde{\mathcal{D}}$ containing the vertex x_0 and let $\mathcal{Q}_\alpha$ be the element of Σ with an edge joining the two vertices of $\mathfrak{f}$ on either side of x_0. It follows readily that $T(\mathcal{Q}_\alpha) \neq \mathcal{Q}_\alpha$.

Finally, if T is a rotation through π about an axis through the center of an edge e of $\widetilde{\mathcal{D}}$, we can let $\mathcal{Q}_\alpha$ be an element of Σ with a vertex at one end of this edge, and see that $T(\mathcal{Q}_\alpha) \neq \mathcal{Q}_\alpha$. This proves Lemma 6.1.

Given Lemma 6.1, we are in position to establish the main result of this section.

Proposition 6.2. *We have an isomorphism*

$$\gamma : \mathcal{G}_+ \xrightarrow{\approx} \mathcal{A}_5. \tag{6.2}$$

Proof. Since $\gamma(\mathcal{G}_+)$ is a subgroup of $\mathcal{S}_5$ of order 60, it suffices to show that $\gamma(\mathcal{G}_+) \subset \mathcal{A}_5$. Thus if we set

$$\sigma : \mathcal{G}_+ \longrightarrow \{1, -1\}, \quad \sigma(T) = \operatorname{sgn} \gamma(t), \tag{6.3}$$

it suffices to show that $\operatorname{Ker}\sigma = \mathcal{G}_+$. Of course $\operatorname{Ker}\sigma$ is a subgroup of $\mathcal{G}_+$, so its order divides 60.

Clearly if $T \in \mathcal{G}_+$ is a rotation about the axis through the center of a face of $\widetilde{\mathcal{D}}$, then $T^5 = I \Rightarrow \sigma(T)^5 = 1 \Rightarrow \sigma(T) = 1$, so this gives $\sigma(T) = 1$ for 25 elements of $\mathcal{G}_+$. Also if $T \in \mathcal{G}_+$ is a rotation about the axis through a vertex of $\widetilde{\mathcal{D}}$, then $T^3 = I \Rightarrow \sigma(T)^3 = 1 \Rightarrow \sigma(T) = 1$, so this gives $\sigma(T) = 1$ for an additional 20 elements of $\mathcal{G}_+$, hence for at least 45 elements of $\mathcal{G}_+$. This is enough to show that $\sigma(T) = 1$ for all $T \in \mathcal{G}_+$, and completes the proof of Proposition 6.2.

A. Geometric solution of $x^2 = ab$

Let a and b be positive numbers. We want to describe the geometric approach to solving

$$x^2 = ab, \tag{A.1}$$

for $x > 0$, a la Euclid, but woth the advantage of the use of algebra. To start, (A.1) is equivalent to

$$\frac{x}{a} = \frac{b}{x}. \tag{A.2}$$

To solve (A.2), we will find two similar right triangles, one with sides of length x and a, the other with sides of length b and x. We describe a construction depicted in Fig. A.1.

With $\alpha = (a + b)/2$, consider the line segment in the real axis of $\mathbb{C}$ from $-\alpha$ to α, of total length $a + b$. Mark off $\beta = -\alpha + a$, so the segment from $-\alpha$ to β has length a and the segment from β to α has length b. Draw the circle of radius α, centered at the origin 0.

Draw the vertical line segment up from β. Say it hits this circle at ω. We have three triangles:

$$\begin{aligned} &\triangle_1, \quad \text{vertices} \quad -\alpha, \omega, \alpha, \\ &\triangle_2, \quad \text{vertices} \quad -\alpha, \beta, \omega, \\ &\triangle_3, \quad \text{vertices} \quad \alpha, \beta, \omega. \end{aligned} \tag{A.3}$$

We claim they are all similar, i.e., they all have the same angles. In particular, we claim that

$$\triangle_2 \ \text{ is similar to } \ \triangle_3, \tag{A.4}$$

which implies that (A.2) holds with

$$x = \operatorname{Im} \omega. \tag{A.5}$$

This is the geometrical construction of the solution to (A.1).

Now (A.4) holds provided

$$\triangle_2 \text{ is similar to } \triangle_1 \tag{A.6}$$

and

$$\triangle_3 \text{ is similar to } \triangle_1. \tag{A.7}$$

Furthermore, each of these assertions follows provided

$$\varphi_2 + \theta_2 \text{ is a right angle,}$$

i.e., provided

$$\omega + \alpha \text{ is orthogonal to } \omega - \alpha. \tag{A.8}$$

It remains to establish (A.8).

Now if $z = x + iy$, $w = u + iv \in \mathbb{C}$, with $x, y, u, v \in \mathbb{R}$, the Euclidean inner product of z and w is given by

$$(z, w) = \operatorname{Re} z\overline{w} = xu + yv. \tag{A.9}$$

In the case of (A.8), since $\alpha > 0$ and $|\omega|^2 = \alpha^2$,

$$\begin{aligned} (\omega + \alpha, \omega - \alpha) &= \operatorname{Re}(\omega + \alpha)(\overline{\omega} - \alpha) \\ &= \operatorname{Re}(|\omega|^2 + \alpha\overline{\omega} - \alpha\omega - \alpha^2) \\ &= \alpha \operatorname{Re}(\overline{\omega} - \omega) \\ &= 0. \end{aligned} \tag{A.10}$$

This ends the proof of (A.8), hence of (A.6)–(A.7), so we do have the positive solution to (A.1), given by (A.5).

References

[G] L. Grove, Groups and Characters, Wiley-Interscience, New York, 1997.

[K] F. Klein, The Icosahedron, Dover, New York, 1956.