DeRham Cohomology of Compact Symmetric Spaces
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A Riemannian manifold $M$ is a homogeneous space provided its isometry group acts transitively on $M$. It is a symmetric space if, in addition, given $p \in M$, there exists an isometry $\iota_p : M \to M$ such that

(1) \[ \iota_p(p) = p, \quad \text{and} \quad d\iota_p(p) = -I \quad \text{on} \quad T_pM. \]

Let $M$ be a compact symmetric space, and let $G$ be the connected component of the identity in its isometry group. (Sometimes $\iota_p \in G$ and sometimes $\iota_p \notin G$.) For $0 \leq k \leq n = \dim M$, let

(2) \[ B_k = \{ \beta \in \Lambda^k(M) : g^*\beta = \beta, \; \forall g \in G \}. \]

Let $H_k$ denote the space of harmonic $k$-forms on $M$. Basic Hodge theory gives a natural isomorphism

(3) \[ \mathcal{H}_k \cong \mathcal{H}^k(M), \]

where the right side of (3) is the $k$th deRham cohomology group of $M$. We aim to prove the following.

**Theorem 1.** If $M$ is a compact, connected symmetric space,

(4) \[ B_k = H_k. \]

To begin, we note that, since $G$ is connected, $\beta \in \mathcal{H}_k \implies g^*\beta$ is both harmonic and cohomologous to $\beta$, hence equal to $\beta$, so clearly

(5) \[ \mathcal{H}_k \subset B_k. \]

It remains to prove that $B_k \subset \mathcal{H}_k$. We bring in some lemmas.

**Lemma 2.** Given the isometry $\iota_p$ as in (1),

(6) \[ g \in G \implies \iota_p g \iota_p^{-1} \in G. \]

**Proof.** Take a continuous $\gamma : [0,1] \to G$, $\gamma(0) = e$ (the identity element), $\gamma(1) = g$. Then $\iota_p \gamma(t) \iota_p^{-1} \in G$ for all $t \in [0,1]$. 

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Lemma 3. We have

\[ \beta \in B_k \implies \iota_p^* \beta \in B_k. \]  

Proof. Given \( g \in G \), Lemma 2 implies \( \iota_p g \iota_p^{-1} = \tilde{g} \in G \). Now, for \( \beta \in B_k \),

\[ g^* \iota_p^* \beta = (\iota_p g)^* \beta = (\tilde{g} \iota_p)^* \beta = \iota_p^* \tilde{g}^* \beta = \iota_p^* \beta, \]

so (7) holds.

Lemma 4. We have

\[ \beta \in B_k \implies d\beta = 0. \]  

Proof. If \( \beta \in B_k \), we have that \( \iota_p^* \beta \in B_k \), and, for each \( p \in M \),

\[ \iota_p^* \beta(p) = (-1)^k \beta(p). \]

Since \( \beta \) and \( \iota_p^* \beta \) belong to \( B_k \), (10) plus the transitivity of \( G \) on \( M \) imply

\[ \iota_p^* \beta = (-1)^k \beta \text{ on } M. \]

Also \( d\beta \in B_{k+1} \), so

\[ \iota_p^* d\beta = (-1)^{k+1} d\beta \text{ on } M. \]

But \( \iota_p^* d\beta = d\iota_p^* \beta = (-1)^k d\beta \) on \( M \), by (11), so \( (-1)^{k+1} d\beta = (-1)^k d\beta \) on \( M \), which forces \( d\beta = 0 \).

The following complement to Lemma 4 establishes the reverse inclusion \( B_k \subset H_k \),
and completes the proof of Theorem 1.

Lemma 5. For \( \delta = d^* \), we have

\[ \beta \in B_k \implies \delta \beta = 0. \]  

Proof. If \( M \) is oriented, it has a Hodge \( * \)-operator \( * : \Lambda^k(M) \to \Lambda^{n-k}(M) \). Since each \( g \in G \) is an orientation-preserving isometry, \( * \) commutes with the action of such \( g \) on forms, so

\[ * : B_k \to B_{n-k}, \]

and, since \( \delta = \pm d^* \), (13) follows from (9).

If \( M \) is not orientable, we use the following argument (thanks to S. Kumar).
While \( d \) commutes with all pull-backs \( \varphi^* \) for smooth \( \varphi : M \to M \), \( \delta = d^* \) commutes
with $\varphi^*$ as long as $\varphi$ is an isometry. In such a case, the action of $\varphi^*$ on forms is unitary (say $U$), and we have

$$dU = Ud \Rightarrow U^{-1}d^* = d^*U^{-1} \Rightarrow d^*U = Ud^*. \tag{15}$$

Thus the proof of Lemma 4 extends as follows. If $\beta \in B_k$, then for each $p \in M$, $t_p^*\beta \in B_k$, and (11) holds. Also, since $g^*\delta = \delta g^*$ for $g \in G$, $\delta \beta \in B_{k-1}$. Hence $t_p^*\delta \beta = \delta t_p^*\beta$ equals both $(-1)^k\delta \beta$ and $(-1)^{k-1}\delta \beta$, forcing $\delta \beta = 0$.

An alternative endgame for the proof of Theorem 1 (noted by S. Kumar) is to use $\beta \in B_k \Rightarrow \delta \beta \in B_k$ and apply Lemma 4 to deduce that $d\delta \beta = 0$, hence $\Delta \beta = -(d\delta \beta + \delta d\beta) = 0$.

It is desirable to say some more about $B_k$. So take $p \in M$, and let

$$K = \{g \in G : gp = p\}, \tag{16}$$

so $K$ is a closed subgroup of $G$ and

$$M = G/K. \tag{17}$$

Given $g \in K$, we have the representation

$$\pi(g) = Dg(p) : T_pM \rightarrow T_pM, \tag{18}$$

of $K$ on $T_pM$, yielding representations $\Lambda^k \pi$ of $K$ on $\Lambda^k T_pM$.

**Proposition 6.** For $0 \leq k \leq n$, we have $B_k$ isomorphic to

$$\{v \in \Lambda^k T_pM : \Lambda^k \pi(g)v = v, \forall g \in K\}. \tag{19}$$

Hence such a linear space is isomorphic to $\mathcal{H}^k(M)$.

We can denote the space (19) as $I^k(T_pM, K)$, where, generally, if $V$ is a finite-dimensional real inner-product space and $K$ a closed subgroup of $O(V)$,

$$I^k(V, K) = \{v \in \Lambda^k V : \Lambda^k g v = v, \forall g \in K\}. \tag{20}$$

Note that

$$I^*(V, K) = \bigoplus_k I^k(V, K) \tag{21}$$

has a natural ring structure, and, in the setting of Proposition 6, $I^*(T_pM, K)$ is isomorphic to the cohomology ring $\mathcal{H}^*(M)$. 