DeRham Cohomology of Compact Symmetric Spaces

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A Riemannian manifold M is a homogeneous space provided its isometry group acts transitively on M. It is a symmetric space if, in addition, given $p \in M$, there exists an isometry $\iota_p : M \to M$ such that

$$\iota_p(p) = p, \quad \text{and} \quad d\iota_p(p) = -I \quad \text{on} \quad T_pM. \tag{1}$$

Let M be a compact symmetric space, and let G be the connected component of the identity in its isometry group. (Sometimes $\iota_p \in G$ and sometimes $\iota_p \notin G$.) For $0 \le k \le n = \dim M$, let

$$\mathcal{B}_k = \{\beta \in \Lambda^k(M) : g^*\beta = \beta, \ \forall \, g \in G\}. \tag{2}$$

Let $\mathcal{H}_k$ denote the space of harmonic k-forms on M. Basic Hodge theory gives a natural isomorphism

$$\mathcal{H}_k \approx \mathcal{H}^k(M), \tag{3}$$

where the right side of (3) is the kth deRham cohomology group of M. We aim to prove the following.

Theorem 1. *If M is a compact, connected symmetric space,*

$$\mathcal{B}_k = \mathcal{H}_k. \tag{4}$$

To begin, we note that, since G is connected, $\beta \in \mathcal{H}_k \Rightarrow g^*\beta$ is both harmonic and cohomologous to β, hence equal to β, so clearly

$$\mathcal{H}_k \subset \mathcal{B}_k. \tag{5}$$

It remains to prove that $\mathcal{B}_k \subset \mathcal{H}_k$. We bring in some lemmas.

Lemma 2. *Given the isometry ι_p as in (1),*

$$g \in G \Longrightarrow \iota_p g \iota_p^{-1} \in G. \tag{6}$$

Proof. Take a continuous $\gamma : [0,1] \to G$, $\gamma(0) = e$ (the identity element), $\gamma(1) = g$. Then $\iota_p \gamma(t) \iota_p^{-1} \in G$ for all $t \in [0,1]$.

Lemma 3. *We have*

$$\beta \in \mathcal{B}_k \Longrightarrow \iota_p^* \beta \in \mathcal{B}_k. \tag{7}$$

Proof. Given $g \in G$, Lemma 2 implies $\iota_p g \iota_p^{-1} = \tilde{g} \in G$. Now, for $\beta \in \mathcal{B}_k$,

$$g^* \iota_p^* \beta = (\iota_p g)^* \beta = (\tilde{g} \iota_p)^* \beta = \iota_p^* \tilde{g}^* \beta = \iota_p^* \beta, \tag{8}$$

so (7) holds.

Lemma 4. *We have*

$$\beta \in \mathcal{B}_k \Longrightarrow d\beta = 0. \tag{9}$$

Proof. If $\beta \in \mathcal{B}_k$, we have that $\iota_p^* \beta \in \mathcal{B}_k$, and, for each $p \in M$,,

$$\iota_p^* \beta(p) = (-1)^k \beta(p). \tag{10}$$

Since β and $\iota_p^* \beta$ belong to $\mathcal{B}_k$, (10) plus the transitivity of G on M imply

$$\iota_p^* \beta = (-1)^k \beta \quad \text{on} \quad M. \tag{11}$$

Also $d\beta \in \mathcal{B}_{k+1}$, so

$$\iota_p^* d\beta = (-1)^{k+1} d\beta \quad \text{on} \quad M. \tag{12}$$

But $\iota_p^* d\beta = d\iota_p^* \beta = (-1)^k d\beta$ on M, by (11), so $(-1)^{k+1} d\beta = (-1)^k d\beta$ on M, which forces $d\beta = 0$.

The following complement to Lemma 4 establishes the reverse inclusion $\mathcal{B}_k \subset \mathcal{H}_k$, and completes the proof of Theorem 1.

Lemma 5. *For $\delta = d^*$, we have*

$$\beta \in \mathcal{B}_k \Longrightarrow \delta\beta = 0. \tag{13}$$

Proof. If M is oriented, it has a Hodge $*$-operator $* : \Lambda^k(M) \to \Lambda^{n-k}(M)$. Since each $g \in G$ is an orientation-preserving isometry, $*$ commutes with the action of such g on forms, so

$$* : \mathcal{B}_k \longrightarrow \mathcal{B}_{n-k}, \tag{14}$$

and, since $\delta = \pm * d *$, (13) follows from (9).

If M is not orientable, we use the following argument (thanks to S. Kumar). While d commutes with all pull-backs φ^* for smooth $\varphi : M \to M$, $\delta = d^*$ commutes with φ^* as long as φ is an isometry. In such a case, the action of φ^* on forms is unitary (say U), and we have

$$dU = Ud \Rightarrow U^{-1}d^* = d^*U^{-1} \Rightarrow d^*U = Ud^*. \tag{15}$$

Thus the proof of Lemma 4 extends as follows. If $\beta \in \mathcal{B}_k$, then for each $p \in M$, $\iota_p^*\beta \in \mathcal{B}_k$, and (11) holds. Also, since $g^*\delta = \delta g^*$ for $g \in G$, $\delta\beta \in \mathcal{B}_{k-1}$. Hence $\iota_p^*\delta\beta = \delta\iota_p^*\beta$ equals both $(-1)^k\delta\beta$ and $(-1)^{k-1}\delta\beta$, forcing $\delta\beta = 0$.

An alternative endgame for the proof of Theorem 1 (noted by S. Kumar) is to use $\beta \in \mathcal{B}_k \Rightarrow \delta\beta \in \mathcal{B}_k$ and apply Lemma 4 to deduce that $d\delta\beta = 0$, hence $\Delta\beta = -(d\delta\beta + \delta d\beta) = 0$.

It is desirable to say some more about $\mathcal{B}_k$. So take $p \in M$, and let

$$K = \{g \in G : gp = p\}, \tag{16}$$

so K is a closed subgroup of G and

$$M = G/K. \tag{17}$$

Given $g \in K$, we have the representation

$$\pi(g) = Dg(p) : T_pM \longrightarrow T_pM, \tag{18}$$

of K on T_pM, yielding representations $\Lambda^k\pi$ of K on Λ^kT_pM.

Proposition 6. *For $0 \le k \le n$, we have $\mathcal{B}_k$ isomorphic to*

$$\{v \in \Lambda^kT_pM : \Lambda^k\pi(g)v = v, \ \forall\, g \in K\}. \tag{19}$$

Hence such a linear space is isomorphic to $\mathcal{H}^k(M)$.

We can denote the space (19) as $\mathcal{I}^k(T_pM, K)$, where, generally, if V is a finite-dimensional real inner-product space and K a closed subgroup of $\mathrm{O}(V)$,

$$\mathcal{I}^k(V, K) = \{v \in \Lambda^kV : \Lambda^kg\, v = v, \ \forall\, g \in K\}. \tag{20}$$

Note that

$$\mathcal{I}^*(V, K) = \bigoplus_k \mathcal{I}^k(V, K) \tag{21}$$

has a natural ring structure, and, in the setting of Proposition 6, $\mathcal{I}^*(T_pM, K)$ is isomorphic to the cohomology ring $\mathcal{H}^*(M)$.