The Dirichlet-to-Neumann Map
And Fractal Variants

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Abstract. These notes develop the study of the Dirichlet-to-Neumann map N associated to a domain Ω in a compact Riemannian manifold M, as a positive, self-adjoint operator on $L^2(\partial\Omega, \mu)$, in a variety of situations. We start with the classical case of a smoothly bounded domain, with μ given by surface measure on $\partial\Omega$, then consider Lipschitz domains.

We move on to rougher finite perimeter domains. The study here includes special results for uniformly rectifiable domains. We also examine finite perimeter domains with rectifiable "inclusions" S, for which $\partial\Omega = \partial\mathcal{O} \cup S$, and see that it is useful to construct N as a self adjoint operator on $L^2(\widetilde{\partial}\Omega, \mu)$, where $\widetilde{\partial}\Omega = \partial\mathcal{O} \cup S_1 \cup S_2$, with S_1 and S_2 denoting two copies of S.

Finally we take up more exotic cases, in which $\partial\Omega$ has a fractal character (it might be totally disconnected), and μ is completely different from surface area.

One enduring theme is the analysis of the semigroup $\{e^{-tN} : t \geq 0\}$, as a semigroup of positivity-preserving operators, shown in various settings to have the property of "irreducibility." These semigroups give rise to Markov processes on $\partial\Omega$. In the fractal cases arising in Chapters 5 and 7, these can be seen as providing new Markov processes on fractals.

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Contents

Preface

Two of the most basic boundary problems in PDE are the Dirichlet problem,

$$\Delta u = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega, \tag{0.1}$$

and the Neumann problem,

$$\Delta u = 0 \text{ on } \Omega, \quad \partial_\nu u = g \text{ on } \partial\Omega. \tag{0.2}$$

Here Δ is the Laplace operator, which on Euclidean space $\mathbb{R}^n$ is

$$\Delta = \partial_1^2 + \cdots + \partial_n^2, \tag{0.3}$$

and solutions to $\Delta u = 0$ on Ω are known as harmonic functions. More generally, we take Ω to be a relatively compact domain in a Riemannian manifold M, and Δ to be the Laplace-Beltrami operator on M. In (0.2), $\partial_\nu u$ denotes the normal derivative of u on $\partial\Omega$.

The Dirichlet problem (0.1) is seen to have a unique solution for large classes of domains Ω, the solution denoted

$$u = \mathrm{PI}\, f. \tag{0.4}$$

The Neumann problem (0.2) has a solution u, unique up to an additive constant (assuming Ω is connected), provided g satisfies the condition

$$\int\limits_{\partial\Omega} g\, dS = 0. \tag{0.5}$$

It has long been observed that a map connecting these two boundary problems is of special significance. This is the Dirichlet-to-Neumann map N, defined by

$$Nf = \partial_\nu(\mathrm{PI}\, f) = g. \tag{0.6}$$

One use of the Dirichlet-to-Neumann map is to enable one to analyze solutions to the Neumann problem, given a knowledge of the solutions to the Dirichlet problem, via the representation of a solution to (0.2) as

$$u = \mathrm{PI}\, f, \quad f = N^{-1} g, \tag{0.7}$$

where N^{-1} is defined initially on functions satisfying (0.5), and extended to annihilate constants. Going further, a study of the behavior of N and N^{-1} enables one to relate other boundary problems to the Dirichlet problem, such as the Robin problem,

$$\Delta u = 0 \text{ on } \Omega, \quad (\partial_\nu + a)u = h \text{ on } \partial\Omega, \tag{0.8}$$

where a is a real valued function on $\partial\Omega$, using the identity

$$u = \mathrm{PI}\, f \Longrightarrow (\partial_\nu + a)u = Nf + af, \tag{0.9}$$

and the oblique derivative problem,

$$\Delta u = 0 \text{ on } \Omega, \quad (\partial_\nu + X)u = h \text{ on } \partial\Omega, \tag{0.10}$$

where X is a real vector field on $\partial\Omega$, via the identity

$$u = \mathrm{PI}\, f \Longrightarrow (\partial_\nu + X)u = (N + X)f. \tag{0.11}$$

The Dirichlet-to-Neumann map also arises in models of non-invasive imaging, such as for electrical impedance tomography. One such model leads to a problem posed by A. P. Calderón. Given a bounded region Ω, with an unknown metric tensor, one has information on the behavior of the map N on functions on $\partial\Omega$, and desires to obtain information on the metric tensor on Ω. This inverse problem has motivated much work, and for tools to tackle it one seeks information on the direct problem of how the metric tensor affects the behavior of N.

The purpose of this text is to explore the structure of the Dirichlet-to-Neumann map N in a variety of settings. We start with smoothly bounded domains, where N can be analyzed as a first-order pseudodifferential operator. In such a setting we see that the semigroup e^{-tN} is positivity preserving. In subsequent chapters we analyze N on more general sorts of domains, tackling Lipschitz domains, then more general classes of finite perimeter domains. In Chapter 7 we take a further step, considering domains with fractal boundaries. At each step, e^{-tN} is seen to be positivity preserving, hence a symmetric Markov semigroup, which consequently gives rise to a Markov process on $\partial\Omega$. In cases treated in Chapter 7, this introduces a class of Markov processes on sets that might be Cantor sets.

Early parts of the text present results of many researchers, but each chapter includes original results, the ratio of such results increasing with the chapter number. This monograph should prove useful to people doing research on the Dirichlet-to-Neumann map. It could also be used in a topics course, following a basic graduate course in PDE. Students can find sufficient background in several sources, including Chapters 5 and 7 of [T1] and Chapter 4 of [T2], or, alternatively, Chapters 1 and 3 of [T4].

1. Introduction

Let Ω be an open, connected subset of a compact, connected Riemannian manifold M, satisfying

$$M \setminus \overline{\Omega} \neq \emptyset. \tag{1.1}$$

If $\partial\Omega$ has some smoothness, the Dirichlet-to-Neumann map N is defined on an appropriate class of functions on $\partial\Omega$ by

$$Nf = g, \tag{1.2}$$

where

$$g = \partial_\nu u, \quad \Delta u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = f. \tag{1.3}$$

Here $\partial_\nu u = \langle \nu, \nabla u \rangle$, and ν is the outward pointing unit normal to $\partial\Omega$. It has an approximate inverse, the Neumann-to-Dirichlet map

$$Kg = f, \tag{1.4}$$

defined also by (1.3) when $\int_{\partial\Omega} g\, dS = 0$, and one normalizes f to satisfy $\int_{\partial\Omega} f\, dS = 0$. (We set $K1 = 0$.)

Our goal here is to analyze the operator N for a variety of classes of domains Ω, starting with domains Ω for which the characterization (1.2)–(1.3) has a straightforward realization, and proceeding to classes of domains for which significant additional work is required to define N.

In Chapter 2 we deal with regions Ω with smooth boundary, and we also assume M carries a smooth metric tensor. In this classical case, basic L^2-Sobolev space analysis of the Dirichlet and Neumann problems readily yields that N and K are well defined by (1.2)–(1.4) and satisfy

$$N : H^{s+1}(\partial\Omega) \longrightarrow H^s(\partial\Omega), \quad K : H^s(\partial\Omega) \longrightarrow H^{s+1}(\partial\Omega), \tag{1.5}$$

for $s > 0$. We see that K and N are symmetric, and then duality yields (1.5) for all $s \in \mathbb{R}$. In this case, one can also analyze N as a pseudodifferential operator,

$$N \in OPS^1(\partial\Omega). \tag{1.6}$$

This property allows one to extend (1.5) to mapping properties on L^p-Sobolev spaces, and a variety of other spaces. The operator N is elliptic, and we can compare it with $\Lambda = \sqrt{-\Delta_X}$, where Δ_X is the Laplace operator on $\partial\Omega$. We also study the semigroup $\{e^{-tN} : t \geq 0\}$. In §2.3 we give an elementary proof of the positivity property that, for $f \in L^2(\partial\Omega)$,

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0, \quad \forall\, t > 0. \tag{1.7}$$

Such positivity has been established in rougher situations in [AM] and [AtE], using more sophisticated techniques, involving the Beurling-Deny positivity criterion. We will return to this issue in subsequent chapters, and seek further settings where (1.7) holds. We also look at a stronger version of (1.7), which leads to the concept of *irreducibility* of e^{-tN}.

In Chapter 3 we treat Lipschitz domains. A number of results on N for this class of domains have been derived in [AM] and [BtE]. Here, making use of results on the Dirichlet and Neumann problems from [Dah], [DK], [Ver], and [MT1]–[MT4], we establish further results. These include allowing for fairly rough metric tensors on M; cf. (3.0.1)–(3.0.2). We show that there exists $q(\Omega) > 2$ such that

$$N : H^{1,p}(\partial\Omega) \to L^p(\partial\Omega), \quad K : L^p(\partial\Omega) \to H^{1,p}(\partial\Omega), \quad 1 < p < q(\Omega). \tag{1.8}$$

We do not have (1.6) in this setting, but layer potentials do provide useful tools for the analysis of N, as discussed in §3.2. We also discuss (1.7) in this setting, and further operator estimates on e^{-tN}. Using these estimates, we establish results on the spectrum of N, extending to the class of Lipschitz domains results that are known for smooth domains as a consequence of (1.6).

In Chapter 4 we treat finite perimeter domains, defined in (4.0.2)–(4.0.4). This class of domains contains the class studied in [AtE], and a number of analytical techniques used in Chapter 4 are adapted from those in that paper. One such technique involves an extension of the Friedrichs method of producing a self-adjoint operator on a Hilbert space from a quadratic form. We give a version of this in Appendix A. Here our Hilbert space is $L^2(\partial\Omega, \mu)$, where μ is a finite Borel measure on $\partial\Omega$ satisfying $\mu \geq \sigma$, with σ as in (4.0.4), surface area on the measure theoretic boundary $\partial_*\Omega$. If $\partial\Omega$ contains big pieces that are disjoint from $\partial_*\Omega$, μ might be quite different from surface area on these pieces. The self-adjoint operator N constructed by this process has the property that

$$\mathrm{Lip}(\partial\Omega) \ \text{ is dense in } \ \mathcal{D}(N^{1/2}). \tag{1.9}$$

We verify (1.7) in this setting, as in [AM] and [AtE] by showing that the Beurling-Deny criterion applies. Another topic treated in Chapter 4 is the Dirichlet problem, from (1.3). We construct two "solution operators,"

$$\mathrm{PI} : \mathcal{D}(N^{1/2}) \longrightarrow H^1(\Omega), \quad \mathrm{PI}_0 : C(\partial\Omega) \longrightarrow L^\infty(\Omega), \tag{1.10}$$

the first by a process closely related to the construction of N, the second by a classical method. We examine conditions under which these two operators can be shown to coincide on their common domain (which contains $\mathrm{Lip}(\partial\Omega)$).

In most of Chapter 4 we assume M has a C^∞ metric tensor. In §4.6 we extend the analysis to include rough metric tensors.

In Chapter 5 we continue the study of N on finite perimeter domains, concentrating on domains Ω for which $\partial\Omega$ has "inclusions," which are pieces that are disjoint from $\partial_*\Omega$, and typically lie in the interior of $\overline{\Omega}$. An example is the slit in a slit disk. More exotic examples include fractal inclusions, of Hausdorff dimension $s \in (n-2, n)$ (where $n = \dim \Omega$). In the former case, i.e., for inclusions K that are $(n-1)$-dimensional surfaces, we look at a variant of the operator N constructed in Chapter 4, which pays attention to the varying behavior of a harmonic function on Ω as one approaches the two separate sides of K. One approach to this, discussed in §5.1, is to cut along K and produce a blow-up. Another, much more broadly applicable approach, developed in §5.2, involves taking a finite perimeter domain $\mathcal{O} \subset M$, partitioning it into two finite perimeter domains Ω_1 and Ω_2, and taking $\Omega = \mathcal{O} \setminus S$, where the inclusion S is a subset of $\partial\Omega_1 \cap \partial\Omega_2$. Section 5.3 deals with fractal inclusions. There, a goal is to produce measures μ that give rise to operators N whose behavior appropriately reflects the appearance of K. This study is naturally done under the hypothesis that $\mathrm{Cap}(K) > 0$.

In Chapter 6 we treat N on an important class of of finite perimeter domains known as uniformly rectifiable domains. This is a maximal class of domains on which there is a viable theory of singular integral operators that allows one to apply layer potential methods to the study of N. Using these tools, we show in §6.3 that the construction of N for Lipschitz domains in Chapter 3 is consistent with the construction in Chapter 4. We also discuss results on N for another important class of domains, which we call regular SKT domains.

In Chapter 7 we pass beyond the class of finite perimeter domains, to open sets $\Omega \subset M$ with a much wilder boundary $\partial\Omega$, assumed to carry a positive finite measure μ satisfying a certain estimate (cf. (7.0.1)) that allows us to push the methods of Chapter 4 to construct N as a self adjoint operator on $L^2(\partial\Omega, \mu)$, and analyze e^{-tN} as a semigroup satisfying (1.7). In this setting, with μ unrelated to "surface area," one might refer to N, not as a Dirichlet-to-Neumann map in a straightforward way involving (1.3), but as some "generalized" D-to-N map. We mention that here we also drop the hypothesis (1.1), and consider cases where $\partial\Omega = M \setminus \Omega$, including cases where $\partial\Omega$ is totally disconnected. In §7.3 we focus on a beautiful special case, in which $M = \widehat{\mathbb{C}}$ is the Riemann sphere and $\partial\Omega$ is the Julia set of a rational map on $\widehat{\mathbb{C}}$, typically a marvelously fractal object.

At the end are some appendices. As mentioned above, Appendix A describes the construction of a self-adjoint operator on a Hilbert space via a quadratic form, starting with the classical Friedrichs method and then bringing in a recent variant of [AtE], of use in Chapter 4. Appendix B discusses a version of Green's formula, valid for the class of finite perimeter domains known as Ahlfors regular domains (which in turn contains the class of UR domains), established in [HMT], and of use here in Chapter 6.

2. Smoothly bounded domains

Let $\overline{\Omega}$ be a compact, connected, n-dimensional Riemannian manifold with smooth boundary. For consistency with later sections, we take Ω to be a smoothly bounded, open subset of a compact, conected Riemannian manifold M, without boundary. Our hypothesis on Ω entails

$$M \setminus \overline{\Omega} \neq \emptyset. \tag{2.0.1}$$

In this chapter we also assume the metric tensor on M is smooth.

The Dirichlet-to-Neumann map N is very well studied in this situation. Our description of N in this chapter will largely serve as a guide to what to look for in subsequent chapters.

In §2.1 we introduce

$$Nf = \partial_\nu \, \mathrm{PI}\, f\big|_{\partial\Omega}, \tag{2.0.2}$$

and derive basic properties that follow from the L^2-Sobolev space theory of the Dirichlet and Neumann boundary problems. These include the mapping properties

$$N : H^{s+1}(\partial\Omega) \longrightarrow H^s(\partial\Omega), \quad s \in \mathbb{R}, \tag{2.0.3}$$

with approximate inverse $K : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ given by solving the Neumann problem. Also, N is a positive semidefinite self-adjoint operator on $L^2(\partial\Omega)$, with

$$\mathcal{D}(N) = H^1(\partial\Omega). \tag{2.0.4}$$

It follows that $-N$ generates a contraction semigroup e^{-tN} on $L^2(\partial\Omega)$.

In §2.2 we show that N is a pseudodifferential operator on $\partial\Omega$, and its principal symbol is the same as that of $\Lambda = \sqrt{-\Delta_X}$, where Δ_X is the Laplace operator on $X = \partial\Omega$. Hence

$$N, \Lambda \in OPS^1(\partial\Omega), \quad N - \Lambda = B \in OPS^0(\partial\Omega). \tag{2.0.5}$$

One approach to (2.0.5) involves the identity

$$SN = -\frac{1}{2} I + A, \tag{2.0.6}$$

where S and A are given by a single and a double layer potential; cf. (2.2.3). The identity (2.0.6) will continue to play a role for Lipschitz domains, in Chapter 3, though it will no longer lead to the representation of N as a pseudodifferential operator.

In §2.3 we show that e^{-tN} has the positivity property: given $f \in L^2(\partial\Omega)$,

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0. \tag{2.0.7}$$

Hence $\{e^{-tN}\}$ is a symmetric Markov semigroup, and is therefore associated with a stochastic process. Furthermore, for each $y \in \partial\Omega$,

$$e^{-tN} \delta_y(x) > 0, \quad \forall\, t > 0,\ x \in \partial\Omega. \tag{2.0.8}$$

One consequence of (2.0.8) is irreducibility of $\{e^{-tN}\}$: there are so proper invariant subspaces of the form $L^2(\Sigma)$, with $\Sigma \subset \partial\Omega$, even when $\partial\Omega$ is not connected. The proofs of (2.0.7) and (2.0.8) are fairly easy consequences of the weak maximum principle for (2.0.7) and of Zaremba's principle for (2.0.8). Extensions of (2.0.7) and the irreducibility of $\{e^{-tN}\}$ to rougher domains require more effort, and more powerful tools, as will be seen in subsequent sections.

The results in (2.0.5) suggest comparing e^{-tN} and $e^{-t\Lambda}$. We do this in §2.4, showing that, for each $K \in \mathbb{N}$, we can write

$$e^{-tN} = (I + P_1 t + \cdots + P_K t^K) e^{-t\Lambda} - Q_K(t), \quad t > 0, \tag{2.0.9}$$

with $P_j \in OPS^0(\partial\Omega)$ and, for each $j \in \{0, 1, \ldots, K\}$,

$$Q_K(t) = O(t^j) \text{ in } OPS^{-(K-j)}_{1,0}(\partial\Omega), \quad \text{as } t \searrow 0. \tag{2.0.10}$$

We also produce results on the integral kernels of $e^{-t\Lambda}$ and e^{-tN}.

In §2.5 we show by example that e^{-tN^2} need not be positivity preserving.

2.1. N and K on smoothly bounded domains

Here Ω is an open connected subset of a compact Riemannian manifold, and we assume $\partial\Omega$ is smooth and nonempty. Recall that the Poisson integral $\mathrm{PI}\, f$ solves the Dirichlet problem

$$\Delta u = 0 \quad \text{on} \quad \Omega, \qquad u\big|_{\partial\Omega} = f, \qquad \text{for} \quad u = \mathrm{PI}\, f. \tag{2.1.1}$$

A classical L^2-Sobolev space analysis, described, e.g., in §5.1 of [T1], gives

$$\mathrm{PI} : H^s(\partial\Omega) \longrightarrow H^{s+1/2}(\Omega), \qquad s \geq \frac{1}{2}. \tag{2.1.2}$$

Stronger results are available, but we will stick with (2.1.2) for now. Consequently, if ν is the unit outward-pointing normal to $\partial\Omega$, we have

$$Nf = \partial_\nu \, \mathrm{PI}\, f\big|_{\partial\Omega}, \tag{2.1.3}$$

defining

$$N : H^{s+1}(\partial\Omega) \longrightarrow H^s(\partial\Omega), \qquad s > 0. \tag{2.1.4}$$

Now if $u = \mathrm{PI}\, f, v = \mathrm{PI}\, g,\ f, g \in H^{s+1}(\partial\Omega),\ s > 0$, then Green's formula gives

$$(\nabla u, \nabla v) = \int_{\Omega} \partial_\nu u \, \overline{v} \, dS = \langle Nf, g \rangle, \tag{2.1.5}$$

where $(\ ,\ )$ denotes the inner product on $L^2(\Omega)$ and $\langle\ ,\ \rangle$ the inner product on $L^2(\partial\Omega)$. It follows that

$$\langle Nf, g \rangle = \langle f, Ng \rangle, \tag{2.1.6}$$

for $f, g \in H^{s+1}(\partial\Omega),\ s > 0$. Thus, by duality

$$N : H^{-s}(\partial\Omega) \longrightarrow H^{-s-1}(\partial\Omega), \qquad s > 0. \tag{2.1.7}$$

Interpolation with (2.1.4) then yields

$$N : H^{s+1}(\partial\Omega) \longrightarrow H^s(\partial\Omega), \qquad s \in \mathbb{R}. \tag{2.1.8}$$

Next, we recall results from §5.7 of [T1] on the Neumann boundary problem

$$\Delta u = 0 \quad \text{on} \quad \Omega, \qquad \partial_\nu u\big|_{\partial\Omega} = g. \tag{2.1.9}$$

It is shown in Proposition 7.7 in Chapter 5 of [T1] that if

$$g \in H^{k+1/2}(\partial\Omega), \quad k \in \mathbb{Z}^+, \quad \int\limits_{\partial\Omega} g\, dS = 0, \tag{2.1.10}$$

then there is a solution $u \in H^{k+2}(\Omega)$ to (2.1.9), unique up to an additive constant. We set

$$Kg = u\big|_{\partial\Omega}, \quad \text{normalized so that} \quad \int\limits_{\partial\Omega} Kg\, dS = 0, \tag{2.1.11}$$

if g satisfies (2.1.10), and we set

$$K1 = 0. \tag{2.1.12}$$

Then we have

$$K : H^{k+1/2}(\partial\Omega) \longrightarrow H^{k+3/2}(\partial\Omega), \quad k \in \{0, 1, 2, \dots\}, \tag{2.1.13}$$

and interpolation yields

$$K : H^s(\partial\Omega) \longrightarrow H^{s+1}(\partial\Omega), \quad s \geq \frac{1}{2}. \tag{2.1.14}$$

To continue, it is useful to bring in the notation

$$f = f^\# + f^b, \quad f^b = \text{const.}, \quad \int\limits_{\partial\Omega} f^\#\, dS = 0. \tag{2.1.15}$$

We have

$$Kf = Kf^\#, \quad NKf = f^\#, \quad \forall\, f \in H^s(\partial\Omega),\ s \geq \frac{1}{2}, \tag{2.1.16}$$

and also

$$KNf = f^\#, \quad f \in H^s(\partial\Omega),\ s \geq \frac{3}{2}. \tag{2.1.17}$$

It follows that, if $f, g \in H^s(\partial\Omega),\ s \geq 1/2$,

$$\begin{aligned} \langle Kf, g\rangle &= \langle Kf, g^\#\rangle = \langle Kf, NKg\rangle \\ &= \langle NKf, Kg\rangle = \langle f^\#, Kg\rangle \\ &= \langle f, Kg\rangle. \end{aligned} \tag{2.1.18}$$

Thus, by duality, we proceed from (2.1.14) to

$$K : H^{-s-1}(\partial\Omega) \longrightarrow H^{-s}(\partial\Omega), \quad s \geq \frac{1}{2}, \tag{2.1.19}$$

and then interpolation with (2.1.14) gives

$$K : H^{s}(\partial\Omega) \longrightarrow H^{s+1}(\partial\Omega), \quad s \in \mathbb{R}. \tag{2.1.20}$$

The identities (2.1.16)–(2.1.17) then extend to all $f \in H^s(\partial\Omega),\ s \in \mathbb{R}$. An example of (2.1.20) is

$$K : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega). \tag{2.1.21}$$

Since $K = K^*$, we have in particular that

$$K : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \ \text{ is compact and self adjoint.} \tag{2.1.22}$$

It follows that $L^2(\partial\Omega)$ has an orthonormal basis $\{\varphi_j : j = 0, 1, 2, \dots\}$, satisfying

$$K\varphi_j = \mu_j\varphi_j, \quad \mu_0 = 0, \ \ \mu_j \searrow 0 \text{ for } j \geq 1. \tag{2.1.23}$$

The positivity follows from (2.1.5), which in turn implies $\langle Kf, f\rangle = \langle NKf, Kf\rangle \geq 0$ for $f \in H^1(\partial\Omega)$, hence for $f \in L^2(\partial\Omega)$. From (2.1.20), we have

$$\varphi_j \in C^\infty(\partial\Omega), \quad \forall\, j. \tag{2.1.24}$$

From (2.1.16)–(2.1.17) we have

$$N\varphi_j = \lambda_j\varphi_j, \quad \lambda_0 = 0, \ \lambda_j = \frac{1}{\mu_j} \nearrow +\infty \text{ for } j \geq 1. \tag{2.1.25}$$

We see that N is a positive semi-definite self adjoint operator on $L^2(\partial\Omega)$, with domain

$$\mathcal{D}(N) = H^1(\partial\Omega). \tag{2.1.26}$$

Furthermore, by our identities for KN and NK,

$$\mathcal{D}(N^k) = H^k(\partial\Omega), \quad k \in \mathbb{N}. \tag{2.1.27}$$

2.2. N as a pseudodifferential operator

A classical attack on the representation of N and K uses single and double layer potentials,

$$\begin{aligned} \mathcal{S}f(x) &= \int\limits_{\partial\Omega} E(x,y)f(y)\,dS(y), \\ \mathcal{D}f(x) &= \int\limits_{\partial\Omega} \partial_{\nu_y} E(x,y)f(y)\,dS(y), \quad x \in \Omega, \end{aligned} \tag{2.2.1}$$

where $E(x,y)$ is a fundamental solution for Δ on a neighborhood $\mathcal{O}$ of $\overline{\Omega}$. More precisely, $E(x,y)$ denotes the integral kernel of $(\Delta - V)^{-1}$, where $V \in C^\infty(M)$ satisfies $V \geq 0$, $V = 0$ on a neighborhood of $\overline{\Omega}$, and $V > 0$ on a nonempty subset of each connected component of $M \setminus \overline{\Omega}$. One has limits

$$\begin{aligned} \mathcal{S}f_\pm(x) &= Sf(x), \\ \mathcal{D}f_\pm(x) &= \Bigl(\pm\frac{1}{2}I + A\Bigr)f(x), \quad x \in \partial\Omega, \end{aligned} \tag{2.2.2}$$

where the limits are taken from within $\Omega_+ = \Omega$ and from within $\Omega_- = M \setminus \overline{\Omega}$, respectively. Here,

$$\begin{aligned} Sf(x) &= \int\limits_{\partial\Omega} E(x,y)f(y)\,dS(y), \\ Af(x) &= \int\limits_{\partial\Omega} \partial_{\nu_y} E(x,y)f(y)\,dS(y). \end{aligned} \tag{2.2.3}$$

When $\partial\Omega$ is smooth, the operators S and A are pseudodifferential operators of negative order,

$$S, A \in OPS^{-1}(\partial\Omega). \tag{2.2.4}$$

See §3 for a discussion of the behavior for rougher boundaries.

The following provides a neat connection to the Dirichlet-to-Neumann map N. Namely, one can deduce from Green's formula that

$$\begin{aligned} \mathcal{D}f(x) - \mathcal{S}Nf(x) = \mathrm{PI}\,f(x), &\quad x \in \Omega, \\ 0, &\quad x \in M \setminus \overline{\Omega}. \end{aligned} \tag{2.2.5}$$

Consequently, (2.2.2) gives

$$SN = -\frac{1}{2}I + A. \tag{2.2.6}$$

Going further, an examination of the integral kernel of S in (2.4.3) allows one to calculate the principal symbol of S. One obtains

$$\sigma_S(x,\xi) = -\frac{1}{2}|\xi|_x^{-1}, \quad x \in \partial\Omega,\ \xi \in T_x^*(\partial\Omega), \tag{2.2.7}$$

where $|\xi|_x^2$ is the square-norm on $T_x^*(\partial\Omega)$ induced from the natural Riemann metric tensor on $\partial\Omega$. Hence S is an elliptic operator in $OPS^{-1}(\partial\Omega)$, and (2.2.6) implies

$$N \in OPS^1(\partial\Omega), \quad \sigma_N(x,\xi) = |\xi|_x. \tag{2.2.8}$$

Another natural operator on $\partial\Omega$ with the same principal symbol as (2.2.8) is $\sqrt{-\Delta_X}$, where Δ_X denotes the Laplace-Beltrami operator on functions on $X = \partial\Omega$. We hence have

$$N = \sqrt{-\Delta_X}, \quad \text{mod } OPS^0(\partial\Omega). \tag{2.2.9}$$

One can go further and show that

$$N = \sqrt{-\Delta_X} + B, \quad B \in OPS^0(\partial\Omega), \tag{2.2.10}$$

with B having principal symbol

$$\sigma_B(x,\xi) = -\frac{1}{2}\mathrm{Tr}(A_N P_\xi^0), \tag{2.2.11}$$

where $A_N : T_x(\partial\Omega) \to T_x(\partial\Omega)$ is the Weingarten map, associated to the second fundamental form of $\partial\Omega \subset M$, and, for nonzero $\xi \in T_x^*(\partial\Omega)$, P_ξ^0 is the orthogonal projection of $T_x(\partial\Omega)$ onto the subspace annihilated by ξ. See Chapter 12, Appendix C of [T1] for a proof (and connection to the $\overline{\partial}$-Neumann problem).

Example. If Ω is the unit ball in $\mathbb{R}^n$, with boundary $X = \partial\Omega = S^{n-1}$, then it follows from (4.5)–(4.6) in Chapter 8 of [T1] (which, however, used a different sign convention) that

$$N = \sqrt{-\Delta_X + c_n^2} - c_n, \quad c_n = \frac{n-2}{2}. \tag{2.2.12}$$

As for K, comparison of (2.2.6) and (2.1.16)–(2.1.17) shows that

$$K \in OPS^{-1}(\partial\Omega), \quad \text{and} \quad K = -2S, \quad \text{mod} \quad OPS^{-2}(\partial\Omega). \tag{2.2.13}$$

The operator results (2.2.8) and (2.2.13) allow us to extend the mapping properties (2.1.8) and (2.1.20) considerably. For example, we have L^p-Sobolev results

$$N : H^{s+1,p}(\partial\Omega) \to H^{s,p}(\partial\Omega), \quad K : H^{s,p} \to H^{s+1,p}(\partial\Omega), \quad s \in \mathbb{R},\ p \in (1,\infty), \tag{2.2.14}$$

and also Besov space results, and endpoint results, involving Hardy spaces and bmo.

2.3. e^{-tN} as a symmetric Markov semigroup

The operator $-N$ generates a one-parameter contraction semigroup of self adjoint operators on $L^2(\partial\Omega)$, specified by

$$e^{-tN}\sum_{j\ge 0} c_j\varphi_j = \sum_{j\ge 0} c_j e^{-t\lambda_j}\varphi_j, \quad t>0, \tag{2.3.1}$$

with φ_j and λ_j as in (2.1.23)–(2.1.25). We note that

$$e^{-tN}1\equiv 1. \tag{2.3.2}$$

Another useful formula for e^{-tN} is

$$e^{-tN}f=\lim_{k\to\infty}\Bigl(I+\frac{t}{k}N\Bigr)^{-k}f, \tag{2.3.3}$$

the limit existing in L^2-norm, for each $f\in L^2(\partial\Omega)$.

Our goal here is to establish some positivity results, starting with the following.

Proposition 2.3.1. *For $f\in L^2(\partial\Omega)$,*

$$f\ge 0 \Longrightarrow e^{-tN}f\ge 0, \quad \forall\, t>0. \tag{2.3.4}$$

Proof. By (2.3.3), it suffices to prove that, for $f\ge 0,\ \lambda>0$,

$$(\lambda+N)^{-1}f\ge 0. \tag{2.3.5}$$

It also suffices to prove (2.3.5) for smooth $f>0$. Then, denoting the left side of by g, we have $g\in C^\infty(\partial\Omega)$ satisfying

$$Ng+\lambda g>0, \tag{2.3.6}$$

and if (2.3.5) fails, there exists such g that has a *negative* minimum, say at $x_0\in\partial\Omega$. Let $u=\mathrm{PI}\,g$, so $u\in C^\infty(\overline{\Omega})$. Clearly, $\partial_\nu u(x_0)\le 0$, so $Ng(x_0)\le 0$. But $\lambda>0\Rightarrow \lambda g(x_0)<0$. This contradicts (2.3.6), so the proposition is proved.

It follows that e^{-tN} is a strongly continuous, positivity preserving, contraction semigroup on $C(\partial\Omega)$. By duality, it also yields a weak* continuous contraction semigroup on $\mathcal{M}(\partial\Omega)$, the space of finite (signed) measures on $\partial\Omega$. The following result strengthens Proposition 2.3.1. Recall that we assume Ω is connected, but we do not assume $\partial\Omega$ is connected.

Proposition 2.3.2. *If μ is a positive measure on $\partial\Omega$ and $\mu \neq 0$, then*

$$e^{-tN}\mu(x) > 0, \quad \forall\, x \in \partial\Omega,\ t > 0. \tag{2.3.7}$$

Proof. We know that $f(t,x) = e^{-tN}\mu(x)$ is C^∞ and ≥ 0 on $(0,\infty) \times \partial\Omega$. If (2.3.7) fails, there exists $t_0 > 0$ and $x_0 \in \partial\Omega$ such that $f(t_0, x_0) = 0$. Then $\partial_t f(t_0, x_0) = 0$. Let $u(x) = \mathrm{PI}\, f(t_0, x)$, so $u \in C^\infty(\overline{\Omega})$ and

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad u(x) = f(t_0, x) \text{ for } x \in \partial\Omega. \tag{2.3.8}$$

We have $u \geq 0$ on $\overline{\Omega}$ and $u(x_0) = 0$, so Zaremba's principle implies $\partial_\nu u(x_0) < 0$, hence $Nf(t_0, x_0) < 0$, contradicting the equation $\partial_t f = -Nf$.

Since $\partial\Omega$ can have several connected components, the result (2.3.7) is quite interesting. Here is an illustration, in the simplest possible case. Namely, take

$$\Omega = (0, 1). \tag{2.3.9}$$

Thus $L^2(\partial\Omega) \approx \mathbb{R}^2$, via $f \mapsto (f(0), f(1))^t$. In this case,

$$\mathrm{PI}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1, \quad \mathrm{PI}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x, \tag{2.3.10}$$

so

$$N\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad N\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{2.3.11}$$

hence $N(1,0)^t = (1,-1)^t$, so

$$N = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{2.3.12}$$

and hence

$$e^{-tN} = e^{-t}\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \tag{2.3.13}$$

which clearly satisfies (2.3.7) for $t > 0$. The associated Markov process is a special case of Markov processes treated in Chapter 4 of [Str].

Remark. As will be explained around Proposition 3.3.7, Proposition 2.3.2 implies that $\{e^{-tN} : t \geq 0\}$ is *irreducible.*

2.4. Comparison of e^{-tN} and $e^{-t\Lambda}$

Here we compare the semigroups e^{-tN} and $e^{-t\Lambda}$, where

$$\Lambda = \sqrt{-\Delta_X}, \quad X = \partial\Omega. \tag{2.4.1}$$

As stated in (2.2.10),

$$N = \Lambda + B, \quad B \in OPS^0(\partial\Omega), \tag{2.4.2}$$

and the principal symbol of B is given in (2.2.11). We will construct a smooth family $P(t)$ of elements of $OPS^0(\partial\Omega)$ with the property that

$$e^{-tN} \sim P(t)e^{-t\Lambda}, \tag{2.4.3}$$

in the sense (to be made precise in (2.4.14) below) that the right side of (2.4.3) is an accurate approximation to e^{-tN} as $t \searrow 0$.

To get $P(t)$, we apply d/dt to (2.4.3):

$$-(\Lambda + B)e^{-tN} \sim [-P(t)\Lambda + P'(t)]e^{-t\Lambda}, \tag{2.4.4}$$

suggesting that $P(t)$ satisfy $P'(t) - P(t)\Lambda \sim -(\Lambda + B)P(t)$, hence

$$P'(t) \sim [P(t), \Lambda] - BP(t), \quad P(0) = I. \tag{2.4.5}$$

Thus we construct $P(t)$ as follows. Pick $K \in \mathbb{N}$ (large) and set

$$P(t) = I + P_1 t + \cdots + P_K t^K, \quad P_j \in OPS^0(\partial\Omega), \tag{2.4.6}$$

with the coefficients to be determined. Plugging (2.4.6) into (2.4.5) uniquely specifies P_j, for $1 \le j \le K$, such that

$$\begin{aligned} P'(t) &= [P(t), \Lambda] - BP(t) + R_K(t), \\ R_K(t) &= R_K t^K, \quad R_K \in OPS^0(\partial\Omega). \end{aligned} \tag{2.4.7}$$

For example,

$$P_1 = -B, \quad P_2 = \frac{1}{2}\Big([\Lambda, B] + B^2\Big). \tag{2.4.8}$$

With $P(t)$ so specified, we compare e^{-tN} with $P(t)e^{-t\Lambda}$, using

$$\begin{aligned} \frac{d}{dt}P(t)e^{-t\Lambda} &= -(\Lambda + B)P(t)e^{-t\Lambda} + R_K(t)e^{-t\Lambda}, \\ \frac{d}{dt}e^{-tN} &= -(\Lambda + B)e^{-tN}. \end{aligned} \tag{2.4.9}$$

Thus

$$Q(t) = P(t)e^{-t\Lambda} - e^{-tN} \tag{2.4.10}$$

satisfies

$$Q'(t) = -(\Lambda + B)Q(t) + R_K(t)e^{-t\Lambda}, \quad Q(0) = 0, \tag{2.4.11}$$

so DuHamel's formula gives

$$Q(t) = \int_0^t e^{-(t-s)(\Lambda+B)} R_K(s) e^{-s\Lambda}\, ds. \tag{2.4.12}$$

Now $\{e^{-tN} : t \geq 0\}$ and $\{e^{-t\Lambda} : t \geq 0\}$ are both bounded subsets of $OPS^0_{1,0}(\partial\Omega)$, and furthermore, for $j, k \geq 0$,

$$t^{j+k} e^{-t\Lambda} \ \text{ is } \ O(t^j) \ \text{ in } \ OPS^{-k}_{1,0}(\partial\Omega), \ \text{ as } \ t \searrow 0. \tag{2.4.13}$$

See [T0], Chapter 12. It follows that, for each $j \in \{0, 1, \dots, K\}$,

$$P(t)e^{-t\Lambda} - e^{-tN} \text{ is } O(t^j) \text{ in } OPS^{-(K-j)}_{1,0}(\partial\Omega), \ \text{ as } \ t \searrow 0. \tag{2.4.14}$$

One has parametrix constructions of both e^{-tN} and $e^{-t\Lambda}$, via (a simplified version of) complex geometrical optics; cf. [T0], Chapters 8 and 12. However, one can obtain a parametrix for $e^{-t\Lambda}$ by a more elementary construction, as follows. Picking $p \in X = \partial\Omega$, set

$$u_p(t, x) = (\operatorname{sgn} t) e^{-|t|\Lambda} \delta_p(x). \tag{2.4.15}$$

Then

$$\partial_t u_p(t, x) = -\Lambda e^{-|t|\Lambda} \delta_p(x) + 2\delta(t)\delta_p(x), \tag{2.4.16}$$

and

$$\begin{aligned} \partial_t^2 u_p(t, x) &= (\operatorname{sgn} t)\Lambda^2 e^{-|t|\Lambda} \delta_p(x) + 2\delta'(t)\delta_p(x) \\ &= -\Delta_X u_p(t, x) + 2\delta'(t)\delta_p(x), \end{aligned} \tag{2.4.17}$$

hence

$$(\partial_t^2 + \Delta_X) u_p = 2\partial_t \delta_{(0,p)}. \tag{2.4.18}$$

Consequently, if we fix $T > 0$, we can say that, for $0 < t \leq T$,

$$e^{-t\Lambda} \delta_p(x) = 2\partial_t G_p(t, x), \tag{2.4.19}$$

where $G_p(t,x)$ is specified on $[-T,T]\times X$ as the solution to

$$(\partial_t^2+\Delta_X)G_p=\delta_{(0,p)}, \tag{2.4.20}$$

with Neumann boundary data

$$\partial_t G_p(\pm T,x)=\pm\frac{1}{2}e^{-T\Lambda}\delta_p(x). \tag{2.4.21}$$

These conditions specify G_p uniquely, up to an additive constant, and hence specify u_p uniquely, on $[-T,T]\times\partial\Omega$. Of course, we do not have an explicit formula for the boundary data in (2.4.21), so what we get out of this is that G_p is a solution to (2.4.20) that is even in t, and as such is uniquely determined up to a harmonic function on $[-T,T]\times\partial\Omega$ that is even in t, hence u_p is uniquely specified up to a harmonic function on $[-T,T]\times\partial\Omega$ that is odd in t (hence vanishes at $t=0$).

A standard parametrix construction for $L=\partial_t^2+\Delta_X$ yields, in a coordinate system about $(0,p)$ in $\mathbb{R}\times\partial\Omega$,

$$G_p(z)\sim\sum_{\ell\ge 0}\Bigl(E_{p\ell}(z)+q_{p\ell}(z)\log|z|\Bigr),\quad z=(t,x-p), \tag{2.4.22}$$

where $E_{p\ell}\in C^\infty(\mathbb{R}^n\setminus 0)$ is homogeneous of degree $2-n+\ell$, and $q_{p\ell}$ is a polynomial, homogeneous of degree $2-n+\ell$, which appears only for $\ell\ge n-2$. If one uses an exponential coordinate system centered at p on a neighborhood $\mathcal{O}$ of p in $\partial\Omega$, and then product coordinates on $\mathbb{R}\times\mathcal{O}$, we can identify the leading term and write

$$G_p(t,x)\sim -c_n\bigl(t^2+|x-p|^2\bigr)^{-(n-2)/2}+\cdots,\quad \text{if }\ n\ge 3, \tag{2.4.23}$$

where $c_n=(n-2)\operatorname{Area}S^{n-1}$, and

$$G_p(t,x)\sim\frac{1}{2\pi}\log\bigl(t^2+|x-p|^2\bigr)^{1/2}+\cdots,\quad \text{if }\ n=2. \tag{2.4.24}$$

Consequently,

$$u_p(t,x)\sim c_n' t\bigl(t^2+|x-p|^2\bigr)^{-n/2}+\cdots,\quad c_n'>0. \tag{2.4.25}$$

Given the coordinate system we are using,

$$|x-p|=\operatorname{dist}(x,p), \tag{2.4.26}$$

the geodesic distance in $\partial\Omega$ from p to x.

We next look at the behavior of

$$v_p(t,x)=Qe^{-t\Lambda}\delta_p(x),\quad Q\in OPS^0(\partial\Omega). \tag{2.4.27}$$

The principal term in (2.4.25) contributes to (2.4.27) a positive constant times

$$\int_{\mathbb{R}^{n-1}} q(x,\xi)e^{-t|\xi|}e^{ix\cdot\xi}\,d\xi, \tag{2.4.28}$$

where $q(x,\xi)$ is the principal symbol of Q, which is homogeneous of degree 0 in ξ. This is is an exponential coordinate system centered at p (and here we take $p=0$). This can be rewritten as

$$\begin{aligned} t^{-(n-1)} \int_{\mathbb{R}^{n-1}} & q(x,\xi)e^{-|\xi|}e^{i(x/t)\cdot\xi}\,d\xi \\ &= t^{-(n-1)}\hat{q}_1\Bigl(x,\frac{x}{t}\Bigr), \quad q_1(x,\xi)=q(x,\xi)e^{-|\xi|}, \end{aligned} \tag{2.4.29}$$

where $\hat{q}_1$ denotes the partial Fourier transform of $q_1(x,\xi)$ with respect to the second argument, i.e.,

$$\hat{q}_1(x,y) = \int_{\mathbb{R}^{n-1}} q_1(x,\xi)e^{iy\cdot\xi}\,d\xi. \tag{2.4.30}$$

Since $q_1(x,\xi)$ is smooth and rapidly decreasing in $\xi \in \mathbb{R}^n \setminus 0$, $\hat{q}_1(x,y)$ is smooth on $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$. The singularity of $q_1(x,\xi)$ at $\xi=0$ controls the asymptotic behavior of $\hat{q}_1(x,y)$ as $y\to\infty$, yielding

$$\hat{q}_1(x,y) \in S^{-(n-1)}_{\mathrm{cl}}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}). \tag{2.4.31}$$

In particular, we can bound the absolute value of (2.4.29):

$$\begin{aligned} t^{-(n-1)}\Bigl|\hat{q}_1\Bigl(x,\frac{x}{t}\Bigr)\Bigr| &\le Ct^{-(n-1)}\Bigl(1+\Bigl|\frac{x}{t}\Bigr|\Bigr)^{-(n-1)} \\ &= C\bigl(t+|x|\bigr)^{-(n-1)}. \end{aligned} \tag{2.4.32}$$

This leads to a bound

$$|tP_1e^{-t\Lambda}\delta_p(x)| \le Ct\bigl(t^2+|x-p|^2\bigr)^{-(n-1)/2}, \tag{2.4.33}$$

where we have resurrectd the label p, to exhibit the parallel with (2.4.25). As one should expect, (2.4.33) exhibits a weaker blow-up very near $x=p$ as $t \searrow 0$ than (2.4.25) does. On the other hand, away from $x=p$, say for $|x-p|\ge\delta>0$, both quantities tend to 0 at the same rate. There is a simple explanation of why this must be the case. In fact,

$$\frac{d}{dt}\Bigl(e^{-tN}\delta_p(x)-e^{-t\Lambda}\delta_p(x)\Bigr)\Big|_{t=0} = -(N-\Lambda)\delta_p(x) = -B\delta_p(x), \tag{2.4.34}$$

and $B\delta_p(x)\in C^\infty(\partial\Omega\setminus p)$ is typically not 0, except in special cases in which $B\in OPS^0(\partial\Omega)$ is a local operator. In particular, this would require that the Weingarten map $A_N : T_p(\partial\Omega)\to T_p(\partial\Omega)$ be scalar for all p. Of course, if the principal symbol of B is independent of ξ, then the main singularity at $\xi=0$ of the Fourier integral (2.4.30), with $Q=P_1=-B$, is ameliorated, and arises from the singularity of $e^{-t|\xi|}$ at $\xi=0$ rather than from that of $\sigma_B(x,\xi)$.

We refer to [EO] and [GG] for other approaches to the analysis of $e^{-tN}\delta_p(x)$, and further results on related material.

2.5. e^{-tN^2} need not be positivity preserving

Here we give an example of a Dirichlet-to-Neumann map N with the property that e^{-tN^2} is not positivity preserving for all $t > 0$. In this example, $\Omega = B^n$ is the unit ball in $\mathbb{R}^n$, with boundary $X = \partial B^n = S^{n-1}$, and, as in (2.2.12),

$$N = \sqrt{-\Delta_X + c_n^2} - c_n, \quad c_n = \frac{n-2}{2}. \tag{2.5.1}$$

A calculation gives

$$\begin{aligned} N^2 &= -\Delta_X + 2c_n^2 - 2c_n\sqrt{-\Delta_X + c_n^2} \\ &= -\Delta_X - 2c_n N. \end{aligned} \tag{2.5.2}$$

Hence, for $t > 0$,

$$e^{-tN^2} = e^{t(\Delta_X + 2c_n N)}, \tag{2.5.3}$$

or, equivalently,

$$e^{t\Delta_X} = e^{-tN^2} e^{-2c_n tN}. \tag{2.5.4}$$

We will use this identity together with known behavior of the heat semigroup $e^{t\Delta_X}$ to establish:

Proposition 2.5.1. *Assume $n \geq 3$. Then, for N given by (2.5.1), there exists $t > 0$ such that e^{-tN^2} is not positivity preserving.*

To start the proof, we recall two formulas for the Poisson integral on B^n, solving

$$\Delta u = 0 \text{ on } B^n, \quad u\big|_{\partial B^n} = f. \tag{2.5.5}$$

One is

$$u(r\omega) = r^N f(\omega), \quad \omega \in S^{n-1}, \tag{2.5.6}$$

and the other is

$$u(r\omega) = \frac{1-r^2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{|r\omega - y|^n}\, dS(y). \tag{2.5.7}$$

See [T1], Chapter 8, §4. Let us fix $p \in S^{n-1}$ and take $f = \delta_p$. We have

$$r^N \delta_p(\omega) = \frac{1-r^2}{A_{n-1}} |r\omega - p|^{-n}, \tag{2.5.8}$$

hence

$$\begin{aligned}\min_{\omega\in S^{n-1}} r^N \delta_p(\omega) &= \frac{1-r^2}{A_{n-1}} \min_{\omega\in S^{n-1}} |r\omega - p|^{-n} \\ &= \frac{1-r^2}{A_{n-1}}(1+r)^{-n}.\end{aligned} \tag{2.5.9}$$

It follows that

$$e^{-2c_n tN}\delta_p(\omega) \geq \alpha_n(1-r), \quad \forall\, \omega \in S^{n-1}, \tag{2.5.10}$$

with $\alpha_n = 1/(2^{n-1}A_{n-1})$ and

$$r = e^{-2c_n t}. \tag{2.5.11}$$

Consequently, if e^{-tN^2} is positivity preserving, one has from (2.5.4) that

$$e^{t\Delta_X}\delta_p(\omega) \geq \alpha_n(1-e^{-2c_n t}), \quad \forall\, \omega \in S^{n-1}. \tag{2.5.12}$$

This, however, contradicts the readily established rapid decay of $e^{t\Delta_X}\delta_p(\omega)$ as $t \searrow 0$, when ω is bounded away from p. Thus the positivity preserving property for e^{-tN^2} must fail, at least for $t > 0$ sufficiently small.

Remark. For $n = 2$, we have $c_2 = 0$, hence $N^2 = -\Delta_X$, so e^{-tN^2} is positivity preserving for all $t > 0$ in that case. Furthermore, for $n = 1$, N is given by a 2×2 matrix, somewhat like (2.3.12) (up to a factor of 1/2), and again e^{-tN^2} is positivity preserving.

3. Lipschitz domains

Let M be a compact, connected Riemannian manifold and $\Omega \subset M$ an open subset. Assume $M \setminus \overline{\Omega} \neq \emptyset$. We say Ω is a Lipschitz domain if, for each $p \in \partial\Omega$, there exists a coordinate neighborhoof U of p in M such that $\partial\Omega \cap U$ is the graph of a Lipschitz function, and $\Omega \cap U$ lies on one side of this graph.

Here we analyze the Dirichlet-to-Neumann map on such domains, building on work of [AM] and [BtE]. We make use of fundamental results on the Dirichlet and Neumann boundary problems for the Laplace operator Δ on Ω. Seminal work on these boundary problems was done in [Dah], [JK], and [Ver], the latter making use of the work of [CMM] on singular integral operators on Lipschitz surfaces. These works were done in the setting of Lipschitz domains in Euclidean space. An extension to domains in Riemannian manifolds, with non-flat metric tensors, was accomplished in [MT1]-[MT4]. These papers covered progressively rougher metric tensors, from those that are C^1 in local coordinates in [MT1] to those in [MT4] that have a modulus of continuity satisfying a Dini-type condition in local coordinates, namely

$$|g_{jk}(x) - g_{jk}(y)| \le C\omega(|x-y|), \tag{3.0.1}$$

with

$$\int_0^1 \frac{\sqrt{\omega(t)}}{t}\, dt < \infty. \tag{3.0.2}$$

In §3.1 we use results on the Dirichlet and Neumann problems to construct the Dirichlet-to-Neumann and Neumann-to-Dirichlet maps

$$N : H^1(\partial\Omega) \longrightarrow L^2(\partial\Omega), \quad K : L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega), \tag{3.0.3}$$

inverses to each other on functions that integrate to 0. We show that K is compact and self adjoint on $L^2(\partial\Omega)$, and deduce that N is a positive semidefinite self-adjoint operator on $L^2(\partial\Omega)$, with

$$\mathcal{D}(N) = H^1(\partial\Omega). \tag{3.0.4}$$

The mapping properties (3.0.3) extend to L^p-Sobolev space mapping properties. Namely, there exists $q(\Omega) > 2$ such that

$$N : H^{1,p}(\partial\Omega) \to L^p(\partial\Omega), \quad K : L^p(\partial\Omega) \to H^{1,p}(\partial\Omega), \quad 1 < p < q(\Omega), \tag{3.0.5}$$

and duality and interpolation yield further mapping properties.

In §3.2 we produce formulas that relate N and K to singular integral operators arising from certain single and double layer potentials.

In §3.3 we study the semigroup e^{-tN}, which is a contraction semigroup of self-adjoint operators in $L^2(\partial\Omega)$. Results from §3.1 yield mapping properties

$$e^{-tN} : L^{p_0}(\partial\Omega) \longrightarrow L^{p_1}(\partial\Omega), \quad t > 0,\ 1 < p_0 < p_1 < \infty, \tag{3.0.6}$$

and

$$e^{-tN} : L^{p_0}(\partial\Omega) \longrightarrow H^{1,p_1}(\partial\Omega), \quad t > 0,\ 1 < p_0 < p_1 < q(\Omega). \tag{3.0.7}$$

We then bring in the positivity result, established in [AM], that, given $f \in L^2(\partial\Omega)$,

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0, \quad \forall\, t > 0. \tag{3.0.8}$$

Since $e^{-tN}1 \equiv 1$, this makes $\{e^{-tN}\}$ a symmetric Markov semigroup. On general principles, this implies $\{e^{-tN}\}$ is a strongly continuous contraction semigroup on $L^p(\partial\Omega)$ for $1 \leq p < \infty$. Also, results of [Var] and [EO] yield quantitative estimates on the operator norms of e^{-tN} in (3.0.6), and we supplement these with operator norm estimates on e^{-tN} in (3.0.7).

In §3.4 we consider the spectrum of N, which thanks to (3.0.4) must be discrete, so N has eigenvalues $\lambda_j, j \geq 0$, satisfying $0 = \lambda_0 \leq \lambda_1 \leq \cdots \nearrow +\infty$. Estimates on e^{-tN} from §3.3 imply

$$\sum_{j \geq 0} e^{-t\lambda_j} \leq C(t \wedge 1)^{-(n-1)}, \tag{3.0.8}$$

where $n = \dim \Omega$, which in turn yields estimates on $\sum_{j\geq 0}(\lambda_j + 1)^{-s}$ for $s > n-1$. This series is seen to diverge when $s = n-1$. These results agree with those for the case when $\partial\Omega$ is smooth, though pseudodifferential operator methods from §2 are not available to produce finer asymptotic expansions.

In §3.5 we discuss proofs of the positivity result (3.0.8), describing the approach in [AM], via the Beurling-Deny positivity criterion, and also an alternative approach.

Remark. The quantity $q(\Omega)$ that appears in (3.0.5) and (3.0.7), and elsewhere in this section, depends on the Lipschitz character of $\partial\Omega$ (and also on M and its metric tensor). It is the case that

$$\Omega \text{ is a } C^1 \text{ domain } \Longrightarrow q(\Omega) = \infty. \tag{3.0.9}$$

In §6 we will obtain such results for a more general class of domains, namely regular SKT domains.

3.1. N and K on Lipschitz domains

Here Ω is a connected Lipschitz domain, with nonempty boundary $\partial\Omega$, in a compact Riemannian manifold M. The Poisson integral $\mathrm{PI}\, f$ solves the Dirichlet problem

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = f, \quad u = \mathrm{PI}\, f. \tag{3.1.1}$$

It follows from material in §7 of [MT1] for C^1 metric tensors, and in §5 of [MT4] for metric tensors satisfying (3.0.1)–(3.0.2), that

$$\begin{aligned} f \in H^1(\partial\Omega) \Rightarrow (\nabla u)^* \in L^2(\partial\Omega), \ \text{ and} \\ \lim_{x\to z} \nabla u(x) \text{ exists for a.e. } z \in \partial\Omega. \end{aligned} \tag{3.1.2}$$

Here $(\nabla u)^*$ denotes the nontangential maximal function of ∇u, and the limit is taken nontangentially. This is proven by representing u in (3.1.1) in terms of a single layer potential (cf. (7.32) of [MT1]), together with basic properties of such layer potentials (see (3.2.10)). Consequently, we have

$$Nf = \partial_\nu \, \mathrm{PI}\, f\big|_{\partial\Omega}, \tag{3.1.3}$$

defining

$$N : H^1(\partial\Omega) \longrightarrow L^2(\partial\Omega). \tag{3.1.4}$$

Now if $u = \mathrm{PI}\, f, v = \mathrm{PI}\, g,\ f, g \in H^1(\partial\Omega)$, one has, by Green's formula,

$$(\nabla u, \nabla v) = \int\limits_{\partial\Omega} \partial_\nu u \, \overline{v} \, dS = \langle Nf, g\rangle, \tag{3.1.5}$$

where, as in §2, $(\ ,\ )$ denotes the inner product on $L^2(\Omega)$ and $\langle\ ,\ \rangle$ that on $L^2(\partial\Omega)$. See Appendix B for a discussion of Green's formula in a setting broad enough to apply here.

It follows from (3.1.5) that

$$\langle Nf, g\rangle = \langle f, Ng\rangle, \quad \text{for } \ f, g \in H^1(\partial\Omega). \tag{3.1.6}$$

Then, by duality

$$N : L^2(\partial\Omega) \longrightarrow H^{-1}(\partial\Omega), \tag{3.1.7}$$

and interpolation with (3.1.4) gives

$$N : H^{s+1}(\partial\Omega) \longrightarrow H^s(\partial\Omega), \quad -1 \le s \le 0. \tag{3.1.8}$$

Next, we look at the Neumann boundary problem

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = g. \tag{3.1.9}$$

As shown in §6 of [MT1] for C^1 metric tensors, and in §5 of [MT4] for Dini-type metric tensors, if

$$g \in L^2(\partial\Omega), \quad \int\limits_{\partial\Omega} g\, dS = 0, \tag{3.1.10}$$

this has a solution u, satisfying

$$(\nabla u)^* \in L^2(\partial\Omega), \quad \text{and} \ \lim_{x\to z} \nabla u(x) \text{ exists for a.e. } z \in \partial\Omega, \tag{3.1.11}$$

parallel to (3.1.2). The solution is unique up to an additive constant. We set

$$Kg = u\big|_{\partial\Omega}, \ \text{ normalized so that } \int\limits_{\partial\Omega} Kg\, dS = 0, \tag{3.1.12}$$

parallel to (2.1.11), if g satisfies (3.1.10), and we also set

$$K1 = 0. \tag{3.1.13}$$

Thus we have

$$K : L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega). \tag{3.1.14}$$

Parallel to (2.1.16)–(2.1.17),

$$Kf = Kf^{\#}, \quad NKf = f^{\#}, \quad \forall\, f \in L^2(\partial\Omega), \tag{3.1.15}$$

and

$$KNf = f^{\#}, \quad f \in H^1(\partial\Omega). \tag{3.1.16}$$

Then a calculation parallel to (2.1.18) gives

$$\langle Kf, g\rangle = \langle f, Kg\rangle, \tag{3.1.17}$$

for $f, g \in L^2(\partial\Omega)$. Thus, by duality, we have from (3.1.14) that

$$K : H^{-1}(\partial\Omega) \longrightarrow L^2(\partial\Omega), \tag{3.1.18}$$

and then interpolation with (3.1.14) gives

$$K : H^s(\partial\Omega) \longrightarrow H^{s+1}(\partial\Omega), \quad -1 \le s \le 0. \tag{3.1.19}$$

The identity (3.1.15) then extends to $f \in H^s$ for $-1 \le s \le 0$, and (3.1.16) extends to $f \in H^{s+1}(\partial\Omega)$ for $-1 \le s \le 0$.

Since $K = K^*$, we have from (3.1.14) that

$$K : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \text{ is compact and self adjoint.} \tag{3.1.20}$$

It follows that $L^2(\partial\Omega)$ has an orthonormal basis $\{\varphi_j : j = 0, 1, 2, \dots\}$ satisfying

$$K\varphi_j = \mu_j \varphi_j, \quad \mu_0 = 0, \quad \mu_j \searrow 0 \text{ for } j \ge 1. \tag{3.1.21}$$

From (3.1.17) we have

$$\varphi_j \in H^1(\partial\Omega), \quad \forall\, j. \tag{3.1.22}$$

From (3.1.15)–(3.1.16), we have

$$N\varphi_j = \lambda_j \varphi_j, \quad \lambda_0 = 0, \quad \lambda_j = \frac{1}{\mu_j} \nearrow +\infty \text{ for } j \ge 1. \tag{3.1.23}$$

We see that N is a positive semi-definite self adjoint operator on $L^2(\partial\Omega)$, and, by (3.1.15)–(3.1.16),

$$\mathcal{D}(N) = H^1(\partial\Omega). \tag{3.1.24}$$

Note that interpolation applied to (3.1.24) gives

$$\mathcal{D}(N^{1/2}) = H^{1/2}(\partial\Omega). \tag{3.1.25}$$

As shown in §5 of [MT1] (for C^1 metrics),

$$\mathrm{PI} : H^{1/2}(\partial\Omega) \longrightarrow H^1(\Omega), \tag{3.1.26}$$

complementing the trace result

$$u \in H^1(\Omega) \Longrightarrow u\big|_{\partial\Omega} \in H^{1/2}(\partial\Omega), \tag{3.1.27}$$

valid for Lipschitz domains. In concert with (3.1.5), it follows that the quadratic form

$$Q(f,g) = (\nabla u, \nabla v), \quad u = \mathrm{PI}\, f,\ v = \mathrm{PI}\, g, \tag{3.1.28}$$

with form domain $\mathcal{D}(Q) = H^{1/2}(\partial\Omega)$, gives rise, via the Friedrichs construction, to the non-negative self-adjoint operator N, and that

$$\mathcal{D}(N) = \{f \in \mathcal{D}(Q) : Q(f,g) \le C\|g\|_{L^2(\partial\Omega)},\ \forall\, g \in \mathcal{D}(Q)\}. \tag{3.1.29}$$

This agrees with the characterization of N as a self-adjoint operator given at the end of §2.2 in [AM].

There are also L^p-Sobolev space results for N and K, though not as general as those given in (2.4.14) for the case of smooth boundary. As shown in [MT3] and [MT4], following work on domains in Euclidean space in [Ver] and [DK], one has the following results on the Dirichlet problem (3.1.1) and the Neumann problem (3.1.9). Namely, there exists

$$q(\Omega) > 2, \quad \text{such that, if} \ \ 1 < p < q(\Omega), \tag{3.1.30}$$

then, in (3.1.1), the solution u satisfies

$$(\nabla u)^* \in L^p(\partial\Omega), \quad \text{and} \ \ \lim_{x\to z} \nabla u(x) \text{ exists for a.e. } z \in \partial\Omega, \tag{3.1.31}$$

provided

$$f \in H^{1,p}(\partial\Omega), \tag{3.1.32}$$

and, in (3.1.9), with $\int_{\partial\Omega} g\, dS = 0$, the solution u satisfies (3.1.31) provided

$$g \in L^p(\partial\Omega). \tag{3.1.33}$$

In follows that

$$\begin{aligned} N &: H^{1,p}(\partial\Omega) \longrightarrow L^p(\partial\Omega), \\ K &: L^p(\partial\Omega) \longrightarrow H^{1,p}(\partial\Omega), \quad \text{for} \ \ 1 < p < q(\Omega). \end{aligned} \tag{3.1.34}$$

Then duality arguments, parallel to those used in (3.1.7) and (3.1.18), give

$$\begin{aligned} N &: L^{p'}(\partial\Omega) \longrightarrow H^{-1,p'}(\partial\Omega), \\ K &: H^{-1,p'}(\partial\Omega) \longrightarrow L^{p'}(\partial\Omega), \end{aligned} \tag{3.1.35}$$

for such p, and one can interpolate to get other mapping properties.

The identities (3.1.15)–(3.1.16) extend, to

$$NK = I - P_0 \ \text{ on } \ L^p(\partial\Omega), \quad KN = I - P_0 \ \text{ on } \ H^{1,p}(\partial\Omega), \tag{3.1.36}$$

for $1 < p < q(\Omega)$, where

$$P_0 f = \alpha\langle f, 1\rangle 1, \quad \alpha = (\text{Area } \partial\Omega)^{-1}. \tag{3.1.37}$$

A useful variant is

$$(N+P_0)(K+P_0) = I \ \text{ on } \ L^p(\partial\Omega), \quad (K+P_0)(N+P_0) = I \ \text{ on } \ H^{1,p}(\partial\Omega), \tag{3.1.38}$$

for $1 < p < q(\Omega)$, hence

$$\begin{aligned} N + P_0 : H^{1,p}(\partial\Omega) &\xrightarrow{\approx} L^p(\partial\Omega), \quad \text{with inverse} \\ K + P_0 : L^p(\partial\Omega) &\xrightarrow{\approx} H^{1,p}(\partial\Omega), \quad \text{for } \ 1 < p < q(\Omega). \end{aligned} \tag{3.1.39}$$

It follows that, for each $\lambda \in \mathbb{R}$,

$$N + \lambda : H^{1,p}(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is Fredholm, of index 0,} \tag{3.1.40}$$

for $1 < p < q(\Omega)$. A standard monotonicity argument implies that (for fixed λ) the kernel and cokernel of $(N + \lambda)|_{H^{1,p}(\partial\Omega)}$ are independent of $p \in (1, q(\Omega))$. Clearly Ker $(N + \lambda) = 0$ if $p = 2$ and $\lambda > 0$, so

$$N + \lambda : H^{1,p}(\partial\Omega) \xrightarrow{\approx} L^p(\partial\Omega), \ \text{ for } 1 < p < q(\Omega), \ \lambda > 0, \tag{3.1.41}$$

hence

$$(N + \lambda)^{-1} : L^p(\partial\Omega) \longrightarrow H^{1,p}(\partial\Omega), \quad 1 < p < q(\Omega), \ \lambda > 0. \tag{3.1.42}$$

Parallel to (3.1.35), duality gives

$$(N + \lambda)^{-1} : H^{-1,p'}(\partial\Omega) \longrightarrow L^{p'}(\partial\Omega), \quad 1 < p < q(\Omega), \ \lambda > 0. \tag{3.1.43}$$

Equivalently, the range of p' in (3.1.43) is

$$q'(\Omega) < p' < \infty, \quad \text{with } \ q'(\Omega) < 2. \tag{3.1.44}$$

For later use, we record implications of iterating (3.1.42) and (3.1.43). Fix $\lambda > 0$. Take $q_0 > q'(\Omega)$, $f \in H^{-1,q_0}(\partial\Omega)$. Then

$$(N + \lambda)^{-1} f \in L^{q_0}(\partial\Omega) \subset H^{-1,q_1}(\partial\Omega),$$

with $q_1 > q_0$, given by the Sobolev embedding theorem. Consequently,

$$(N + \lambda)^{-2} f \in L^{q_1}(\partial\Omega) \subset H^{-1,q_2}(\partial\Omega),$$

with $q_2 > q_1$, also given by the Sobolev embedding theorem. Continuing, we get the following.

Proposition 3.1.1. *Given* $q_0 > q'(\Omega),\ p_1 < \infty$*, there exists* $k = k(q_0, p_1, n)$ *such that*

$$(N+\lambda)^{-k} : H^{-1,q_0}(\partial\Omega) \longrightarrow L^{p_1}(\partial\Omega), \tag{3.1.45}$$

for $\lambda > 0$. *A fortiori,*

$$(N+\lambda)^{-k} : L^2(\partial\Omega) \longrightarrow L^{p_1}(\partial\Omega). \tag{3.1.46}$$

By duality, we have the following.

Proposition 3.1.2. *Given* $q_0 > 1,\ p_1 \in [2, q(\Omega))$*, there exists* $k = k(q_0, p_1, n)$ *such that*

$$(N+\lambda)^{-k} : L^{q_0}(\partial\Omega) \longrightarrow H^{1,p_1}(\partial\Omega), \tag{3.1.47}$$

for $\lambda > 0$. *A fortiori,*

$$(N+\lambda)^{-k} : L^{q_0}(\partial\Omega) \longrightarrow L^2(\partial\Omega). \tag{3.1.48}$$

3.2. Connections with singular integral operators

Let $\Omega \subset M$ be a connected Lipschitz domain with nonempty boundary, as in §3.1. If $f \in H^1(\partial\Omega)$, the result (2.2.5), i.e.,

$$\mathcal{D}f(x) - \mathcal{S}Nf(x) = \begin{cases} \mathrm{PI}\, f(x), & x \in \Omega, \\ 0, & x \in M \setminus \Omega, \end{cases} \tag{3.2.1}$$

extends to this setting, with $\mathcal{D}$ and $\mathcal{S}$ defined as in (2.2.1); cf. (7.37) of [MT1]. One continues to have the limiting results (2.2.2), with S as in (2.2.3), but the formula for A in (2.2.3) needs to be rewritten as

$$Af(x) = \mathrm{PV} \int\limits_{\partial\Omega} \partial_{\nu_y} E(x,y) f(y)\, dS(y). \tag{3.2.2}$$

The necessary distinction arises as follows. The kernel $\nabla_y E(x,y)$ has leading part blowing up like $d(x,y)^{-(n-1)}$ as $y \to x$, if $n = \dim M$. Now taking $\partial_{\nu_y} E(x,y) = \langle \nu(y), \nabla_y E(x,y)\rangle$ cancels the leading part of this singularity on $\partial\Omega \times \partial\Omega$ if $\partial\Omega$ is C^2, leading to an easy proof that A is bounded on $L^p(\partial\Omega)$ in such a case. This fails when $\partial\Omega$ is C^1 and fails even more severely when $\partial\Omega$ is merely Lipschitz. The fact that

$$A : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < \infty \tag{3.2.3}$$

when Ω is a Lipschitz domain, and the accompanying maximal function estimate

$$\|(\mathcal{D}f)^*\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{3.2.4}$$

were established in [CMM] in the Euclidean space setting, with $E(x,y) = E(x-y)$. Variable coefficient extensions, valid in the manifold setting, were estabished in [MT1] (with improvements for successively less regular metric tensors in [MT2]–[MT4]). From these results, the boundary trace results (2.2.2) follow, the limits being nontangential limits, existing a.e. on $\partial\Omega$.

In the Lipschitz setting, (2.2.4) fails. We do, however, have

$$S : L^p(\partial\Omega) \longrightarrow H^{1,p}(\partial\Omega), \quad 1 < p < \infty, \tag{3.2.5}$$

and the associated nontangential maximal function estimate

$$\|(\nabla \mathcal{S}f)^*\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{3.2.6}$$

In connection with this, it is shown in §3 of [MT1] that

$$\partial_\nu \mathcal{S} f_\pm(x) = \Bigl(\mp\frac{1}{2}I + A^*\Bigr) f(x), \quad \text{for a.e. } x \in \partial\Omega, \tag{3.2.7}$$

given $f \in L^p(\partial\Omega),\ 1 < p < \infty$, the limit taken nontangentially.

Another important property of S, established in Proposition 7.5 of [MT1] for C^1 metric tensors, and in Theorem 3.5 of [MT4] for Dini-type metric tensors, is that

$$S : L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega) \ \text{ is an isomorphism.} \tag{3.2.8}$$

It follows that the solution to

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = f \in H^1(\partial\Omega), \tag{3.2.9}$$

is given by

$$u = \mathcal{S}(S^{-1}f). \tag{3.2.10}$$

The result (3.1.4) on the map N follows from this, together with (3.2.6)–(3.2.7). That is, one has

$$N : H^1(\partial\Omega) \longrightarrow L^2(\partial\Omega). \tag{3.2.11}$$

Then, applying results related to (3.2.3)–(3.2.4) to (3.2.1), we have (as in (2.4.6)),

$$SNf = \Bigl(-\frac{1}{2}I + A\Bigr) f, \quad f \in H^1(\partial\Omega). \tag{3.2.12}$$

One consequence of (3.2.12) and (3.2.11) is that we can complement (3.2.3) by

$$A : H^1(\partial\Omega) \longrightarrow H^1(\partial\Omega). \tag{3.2.13}$$

Using (3.2.8), we can write

$$N = S^{-1}\Bigl(-\frac{1}{2}I + A\Bigr), \quad \text{on } \ H^1(\partial\Omega). \tag{3.2.14}$$

By comparison, (3.2.10) plus (3.2.7) give

$$N = \Bigl(-\frac{1}{2}I + A^*\Bigr) S^{-1}, \quad \text{on } \ H^1(\partial\Omega). \tag{3.2.15}$$

As seen in Proposition 4.6 of [MT1] for C^1 metric tensors, and in Theorem 3.5 of [MT4] for Dini-type metric tensors,

$$-\frac{1}{2}I + A^* : L^2_0(\partial\Omega) \longrightarrow L^2_0(\partial\Omega) \ \text{ is an isomorphism,} \tag{3.2.16}$$

where $L_0^2(\partial\Omega)$ consists of elements of $L^2(\partial\Omega)$ with mean value 0 over $\partial\Omega$.

In fact, with $g \in L_0^2(\partial\Omega)$, the solution to the Neumann problem (3.1.9), satisfying (3.1.11), is given, up to an additive constant, by

$$u = \mathcal{S}\Bigl(-\frac{1}{2}I + A^*\Bigr)^{-1} g \tag{3.2.17}$$

(cf. [MT1], Theorem 6.1, and [MT4], Theorem 4.2). Hence, consistent with (3.2.15),

$$Kg = (I - P_0)S\Bigl(-\frac{1}{2}I + A^*\Bigr)^{-1} g, \quad g \in L_0^2(\partial\Omega), \tag{3.2.18}$$

where P_0 is the orthogonal projection of $L^2(\partial\Omega)$ onto $L_0^2(\partial\Omega)$.

Remark. Complementing (3.2.8) and (3.2.16), we have isomorphisms

$$\begin{gathered} S : L^p(\partial\Omega) \xrightarrow{\approx} H^{1,p}(\partial\Omega), \quad \text{and} \\ -\frac{1}{2}I + A^* : L_0^p(\partial\Omega) \xrightarrow{\approx} L_0^p(\partial\Omega), \quad \text{for} \;\; 1 < p < q(\Omega), \end{gathered} \tag{3.2.19}$$

with $q(\Omega)$ as in §3.1, as shown in [MT3], Theorems 7.1 and 7.3, and [MT4], Corollary 5.4, for metric tensors that are Lipschitz, or satisfy (3.0.1)–(3.0.2), respectively.

3.3. The semigroup e^{-tN}

As we have seen, N is a positive self-adjoint operator on $L^2(\partial\Omega)$, with domain $\mathcal{D}(N) = H^1(\partial\Omega)$. It follows that

$$e^{-tN} : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{3.3.1}$$

is, for $t \geq 0$, a strongly continuous contraction semigroup of positive self-adjoint operators. For $t > 0$, e^{-tN} has additional mapping properties, not as extensive for general Lipschitz domains as for those that are smoothly bounded, but worth noting. First, if we write

$$e^{-tN} = (N+1)^{-k}\Big[(N+1)^k e^{-tN}\Big], \tag{3.3.2}$$

and note that the factor in brackets is bounded on $L^2(\partial\Omega)$ for each $t > 0$, we deduce from (3.1.46) that

$$e^{-tN} : L^2(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \forall\, t > 0,\ p < \infty. \tag{3.3.3}$$

By duality,

$$e^{-tN} : L^p(\partial\Omega) \longrightarrow L^2(\partial\Omega), \quad \forall\, t > 0,\ p > 1, \tag{3.3.4}$$

and then, via $e^{-tN} = e^{-(t/2)N}e^{-(t/2)N}$,

$$e^{-tN} : L^{p_0}(\partial\Omega) \longrightarrow L^{p_1}(\partial\Omega), \quad t > 0,\ 1 < p_0 < p_1 < \infty. \tag{3.3.5}$$

We can also use (3.3.2) in conjunction with (3.1.47) to obtain

$$e^{-tN} : L^2(\partial\Omega) \longrightarrow H^{1,p_1}(\partial\Omega), \quad \forall\, t > 0,\ p_1 < q(\Omega), \tag{3.3.6}$$

and then use this plus (3.3.4) and $e^{-tN} = e^{-(t/2)N}e^{-(t/2)N}$ to get

$$e^{-tN} : L^{p_0}(\partial\Omega) \longrightarrow H^{1,p_1}(\partial\Omega), \quad t > 0,\ 1 < p_0 < p_1 < q(\Omega). \tag{3.3.7}$$

If $n = \dim \Omega$, we can deduce from (3.3.7) that

$$q(\Omega) > n-1 \Rightarrow e^{-tN} : L^{p_0}(\partial\Omega) \longrightarrow C(\partial\Omega), \quad \forall\, t > 0,\ p_0 > 1, \tag{3.3.8}$$

and then, by duality plus $e^{-tN} = e^{-(t/2)N}e^{-(t/2)N}$,

$$q(\Omega) > n-1 \Rightarrow e^{-tN} : \mathcal{M}(\partial\Omega) \longrightarrow C(\partial\Omega), \quad t > 0, \tag{3.3.9}$$

where $\mathcal{M}(\partial\Omega)$ denotes the space of finite signed measures on $\partial\Omega$. Note that the hypothesis in (3.3.8)–(3.3.9) holds for all Lipschitz domains of dimension $n \leq 3$.

The following positivity result was established in [AM].

Proposition 3.3.1. *Given $f \in L^2(\partial\Omega)$,*

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0, \quad \forall\, t > 0. \tag{3.3.10}$$

See §3.5 for a proof. Given this, and noting that

$$e^{-tN} 1 \equiv 1, \tag{3.3.11}$$

we see that if $f \in L^2(\partial\Omega)$ and $a < b \in \mathbb{R}$,

$$a \leq f \leq b \Longrightarrow a \leq e^{-tN} f \leq b, \tag{3.3.12}$$

hence

$$e^{-tN} : L^\infty(\partial\Omega) \longrightarrow L^\infty(\partial\Omega), \quad t \geq 0, \tag{3.3.13}$$

and

$$\|e^{-tN} f\|_{L^\infty(\partial\Omega)} \leq \|f\|_{L^\infty(\partial\Omega)}. \tag{3.3.14}$$

Note that interpolation with $\|e^{-tN} f\|_{L^2} \leq \|f\|_{L^2}$ gives

$$\|e^{-tN} f\|_{L^p(\partial\Omega)} \leq \|f\|_{L^p(\partial\Omega)}, \quad \text{for } \ 2 \leq p \leq \infty. \tag{3.3.15}$$

By duality,

$$\|e^{-tN} f\|_{L^p(\partial\Omega)} \leq \|f\|_{L^p(\partial\Omega)}, \quad \text{for } \ 1 < p \leq \infty. \tag{3.3.16}$$

We are now prepared to prove the following.

Proposition 3.3.2. *The family $\{e^{-tN} : t \geq 0\}$ is a strongly continuous semigroup on $L^p(\partial\Omega)$ for each $p \in (1, \infty)$.*

Proof. We already have this for $p = 2$. For $p \in (1, 2)$, the result follows from the uniform bound (3.3.16) together with the denseness of $L^2(\partial\Omega)$ in $L^p(\partial\Omega)$ and the strong continuity on $L^2(\partial\Omega)$. For $p \in (2, \infty)$, it follows from the estimate (valid for $f \in L^\infty(\partial\Omega)$)

$$\|e^{-t_j N} f - e^{-tN} f\|_{L^p(\partial\Omega)} \leq \|e^{-t_j N} f - e^{-tN} f\|_{L^2(\partial\Omega)}^{\theta} \|e^{-t_j N} f - e^{-tN} f\|_{L^\infty(\partial\Omega)}^{1-\theta}, \tag{3.3.17}$$

with $\theta = \theta(p) \in (0, 1)$, that strong continuity in L^p-norm holds for $f \in L^\infty(\partial\Omega)$, and then strong continuity for all $f \in L^p(\partial\Omega)$ follows from denseness of $L^\infty(\partial\Omega)$ in $L^p(\partial\Omega)$, together with the uniform operator bounds in (3.3.16).

We next establish the left endpoint case of Proposition 3.3.2.

Proposition 3.3.3. *The family* $\{e^{-tN} : t \ge 0\}$ *is a strongly continuous semigroup on* $L^1(\partial\Omega)$.

Proof. Given $f \in L^2(\partial\Omega),\ t \ge 0$, we have

$$\begin{aligned} \|e^{-tN} f\|_{L^1(\partial\Omega)} &= \sup\{|\langle e^{-tN} f, g\rangle| : \|g\|_{L^\infty(\partial\Omega)} \le 1\} \\ &= \sup\{|\langle f, e^{-tN} g\rangle| : \|g\|_{L^\infty(\partial\Omega)} \le 1\} \\ &\le \sup\{\|f\|_{L^1(\partial\Omega)} \|e^{-tN} g\|_{L^\infty(\partial\Omega)} : \|g\|_{L^\infty(\partial\Omega)} \le 1\} \\ &\le \|f\|_{L^1(\partial\Omega)}, \end{aligned} \tag{3.3.18}$$

the last inequality by the contraction property of e^{-tN} on $L^\infty(\partial\Omega)$. Since $L^2(\partial\Omega)$ is dense in $L^1(\partial\Omega)$, it follows that e^{-tN} has a unique continuous extension

$$e^{-tN} : L^1(\partial\Omega) \longrightarrow L^1(\partial\Omega), \tag{3.3.19}$$

for each $t \ge 0$, and each e^{-tN} has L^1-operator norm ≤ 1. The semigroup property follows from the denseness of $L^2(\partial\Omega)$ in $L^1(\partial\Omega)$, together with the uniform operator norm bounds, as does the strong continuity of the semigroup on $L^1(\partial\Omega)$.

The arguments proving Propositions 3.3.2–3.3.3 apply quite generally when (3.3.10)–(3.3.11) hold, as long as $(\partial\Omega, dS)$ is a finite measure space. The following results make use of properties special to Lipschitz domains, starting with the following consequence of an application of (3.1.41) to Proposition 3.3.2.

Proposition 3.3.4. *The family* $\{e^{-tN} : t \ge 0\}$ *is a strongly continuous semigroup on*

$$H^{1,p}(\partial\Omega), \quad \textit{for each} \ \ p \in (1, q(\Omega)). \tag{3.3.20}$$

A Sobolev embedding argument plus (3.3.14) leads to the following.

Corollary 3.3.5. *The family* $\{e^{-tN} : t \ge 0\}$ *is a strongly continuous contraction semigroup on* $C(\partial\Omega)$, *provided* $\dim \Omega = n$ *and* $q(\Omega) > n - 1$. *In particular, this holds when* $\dim \Omega \le 3$.

From here, a duality argument gives the following.

Corollary 3.3.6. *If* $q(\Omega) > n - 1$, *then*

$$e^{-tN} : \mathcal{M}(\partial\Omega) \longrightarrow \mathcal{M}(\partial\Omega), \quad t \ge 0, \tag{3.3.21}$$

is a weak continuous contraction semigroup.*

We next discuss *irreducibility* of the semigroup $\{e^{-tN}\}$, defined as follows. Given measurable $\Gamma_0 \subset \partial\Omega$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, set $L^2(\Gamma_j) = \{f \in L^2(\partial\Omega) : \operatorname{supp} f \subset \Gamma_j\}$. We say $\{e^{-tN}\}$ is reducible if such Γ_j exist, both with positive measure, and

$$e^{-tN} : L^2(\Gamma_0) \longrightarrow L^2(\Gamma_0), \quad \forall\, t \ge 0. \tag{3.3.22}$$

Note that if (3.3.22) holds, then, by self adjointness, e^{-tN} also preserves the orthogonal complement of $L^2(\Gamma_0)$, which is $L^2(\Gamma_1)$. If no such Γ_j exist, we say $\{e^{-tN}\}$ is irreducible. The next result is contained in Theorem 4.2 of [AM]. See also Proposition 2.2 of [AtE].

Proposition 3.3.7. *Assume that Ω is connected. Then $\{e^{-tN} : t \geq 0\}$ is irreducible.*

Proof. Take measurable $\Gamma_0 \subset \partial\Omega$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, and assume (3.3.22) holds. Set $\chi_j = \chi_{\Gamma_j}$. Using the facts

$$\text{(3.3.23)} \quad \operatorname{supp} e^{-tN}\chi_j \subset \Gamma_j, \quad 0 \leq e^{-tN}\chi_j \leq 1, \quad e^{-tN}\chi_0 + e^{-tN}\chi_1 = e^{tN}1 \equiv 1,$$

we deduce that

$$\text{(3.3.24)} \qquad e^{-tN}\chi_j \equiv \chi_j, \quad \forall\, t \geq 0.$$

It follows that

$$\text{(3.3.25)} \qquad \chi_j \in \mathcal{D}(N) = H^1(\partial\Omega), \quad \text{and} \quad N\chi_j = 0.$$

Then, thanks to (3.1.2), $u = \mathrm{PI}\,\chi_0$ satisfies (3.1.9)–(3.1.11) with $g = 0$. Consequently uniqueness for the Neumann problem implies u is a constant, say $u = c$. Meanwhile, $u|_{\partial\Omega} = \chi_0$. Hence either $c = 0$ and $\chi_0 = 0$ or $c = 1$ and $\chi_1 = 0$, i.e., either Γ_0 or Γ_1 has measure 0. This proves irreducibility.

Remark. In case $\partial\Omega$ is smooth, Proposition 2.3.2 implies irreducibility. It is tempting to ask if Proposition 2.3.2 continues to work for Lipschitz domains, at least in the setting of Corollary 3.3.6 (where we also have (3.3.9)).

We return to mapping properties of e^{-tN}, such as (3.3.3)–(3.3.7), and seek quantitative estimates, particularly on the rate of blow-up as $t \searrow 0$. To start, we have

$$\text{(3.3.26)} \qquad \begin{aligned} \|e^{-tN}f\|_{H^1(\partial\Omega)} &\leq Ce^t\|(N+1)e^{-t(N+1)}f\|_{L^2(\partial\Omega)} \\ &\leq C\frac{e^t}{t}\|f\|_{L^2(\partial\Omega)}, \end{aligned}$$

since $t(N+1)e^{-t(N+1)}$ has uniformly bounded L^2-operator norm. More generally, we have

$$\text{(3.3.27)} \qquad \mathcal{D}((N+1)^s) = [L^2(\partial\Omega), H^1(\partial\Omega)]_s = H^s(\partial\Omega), \quad 0 \leq s \leq 1,$$

hence

$$\text{(3.3.28)} \qquad \|e^{-tN}f\|_{H^s(\partial\Omega)} \leq C\frac{e^t}{t^s}\|f\|_{L^2(\partial\Omega)}, \quad 0 \leq s \leq 1.$$

It is an exercise to improve this for $t \geq 1$, and replace $e^t t^{-s}$ by $(t \wedge 1)^{-s}$. Next, we can use Sobolev embedding results,

$$\text{(3.3.29)} \qquad H^s(\partial\Omega) \subset L^{2(n-1)/(n-1-2s)}(\partial\Omega), \quad 0 \leq s < \frac{n-1}{2}, \ s \leq 1.$$

For example,

$$
\begin{aligned}
(3.3.30) \qquad H^1(\partial\Omega) &\subset L^{2(n-1)/(n-3)}(\partial\Omega), \quad n > 3,\\
H^{1/2}(\partial\Omega) &\subset L^{2(n-1)/(n-2)}(\partial\Omega), \quad n > 2,\\
H^{1/4}(\partial\Omega) &\subset L^{2(n-1)/(n-3/2)}(\partial\Omega), \quad n > 1.
\end{aligned}
$$

From (3.3.28) (with $e^t t^{-s}$ replaced by $(t \wedge 1)^{-s}$) and (3.3.29), one has estimates of the form

$$
(3.3.31) \qquad \|e^{-tN} f\|_{L^p(\partial\Omega)} \le C(t \wedge 1)^{-s} \|f\|_{L^2(\partial\Omega)},
$$

with

$$
(3.3.32) \qquad p = \frac{2(n-1)}{n-1-2s}, \quad \text{provided } \ 0 \le s < \frac{n-1}{2} \text{ and } s \le 1,
$$

or equivalently

$$
(3.3.33) \qquad s = (n-1)\Big(\frac{1}{2} - \frac{1}{p}\Big), \quad \text{provided } \ 0 \le s < \frac{n-1}{2} \text{ and } s \le 1.
$$

Since e^{-tN} satisfies the Markovian conditions (3.3.10)–(3.3.11), results of [Var] and [EO] allow us to extend the scope of (3.3.31) as follows.

Proposition 3.3.8. *For each* $p \in [2, \infty]$,

$$
(3.3.34) \qquad \|e^{-tN} f\|_{L^p(\partial\Omega)} \le C_p (t \wedge 1)^{-(n-1)(1/2-1/p)} \|f\|_{L^2(\partial\Omega)}.
$$

Proof. First we treat the case $n > 2$. As shown in §2 of [Var], with complements in §7, the following are equivalent:

$$
(3.3.35) \qquad \|f\|^2_{L^{2\sigma/(\sigma-1)}(\partial\Omega)} \le C_1 \Big[\langle Nf, f\rangle + \|f\|^2_{L^2(\partial\Omega)} \Big], \quad \forall f \in \mathcal{D}(N),
$$

and

$$
(3.3.36) \qquad \|e^{-tN} f\|_{L^\infty(\partial\Omega)} \le C_2 t^{-\sigma/2} \|f\|_{L^2(\partial\Omega)}, \quad 0 < t \le 1,
$$

as long as $\sigma > 1$. The estimate (3.3.35) is equivalent to

$$
(3.3.37) \qquad \|f\|^2_{L^p(\partial\Omega)} \le \|f\|^2_{H^{1/2}(\partial\Omega)}, \quad p = \frac{2\sigma}{\sigma - 1},
$$

which, by (3.3.30), holds with $\sigma = n - 1$. Consequently, the results of [Var] yield

$$
(3.3.38) \qquad \|e^{-tN} f\|_{L^\infty(\partial\Omega)} \le C t^{-(n-1)/2} \|f\|_{L^2(\partial\Omega)}, \quad 0 < t \le 1,
$$

provided $n > 2$. This plus interpolation gives (3.3.34), since estimates for $t \geq 1$ are elementary.

When $n = 2$, these results of [Var] do not apply. Note that in this case (3.3.31)–(3.3.33) hold for $0 \leq s < 1/2$, and hence we already have (3.3.34) for $2 \leq p < \infty$.

To get the endpoint case $p = \infty$ when $n = 2$, we can follow [EO] and use the fact that (again thanks to the Markov property), estimates of the form (3.3.34) extrapolate, from a given $p = p_0 \in (2, \infty)$ to all $p \in (2, \infty]$, via a result of [Cou].

Here is another approach to the endpoint case $p = \infty$ of (3.3.34) when $n = 2$. In that case, we can use the estimate

$$\|g\|^2_{L^\infty(\partial\Omega)} \leq C \|g\|_{L^2(\partial\Omega)} \|g\|_{H^1(\partial\Omega)}, \quad \dim \partial\Omega = 1. \tag{3.3.39}$$

We get

$$\begin{aligned} \|e^{-tN} f\|^2_{L^\infty(\partial\Omega)} &\leq C \|e^{-tN} f\|_{L^2(\partial\Omega)} \|e^{-tN} f\|_{H^1(\partial\Omega)} \\ &\leq C (t \wedge 1)^{-1} \|f\|^2_{L^2(\partial\Omega)}, \end{aligned} \tag{3.3.40}$$

the last inequality by (3.3.21) (with $e^t t^{-1}$ replaced by $(t \wedge 1)^{-1}$). This gives (3.3.34) for $n = 2,\ p = \infty$.

Let us linger on the case $n = 3$ of Proposition 3.3.8. In that case, the range of s in (3.3.33) is $0 \leq s < 1$, so the range of p in (3.3.32) is $2 \leq p < \infty$. Again, the results of (3.3.31)–(3.3.33) cover all of (3.3.34) except the endpoint case, $p = \infty$. We will provide an alternative treatment of this case shortly, after looking into $H^{1,p}(\partial\Omega)$-estimates on $e^{-tN} f$, for $p \in (2, q(\Omega))$.

We resume study of the general case $\dim \Omega = n$. We have, for $2 < p < q(\Omega)$,

$$\begin{aligned} \|e^{-tN} f\|_{H^{1,p}(\partial\Omega)} &\leq C \|(N+1) e^{-tN} f\|_{L^p(\partial\Omega)} \\ &= C \|e^{-tN} (N+1) f\|_{L^p(\partial\Omega)} \\ &\leq C (t \wedge 1)^{-(n-1)(1/2-1/p)} \|(N+1) f\|_{L^2(\partial\Omega)} \\ &\leq C (t \wedge 1)^{-(n-1)(1/2-1/p)} \|f\|_{H^1(\partial\Omega)}, \end{aligned} \tag{3.3.41}$$

the third line by (3.3.34). Replacing f by $e^{-tN} f$ and rescaling t, we have

$$\begin{aligned} \|e^{-tN} f\|_{H^{1,p}(\partial\Omega)} &\leq C (t \wedge 1)^{-(n-1)(1/2-1/p)} \|e^{-(t/2)N} f\|_{H^1(\partial\Omega)} \\ &\leq C (t \wedge 1)^{-1-(n-1)(1/2-1/p)} \|f\|_{L^2(\partial\Omega)}, \end{aligned} \tag{3.3.42}$$

for $2 < p < q(\Omega)$.

Returning to the case $n = \dim \Omega = 3$, we bring in the following Gagliardo-Nirenberg type inequality:

$$\|g\|_{L^\infty(\partial\Omega)} \leq C \|g\|^{1-2/p}_{L^p(\partial\Omega)} \|g\|^{2/p}_{H^{1,p}(\partial\Omega)}, \quad \dim \partial\Omega = 2,\ p > 2. \tag{3.3.43}$$

Taking $p \in (2, q(\Omega))$ and $g = e^{-tN} f$, we have

$$
\begin{aligned}
&\|e^{-tN} f\|_{L^\infty(\partial\Omega)} \\
&\le C\|e^{-tN} f\|_{L^p(\partial\Omega)}^{1-2/p} \|e^{-tN} f\|_{H^{1,p}(\partial\Omega)}^{2/p} \\
&\le C(t\wedge 1)^{-(n-1)(1/2-1/p)(1-1/p)} (t\wedge 1)^{-(2/p)(1+(n-1)(1/2-1/p))} \|f\|_{L^2(\partial\Omega)} \\
&= C(t\wedge 1)^{-1} \|f\|_{L^2(\partial\Omega)},
\end{aligned}
\tag{3.3.44}
$$

thereby yielding another proof of (3.3.34) for $p = \infty,\ n = 3$.

3.4. On the spectrum of N

Since $\mathcal{D}(N) = H^1(\partial\Omega)$, we know that N has compact resolvent and discrete spectrum. As in §3.1, we have an orthonormal basis $\{\varphi_j\}$ of $L^2(\partial\Omega)$ consisting of eigenfunctions of N with

$$N\varphi_j = \lambda_j \varphi_j, \quad 0 \le \lambda_j \nearrow +\infty. \tag{3.4.1}$$

In case $\partial\Omega$ is smooth, the fact that N is an elliptic pseudodifferential operator in $OPS^1(\partial\Omega)$ yields strong results on the behavior of λ_j as $j \nearrow +\infty$. Namely, we have

$$\lambda_j \sim \alpha j^{1/(n-1)} \quad \text{as} \quad j \nearrow +\infty, \tag{3.4.2}$$

where, recall $n = \dim \Omega$. One way to get this is to construct a parametrix for e^{-tN} and show that

$$\operatorname{Tr} e^{-tN} \sim A t^{-(n-1)}, \quad t \searrow 0, \tag{3.4.3}$$

with

$$A = (\text{Area } S^{n-2})\,(\text{Area } \partial\Omega). \tag{3.4.4}$$

Alternatively, one can use wave equation techniques to get finer results.

When Ω is a general Lipschitz domain, such an analysis is not available. Our goal here is to derive results on how $\lambda_j \nearrow +\infty$ that, while weaker than (3.4.2), have common features.

We start with results on $\operatorname{Tr} e^{-tN}$ that are derivable from (3.3.34), with $p = \infty$, which allows us to write

$$e^{-tN} f(x) = \int_{\partial\Omega} K_t(x,y) f(y)\, dS(y), \tag{3.4.5}$$

with

$$\int_{\partial\Omega} |K_t(x,y)|^2\, dS(y) \le C(t \wedge 1)^{-(n-1)}. \tag{3.4.6}$$

It follows that the Hilbert-Schmidt norm of e^{-tN} has the estimate

$$\|e^{-tN}\|_{HS}^2 \le C t^{-(n-1)}, \quad t \le 1, \tag{3.4.7}$$

and hence

$$\operatorname{Tr} e^{-tN} = \|e^{-(t/2)N}\|_{HS}^2 \le Ct^{-(n-1)}, \quad t \le 1, \tag{3.4.8}$$

which is a good estimate, in light of the result (3.4.3) for the smooth case. In terms of the eigenvalues $\{\lambda_j\}$, this becomes

$$\sum_{j\ge 0} e^{-t\lambda_j} \le C(t \wedge 1)^{-(n-1)}. \tag{3.4.9}$$

If we multiply both sides of (3.4.9) by $e^{-t}t^{s-1}$ and integrate over $t \in (0,\infty)$, we get

$$\sum_{j\ge 0} (\lambda_j + 1)^{-s} \le C + \frac{C}{s-(n-1)}, \quad s > n-1. \tag{3.4.10}$$

Furthermore, one can readily show that $\operatorname{Tr}(N+1)^{-s}$ is holomorphic in the half plane $\operatorname{Re} s > n-1$. When $\partial\Omega$ is smooth, (3.4.3), supplemented by the next term in the asymptotic expansion, implies a stronger result, namely $\operatorname{Tr}(N+1)^{-s}$ is meromorphic in the half space $\operatorname{Re} s > n-2$, with one pole, at $s = n-1$. For general Lipschitz domains, we do not establish such a result, but we will prove the following.

Proposition 3.4.1. *When Ω is an n-dimensional Lipschitz domain,*

$$\sum_{j\ge 0} (\lambda_j + 1)^{-(n-1)} = +\infty. \tag{3.4.11}$$

To show this, we bring in the space $\mathcal{I}_p(\partial\Omega)$ of bounded linear operators B on $L^2(\partial\Omega)$ satisfying

$$\operatorname{Tr}(B^*B)^{p/2} < \infty. \tag{3.4.12}$$

The content of (3.4.11) is that

$$(N+1)^{-1} \notin \mathcal{I}_{n-1}(\partial\Omega). \tag{3.4.13}$$

By comparison, (3.4.10) implies

$$(N+1)^{-1} \in \mathcal{I}_p(\partial\Omega), \quad \forall\, p > n-1. \tag{3.4.14}$$

Thus, Proposition 3.4.1 is a consequence of the following.

Proposition 3.4.2. *Let Ω be an n-dimensional Lipschitz domain. If $B \in \mathcal{L}(L^2(\partial\Omega))$ and*

$$B : L^2(\partial\Omega) \xrightarrow{\approx} H^1(\partial\Omega), \tag{3.4.15}$$

then

$$B \in \mathcal{I}_p(\partial\Omega) \Longleftrightarrow p > n-1. \tag{3.4.16}$$

Proof. There exists a smooth compact $(n-1)$-dimensional manifold X that is bi-Lipschitz equivalent to $\partial\Omega$. Under this equivalence, operators in $\mathcal{L}(L^2(\partial\Omega))$ are taken to operators in $\mathcal{L}(L^2(X))$. This correspondence takes $\mathcal{I}_p(\partial\Omega)$ isomorphically to $\mathcal{I}_p(X)$, and since the bi-Lipschitz map takes $H^1(\partial\Omega)$ isomorphically to $H^1(X)$, it takes operators B satisfying (3.4.15) to $\widetilde{B} : L^2(X) \xrightarrow{\approx} H^1(X)$. Note that

$$B \in \mathcal{I}_p(\partial\Omega) \Longleftrightarrow \widetilde{B} \in \mathcal{I}_p(X). \tag{3.4.17}$$

Now take a positive, self-adjoint $\Lambda \in OPS^1(X)$. By the discussion above,

$$(\Lambda+1)^{-1} \in \mathcal{I}_p(X) \Longleftrightarrow p > n-1.$$

Also $(\Lambda+1)^{-1} : L^2(X) \xrightarrow{\approx} H^1(X)$. It follows that, if B satisfies (3.4.15),

$$(\Lambda+1)^{-1} = \widetilde{B}V, \quad V \in \mathcal{L}(L^2(X)), \text{ invertible.} \tag{3.4.18}$$

Now $\mathcal{I}_p(X)$ is a two-sided ideal in $\mathcal{L}(L^2(X))$ (cf. [Si]), so (3.4.18) implies $(\Lambda+1)^{-1} \in \mathcal{I}_p(X)$ if and only if $\widetilde{B} \in \mathcal{I}_p(X)$. This proves (3.4.16).

3.5. Approaches to the proof of Proposition 3.3.1

Proposition 3.3.1, from [AM], asserts that if $\Omega \subset M$ is a Lipschitz domain, then, for $f \in L^2(\partial\Omega)$,

$$f \ge 0 \Longrightarrow e^{-tN} f \ge 0, \quad \forall\, t > 0. \tag{3.5.1}$$

If $\partial\Omega$ is smooth, this result has a simple proof, as seen in §2.3, but the proof given there does not work on general Lipschitz domains. Here we discuss the more subtle argument used in [AM], and some variants.

A key tool used in the proof is the following Beurling-Deny positivity criterion. Let H be a positive semidefinite self-adjoint operator on $L^2(\mathfrak{X}, \mu)$, and set

$$Q(f,g) = \langle H^{1/2} f, H^{1/2} g\rangle, \quad f, g \in \mathcal{D}(Q) = \mathcal{D}(H^{1/2}). \tag{3.5.2}$$

Here $\langle\ ,\ \rangle$ denotes the inner product in $L^2(\mathfrak{X}, \mu)$. Assume

$$\begin{aligned} f \in \mathcal{D}(Q) \Longrightarrow |f| \in \mathcal{D}(Q),&\ \text{and} \\ Q(|f|, |f|) &\le Q(f,f). \end{aligned} \tag{3.5.3}$$

Then

$$f \in L^2(\mathfrak{X}, \mu),\ f \ge 0 \Longrightarrow e^{-tH} f \ge 0. \tag{3.5.4}$$

See [Dav], Theorem 1.3.2.

In the current setting, where $\mathfrak{X} = \partial\Omega$ is the boundary of a Lipschitz domain and $H = N$, we have $\mathcal{D}(Q) = H^{1/2}(\partial\Omega)$, by (3.1.25), and the first condition in (3.5.3) is

$$f \in H^{1/2}(\partial\Omega) \Longrightarrow |f| \in H^{1/2}(\partial\Omega). \tag{3.5.5}$$

This is a special case of the result of [CW] that if $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, with Lipschitz constant K, and $\Phi(0) = 0$, then, for $s \in (0,1),\ p \in (1,\infty)$,

$$\|\Phi \circ f\|_{H^{s,p}(\partial\Omega)} \le C_{sp} K \|f\|_{H^{s,p}(\partial\Omega)}. \tag{3.5.6}$$

See also [T2], Chapter 2, §4.

It is useful to know that one can replace the hypothesis (3.5.3) by the following, which can be easier to check. Namely, let $V \subset \mathcal{D}(H^{1/2})$ be a dense linear subspace, and assume that

$$f \in V \Longrightarrow |f| \in \mathcal{D}(H^{1/2}) \ \text{ and } \ Q(|f|, |f|) \le Q(f,f). \tag{3.5.7}$$

Then (3.5.4) holds. See Lemma 1.3.4 of [Dav].

In light of this result, we see that, to establish (3.5.1), it suffices to show that

$$f \in \mathrm{Lip}(\partial\Omega) \Longrightarrow \langle N|f|, |f|\rangle \le \langle Nf, f\rangle. \tag{3.5.8}$$

To restate this condition, let us set

$$f = f^+ - f^-, \quad |f| = f^+ + f^-, \quad f^+ = \max(f, 0). \tag{3.5.9}$$

Then

$$\begin{aligned} Q(|f|, |f|) &= Q(f^+ + f^-, f^+ + f^-) \\ &= Q(f^+, f^+) + Q(f^-, f^-) + 2Q(f^+, f^-) \\ &= Q(f, f) + 4Q(f^+, f^-), \end{aligned} \tag{3.5.10}$$

so

$$Q(|f|, |f|) \le Q(f, f) \Longleftrightarrow Q(f^+, f^-) \le 0. \tag{3.5.11}$$

Hence, to establish (3.5.1), it suffices to show that

$$f \in \mathrm{Lip}(\partial\Omega) \Longrightarrow \langle Nf^+, f^-\rangle \le 0. \tag{3.5.12}$$

Here is one further simplifying construction. Define $\Phi_j : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \Phi_j(s) = \quad & 0 \qquad \text{for} \quad |s| \le \frac{1}{j}, \\ & s - \frac{1}{j} \quad \text{for} \quad s \ge \frac{1}{j}, \\ & s + \frac{1}{j} \quad \text{for} \quad -s \ge \frac{1}{j}. \end{aligned} \tag{3.5.13}$$

Given $f \in \mathrm{Lip}(\partial\Omega)$, $f_j = \Phi_j \circ f$ is a bounded family of Lipschitz functions, and we have both $f_j \to f$ and $|f_j| \to |f|$ in $H^1(\partial\Omega)$-norm. (See [T2], Chapter 2, Proposition 6.5.) Therefore, (3.5.8) follows from $\langle N|f_j|, |f_j|\rangle \le \langle Nf_j, f_j\rangle$, and hence from $\langle Nf_j^+, f_j^-\rangle \le 0$. We have demonstrated the following.

Lemma 3.5.1. *Let $\Omega \subset M$ be a Lipschitz domain. Assume you know that $\langle Nf^+, f^-\rangle \le 0$ whenever $f^+, f^- \in \mathrm{Lip}(\partial\Omega)$ satisfy*

$$f^+, f^- \ge 0, \quad \textit{and are supported on disjoint compact subsets of } \partial\Omega. \tag{3.5.14}$$

Then the positivity result (3.5.1) holds.

To proceed, assume

$$f^+ \in \mathrm{Lip}(\partial\Omega), \quad f^+ \ge 0, \quad f^+ = 0 \ \text{ on } \mathcal{O}, \tag{3.5.15}$$

where $\mathcal{O} \subset \partial\Omega$ is open. Set

$$u^+ = \mathrm{PI}\, f^+, \tag{3.5.16}$$

so (3.1.2) applies to u^+. We also have

$$u^+(x) \geq 0, \quad \forall\, x \in \Omega, \quad u^+\Big|_{\mathcal{O}} = 0, \tag{3.5.17}$$

the latter in the sense of a nontangential trace, a.e. on $\mathcal{O}$. These results lead to

$$\partial_\nu u^+\Big|_{\mathcal{O}} \leq 0, \quad \text{a.e.}, \tag{3.5.18}$$

which implies $\langle Nf^+, f^-\rangle \leq 0$ in the setting of Lemma 3.5.1.

The proof of (3.5.1) in [AM] did not make use of reductions such as (3.5.8). Rather, it took $f \in \mathcal{D}(Q)$ and verified (3.5.3), in the form (3.5.11), as follows. To fix notation, we set

$$u = \mathrm{PI}\, f, \quad u^\pm = \mathrm{PI}\, f^\pm, \tag{3.5.19}$$

and we also write

$$u = u^p - u^m, \quad u^p = \max(u, 0). \tag{3.5.20}$$

Thus $u^p, u^m \in H^1(\Omega)$ and $u^p|_{\partial\Omega} = f^+$, $u^m|_{\partial\Omega} = f^-$. As noted in [AM], this directly gives $f^\pm \in \mathcal{D}(Q)$. We also set

$$u^0 = u^+ - u^p, \quad \text{so} \;\; u^0 \in H^1_0(\Omega), \;\; \Delta u^0 = -\Delta u^p \in H^{-1}(\Omega). \tag{3.5.21}$$

Note that $u^m - u^-$ has the same properties, so

$$u^m - u^- = u^0. \tag{3.5.22}$$

We now compute:

$$\begin{aligned} Q(f^+, f^-) &= (\nabla u^+, \nabla u^-) \\ &= (\nabla u^p + \nabla u^0, \nabla u^m - \nabla u^0) \\ &= (\nabla u^p, \nabla u^m) + (\nabla u, \nabla u^0) - \|\nabla u^0\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.5.23}$$

Now

$$(\nabla u^p, \nabla u^m) = 0 \;\; \text{and} \;\; (\nabla u, \nabla u^0) = 0, \tag{3.5.24}$$

so

$$Q(f^+, f^-) = -\|\nabla u^0\|^2_{L^(\Omega)} \leq 0, \tag{3.5.25}$$

as desired.

One advantage of this argument of [AM] over the reduction argument, via Lemma 3.5.1, is its extendability to rougher settings, as pursued in [AtE]. We discuss this further in §4.3.

4. Finite perimeter domains

Let M be a compact, connected, n-dimensional Riemannian manifold and $\Omega \subset M$ a nonempty open subset. We will assume

$$M \setminus \overline{\Omega} \neq \emptyset. \tag{4.0.1}$$

We say Ω is a finite perimeter domain if and only if its characteristic function χ_Ω has the property that $\nabla \chi_\Omega$, defined a priori as a distribution on M, is a finite (vector valued) measure. In such a case, we can write

$$\nabla \chi_\Omega = -\nu \sigma, \tag{4.0.2}$$

where σ is a positive measure on $\partial\Omega$ and $\nu(x) \in T_x M$ satisfies $|\nu(x)| = 1$ for σ-a.e. x. A useful characterization is that Ω is a finite perimeter domain if and only if

$$\mathcal{H}^{n-1}(\partial_* \Omega) < \infty, \tag{4.0.3}$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure and $\partial_*\Omega$ is the measure-theoretic boundary of Ω, consisting by definition of all points $p \in \partial\Omega$ at which both Ω and $M \setminus \Omega$ have positive density. It is an important structural result that if Ω has finite perimeter, then $\partial_*\Omega$ is countably rectifiable and

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial_* \Omega. \tag{4.0.4}$$

Proofs of these results can be found in Chapter 5 of [EG], in the case of domains in Euclidean space, with complements in [HMT] for domains in Riemannian manifolds. One consequence of these results is that Ω is a finite perimeter domain whenever $\mathcal{H}^{n-1}(\partial\Omega) < \infty$. On the other hand, there are finite perimeter domains for which $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = \infty$.

An important property of finite perimeter domains is the divergence formula, derived as follows. Let X be a smooth vector field on M. Taking the inner product (in the distributional sense) of both sides of (4.0.2) with X and passing from ∇, acting on χ_Ω, to its adjoint, $-\operatorname{div}$, acting on X, yields

$$\int_\Omega \operatorname{div} X \, dV = \int_{\partial_*\Omega} \langle \nu, X \rangle \, d\sigma. \tag{4.0.5}$$

A straightforward approximation argument (cf. [HMT], §2.2 and §5.3) extends the scope of this identity from vector fields $X \in C^\infty(M)$ to

$$X \in C(M) \ \text{ such that } \ \operatorname{div} X \in L^1(M). \tag{4.0.6}$$

In particular, (4.0.5) holds for vector fields $X \in \mathrm{Lip}(\overline{\Omega})$.

In this chapter we pick a measure μ on $\partial\Omega$ such that $\mu \geq \sigma$ and construct the Dirichlet-to-Neumann map N as a positive semidefinite self-adjoint operator on $L^2(\partial\Omega, \mu)$, following [AtE], which considered

$$\Omega \subset \mathbb{R}^n, \quad \mathcal{H}^{n-1}(\partial\Omega) < \infty, \quad \mu = \mathcal{H}^{n-1} \lfloor \partial\Omega. \tag{4.0.7}$$

The approach of [AtE] involved a generalization of the Friedrichs method of producing a positive self-adjoint operator from a quadratic form. We give an abstract version of such a result in Appendix A of this work. Another important ingredient in [AtE] is an estimate of the L^2-norm of a function u on Ω in terms of the L^2-norm of ∇u on Ω and the L^2-norm of u on $\partial\Omega$, due to [Maz], in case $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}^{n-1}(\partial\Omega) < \infty$. We give an extension of this result in §4.1, and proceed in §4.2 to use this to implement the general method described in Appendix A to define N. A key object in the construction is the space $W^\#$, the closure in $H^1(\Omega) \oplus L^2(\partial\Omega, \mu)$ of $\{(u, u|_{\partial\Omega}) : u \in \mathrm{Lip}(\overline{\Omega})\}$. The restriction map $\tau : \mathrm{Lip}(\overline{\Omega}) \to \mathrm{Lip}(\partial\Omega)$ given by $\tau u = u|_{\partial\Omega}$ gives rise to a continuous map $\tau^\# : W^\# \to L^2(\partial\Omega, \mu)$, and we have

$$\mathcal{D}(N^{1/2}) = \tau^\#(W^\#) \subset L^2(\partial\Omega, \mu). \tag{4.0.8}$$

Also $\mathcal{D}(N^{1/2})$ is the isomorphic image under $\tau^\#$ of the space $V \subset W^\#$, the orthogonal complement of $\operatorname{Ker} \tau^\#$ in $W^\#$, with respect to the inner product

$$\beta\big((u, f), (v, g)\big) = \int_\Omega \langle \nabla u, \nabla v \rangle \, dV + \int_{\partial\Omega} fg \, d\mu. \tag{4.0.9}$$

In §4.3 we consider e^{-tN} and show that this is a symmetric Markov semigroup. As in [AtE], the crux is to show that the Beurling-Deny criterion is applicable.

In §4.4 we consider the Poisson integral. We define

$$\mathrm{PI} : \mathcal{D}(N^{1/2}) \longrightarrow H^1(\Omega) \tag{4.0.10}$$

by

$$\mathrm{PI}\, f = u, \quad \text{where} \quad (u, f) \in V. \tag{4.0.11}$$

We note that $C^\infty(\partial\Omega)$ is dense in $\mathcal{D}(N^{1/2})$, and compare $\mathrm{PI}\,|_{C^\infty(\partial\Omega)}$ as defined above with a common alternative, described in various places, including [GT] and [T1]. We see that the two maps coincide provided $\overset{\circ}{H}{}^1(\Omega) = H^1_0(\Omega)$, where $H^1_0(\Omega)$ is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, and

$$\overset{\circ}{H}{}^1(\Omega) = \{u \in H^1(\Omega) : (u, 0) \in W^\#\}. \tag{4.0.12}$$

Issues such as whether $\overset{\circ}{H}{}^1(\Omega) = H^1_0(\Omega)$, whether $\mathrm{Lip}(\overline{\Omega})$ is dense in $H^1(\Omega)$, and others, which can be regarded as regularity conditions on Ω, are examined in §4.5.

In §§4.1–4.5, we assume M carries a C^∞ metric tensor. In §4.6 we extend our analysis of finite perimeter $\Omega \subset M$ to cases where the metric tensor on M is rough.

4.1. A Sobolev space estimate

As in the introduction to this section, we assume $\Omega \subset M$ is a finite perimeter domain satisfying (4.0.1). Our goal is to prove the following.

Proposition 4.1.1. *There exists $C = C(\Omega) < \infty$ such that, for all $u \in \mathrm{Lip}(\overline{\Omega})$,*

$$\int_{\Omega} |u|^2 \, dV \le C \int_{\Omega} |\nabla u|^2 \, dV + C \int_{\partial_* \Omega} |u|^2 \, d\sigma. \tag{4.1.1}$$

A result of this nature, taken from [Maz], was used in [AtE], for a more restricted class of domains Ω. There it was assumed that $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, and the boundary integral was taken to be $\int_{\partial\Omega} |u|^2 \, d\mathcal{H}^{n-1}$.

Proof of Proposition 4.1.1. We use the divergence formula (4.0.5), valid for vector fields $X \in \mathrm{Lip}(\overline{\Omega})$, taking

$$X = |u|^2 Y, \quad Y \in C^\infty(M), \quad \operatorname{div} Y = 1 \text{ on } \overline{\Omega}, \tag{4.1.2}$$

which can be arranged, given (4.0.1). For example, take $Y = \nabla v$ with $\Delta v = 1$ on a neighborhood of $\overline{\Omega}$. From (4.1.2) we have

$$\operatorname{div} X = |u|^2 \operatorname{div} Y + 2u\langle \nabla u, Y\rangle, \tag{4.1.3}$$

and (4.0.5) yields

$$\int_{\Omega} |u|^2 \, dV = \int_{\partial_*\Omega} |u|^2 \langle \nu, Y\rangle \, d\sigma - 2\int_{\Omega} u\langle \nabla u, Y\rangle \, dV. \tag{4.1.4}$$

Now

$$2\Big|\int_{\Omega} u\langle \nabla u, Y\rangle \, dV\Big| \le \frac{1}{2}\int_{\Omega} |u|^2 \, dV + 2\int_{\Omega} \langle \nabla u, Y\rangle^2 \, dV, \tag{4.1.5}$$

so (4.1.4) yields

$$\frac{1}{2}\int_{\Omega} |u|^2 \, dV \le 2\int_{\Omega} \langle \nabla u, Y\rangle^2 \, dV + \int_{\partial_*\Omega} |u|^2 \langle \nu, Y\rangle \, d\sigma, \tag{4.1.6}$$

and this implies (4.1.1).

4.2. Construction of N

Let $\Omega \subset M$ be a finite perimeter domain, as in §4.1. We continue to assume (4.0.1). Adapting arguments from [AtE], we will construct the Dirichlet-to-Neumann map N as a positive semidefinite, self-adjoint operator on a Hilbert space H of the form

$$H = L^2(\partial\Omega, \mu), \tag{4.2.1}$$

where μ is a finite measure on $\partial\Omega$ satisfying

$$\mu \geq \sigma, \tag{4.2.2}$$

with σ as in (4.0.2) and (4.0.4). In case $\partial\Omega = \partial_*\Omega$, we are happy to take $\mu = \sigma$, but if $\partial\Omega \setminus \partial_*\Omega$ has big pieces, we might want to add a measure supported on this complement. For example, if $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ but $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) > 0$, we could take $\mu = \mathcal{H}^{n-1}\lfloor\partial\Omega$, as is done in [AtE]. Note that (4.2.2) combined with Proposition 4.1.1 yields

$$\int_\Omega |u|^2 \, dV \leq C \int_\Omega |\nabla u|^2 \, dV + C \int_{\partial\Omega} |u|^2 \, d\mu, \tag{4.2.3}$$

for all $u \in \mathrm{Lip}(\overline{\Omega})$.

We will construct N via the general program described in Appendix A. Ingredients include, in addition to H in (4.2.1), a space W_0 and maps τ and α, as in (A.14)–(A.15). We take

$$W_0 = \mathrm{Lip}(\overline{\Omega}), \quad \tau u = u\big|_{\partial\Omega}. \tag{4.2.4}$$

Clearly the range of τ is dense in $C(\partial\Omega)$, which in turn is dense in $L^2(\partial\Omega, \mu)$, so (A.14) holds. Next, we take

$$\alpha : W_0 \times W_0 \longrightarrow \mathbb{R}, \quad \alpha(u, v) = \int_\Omega \langle \nabla u, \nabla v \rangle \, dV. \tag{4.2.5}$$

Elements of $\mathrm{Ker}\,\alpha$ are constant on each connected component of Ω, so clearly $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\tau = 0$, giving (A.16). Now, as in (A.17), we set

$$\begin{aligned} \beta(u, v) &= \alpha(u, v) + (\tau u, \tau v)_H \\ &= \int_\Omega \langle \nabla u, \nabla v \rangle \, dV + \int_{\partial\Omega} uv \, d\mu. \end{aligned} \tag{4.2.6}$$

Thanks to (4.2.3), this is a positive-definite inner product on W_0, and as in (A.18), we denote by $W^{\#}$ its Hilbert space completion, to which τ extends continuously:

$$\tau^{\#} : W^{\#} \longrightarrow H. \tag{4.2.7}$$

Note that, by (4.2.3), a sequence (u_k) in $\mathrm{Lip}(\overline{\Omega})$ that is Cauchy in the norm defiuned by β is also Cauchy in $L^2(\Omega)$, hence in $H^1(\Omega)$, while simultaneously $(\tau u_k) = (u_k|_{\partial\Omega})$ is Cauchy in $L^2(\partial\Omega, \mu)$. Thus we have a natural map

$$\pi_\Omega : W^{\#} \longrightarrow H^1(\Omega), \tag{4.2.8}$$

such that if (u_k) is such a Cauchy sequence, $u_k \to U$ in $W^{\#}$, $u_k \to u$ in $H^1(\Omega)$, and $\tau u_k \to f$ in $L^2(\partial\Omega, \mu)$, then

$$\pi_\Omega U = u, \quad \tau^{\#} U = f, \tag{4.2.9}$$

and

$$\pi_\Omega \oplus \tau^{\#} : W^{\#} \longrightarrow H^1(\Omega) \oplus L^2(\partial\Omega, \mu) \tag{4.2.10}$$

is a topological isomorphism of $W^{\#}$ onto a closed linear subspace of $H^1(\Omega) \oplus L^2(\partial\Omega, \mu)$.

If Ω is a Lipschitz domain and $\mu = \sigma$, then π_Ω in (4.2.8)–(4.2.9) is an isomorphism, there is a trace map $\mathrm{Tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, and $\tau^{\#} = \mathrm{Tr} \circ \pi_\Omega$. However, for rougher domains these properties can fail, as has been noted in [AtE].

It is natural to identify $W^{\#}$ with the closure of

$$\mathcal{G} = \{(u, u|_{\partial\Omega}) : u \in \mathrm{Lip}(\overline{\Omega})\} \ \text{ in } \ H^1(\Omega) \oplus L^2(\partial\Omega, \mu), \tag{4.2.11}$$

and note that inherited from (4.2.3) is

$$\begin{aligned} &(u, f) \in \overline{\mathcal{G}} = W^{\#} \\ &\qquad \Longrightarrow \int_\Omega |u|^2 \, dV \le C \int_\Omega |\nabla u|^2 \, dV + C \int_{\partial\Omega} |f|^2 \, d\mu. \end{aligned} \tag{4.2.12}$$

Thus

$$\beta\big((u, f), (v, g)\big) = \int_\Omega \langle \nabla u, \nabla v \rangle \, dV + \int_{\partial\Omega} f g \, d\mu \tag{4.2.13}$$

is a Hilbert space inner product on $W^{\#}$.

To continue, as in (A.19) we form

$$V = W^{\#} / \operatorname{Ker} \tau^{\#}, \tag{4.2.14}$$

which inherits from $\tau^{\#}$ a continuous injection

$$J : V \longrightarrow L^2(\partial\Omega, \mu), \tag{4.2.15}$$

whose image contains $\mathrm{Lip}(\partial\Omega)$ and hence is dense. The space V has a Hilbert space structure, naturally isomorphic to the orthogonal complement of $\operatorname{Ker}\tau^{\#}$ in $W^{\#}$, with respect to the inner product (4.2.13):

$$V = \{(u, f) \in W^{\#} : \beta((u, f), (v, g)) = 0,\ \forall\,(v, g) \in \operatorname{Ker}\tau^{\#}\}. \tag{4.2.16}$$

Of course,

$$\begin{aligned} (v, g) \in \operatorname{Ker}\tau^{\#} &\Longrightarrow g = 0 \\ &\Longrightarrow \beta\big((u, f), (v, g)\big) = \int_\Omega \langle \nabla u, \nabla v\rangle\, dV. \end{aligned} \tag{4.2.17}$$

Consequently, we can rewrite (4.2.16) as

$$V = \Big\{(u, f) \in W^{\#} : \int_\Omega \langle \nabla u, \nabla v\rangle\, dV = 0,\ \forall\, v \in \overset{\circ}{H}{}^1(\Omega)\Big\}, \tag{4.2.18}$$

where (cf. (4.2.11))

$$\overset{\circ}{H}{}^1(\Omega) = \{v \in H^1(\Omega) : (v, 0) \in W^{\#}\}. \tag{4.2.18A}$$

Note that

$$H^1_0(\Omega) \subset \overset{\circ}{H}{}^1(\Omega), \tag{4.2.18B}$$

where $H^1_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Consequently,

$$\begin{aligned} (u, f) \in V &\Longrightarrow \int_\Omega \langle \nabla u, \nabla v\rangle\, dV = 0, \quad \forall\, v \in H^1_0(\Omega) \\ &\Longrightarrow \Delta u = 0 \ \text{ on } \ \Omega. \end{aligned} \tag{4.2.19}$$

We also mention that

$$H^1_0(\Omega) = \overset{\circ}{H}{}^1(\Omega) \Longrightarrow V = \{(u, f) \in W^{\#} : \Delta u = 0 \text{ on } \Omega\}. \tag{4.2.19A}$$

From the injection (4.2.15), the Friedrichs method yields a positive self-adjoint operator B on $H = L^2(\partial\Omega, \mu)$, as described in Appendix A. To recap, the adjoint $J^t : H \to V'$ is also injective, with dense range, yielding

$$V \hookrightarrow H \hookrightarrow V'. \tag{4.2.20}$$

The inner product β, given by (4.2.13), restricted to $V = (\operatorname{Ker}\tau^{\#})^{\perp}$, yields an isomorphism $B_\beta : V \overset{\approx}{\to} V'$, with inverse $T_\beta : V' \to V$. We restrict T_β to H, to get a bounded linear operator $T : H \to H$, which is seen to be self-adjoint and injective, hence with dense range $\mathcal{R}(T) \subset H$. Then the inverse B of T is densely defined,

$$B : H \longrightarrow H, \quad \mathcal{D}(B) = \mathcal{R}(T) = \{f \in J(V) : B_\beta f \in H\}, \tag{4.2.21}$$

and B is self-adjoint. We have

$$\begin{aligned} &(u,f),(v,g) \in V, \quad f,\, g \in \mathcal{D}(B) \\ &\Longrightarrow \langle Bf, g\rangle = \int_{\Omega} \langle \nabla u, \nabla v\rangle\, dV + \int_{\partial\Omega} fg\, d\mu, \end{aligned} \tag{4.2.22}$$

where $\langle Bf, g\rangle$ is the inner product of Bf and g in $H = L^2(\partial\Omega, \mu)$. In particular, $\operatorname{Spec} B \subset [1,\infty)$. We set

$$N = B - 1, \tag{4.2.23}$$

so N is self-adjoint and $\operatorname{Spec} N \subset [0,\infty)$. In the setting of (4.2.22),

$$\langle Nf, g\rangle = \int_{\Omega} \langle \nabla u, \nabla v\rangle\, dV. \tag{4.2.24}$$

Also, as noted in (A.24),

$$\mathcal{D}(N^{1/2}) = \tau^{\#}(W^{\#}). \tag{4.2.25}$$

We restate this last part.

Proposition 4.2.1. *Given $f \in L^2(\partial\Omega, \mu)$, we have $f \in \mathcal{D}(N^{1/2})$ if and only if there exist $u_k \in \operatorname{Lip}(\overline{\Omega})$ and $u \in H^1(\Omega)$ such that*

$$u_k\Big|_{\partial\Omega} \longrightarrow f \quad \textit{in} \quad L^2(\partial\Omega, \mu) \tag{4.2.26}$$

and

$$u_k \longrightarrow u \quad \textit{in} \quad H^1(\Omega). \tag{4.2.27}$$

In such a case, we can pick (u_k) such that

$$\Delta u = 0 \quad \textit{on} \quad \Omega. \tag{4.2.28}$$

By construction, for $f \in \mathcal{D}(B^{1/2}) = \mathcal{D}(N^{1/2})$,

$$\langle B^{1/2} f, B^{1/2} f\rangle = \inf \Big\{ \int_{\Omega} |\nabla u|^2\, dV + \int_{\partial\Omega} |f|^2\, d\mu : (u,f) \in W^{\#} \Big\}. \tag{4.2.29}$$

Also

$$\langle N^{1/2}f, N^{1/2}f\rangle = \langle B^{1/2}f, B^{1/2}f\rangle - \langle f, f\rangle. \tag{4.2.30}$$

In fact, for $f \in \mathcal{D}(N)$, the left side is $\langle Nf, f\rangle$, and (4.2.23) yields the right side. Denseness of $\mathcal{D}(N)$ in $\mathcal{D}(N^{1/2})$ then gives (4.2.30) for general $f \in \mathcal{D}(N^{1/2})$. Hence, for $f \in \mathcal{D}(N^{1/2})$,

$$\langle N^{1/2}f, N^{1/2}f\rangle = \inf\Big\{\int_\Omega |\nabla u|^2\, dV : (u, f) \in W^\#\Big\}. \tag{4.2.31}$$

Clearly $(1, 1) \in W^\#$, so we have

$$1 \in \operatorname{Ker} N^{1/2} = \operatorname{Ker} N. \tag{4.2.32}$$

Keeping in mind that the inf on the right side of (4.2.29) is achieved, we see that, for $f \in L^2(\partial\Omega, \mu)$,

$$f \in \operatorname{Ker} N \Leftrightarrow (u, f) \in W^\# \text{ for some } u \in H^1(\Omega) \text{ satisfying } \nabla u \equiv 0. \tag{4.2.33}$$

If Ω is connected, such u is constant on Ω, and we have the following.

Proposition 4.2.2. *If Ω is connected,*

$$\operatorname{Ker} N = \operatorname{Span}(1) + \mathfrak{Z}(\Omega, \mu), \tag{4.2.34}$$

where

$$\mathfrak{Z}(\Omega, \mu) = \{f \in L^2(\partial\Omega, \mu) : (0, f) \in W^\#\}. \tag{4.2.35}$$

If Ω is a Lipschitz domain (and $\mu = \sigma$), $\mathfrak{Z}(\Omega, \mu) = 0$. In fact, this holds whenever there is an estimate

$$\int_{\partial\Omega} |u|^2\, d\mu \le C \int_\Omega \big(|\nabla u|^2 + |u|^2\big)\, dV, \quad \forall\, u \in \operatorname{Lip}(\overline{\Omega}). \tag{4.2.36}$$

See Proposition 4.5.1 for more general conditions that guarantee $\mathfrak{Z}(\Omega, \sigma) = 0$. However, there exist domains for which $\mathfrak{Z}(\Omega, \mu) \neq 0$, as seen in §4 of [AtE], and in (4.5.19) below.

4.3. The semigroup e^{-tN}

The positive semidefinite, self-adjoint operator N constructed on $H = L^2(\partial\Omega, \mu)$ in §4.2 yields a contraction semigroup e^{-tN} on H. Our first goal here is to prove the following.

Proposition 4.3.1. *Given $f \in L^2(\partial\Omega, \mu)$,*

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0, \quad \forall\, t > 0. \tag{4.3.1}$$

The proof, adapted from that of Proposition 3.7 in [AtE], centers around an application of the Beurling-Deny positivity criterion. That is, we need to show that $f \in \mathcal{D}(N^{1/2}) \Rightarrow |f| \in \mathcal{D}(N^{1/2})$ and

$$\langle N^{1/2}|f|, N^{1/2}|f|\rangle \leq \langle N^{1/2} f, N^{1/2} f\rangle. \tag{4.3.2}$$

As seen in (4.2.31),

$$\langle N^{1/2} f, N^{1/2} f\rangle = \inf \Big\{ \int_\Omega |\nabla u|^2 \, dV : (u, f) \in W^{\#} \Big\}. \tag{4.3.3}$$

Furthermore, we have (cf. [T2], Chapter 2, Proposition 6.4)

$$u_k \to u \text{ in } H^1(\Omega) \Longrightarrow |u_k| \to |u| \text{ in } H^1(\Omega).$$

It follows that

$$(u, f) \in W^{\#} \Longrightarrow (|u|, |f|) \in W^{\#}. \tag{4.3.4}$$

Thus it suffices to show that

$$\int_\Omega |\nabla |u|\,|^2 \, dV \leq \int_\Omega |\nabla u|^2 \, dV, \tag{4.3.5}$$

for each u in the closure of $\mathrm{Lip}(\overline{\Omega})$ in $H^1(\Omega)$. In fact, (4.3.5) holds for all $u \in H^1(\Omega)$, by Lemma 7.6 of [GT].

Since $1 \in \operatorname{Ker} N$, we have

$$e^{-tN} 1 \equiv 1, \tag{4.3.6}$$

so e^{-tN} is a symmetric Markov semigroup. The argument involving (3.3.23)–(3.3.24), used to prove Proposition 3.3.7, adapts readily to prove the following (cf. [AtE], Proposition 2.2).

Proposition 4.3.2. *Assume Ω is connected. If $\mathfrak{Z}(\Omega, \mu) = 0$, then $\{e^{-tN} : t \geq 0\}$ is irreducible.*

For an extension to a class of sets Ω that are not connected, see §7.2.

4.4. The Poisson integral

We take $\Omega \subset M$ as in §4.1. Let us set

$$\overline{H}^1(\Omega) = \text{ closure of } \mathrm{Lip}(\overline{\Omega}) \text{ in } H^1(\Omega), \tag{4.4.1}$$

and recall that

$$\begin{aligned} \mathcal{D}(N^{1/2}) &= \{f \in L^2(\partial\Omega, \mu) : (w, f) \in W^\# \text{ for some } w \in \overline{H}^1(\Omega)\} \\ &= \{f \in L^2(\partial\Omega, \mu) : (u, f) \in V \text{ for some } u \in \overline{H}^1(\Omega)\}. \end{aligned} \tag{4.4.2}$$

We define the continuous linear map

$$\mathrm{PI} : \mathcal{D}(N^{1/2}) \longrightarrow \overline{H}^1(\Omega) \tag{4.4.3}$$

by

$$\mathrm{PI}\, f = u, \quad \text{where } (u, f) \in V. \tag{4.4.4}$$

Recall that V was constructed so that such u is unique. We have

$$\begin{aligned} (u, f) \in V \Longleftrightarrow (u, f) \in W^\# \ \text{ and} \\ \int_\Omega \langle \nabla u, \nabla v \rangle \, dV = 0, \quad \forall\, v \in \overset{\circ}{H}{}^1(\Omega), \end{aligned} \tag{4.4.5}$$

with $\overset{\circ}{H}{}^1(\Omega)$ defined as in (4.2.18A). Also recall that $(u, f) \in W^\#$ if and only if there exist $u_k \in \mathrm{Lip}(\overline{\Omega})$ such that

$$u_k \to u \text{ in } H^1(\Omega) \ \text{ and } \ u_k\big|_{\partial\Omega} \to f \text{ in } L^2(\partial\Omega, \mu). \tag{4.4.6}$$

Also, reiterating the second identity in (4.4.2), we emphasize that

$$\forall\, (w, f) \in W^\#, \quad \exists\,!\ u \in \overline{H}^1(\Omega) \text{ such that } (u, f) \in V. \tag{4.4.7}$$

Furthermore, the construction of V as the orthogonal complement of $\mathrm{Ker}\, \tau^\#$ in $W^\#$ yields, for $(u, f) \in V$,

$$\|\nabla u\|^2_{L^2(\Omega)} = \inf \Big\{ \|\nabla w\|^2_{L^2(\Omega)} : (w, f) \in W^\# \Big\}. \tag{4.4.7A}$$

Clearly $w \in \mathrm{Lip}(\overline{\Omega}) \Rightarrow (w, w|_{\partial\Omega}) \in W^\#$, so we have

$$\mathrm{Lip}(\partial\Omega) \subset \mathcal{D}(N^{1/2}). \tag{4.4.8}$$

Actually, to call this natural map an inclusion, we want to require that

$$\mathrm{Lip}(\partial\Omega) \longrightarrow L^2(\partial\Omega, \mu) \ \text{ is injective,} \tag{4.4.9}$$

which holds if and only if

$$\mathrm{supp}\, \mu = \partial\Omega. \tag{4.4.10}$$

From here on, we make this a requirement.

Since $\{(u, u|_{\partial\Omega}) : u \in \mathrm{Lip}(\overline{\Omega})\}$ is dense in $W^\#$, it is clear that $\mathrm{Lip}(\partial\Omega)$ is dense in $\mathcal{D}(N^{1/2})$. The following stronger result is useful. Let $C^\infty(\partial\Omega)$ denote $\{u|_{\partial\Omega} : u \in C^\infty(M)\}$.

Proposition 4.4.1. *The space $C^\infty(\partial\Omega)$ is dense in $\mathcal{D}(N^{1/2})$.*

Proof. It suffices to show that $\{(u|_{\overline{\Omega}}, u|_{\partial\Omega} : u \in C^\infty(M)\}$ is dense in $\{(u, u|_{\partial\Omega}) : u \in \mathrm{Lip}(\overline{\Omega})\}$, in the $W^\#$ norm, given by

$$\|(u,f)\|^2_{W^\#} = \|u\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(\partial\Omega,\mu)}. \tag{4.4.11}$$

To get this, take $u \in \mathrm{Lip}(\overline{\Omega})$ and extend it to $v \in \mathrm{Lip}(M)$. Apply a standard mollifier to obtain $v_k \to C^\infty(M)$ such that

$$\begin{aligned} v_k &\longrightarrow v \text{ in } H^{1,p}(M), \quad \forall\, p < \infty, \\ v_k\big|_{\partial\Omega} &\longrightarrow v\big|_{\partial\Omega}, \quad \text{uniformly.} \end{aligned} \tag{4.4.12}$$

This proves the denseness.

It follows that PI in (4.4.3) is uniquely determined by its restriction to $C^\infty(\partial\Omega)$:

$$\mathrm{PI} : C^\infty(\partial\Omega) \longrightarrow \overline{H}^1(\Omega). \tag{4.4.13}$$

It is natural to compare (4.4.13) with a variant,

$$\mathrm{PI}_0 : C^\infty(\partial\Omega) \longrightarrow \overline{H}^1(\Omega), \tag{4.4.14}$$

defined as follows. Given $f \in C^\infty(\partial\Omega)$, pick $F \in C^\infty(M)$ such that $f = F|_{\partial\Omega}$, and set

$$\mathrm{PI}_0\, f = F + v, \tag{4.4.15}$$

where v is defined by

$$\Delta v = -\Delta F \in L^2(\Omega), \quad v \in H^1_0(\Omega). \tag{4.4.16}$$

This operator is analyzed in §5.5 of [T1]. (For this, Ω can be any open subset of M satisfying (4.0.1).) It is shown that $\mathrm{PI}_0\, f$ is independent of the choice of extension F and that, given constants $a, b \in \mathbb{R}$,

$$a \le f \le b \Longrightarrow a \le \mathrm{PI}_0\, f \le b. \tag{4.4.17}$$

Thus PI_0 has a unique extension from (4.4.14) to

$$\mathrm{PI}_0 : C(\partial\Omega) \longrightarrow L^\infty(\Omega), \quad \Delta\, \mathrm{PI}_0\, f = 0 \text{ on } \Omega. \tag{4.4.18}$$

To compare (4.4.13) and (4.4.14), we note that

$$u = \mathrm{PI}\, f, \quad u_0 = \mathrm{PI}_0\, f \tag{4.4.19}$$

have the following characterizations:

$$\begin{aligned} u - F \in \mathring{H}^1(\Omega), &\quad \int\limits_\Omega \langle \nabla u, \nabla w\rangle\, dV = 0, \quad \forall\, w \in \mathring{H}^1(\Omega), \\ u_0 - F \in H^1_0(\Omega), &\quad \int\limits_\Omega \langle \nabla u_0, \nabla w\rangle\, dV = 0, \quad \forall\, w \in H^1_0(\Omega). \end{aligned} \tag{4.4.20}$$

Consequently we have the following.

Proposition 4.4.2. *Take $\Omega \subset M$ as in §4.1 and assume μ is a measure on $\partial\Omega$ satisfying (4.2.2). Then*

$$\overset{\circ}{H}{}^1(\Omega) = H_0^1(\Omega) \Longrightarrow \mathrm{PI}\, f = \mathrm{PI}_0\, f, \quad \forall\, f \in C^\infty(\partial\Omega). \tag{4.4.21}$$

We prepare to give an example of a domain Ω for which the hypothesis and the conclusion in (4.4.21) fail. Before doing so, we introduce notation that records how the objects $W^\#$, $\overset{\circ}{H}{}^1(\Omega)$, V, and PI all depend on the choice of measure μ on $\partial\Omega$. In detail, we use the (temporary) notation

$$W^\#(\mu), \quad \overset{\circ}{H}{}^1(\Omega,\mu), \quad V(\mu), \quad \text{and} \quad \mathrm{PI}_{(\mu)}, \tag{4.4.22}$$

respectively. Here $W^\#(\mu)$ is the completion of $W_0 = \mathrm{Lip}(\overline{\Omega})$ with respect to the norm arising from (4.2.6), $\overset{\circ}{H}{}^1(\Omega,\mu)$ is defined as in (4.2.18A), with $W^\#(\mu)$ for $W^\#$, $V(\mu)$ is as in (4.2.18), with $\overset{\circ}{H}{}^1(\Omega,\mu)$ for $\overset{\circ}{H}{}^1(\Omega)$, and $\mathrm{PI}_{(\mu)}$ is as in (4.4.8), with $V(\mu)$ for V. (In concert with this, we might also use the notation N_μ, given as in (4.2.20)–(4.2.23), with $V(\mu)$ for V.)

Now, consider the following example. Take

$$D = \{x \in \mathbb{R}^2 : |x| < 1\}, \quad \Omega = D \setminus \gamma, \quad \gamma = \{(t,0) : -1 < t \le 0\}. \tag{4.4.23}$$

Then

$$\partial\Omega = \partial D \cup \gamma, \quad \partial_*\Omega = \partial D. \tag{4.4.24}$$

We use the measures

$$\sigma = \mathcal{H}^1\lfloor\partial D, \quad \mu = \mathcal{H}^1\lfloor\partial\Omega. \tag{4.4.25}$$

In these cases,

$$\begin{aligned} \overset{\circ}{H}{}^1(\Omega,\sigma) &= \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}, \\ \overset{\circ}{H}{}^1(\Omega,\mu) &= H_0^1(\Omega), \end{aligned} \tag{4.4.26}$$

so (4.4.21) holds for $\overset{\circ}{H}{}^1(\Omega) = \overset{\circ}{H}{}^1(\Omega,\mu)$, and we have

$$\mathrm{PI}_{(\mu)}\, f = \mathrm{PI}_0\, f. \tag{4.4.27}$$

However, the hypothesis of (4.4.21) fails for $\overset{\circ}{H}{}^1(\Omega,\sigma)$, and so does the conclusion. In fact, given $f \in C^\infty(\partial\Omega)$,

$$\mathrm{PI}_{(\sigma)}\, f = \mathrm{PI}_D(f|_{\partial D}), \tag{4.4.28}$$

where PI_D is the Poisson integral on D. Also, the hypothesis (4.4.10) fails for σ, so we bring in another measure, for which (4.4.10) holds. Namely, let $\{p_j\}$ be a countable dense subset of γ, and set

$$\lambda = \sigma + \sum_{j\geq 1} 2^{-j}\delta_{p_j}. \tag{4.4.29}$$

Then $\operatorname{supp}\lambda = \partial\Omega$, and we again have

$$\mathrm{PI}_{(\lambda)} = \mathrm{PI}_D\,. \tag{4.4.30}$$

We also mention that (4.4.1) fails for Ω in (4.4.23). The example (4.4.23) will arise again in §5.1.

4.5. Domains satisfying mild regularity conditions

Take $\Omega \subset M$ to be a finite perimeter domain satisfying (4.0.1). Here are some properties that hold if Ω is a Lipschitz domain, and $\mu = \sigma$.

$$H^1(\Omega) = \overline{H}^1(\Omega), \tag{4.5.1}$$

$$H^1_0(\Omega) = \overset{\circ}{H}{}^1(\Omega), \tag{4.5.2}$$

$$\mathfrak{Z}(\Omega, \sigma) = 0. \tag{4.5.3}$$

We want to identify broader classes of domains for which these properties hold.

First, it is well known and elementary that (4.5.1) holds whenever $\partial\Omega$ is locally the graph of a continuous function, with the property that, for each $p \in \partial\Omega$, there is a neighborhood U of p in M and coordinates on U such that $\Omega \cap U$ is given by

$$x_n < g(x'), \tag{4.5.4}$$

where $x' = (x_1, \dots, x_{n-1})$ and g is continuous. For this, Ω need not have finite perimeter. On the other hand, as shown in §§2.2 and 5.3 of [HMT], Ω has finite perimeter provided one can cover $\partial\Omega$ with such open sets and take g in (4.5.4) to satisfy

$$g \ \text{ continuous and } \ \nabla g \in L^1. \tag{4.5.5}$$

In these coordinates,

$$\nu(x) = (1 + |\nabla g(x')|^2)^{-1/2}(-\nabla g(x'), 1), \quad \sigma\text{-a.e. on } \ U \cap \partial\Omega, \tag{4.5.6}$$

up to a smooth positive factor if M has a non-flat metric. Furthermore, for this class of domains,

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{4.5.7}$$

Using (4.5.6) and a partition of unity, we can construct a smooth vector field Y on M such that

$$\langle \nu, Y \rangle > 0, \quad \sigma\text{-a.e. on } \partial_*\Omega, \tag{4.5.8}$$

when Ω is a domain whose behavior near its boundary is locally described by (4.5.4)–(4.5.5). Consequently the following result applies.

Proposition 4.5.1. *Let $\Omega \subset M$ be a finite perimeter domain satisfying (4.0.1). Assume there is a smooth vector field Y on M such that (4.5.8) holds. Then $\mathfrak{Z}(\Omega, \sigma) = 0$.*

Proof. From (4.0.5) with $X = |u|^2 Y$, we have the following variant of (4.1.4):

$$\int_{\partial_* \Omega} |u|^2 \langle \nu, Y \rangle \, d\sigma = \int_{\Omega} |u|^2 (\operatorname{div} Y) \, dV + 2 \int_{\Omega} u \langle \nabla u, Y \rangle \, dV, \tag{4.5.9}$$

for each $u \in \operatorname{Lip}(\overline{\Omega})$. Now suppose $f \in \mathfrak{Z}(\Omega, \sigma)$, so there exist $u_k \in \operatorname{Lip}(\overline{\Omega})$ such that

$$u_k \to 0 \text{ in } H^1(\Omega), \quad u_k\big|_{\partial\Omega} \to f \text{ in } L^2(\partial\Omega, \sigma). \tag{4.5.10}$$

By (4.5.9),

$$\int_{\partial_* \Omega} |u_k|^2 \langle \nu, Y \rangle \, d\sigma \le C \|u_k\|^2_{H^1(\Omega)} \longrightarrow 0. \tag{4.5.11}$$

But the left side of (4.5.11) tends to the limit

$$\int_{\partial_* \Omega} \langle \nu, Y \rangle |f|^2 \, d\sigma. \tag{4.5.12}$$

If (4.5.8) holds, this forces $f = 0$.

Remark. Extending the scope of Proposition 4.5.1, we note that, for Y, it suffices to have $Y \in C(M)$, $\operatorname{div} Y \in L^\infty(M)$, and (4.5.8).

Here is a more general class of domains to which Proposition 4.5.1 applies.

Proposition 4.5.2. *Let $\Omega \subset M$ be a finite perimeter domain satisfying (4.0.1). Assume that, for σ-a.e. $p \in \partial_* \Omega$, there is a neighborhood U of p in M and coordinates on U such that $\Omega \cap U$ is given by (4.5.4)–(4.5.5). Then there exists a smooth vector field Y on M such that (4.5.8) holds. Hence $\mathfrak{Z}(\Omega, \sigma) = 0$.*

Proof. Let $\Gamma \subset \partial\Omega$ denote the set of $p \in \partial\Omega$ for which such a neighborhood U exists. Note that Γ is a relatively open subset of $\partial\Omega$. Our hypothesis is that $\sigma(\partial\Omega \setminus \Gamma) = 0$. Let $\Gamma_k \subset \Gamma$ be an increasing sequence of compact subsets of Γ such that $\sigma(\Gamma \setminus \Gamma_k) \searrow 0$. Each Γ_k has a finite cover by such open sets U, and a partition of unity argument yields smooth vector fields Y_k on M such that

$$\langle \nu, Y_k \rangle \ge 0 \text{ on } \partial\Omega, \quad \langle \nu, Y_k \rangle > 0, \ \sigma\text{-a.e. on } \Gamma_k. \tag{4.5.13}$$

Then one can choose $\alpha_k \searrow 0$ sufficiently fast that $Y = \sum_k \alpha_k Y_k$ does the trick.

Example. Let $S = \{p_k\}$ be a sequence of points in the unit square $Q = (0,1)\times(0,1)$ such that

$$\overline{S} = S \cup \gamma, \quad \gamma = \{(1,y) : 0 \le y \le 1\}. \tag{4.5.14}$$

Pick $\varepsilon_k > 0$ such that $\sum_k \varepsilon_k < \infty$ and the open disks $D_k = D_{\varepsilon_k}(p_k)$ are disjoint, and set

$$\Omega = \bigcup_k D_k. \tag{4.5.15}$$

Then Ω is a finite perimeter domain. We have

$$\partial_* \Omega = \bigcup_k \partial D_k, \quad \partial\Omega = \partial_*\Omega \cup \gamma. \tag{4.5.16}$$

Clearly Proposition 4.5.2 applies to Ω, so

$$\mathfrak{Z}(\Omega, \sigma) = 0. \tag{4.5.17}$$

On the other hand, if we take

$$\mu = \mathcal{H}^1 \lfloor \partial\Omega = \sigma + \mathcal{H}^1 \lfloor \gamma, \tag{4.5.18}$$

it is readily verified that

$$\mathfrak{Z}(\Omega, \mu) = L^2(\gamma, \mu). \tag{4.5.19}$$

Note. The set Ω described above is not connected. One can form a connected 3D domain with similar properties by taking the Cartesian product with the interval $(0,1)$ and putting this product on a slab. This yields a domain similar to that depicted in Fig. 1 of [AtE].

For some further results, suppose $\mathcal{O} \subset M$ is another open set, and that

$$\Phi : \overline{\Omega} \longrightarrow \overline{\mathcal{O}}, \quad \Phi : \Omega \longrightarrow \mathcal{O} \tag{4.5.20}$$

is a bi-Lipschitz map. We have the action on functions, $\Phi^* u(x) = u(F(x))$, and the following isomorphisms:

$$\begin{aligned} \Phi^* &: H^1(\mathcal{O}) \longrightarrow H^1(\Omega), & \Phi^* &: \mathrm{Lip}(\overline{\mathcal{O}}) \longrightarrow \mathrm{Lip}(\overline{\Omega}), \\ \Phi^* &: \overline{H}^1(\mathcal{O}) \longrightarrow \overline{H}^1(\Omega), & \Phi^* &: H^1_0(\mathcal{O}) \longrightarrow H^1_0(\Omega), \\ \Phi^* &: \mathring{H}^1(\mathcal{O}) \longrightarrow \mathring{H}^1(\Omega), & \Phi^* &: \mathfrak{Z}(\mathcal{O}, \Phi_*\mu) \longrightarrow \mathfrak{Z}(\Omega, \mu). \end{aligned} \tag{4.5.21}$$

Regarding the action on $\mathring{H}^1(\mathcal{O})$, recall that the space $\mathring{H}^1(\Omega)$ depends on the choice of measure μ on $\partial\Omega$, so we use the push-forward measure $\Phi_*\mu$ on $\partial\mathcal{O}$ to define $\mathring{H}^1(\mathcal{O})$. In light of these isomorphisms, we have the following.

Proposition 4.5.3. *Assume there is a smoothly bounded $\mathcal{O} \subset M$ and a bi-Lipschitz map Φ satisfying (4.5.20). Use $\mu = \sigma$. Then the conditions (4.5.1)–(4.5.3) hold.*

There are domains Ω that are bi-Lipschitz equivalent to the unit ball but are not Lipschitz domains, as $\partial\Omega$ is not everywhere locally given as the graph of a Lipschitz function. One famous example is the "two brick" domain in $\mathbb{R}^3$:

$$\overline{\Omega} = [-2,2] \times [-1,1] \times [0,1] \cup [-1,1] \times [-2,2] \times [-1,0]. \tag{4.5.22}$$

4.6. Rough metric tensors

So far in this chapter we have assumed M carries a smooth metric tensor. This contrasts with Chapter 3, where we dealt with Lipschitz domains contained in compact manifolds endowed with metric tensors which, in local coordinate systems, satisfied a Dini-type modulus of continuity, described by (3.0.1)–(3.0.2). Here we want to deal with finite perimeter domains in a compact manifold M with a rough metric tensor, for example,

$$g_{jk} \in C(U). \tag{4.6.1}$$

One motivating example for considering rough metric tensors here will be described in §5.1. It arises from "blowing up" an inclusion in a finite perimeter domain Ω, assumed to lie in a manifold with a smooth metric tensor. The blow-up will be seen to produce a metric tensor which, in local coordinates, satisfies

$$g_{jk} \in H^{1,p}(U), \quad \forall\, p < \infty. \tag{4.6.2}$$

As stated above, we are also interested in more singular metrics.

We will assume $\Omega \subset M$ where M is a compact manifold, and (4.0.1) holds. We will assume that M is endowed with a C^∞ structure as a differential manifold, but that its metric tensor is not smooth. This is for simplicity. One could work with the fact that metrics with a certain limited smoothness are naturally associated with differential structures with a related degree of smoothness (cf. [T3]), but we will not do this here.

We begin with a preliminary discussion of some of the analytical tools that carry over from the case of smooth metric tensors to more general situations.

For one, the validity of the divergence theorem

$$\int\limits_{\Omega} \operatorname{div} X \, dV = \int\limits_{\partial_* \Omega} \langle \nu, X \rangle \, d\sigma, \tag{4.6.3}$$

for a vector field X on M satisfying

$$X \in C(M), \quad \operatorname{div} X \in L^1(M), \tag{4.6.4}$$

was established in §5.3 of [HMT] in the setting of continuous metric tensors.

In connection with this, we note that, in a local coordinate patch U,

$$\operatorname{div} X = g^{-1/2} \operatorname{div}_0(g^{1/2} X), \tag{4.6.5}$$

where div_0 denotes the divergence operator on flat Euclidean space, and we assume $\operatorname{div}_0(g^{1/2}X) \in L^1$. If (4.6.2) holds, then $g^{1/2} \in H^{1,p}(U)$, for all $p < \infty$, and

$$\operatorname{div} X = \operatorname{div}_0 X + g^{-1/2} \nabla g^{1/2} \cdot X, \tag{4.6.6}$$

so

$$\text{given } X \in C(U), \quad \operatorname{div} X \in L^1(U) \Leftrightarrow \operatorname{div}_0 X \in L^1(U). \tag{4.6.7}$$

Next, we have the following extension of Proposition 4.1.1.

Proposition 4.6.1. *Let $\Omega \subset M$ have finite perimeter and assume (4.0.1) holds. Then the estimate (4.1.1) holds, for all $u \in \mathrm{Lip}(\overline{\Omega})$, provided that the metric tensor is continuous, or even more generally,*

$$g_{jk},\ g^{jk} \in L^\infty. \tag{4.6.8}$$

Proof. Given that M has a C^∞ structure, you can put a smooth metric tensor on M. Now, note that the validity of the estimate (4.1.1) is invariant under passing from a metric tensor satisfying (4.6.8) to such a smooth metric tensor. Hence the conclusion follows from Proposition 4.1.1

To proceed, let M have a metric tensor g satisfying (4.6.8) and also a C^∞ metric tensor h. The inner product on tangent vectors associated to g will be denoted $\langle\ ,\ \rangle$, and that associated to h, $\langle\ ,\ \rangle_h$. Similarly we have dV and dV_h, ∇u and $\nabla_h u$, etc. We also modify μ to μ_h, but note that

$$0 < c_0 < \frac{d\mu}{d\mu_h} \le c_1 < \infty. \tag{4.6.9}$$

The following observation will be key in our analysis.

Proposition 4.6.2. *Let $\Omega \subset M$ be a finite perimeter domain satisfying (4.0.1). The following spaces are the same for M endowed with the metric tensors g and h:*

$$\mathrm{Lip}(\overline{\Omega}),\ H^1(\Omega),\ \overline{H}^1(\Omega),\ \mathring{H}^1(\Omega),\ H^1_0(\Omega),\ W^\#. \tag{4.6.10}$$

The various Hilbert spaces have different inner products, but they yield equivalent norms. Also the spaces $L^2(\partial\Omega, \mu)$ and $L^2(\partial\Omega, \mu_h)$ coincide, with equivalent inner products, and the two metric tensors lead to the same map

$$\tau^\# : W^\# \longrightarrow L^2(\partial\Omega, \mu) = L^2(\partial\Omega, \mu_h). \tag{4.6.11}$$

Furthermore,

$$\mathfrak{Z}(\Omega, \mu) = \mathfrak{Z}(\Omega, \mu_h). \tag{4.6.12}$$

On $W^\#$ we have the two inner products,

$$\begin{aligned}\beta\big((u,f),(v,g)\big) &= \int\limits_{\Omega} \langle \nabla u, \nabla v\rangle\, dV + \int\limits_{\partial\Omega} fg\, d\mu,\\ \beta_h\big((u,f),(v,g)\big) &= \int\limits_{\Omega} \langle \nabla_h u, \nabla_h v\rangle_h\, dV_h + \int\limits_{\partial\Omega} fg\, d\mu_h.\end{aligned} \tag{4.6.13}$$

These lead to two (typically different) spaces

$$
\begin{aligned}
V &= \Big\{(u,f) \in W^{\#} : \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dV = 0, \ \forall \, v \in \overset{\circ}{H}{}^1(\Omega)\Big\}, \\
V_h &= \Big\{(u,f) \in W^{\#} : \int_{\Omega} \langle \nabla_h u, \nabla_h v \rangle_h \, dV_h = 0, \ \forall \, v \in \overset{\circ}{H}{}^1(\Omega)\Big\},
\end{aligned} \tag{4.6.14}
$$

though there is a natural isomorphism between the two, produced by the isomorphism of each with $W^{\#}/\operatorname{Ker}\tau^{\#}$. In particular, they have the same isomorphic image in $L^2(\partial\Omega,\mu) = L^2(\partial\Omega,\mu_h)$, namely $\tau^{\#}(W^{\#})$.

We now have all the ingredients to apply the general construction of Appendix A, as done in §4.2. We obtain two operators, N and N_h, on $L^2(\partial\Omega,\mu) = L^2(\partial\Omega,\mu_h)$, positive semidefinite and self-adjoint for the respective Hilbert space inner products. Note that

$$
\mathcal{D}(N^{1/2}) = \mathcal{D}(N_h^{1/2}) = \tau^{\#}(W^{\#}). \tag{4.6.15}
$$

The operator N_h is of the sort constructed in §4.2, and N represents an extension of that construction, to the broader setting of metric tensors satisfying (4.6.8). We have

$$
\int_{\partial\Omega} (N^{1/2} f)(N^{1/2} g) \, d\mu = \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dV, \quad (u,f),(v,g) \in V, \tag{4.6.16}
$$

and

$$
\int_{\partial\Omega} (N_h^{1/2} f)(N_h^{1/2} g) \, d\mu_h = \int_{\Omega} \langle \nabla_h u_h, \nabla_h v_h \rangle_h \, dV_h, \quad (u_h,f),(v_h,g) \in V_h. \tag{4.6.17}
$$

5. Finite perimeter domains with inclusions

In (4.5.14)–(4.5.19) we have an example of a finite perimeter domain Ω for which $\partial\Omega \setminus \partial_*\Omega$ is sizable but lies "outside" Ω. Here we consider finite perimeter domains for which $\partial\Omega \setminus \partial_*\Omega$ is sizable but lies "inside" Ω, in the sense of lying in the interior of $\overline{\Omega}$. We call the domains considered "domains with inclusions." One point we make is that, while the considerations of Chapter 4 apply to these domains, one can obtain a richer theory by modifying our notion of the Dirichlet and Neumann boundary problems on these domains, to take into account the two-sided nature of these inclusions.

We begin in §5.1 with some examples, involving planar domains. We show how cutting these along the inclusions produces modified domains on which this richer analysis can be performed. In the first case, cutting along the inclusion produces a Lipschitz domain, to which the results of §3 apply. In the second case, cutting along the inclusion and "blowing up" along this cut produces a domain with a cusp, in a Riemannian manifold with a non-smooth metric tensor, to which results of §4.6 apply. We proceed to some more exotic examples. The first is S^2 with a segment γ of an arc removed, a setting that violates (4.0.1). When the inclusion γ is cut and blown up, one obtains a domain with cusps in a manifold with non-smooth metric tensor, to which §4.6 again applies. Another example is the "jelly roll" of [Si2], a two-dimensional domain with an inclusion whose $\mathcal{H}^1$-measure is infinite.

In §5.2 we treat a general class of inclusions that arise from partitioning one finite perimeter domain into finite perimeter subdomains. In more detail, let $\mathcal{O} \subset M$ be a finite perimeter domain in a compact Riemannian manifold M, and let Ω_1 and Ω_2 be nonempty disjoint open subsets of $\mathcal{O}$ that are finite perimeter domains and that "partition" $\mathcal{O}$ (cf. (5.2.2)–(5.2.3)). Then let S be a subset of $\partial\Omega_1 \cap \partial\Omega_2 \cap \mathcal{O}$ that is closed in $\mathcal{O}$, and has positive σ-measure, and let $\Omega = \mathcal{O} \setminus S$. The set S is the inclusion. Under the hypothesis that S is "thick," as defined below (5.2.17), we extend the construction of §4.2 to produce a positive, self-adjoint operator N on $H = L^2(\partial\mathcal{O} \cup S_1 \cup S_2, \sigma)$, where S_1 and S_2 are two copies of S (as seen, respectively, from Ω_1 and from Ω_2). We also show that $\{e^{-tN} : t \geq 0\}$ is a symmetric Markov semigroup on H, and examine conditions for it to be irreducible.

In §5.3 we tackle another class of inclusions, namely those of positive capacity, but generally not of finite, positive $(n-1)$-dimensional measure. Examples include inclusions K of Hausdorff dimension $s \in (n-2, n)$. These cases include many fractals, and are in a number of respects more exotic than the examples considered in §§5.1–5.2. One issue is to determine what sort of measures μ yield interesting results regarding the effect of K on N_μ. Sometimes Hausdorff measure $\mathcal{H}^s$ on K works, sometimes not. Another class that works is the class of harmonic measures. The inclusions treated in §5.3 are as singular as the boundaries treated in Chapter 7. One difference is that here we also require $\partial\Omega$ to have a "regular" (finite perimeter) part, which facilitates the use of results of Chapter 4.

5.1. Some key examples

Here we look at several planar domains with inclusions.
The first two will be subdomains of a disk and an annulus:

$$D = \{z \in \mathbb{C} : |z| < 1\}, \quad A = \{z \in \mathbb{C} : \tfrac{1}{4} < |z| < 1\}. \tag{5.1.1}$$

For convenience we vaccilate between the use of real coordinates (x, y) and complex coordinates $z = x + iy$. We take

$$\gamma_1 = D \cap \{x + i0 : x \le 0\}, \quad \gamma_2 = A \cap \{x + i0 : x \le 0\}, \tag{5.1.2}$$

and set

$$\Omega_1 = D \setminus \gamma_1, \quad \Omega_2 = A \setminus \gamma_2. \tag{5.1.3}$$

Then

$$\begin{aligned} \partial_* \Omega_1 = \partial D, \quad \partial\Omega_1 \setminus \partial_*\Omega_1 = \gamma_1, \\ \partial_* \Omega_2 = \partial A, \quad \partial\Omega_2 \setminus \partial_*\Omega_2 = \gamma_2. \end{aligned} \tag{5.1.4}$$

The measure σ arising as in (4.0.2) is arc-length measure on ∂D and ∂A, in these two cases. For the measure μ, as in (4.2.1)–(4.2.2), we take

$$\mu = \mathcal{H}^1 \lfloor \partial\Omega_j, \tag{5.1.5}$$

so μ is σ plus linear measure on γ_j.
Now define f_j on $\partial\Omega_j$ by

$$f_j = y\big|_{\partial\Omega_j}. \tag{5.1.6}$$

We have

$$u_j = \mathrm{PI}\, f_j = y \;\text{ on }\; \Omega_j. \tag{5.1.7}$$

One can show that

$$N f_j\big|_{\gamma_j} = 0. \tag{5.1.8}$$

On the other hand, if we approach γ_j from above, we get

$$\frac{\partial u_j}{\partial \nu_+} = \lim_{y \searrow 0} -\frac{\partial u_j}{\partial y}(x, y) = -1, \tag{5.1.9}$$

and if we approach γ_j from below, we get

$$\frac{\partial u_j}{\partial \nu_-} = \lim_{y \nearrow 0} \frac{\partial u_j}{\partial y}(x, y) = +1. \tag{5.1.10}$$

Clearly (5.1.9)–(5.1.10) contain information on the boundary behavior of ∇u_j that is lost in (5.1.8).

It is desirable to have modified Dirichlet and Neumann boundary problems, defined on domains $\mathcal{O}_j$, whose interiors coincide with Ω_j, but whose boundaries ramify over γ_j. Here is a construction that works well for Ω_2, but not as well for Ω_1. Let us take

$$\Phi_j : \mathcal{O}_j \longrightarrow \Omega_j, \quad \Phi_j(z) = z^2, \tag{5.1.11}$$

where

$$\begin{aligned} \mathcal{O}_1 &= \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Re} z > 0\}, \\ \mathcal{O}_2 &= \{z \in \mathbb{C} : \tfrac{1}{2} < |z| < 1,\ \operatorname{Re} z > 0\}. \end{aligned} \tag{5.1.12}$$

We put on $\mathcal{O}_j$ not the Euclidean metric, but the metric tensor pulled back via Φ_j from the Euclidean metric tensor on Ω_j, that is to say, $|dz|^2$ on Ω_j pulled back to the metric tensor

$$|\Phi_j'(z)|^2 \cdot |dz|^2 = 4|z|^2 \cdot |dz|^2. \tag{5.1.13}$$

This is a smooth, nondegenerate metric tensor on a neighborhood of $\overline{\mathcal{O}}_2$, but not on $\overline{\mathcal{O}}_1$, due to the degeneracy at $z = 0$.

It is an exercise to cast $\mathcal{O}_2$ as a Lipschitz domain in a compact 2D Riemannian manifold, and then all the material in §3 is applicable.

As for Ω_1, we still want to cut along γ_1, but we need to pull the domain apart in a different fashion. We set

$$\begin{aligned} \beta(x) = e^{1/x}&, \quad x < 0, \\ 0&, \quad x \geq 0, \end{aligned} \tag{5.1.14}$$

and define

$$\varphi : U_1 \longrightarrow \Omega_1 \tag{5.1.15}$$

by

$$\varphi(x, y) = (x, y - \beta(x)(\operatorname{sgn} y)). \tag{5.1.16}$$

We specify U_1 such that (5.1.15) is a diffeomorphism. In particular,

$$(x, y) \in U_1,\ x < 0 \Longrightarrow |y| > \beta(x). \tag{5.1.17}$$

Note that

$$D\varphi(x,y) = \begin{pmatrix} 1 & 0 \\ \beta'(x)(\operatorname{sgn} y) & 1 \end{pmatrix}, \quad \text{for} \quad (x,y) \in U_1. \tag{5.1.18}$$

The metric tensor $\varphi^*(dx^2 + dy^2)$ induced on U_1 via φ has components

$$\begin{aligned} G(x,y) &= D\varphi(x,y)^t D\varphi(x,y) \\ &= \begin{pmatrix} 1+\beta'(x)^2 & \beta'(x)(\operatorname{sgn} y) \\ \beta'(x)(\operatorname{sgn} y) & 1 \end{pmatrix}, \quad (x,y) \in U_1. \end{aligned} \tag{5.1.19}$$

Calculation of the limit

$$G(x, \pm\beta(x)) = \begin{pmatrix} 1+\beta'(x)^2 & \pm\beta'(x) \\ \pm\beta'(x) & 1 \end{pmatrix} \tag{5.1.20}$$

reveals that the components of $G(x,y)$, while extending continuously to $\overline{U}_1$, are not quite Lipschitz continuous, since the off-diagonal elements of (5.1.20) differ by an amount $2\beta'(x)$ across a gap of width $2\beta(x)$, for $x < 0$, but

$$\frac{\beta'(x)}{\beta(x)} = \frac{d}{dx} \log \beta(x) = -\frac{1}{x^2}, \quad \text{for} \quad x < 0. \tag{5.1.21}$$

With this in mind, we can extend $G(x,y)$ by setting

$$G(x,y) = \begin{pmatrix} 1+\beta'(x)^2 & y\beta'(x)/\beta(x) \\ y\beta'(x)/\beta(x) & 1 \end{pmatrix}, \quad |y| \le \beta(x), \ x < 0. \tag{5.1.22}$$

Computing $\partial_x G$ and $\partial_y G$, we have the following.

Lemma 5.1.1. *If U is a bounded open neighborhood of $\overline{U}_1$ in $\mathbb{R}^2$, then G belongs to the L^p-Sobolev space*

$$G \in H^{1,p}(U), \quad \forall\, p < \infty. \tag{5.1.23}$$

Our next example is

$$\Omega = S^2 \setminus \gamma, \tag{5.1.24}$$

where γ is a geodesic arc from one point of S^1 to another. In this case, $\overline{\Omega} = S^2$, so (4.0.1) fails, and results of §§4.1–4.2 do not apply. However, we can "blow up" γ in a fashion parallel to the way done for example 2 in §5.1. We obtain $\mathcal{O} \subset M$, where $\mathcal{O}$ is isometric to Ω above, M carries a metric tensor with components

$$g_{jk} \in H^{1,p}, \quad \forall\, p < \infty, \tag{5.1.25}$$

in local coordinates, and $M \setminus \overline{\mathcal{O}} \neq \emptyset$. Then material in §4.6 applies to this blow-up.

Our fourth example is B. Simon's "jelly roll" (cf. [Si2]):

$$\Omega = A \setminus \gamma, \quad A = \{z \in \mathbb{C} : \tfrac{1}{2} < |z| < 1\}, \tag{5.1.26}$$

where $\gamma : \mathbb{R} \to A$ is given by

$$\gamma(t) = r(t)e^{it}, \quad r(t) = \frac{3}{4} + \frac{1}{2\pi} \tan^{-1} t. \tag{5.1.27}$$

Note that $|\gamma(t)| \to 1/2$ as $t \to -\infty$ and $|\gamma(t)| \to 1$ at $t \to +\infty$. We have

$$\partial_* \Omega = \partial A, \quad \partial \Omega = \partial_* \Omega \cup \gamma, \quad \mathcal{H}^1(\gamma) = \infty. \tag{5.1.28}$$

We place on $\partial \Omega$ a finite measure of the form

$$\mu = \sigma + \gamma_*(\varphi \, dt), \tag{5.1.29}$$

where, as usual, $\sigma = \mathcal{H}^1 \lfloor \partial_* \Omega$, and φ is a positive, continuous, integrable function on $\mathbb{R}$, e.g., $\varphi(t) = (1 + t^2)^{-1}$. Then results of §§4.1–4.5 are applicable.

This time, if we blow up the inclusion γ, we get a domain $\mathcal{O}$ such that $\mathcal{H}^1(\partial_* \mathcal{O}) = \infty$, so results of Chapter 4 are not applicable to this blow-up. On the other hand, $\overline{\mathcal{O}}$ is a complete Riemannian manifold with smooth boundary, a setting in which the Dirichlet-to-Neumann map has been studied; cf. [LTU].

For our fifth example, take D_k and γ as in (4.5.15)–(4.5.16), and set

$$\mathcal{O} = \bigcup_k D_k, \quad \Omega = \widetilde{Q} \setminus \overline{\mathcal{O}}, \tag{5.1.30}$$

where $\widetilde{Q} = (-1, 2) \times (-1, 2)$, so

$$\partial_* \Omega = \partial \widetilde{Q} \cup \bigcup_k \partial D_k, \quad \partial \Omega = \partial_* \Omega \cup \gamma. \tag{5.1.31}$$

In this case, γ is not contained in the interior of $\overline{\Omega}$. It is an "inclusion" in the weaker sense that Ω has density 1 at each point of γ. One can still cut along γ and blow it up, obtaining a domain $\widetilde{\Omega} \subset \widetilde{M}$ to which §4.6 applies.

5.2. Inclusions via finite perimeter partitions

Let M be a conpact, connected, n-dimensional Riemannian manifold, $\mathcal{O} \subset M$ an open subset. Assume that $\mathcal{O}$ is a finite perimeter domain, satisfying (4.0.1), i.e.,

$$M \setminus \overline{\mathcal{O}} \neq \emptyset. \tag{5.2.1}$$

Let Ω_1 and Ω_2 be nonempty open subsets of $\mathcal{O}$ such that

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\mathcal{O}}. \tag{5.2.2}$$

Assume that, for each j,

$$\begin{gathered} \Omega_j \text{ is a finite perimeter domain, and} \\ \mathcal{H}^n(\partial\Omega_j) = 0. \end{gathered} \tag{5.2.3}$$

In particular, $\chi_{\Omega_1} + \chi_{\Omega_2} = \chi_{\mathcal{O}}$ in $L^1(M)$. Then let

$$S \subset (\partial\Omega_1 \cap \partial\Omega_2) \cap \mathcal{O} \ \text{ be closed in } \ \mathcal{O}, \tag{5.2.4}$$

and set

$$\Omega = \mathcal{O} \setminus S. \tag{5.2.5}$$

The set S is the inclusion. Note that

$$S \cap \partial_*\Omega_1 = S \cap \partial_*\Omega_2. \tag{5.2.6}$$

Let us introduce some notation. We denote by π the "partition" of $\mathcal{O}$ described above:

$$\pi = \{\Omega_1, \Omega_2\}, \tag{5.2.7}$$

and say

$$\begin{aligned} &u \in \mathrm{Lip}_\pi(\Omega) \ \text{ provided} \\ &u \in C(\Omega), \quad \nabla u \in L^\infty(\Omega), \quad \text{and} \\ &u_j = u\big|_{\Omega_j} \ \text{ extends to an element of } \ \mathrm{Lip}(\overline{\Omega}_j). \end{aligned} \tag{5.2.8}$$

An element $u \in \mathrm{Lip}_\pi(\Omega)$ could have a jump across S.

We next provide a useful version of the divergence theorem. Before stating it, let us denote by ν_j the measure theoretic unit outer normal to $\partial\Omega_j$, and note that

$$\nu_1 = -\nu_2, \quad \sigma\text{-a.e. on } \ \partial\Omega_1 \cap \partial\Omega_2, \tag{5.2.9}$$

with σ as in (4.0.2)–(4.0.3), adjusted to the current setting, involving both Ω, Ω_1, and Ω_2.

Lemma 5.2.1. *Given a vector field $X \in \mathrm{Lip}_\pi(\Omega)$, we have*

$$\int_\Omega \operatorname{div} X \, dV = \int_{\partial_* \mathcal{O}} \langle \nu, X \rangle \, d\sigma + \int_S \langle \nu_1, X_1 - X_2 \rangle \, d\sigma. \tag{5.2.10}$$

Proof. Here, X_j denotes the restriction of X to Ω_j, extended to a Lipschitz vector field on $\overline{\Omega}_j$. For each j, (4.0.5) gives

$$\int_{\Omega_j} \operatorname{div} X_j \, dV = \int_{\partial_* \Omega_j} \langle \nu_j, X_j \rangle \, d\sigma. \tag{5.2.11}$$

The sum over $j = 1, 2$ of the left side of (5.2.11) is the left side of (5.2.10). The corresponding sum over j of the right side of (5.2.11) is equal to

$$\int_{\partial_* \mathcal{O}} \langle \nu, X \rangle \, d\sigma + \int_{\partial_* \Omega_1 \cap \partial_* \Omega_2} \big\{ \langle \nu_1, X_1 \rangle + \langle \nu_2, X_2 \rangle \big\} \, d\sigma. \tag{5.2.12}$$

Thanks to (5.2.9), the last integral on the right side of (5.2.12) is equal to the last integral on the right side of (5.2.10).

With this result in hand, we can establish the following extension of Proposition 4.1.1.

Proposition 5.2.2. *In the setting of Lemma 5.2.1, assume also that (5.2.1) holds. Then there exists $C = C_\pi(\Omega) < \infty$ such that, for all $u \in \mathrm{Lip}_\pi(\Omega)$,*

$$\int_\Omega |u|^2 \, dV \le C \int_\Omega |\nabla u|^2 \, dV + C \int_{\partial_* \mathcal{O}} |u|^2 \, d\sigma + C \int_S |u_1 - u_2| (|u_1| + |u_2|) \, d\sigma. \tag{5.2.13}$$

Proof. As in the proof of Proposition 4.1.1, we take a vector field $Y \in C^\infty(M)$ such that $\operatorname{div} Y = 1$ on $\overline{\mathcal{O}}$, and set

$$X = |u|^2 Y, \quad \text{so} \quad \operatorname{div} X = |u|^2 \operatorname{div} Y + 2u \langle \nabla u, Y \rangle, \tag{5.2.14}$$

and (5.2.10) yields

$$\begin{aligned} \int_\Omega |u|^2 (\operatorname{div} Y) \, dV = & -2 \int_\Omega u \langle \nabla u, Y \rangle \, dV \\ & + \int_{\partial_* \mathcal{O}} |u|^2 \langle \nu, Y \rangle \, d\sigma \\ & + \int_S \langle \nu_1, (|u_1|^2 - |u_2|^2) Y \rangle \, d\sigma. \end{aligned} \tag{5.2.15}$$

Arguing as in (4.1.5)–(4.1.6), we obtain (5.2.13).

To proceed, parallel to (4.2.4) we take

$$W_0 = \mathrm{Lip}_\pi(\Omega), \quad \tau u = \bigl(u\big|_{\partial\mathcal{O}}, u_1\big|_S, u_2\big|_S\bigr). \tag{5.2.16}$$

We have

$$\begin{aligned} \tau : \mathrm{Lip}_\pi(\Omega) \longrightarrow H &= L^2(\partial\mathcal{O},\sigma) \oplus L^2(S,\sigma) \oplus L^2(S,\sigma) \\ &= L^2(\partial\mathcal{O} \cup S_1 \cup S_2, \sigma), \end{aligned} \tag{5.2.17}$$

where S_j denotes a copy of S, regarded as a part of $\partial\Omega_j$.

Definition. We say the inclusion S is *thick* if τ in (5.2.17) has dense range.

The inclusions γ_1 and γ_2 in (5.1.2) are thick. On the other hand, if we replace γ_j here by a subset $\tilde{\gamma}_j$ that is a Cantor set of positive one-dimensional measure, then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are not thick.

If S is thick, then (A.14) holds. We proceed to define N as a self adjoint operator on the Hilbert space H (defined in (5.2.17)), via the general program described in Appendix A, in a fashion parallel to the method used in §4.2. For the next step, parallel to (4.2.5), we take

$$\alpha : W_0 \times W_0 \longrightarrow \mathbb{R}, \quad \alpha(u,v) = \int_\Omega \langle \nabla u, \nabla v\rangle \, dV. \tag{5.2.18}$$

Again, elements of $\mathrm{Ker}\,\alpha$ are constant on each connected component of Ω, so $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\tau = 0$, and (A.16) holds. Now, as in (A.17) (and parallel to (4.2.6)), we set

$$\begin{aligned} \beta(u,v) &= \alpha(u,v) + (\tau u, \tau v)_H \\ &= \int_\Omega \langle \nabla u, \nabla v\rangle \, dV + \int_{\partial_*\mathcal{O}} uv \, d\sigma \\ &\quad + \int_S u_1 v_1 \, d\sigma + \int_S u_2 v_2 \, d\sigma. \end{aligned} \tag{5.2.19}$$

Thanks to (5.2.13), this is a positive-definite inner product on W_0, and as in (A.18), we denote by $W^\#$ its Hilbert space completion, to which τ extends continuously:

$$\tau^\# : W^\# \longrightarrow H. \tag{5.2.20}$$

Parallel to (4.2.11), we identify $W^\#$ with the closure of

$$\mathcal{G} = \{(u, \tau u) : u \in \mathrm{Lip}_\pi(\Omega)\} \ \text{ in } \ H^1(\Omega) \oplus H, \tag{5.2.21}$$

and note that (5.2.13) yields

$$(u,f)\in\overline{\mathcal{G}}=W^{\#} \implies \int_\Omega |u|^2\,dV \le C\|\nabla u\|^2_{L^2(\Omega)}+C\|f\|^2_H. \tag{5.2.22}$$

Thus

$$\beta\big((u,f),(v,g)\big)=(\nabla u,\nabla v)_{L^2(\Omega)}+(f,g)_H \tag{5.2.23}$$

is a Hilbert space inner product on $W^{\#}$.

To continue, as in (A.19) (and (4.2.14)) we form

$$V=W^{\#}/\operatorname{Ker}\tau^{\#}, \tag{5.2.24}$$

which inherits from $\tau^{\#}$ a continuous injection

$$J:V\longrightarrow H, \tag{5.2.28}$$

whose image contains the image of τ in (5.2.17), and hence is dense (assuming S is thick). The space V has a Hilbert space structure, naturally isomorphic to the orthogonal complement of $\operatorname{Ker}\tau^{\#}$ in $W^{\#}$, with respect to the inner product in (5.2.23). Parallel to (4.2.16)–(4.2.18), we can write

$$V=\Big\{(u,f)\in W^{\#}:\int_\Omega\langle\nabla u,\nabla v\rangle\,dV=0,\ \forall\,v\in\overset{\circ}{H}{}^1(\Omega)\Big\}, \tag{5.2.29}$$

where

$$\overset{\circ}{H}{}^1(\Omega)=\{v\in H^1(\Omega):(v,0)\in W^{\#}\}. \tag{5.2.30}$$

Parallel to (4.2.18B), we have

$$H^1_0(\Omega)\subset\overset{\circ}{H}{}^1(\Omega), \tag{5.2.31}$$

where $H^1_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. As in (4.2.19),

$$(u,f)\in V\Longrightarrow \Delta u=0 \ \text{ on } \ \Omega. \tag{5.2.32}$$

Also, as in (4.2.19A),

$$H^1_0(\Omega)=\overset{\circ}{H}{}^1(\Omega)\Longrightarrow V=\{(u,f)\in W^{\#}:\Delta u=0\text{ on }\Omega\}. \tag{5.2.33}$$

From the injection (5.2.28), the Friedrichs method yields a positive self adjoint operator B on H, as described in Appendix A. We have $\operatorname{Spec}B\subset[1,\infty)$, and we set

$$N=B-1, \tag{5.2.34}$$

as in (4.2.23), so N is self adjoint and $\operatorname{Spec}N\subset[0,\infty)$. As in (4.2.22)–(4.2.24),

$$(u,f),(v,g)\in V\Longrightarrow\langle Nf,g\rangle=\int_\Omega\langle\nabla u,\nabla v\rangle\,dV. \tag{5.2.35}$$

Also, parallel to (4.2.25),

$$\mathcal{D}(N^{1/2})=\tau^{\#}(W^{\#}). \tag{5.2.36}$$

We have the following parallel to Proposition 4.2.1.

Proposition 5.2.3. *Given $f \in H$, we have $f \in \mathcal{D}(N^{1/2})$ if and only if there exist $u_k \in \mathrm{Lip}_\pi(\Omega)$ and $u \in H^1(\Omega)$ such that*

$$\tau u_k \longrightarrow f \quad in \quad H = L^2(\partial\mathcal{O},\sigma) \oplus L^2(S,\sigma) \oplus L^2(S,\sigma), \tag{5.2.37}$$

and

$$u_k \longrightarrow u \quad in \quad H^1(\Omega). \tag{5.2.38}$$

In such a case, we can pick (u_k) such that

$$\Delta u = 0 \quad on \quad \Omega. \tag{5.2.39}$$

Next, parallel to (4.2.31), we have, for $f \in \mathcal{D}(N^{1/2})$,

$$\langle N^{1/2} f, N^{1/2} f\rangle = \inf\Big\{ \int_\Omega |\nabla u|^2 \, dV : (u,f) \in W^\# \Big\}, \tag{5.2.40}$$

where $\langle\ ,\ \rangle$ denotes the inner product in H. Furthermore, as in §4.2, this inf is achieved, so, for $f \in \mathcal{D}(N^{1/2})$,

$$f \in \operatorname{Ker} N \Longleftrightarrow (u,f) \in W^\# \text{ for some } u \in H^1(\Omega) \text{ satisfying } \nabla u \equiv 0. \tag{5.2.41}$$

This gives the following, parallel to Proposition 4.2.2.

Proposition 5.2.4. *If S is thick and Ω is connected, then*

$$\operatorname{Ker} N = \mathrm{Span}(1) + \mathfrak{Z}(\Omega,\sigma), \tag{5.2.42}$$

where

$$\mathfrak{Z}(\Omega,\sigma) = \{ f \in L^2(\partial\mathcal{O} \cup S_1 \cup S_2, \sigma) : (0,f) \in W^\# \}. \tag{5.2.43}$$

Now the results of §4.3 extend readily to the current setting. The operator N yields a contraction semigroup e^{-tN} on $H = L^2(\partial\mathcal{O} \cup S_1 \cup S_2, \sigma)$, and the arguments proving Propositions 4.3.1–4.3.2 extend, to give:

Proposition 5.2.5. *Take Ω as in Proposition 5.2.2, and assume S is thick. Then, given $f \in H = L^2(\partial\mathcal{O} \cup S_1 \cup S_2, \sigma)$,*

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0, \quad \forall\, t > 0. \tag{5.2.44}$$

Furthermore, $e^{-tN} 1 \equiv 1$, so e^{-tN} is a symmetric Markov semigroup. In addition, if Ω is connected and $\mathfrak{Z}(\Omega,\sigma) = 0$, then $\{e^{-tN} : t \geq 0\}$ is irreducible.

Regarding the condition $\mathfrak{Z}(\Omega,\sigma) = 0$, we have the following variant of Proposition 4.5.1.

Proposition 5.2.6. *In the setting of Proposition 5.2.5, assume there is a vector field* $Y \in \mathrm{Lip}_\pi(\Omega)$ *such that*

$$\begin{aligned} \langle \nu, Y \rangle &> 0, \quad \sigma\text{-}a.e. \ on \ \ \partial\mathcal{O}, \\ \langle \nu_j, Y \rangle &> 0, \quad \sigma\text{-}a.e. \ on \ \ S_j. \end{aligned} \tag{5.2.45}$$

Here, the limit $Y|_{S_j}$ *is taken from inside* Ω_j*. Then* $\mathfrak{Z}(\Omega, \sigma) = 0$*.*

Proof. From (5.2.10), with $X = |u|^2 Y$, we have the following variant of (5.2.15):

$$\begin{aligned} &\int\limits_{\partial\mathcal{O}} |u|^2 \langle \nu, Y \rangle \, d\sigma + \int\limits_{S_1} |u|^2 \langle \nu_1, Y \rangle \, d\sigma + \int\limits_{S_2} |u|^2 \langle \nu_2, Y \rangle \, d\sigma \\ &\quad = \int\limits_{\Omega} |u|^2 (\operatorname{div} Y) \, dV + 2 \int\limits_{\Omega} u \langle \nabla u, Y \rangle \, dV, \end{aligned} \tag{5.2.46}$$

for each $u \in \mathrm{Lip}_\pi(\Omega)$. Now suppose $f \in \mathfrak{Z}(\Omega, \sigma)$. Then there exist $u_k \in \mathrm{Lip}_\pi(\Omega)$ such that

$$u_k \longrightarrow 0 \text{ in } H^1(\Omega), \quad \tau u_k \longrightarrow f \text{ in } L^2(\partial\mathcal{O} \cup S_1 \cup S_2, \sigma). \tag{5.2.47}$$

By (5.2.46),

$$\begin{aligned} &\int\limits_{\partial\mathcal{O}} |u_k|^2 \langle \nu, Y \rangle \, d\sigma + \int\limits_{S_1} |u_k|^2 \langle \nu_1, Y \rangle \, d\sigma \\ &\qquad + \int\limits_{S_2} |u_k|^2 \langle \nu_2, Y \rangle \, d\sigma \le C \|u_k\|^2_{H^1(\Omega)} \longrightarrow 0. \end{aligned} \tag{5.2.48}$$

But the left side of (5.2.48) tends to the limit

$$\int\limits_{\partial\mathcal{O}} |f|^2 \langle \nu, Y \rangle \, d\sigma + \int\limits_{S_1} |f|^2 \langle \nu_1, Y \rangle \, d\sigma + \int\limits_{S_2} |f|^2 \langle \nu_2, Y \rangle \, d\sigma. \tag{5.2.49}$$

If (5.2.45) holds, this forces $f = 0$.

5.3. Fractal inclusions with positive capacity

Let $\mathcal{O} \subset M$ be an open subset of the compact, connected, n-dimensional Riemannian manifold M. Assume $\mathcal{O}$ is a finite perimeter domain and $M \setminus \overline{\mathcal{O}} \neq \emptyset$. Take

$$K \subset \mathcal{O}, \quad K \text{ compact}, \quad \mathcal{H}^n(K) = 0, \tag{5.3.1}$$

and set

$$\Omega = \mathcal{O} \setminus K. \tag{5.3.2}$$

Then Ω is a finite perimeter domain,

$$\partial_* \Omega = \partial_* \mathcal{O}, \quad \partial \Omega = \partial \mathcal{O} \cup K, \tag{5.3.3}$$

and $K \subset \overline{\Omega}$ is an inclusion.

We define the capacity

$$\operatorname{Cap}(K) = \inf\{\|u\|^2_{H^1(M)} : u \in \operatorname{Lip}(M),\ u \geq 1 \text{ on a neighborhood of } K\}, \tag{5.3.4}$$

and, following [AW] and [AtE], the relative capacity

$$\operatorname{Cap}_{\overline{\Omega}}(K) = \inf\{\|u\|^2_{H^1(\Omega)} : u \in \operatorname{Lip}(\overline{\Omega}),\ u \geq 1 \text{ on a neighborhood of } K\}. \tag{5.3.5}$$

Clearly

$$\operatorname{Cap}_{\overline{\Omega}}(K) \leq \operatorname{Cap}(K). \tag{5.3.6}$$

Also, in the setting (5.3.1)–(5.3.2), it is straightforward that

$$\operatorname{Cap}(K) = 0 \Longleftrightarrow \operatorname{Cap}_{\overline{\Omega}}(K) = 0, \tag{5.3.7}$$

by considering $u \mapsto \psi u$, with $\psi \in C_0^\infty(\mathcal{O})$, $\psi = 1$ on a neighborhood of K. We remark that, for $n \geq 2$,

$$\begin{aligned} \mathcal{H}^{n-2}(K) < \infty &\Longrightarrow \operatorname{Cap}(K) = 0, \\ \operatorname{Cap}(K) = 0 &\Longrightarrow \mathcal{H}^s(K) = 0, \quad \forall\, s > n-2. \end{aligned} \tag{5.3.8}$$

Cf. [EG], §4.7. In particular,

$$\mathcal{H}^s(K) > 0 \text{ for some } s \in (n-2, n) \Longrightarrow \operatorname{Cap}(K) > 0. \tag{5.3.9}$$

Now let μ be a measure on $\partial\Omega$ satisfying $\mu \geq \sigma$, and assume

$$\operatorname{supp}\mu \supset K. \tag{5.3.10}$$

Construct the self adjoint operator $N_\mu = N$ on $L^2(\partial\Omega, \mu)$ by the methods of §4.2. It is clear that

$$\begin{aligned} \operatorname{Cap}_{\overline{\Omega}}(K) = 0 &\Longrightarrow f\chi_K \in \mathfrak{Z}(\Omega, \mu) \\ &\Longrightarrow N_\mu(f\chi_K) = 0, \quad \forall\, f \in \operatorname{Lip}(K). \end{aligned} \tag{5.3.11}$$

Thus we want to confine our attention to sets K as in (5.3.1) that satisfy $\operatorname{Cap}(K) > 0$, and look for measures $\mu \geq \sigma$ such that $N_\mu(f\chi_K) \neq 0$.

To proceed, given Ω and K as in (5.3.1)–(5.3.2) and a positive, finite Borel measure μ on $\partial\Omega$, we say

$$\begin{aligned} &\mu \text{ is Cap-regular on } K \text{ provided that, if } K_0 \text{ is compact,} \\ &K_0 \subset K, \quad \operatorname{Cap}(K_0) = 0 \Longrightarrow \mu(K_0) = 0. \end{aligned} \tag{5.3.12}$$

If the hypothesis in (5.3.12) holds for all compact $K_0 \subset K$, then it holds for all Borel $K_0 \subset K$, since μ is regular. The following result is essentially a consequence of Theorem 3.3 of [AW], but due to our hypotheses on K, the proof simplifies.

Proposition 5.3.1. *If K and Ω are as in (5.3.1)–(5.3.2), and μ is Cap-regular on K, then*

$$L^2(K, \mu) \cap \mathfrak{Z}(\Omega, \mu) = 0. \tag{5.3.13}$$

Proof. Assume $f \in L^2(K, \mu) \cap \mathfrak{Z}(\Omega, \mu)$. Take

$$u_k \in \operatorname{Lip}(\overline{\Omega}), \quad u_k \to 0 \text{ in } H^1(\Omega), \quad u_k\big|_K \to f \text{ in } L^2(K, \mu). \tag{5.3.14}$$

Since $\mathcal{H}^n(K) = 0$, $\|u_k\|_{H^1(\mathcal{O})} = \|u_k\|_{H^1(\Omega)}$, so

$$u_k \longrightarrow 0 \ \text{ in } \ H^1(\mathcal{O}). \tag{5.3.15}$$

Passing to a subsequence, we can assume

$$u_k \longrightarrow 0, \ \text{ q.e. on } \ \mathcal{O} \tag{5.3.16}$$

(cf. [EG], §4.8), and passing to a further subsequence, we can assume

$$u_k \longrightarrow f, \ \ \mu\text{-a.e. on } \ K. \tag{5.3.17}$$

Consequently, there exists a Borel set $E \subset K$ such that

$$\operatorname{Cap}(E) = 0 \ \text{ and } \ u_k(x) \to 0, \ \ \forall\, x \in K \setminus E. \tag{5.3.18}$$

Our hypotheses imply $\mu(E) = 0$, so $f = 0$, μ-a.e. on K. This proves (5.3.13).

We have the following result on the degree to which the action of N_μ is affected by the presence of K. (Recall from (4.2.25) that $\operatorname{Lip}(K) \subset \mathcal{D}(N_\mu^{1/2})$.)

Proposition 5.3.2. *Take the setting of Proposition 5.3.1, and assume that* Ω *is connected. Then*

$$\dim \operatorname{Ker} N_\mu \cap L^2(K,\mu) \le 1. \tag{5.3.19}$$

Proof. Take two linearly independent $f_j \in L^2(K,\mu)$. If $f_j \in \operatorname{Ker} N_\mu$, then, by Proposition 4.2.2,

$$f_j = c_j + g_j, \quad c_j \in \mathbb{R}, \quad g_j \in \mathfrak{Z}(\Omega,\mu). \tag{5.3.20}$$

If $c_j = 0$, then $g_j \in \mathfrak{Z}(\Omega,\mu) \cap L^2(K,\mu)$, so also $g_j = 0$. If c_1 and c_2 are not both 0, consider

$$c_2 f_1 - c_1 f_2 = c_2 g_1 - c_1 g_2 \in \mathfrak{Z}(\Omega,\mu) \cap L^2(K,\mu). \tag{5.3.21}$$

Again, by Proposition 5.3.1, this forces $c_2 f_1 - c_1 f_2 = 0$, a contradiction.

We now look at some examples to which Proposition 5.3.2 applies. Assume (5.3.1)–(5.3.2). Suppose

$$0 < \mathcal{H}^s(K) < \infty, \quad \text{for some } \ s \in (n-2,n), \tag{5.3.22}$$

and take $\mu \ge \sigma$ such that

$$\mu \lfloor K = \mathcal{H}^s \lfloor K. \tag{5.3.23}$$

If necessary, shrink K so that (5.3.10) holds. The result (5.3.9), applied to compact subsets of K, implies that μ is Cap-regular on such K.

There are many examples of compact sets K satisfying (5.3.22), from various Cantor sets to arcs of Koch snowflakes (cf. [AW], Example 4.4). However, many compact sets with positive capacity, even those with Hausdorff dimension $s \in (n-2,n)$, do not satisfy (5.3.22). It is therefore of interest to know that, whenever K is compact and $\operatorname{Cap}(K) > 0$, there is a nontrivial measure that is Cap-regular on K.

A simple family of examples is provided by harmonic measure on $\partial\Omega$. Given $p \in \Omega$, the harmonic measure $\omega = \omega_p$ is defined by

$$\int_{\partial\Omega} f \, d\omega_p = \mathrm{PI}_0 \, f(p), \quad f \in C(\partial\Omega), \tag{5.3.24}$$

with PI_0 as in (4.4.14)–(4.4.18). As is well known (see §7.1 for cases where more precise results are available),

$$\operatorname{Cap}(K) > 0 \Longrightarrow \omega_p(K) > 0, \tag{5.3.25}$$

and one has the following.

Proposition 5.3.3. *In the setting of Proposition 5.3.1, with* $\operatorname{Cap}(K) > 0$, *it follows that* ω_p *is Cap-regular on* K. *Hence Proposition 5.3.1 applies for* $\mu \ge \sigma$ *satisfying* $\mu \lfloor K = \omega_p \lfloor K$.

6. Uniformly rectifiable domains

As before we assume Ω is an open subset of a compact, n-dimensional Riemannian manifold M, satisfying $M \setminus \overline{\Omega} \neq \emptyset$. Here we work with uniformly rectifiable domains (UR domains, for short), defined as follows. First, we assume Ω is a finite perimeter domain, and that

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{6.0.1}$$

In this case, (4.0.4) becomes $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$. Next, we assume there exist $c_j \in (0, \infty)$ such that

$$c_0 r^{n-1} \leq \sigma\big(B_r(x_0) \cap \partial\Omega\big) \leq c_1 r^{n-1}, \quad \forall\, x_0 \in \partial\Omega, \tag{6.0.2}$$

for $r \in (0, 1]$. We call such Ω an Ahlfors regular domain. An Ahlfors regular domain is said to be a UR domain if, in addition, $\partial\Omega$ contains "big pieces of Lipschitz surfaces," in the sense that there exist $\varepsilon, L \in (0, \infty)$ such that, for each $x \in \partial\Omega$, $R \in (0, 1]$, there is a Lipschitz map $\varphi : B_R^{n-1} \to M$, with Lipschitz constant $\leq L$, such that

$$\mathcal{H}^{n-1}\big(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})\big) \geq \varepsilon R^{n-1}. \tag{6.0.3}$$

Here, B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$.

The class of UR domains is a natural class on which to study layer potentials, thanks to the following results. Assume $G \in OPS^{-1}(M)$ is a pseudodifferential operator of order -1, with *odd* principal symbol, and integral kernel $K(x, y)$, so

$$Gu(x) = \int_M K(x, y) u(y)\, dV(y), \quad u \in C^\infty(M). \tag{6.0.4}$$

(More generally, we can take such G to belong to $OPC^0S^{-1}(M)$; cf. [HMT].) Consider the "principal value" singular integral

$$\begin{aligned} Bf(x) &= \mathrm{PV} \int_{\partial\Omega} K(x, y) f(y)\, d\sigma(y) \\ &= \lim_{\varepsilon \searrow 0} \int_{\partial\Omega \setminus B_\varepsilon(x)} K(x, y) f(y)\, d\sigma(y). \end{aligned} \tag{6.0.5}$$

Then

$$B : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \forall\, p \in (1, \infty). \tag{6.0.6}$$

This was demonstrated in [D] when $M = \mathbb{R}^n$ and G is a convolution operator. Also [D] established associated L^p-estimates on the maximal function

$$\sup_{0<\varepsilon\le 1} \Big| \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} K(x,y) f(y)\, d\sigma(y) \Big|, \tag{6.0.7}$$

in the convolution setting. In [HMT] this was extended to the variable coefficient setting, and to manifolds. Also [HMT] studied the "double layer potential"

$$\mathcal{B} f(x) = \int\limits_{\partial\Omega} K(x,y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{6.0.8}$$

supplemented estimates on (6.0.7) with the nontangential maximal function estimate

$$\|(\mathcal{B} f)^*\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{6.0.9}$$

and established nontangential a.e. convergence

$$\mathcal{B} f\big|_{\partial\Omega}(x) = \frac{1}{2i} \sigma_G(x, \nu(x)) + B f(x), \quad \text{a.e. } x \in \partial\Omega, \tag{6.0.10}$$

where $\sigma_G(x,\xi)$ is the principal symbol of G and B is as in (6.0.5)–(6.0.6).

These results have important consequences for the single layer potential

$$\mathcal{S} f(x) = \int\limits_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{6.0.11}$$

defined as in Chapters 2–3, as follows. The function $E(x,y)$ is the integral kernel of L^{-1}, with $L = \Delta - V$, where Δ is the Laplace operator on M and $V \in C^\infty(M)$ satisfies $V \ge 0$ on M, $V = 0$ on $\overline{\Omega}$, and $V > 0$ on a nonempty subset of each connected component of $M \setminus \overline{\Omega}$. Thus $L^{-1} \in OPS^{-2}(M)$ has even principal symbol, so $\nabla L^{-1} \in OPS^{-1}(M)$ has odd principal symbol, and the results (6.0.9)–(6.0.10) apply to $\mathcal{B} = \nabla \mathcal{S}$. We have

$$\|(\nabla \mathcal{S} f)^*\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{6.0.12}$$

In particular, if X is a smooth vector field on M, $X\mathcal{S}$ is of the form (6.0.8) with $K(x,y) = X_x E(x,y)$, and (6.0.10) yields, for $f \in L^p(\partial\Omega)$, $1 < p < \infty$,

$$X \mathcal{S} f\big|_{\partial\Omega}(x) = -\frac{1}{2} \langle X, \nu(x) \rangle f(x) + B_X f(x), \quad \text{a.e. } x \in \partial\Omega, \tag{6.0.13}$$

since $\sigma_{XL^{-1}}(x,\xi) = -i\langle X,\xi\rangle|\xi|^{-2}$. Here

$$B_X f(x) = \mathrm{PV}\int_{\partial\Omega} X_x E(x,y) f(y)\, d\sigma(y). \tag{6.0.14}$$

Taking linear combinations (with measurable coefficients), we get

$$\partial_\nu \mathcal{S} f\big|_{\partial\Omega}(x) = \Big(-\frac{1}{2}I + A^*\Big) f(x), \quad \text{a.e. } x \in \partial\Omega, \tag{6.0.15}$$

for $f \in L^p(\partial\Omega),\ 1 < p < \infty$, where

$$A^* f(x) = \mathrm{PV}\int_{\partial\Omega} \partial_{\nu_x} E(x,y) f(y)\, d\sigma(y), \tag{6.0.16}$$

which defines the bounded linear operator

$$A^* : L^p(\partial\Omega) \to L^p(\partial\Omega), \quad 1 < p < \infty. \tag{6.0.17}$$

Closely related to $\nabla\mathcal{S}$ and (6.0.15)–(6.0.17) is the following double layer potential:

$$\mathcal{D} f(x) = \int_{\partial\Omega} \partial_{\nu_y} E(x,y) f(y)\, d\sigma(y), \quad x \in \Omega. \tag{6.0.18}$$

We have

$$\|(\mathcal{D}f)^*\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{6.0.19}$$

and nontangential limits, for $f \in L^p(\partial\Omega),\ 1 < p < \infty$, given by

$$\mathcal{D} f\big|_{\partial\Omega}(x) = \Big(\frac{1}{2}I + A\Big) f(x), \quad \text{a.e. } x \in \partial\Omega, \tag{6.0.20}$$

where A is the adjoint of A^* in (6.0.16), as a consequence of the symmetry $E(x,y) = E(y,x)$.

Our goal here is to use these layer potentials to obtain information on the operator N, constructed in Chapter 4, when Ω is a UR domain. This task is tackled in §6.2, following some further layer potential estimates, established in §6.1, which will be useful for the task. In §6.2 we show that, if Ω is a UR domain of dimension n,

$$S : L^p(\partial\Omega,\sigma) \longrightarrow \mathcal{D}(N^{1/2}), \tag{6.0.21}$$

provided $p \geq 2(n-1)/n$ for $n \geq 3$, or $p > 1$ for $n = 2$. Under certain further conditions, namely that $\overset{\circ}{H}{}^1(\Omega) = H_0^1(\Omega)$ and that $S(L^2(\partial\Omega, \sigma))$ is dense in $\mathcal{D}(N^{1/2})$ (for example, contains $\mathrm{Lip}(\partial\Omega)$), we show that

$$S : L^2(\partial\Omega, \sigma) \longrightarrow \mathcal{D}(N), \tag{6.0.22}$$

and

$$N(Sf) = \partial_\nu \mathcal{S} f\big|_{\partial\Omega}, \quad \forall\, f \in L^2(\partial\Omega, \sigma). \tag{6.0.23}$$

We also give a sufficient condition to guarantee that S is an isomorphism in (6.0.22).

In §6.3, we revisit the class of Lipschitz domains. We show that, for this class of domains, and with N defined as in Chapter 4, we have (6.0.22)–(6.0.23), with isomorphism in (6.0.22). We conclude that, for Lipschitz domains, the definition of N given in Chapter 3 is consistent with that given in Chapter 4.

In §6.4 we treat N on another special class of UR domains, called regular SKT domains. This class was introduced in [Se] and [KT], and studied in a number of places, including [HMT] (which proposed our current label). See §6.4 for a definition. We obtain (6.0.22)–(6.0.23) for this class (subject to the hypotheis that $\overset{\circ}{H}{}^1(\Omega) = H_0^1(\Omega)$), and use this to derive further properties of N in this setting, including spectral properties parallel to those obtained for Lipschitz domains in §3.4.

6.1. Further layer potential estimates

We produce further estimates on the single layer potential

$$\mathcal{S}f(x) = \int_{\partial\Omega} E(x,y)f(y)\,d\sigma(y), \tag{6.1.1}$$

defined after (6.0.11). In the next two propositions, it is desirable to take $x \in M$, rather than merely $x \in \Omega$. While we concentrate on UR domains, our first result actually applies more broadly.

Proposition 6.1.1. *Let $\Omega \subset M$ be an Ahlfors regular domain. Then*

$$\mathcal{S} : L^\infty(\partial\Omega) \longrightarrow C^\omega(M), \tag{6.1.2}$$

where, for $h \in (0, 1/e]$,

$$\omega(h) = h \log \frac{1}{h}. \tag{6.1.3}$$

Here, we say

$$u \in C^\omega(M) \Longleftrightarrow |u(x) - u(z)| \le C\omega(d(x,z)), \tag{6.1.4}$$

for $x, z \in M$.

Proof. Take $x, z \in M$, and set $\varepsilon = d(x,z)$. We desire to estimate

$$\mathcal{S}f(x) - \mathcal{S}f(z) = \int_{\partial\Omega} \{E(x,y) - E(z,y)\}f(y)\,d\sigma(y). \tag{6.1.5}$$

We have

$$\begin{aligned}|\mathcal{S}f(x) - \mathcal{S}f(z)| \le \|f\|_{L^\infty(\partial\Omega)} \Big\{ &\int_{\partial\Omega\setminus B_{2\varepsilon}(x)} |E(x,y) - E(z,y)|\,d\sigma(y) \\ &+ \int_{\partial\Omega\cap B_{2\varepsilon}(x)} \big(|E(x,y)| + |E(z,y)|\big)\,d\sigma(y)\Big\}.\end{aligned} \tag{6.1.6}$$

Now, for $x, y \in M$,

$$\begin{aligned}|E(x,y)| &\le Cd(x,y)^{-(n-2)}, \quad && n \ge 3, \\ &\quad C \log \frac{K}{d(x,y)}, && n = 2,\end{aligned} \tag{6.1.7}$$

where $K = 2\,\mathrm{diam}\,M$. Also,

$$|\nabla_x E(x,y)| \le C d(x,y)^{-(n-1)}. \tag{6.1.8}$$

Since $d(\tilde{x},y)$ has size comparable to $d(x,y)$ for all $\tilde{x}$ on the geodesic segment from x to z, when $y \in \partial\Omega \setminus B_{2\varepsilon}(x)$ and $d(x,z) = \varepsilon$, we bound the first integral on the right side of (6.1.6) by

$$C\varepsilon \int\limits_{\partial\Omega\setminus B_{2\varepsilon}(x)} d(x,y)^{-(n-1)}\, d\sigma(y), \tag{6.1.9}$$

and, using Ahlfors regularity, we bound this by

$$C\varepsilon \int_{2\varepsilon}^{K} \frac{ds}{s} = C\varepsilon \log \frac{K}{2\varepsilon}. \tag{6.1.10}$$

A similar analysis applies to the second integral on the right side of (6.1.6), yielding (6.1.2).

Proposition 6.1.2. *Let $\Omega \subset M$ be a UR domain. Then*

$$\mathcal{S} : L^\infty(\partial\Omega) \longrightarrow H^{1,p}(M), \quad \forall\, p < \infty. \tag{6.1.11}$$

Proof. We continue to have (6.0.12) if u^* denotes the nontangential maximal function for $\partial\Omega \subset M$. This result implies that

$$f \in L^\infty(\partial\Omega) \Longrightarrow \nabla\mathcal{S}f \in L^p(\Omega) \oplus L^p(\Omega^-), \quad \forall\, p < \infty, \tag{6.1.12}$$

where $\Omega^- = M \setminus \overline{\Omega}$. Say $g = \nabla\mathcal{S}f|_{\Omega\cup\Omega^-}$. We can regard g as an element of $L^p(M)$, for each $p < \infty$. Meanwhile, Proposition 6.1.1 implies that, if $f \in L^\infty(\partial\Omega)$,

$$\nabla\mathcal{S}f \in H^{-\varepsilon,p}(M), \quad \forall\, \varepsilon > 0,\ p < \infty. \tag{6.1.13}$$

Therefore,

$$v = \nabla\mathcal{S}f - g \Longrightarrow v \in \bigcap_{\varepsilon>0,p<\infty} H^{-\varepsilon,p}(M) \ \text{ and } \ \mathrm{supp}\, v \subset \partial\Omega. \tag{6.1.14}$$

But then

$$\mathcal{H}^{n-1}(\partial\Omega) < \infty \Longrightarrow v = 0 \Longrightarrow \nabla\mathcal{S}f = g, \tag{6.1.15}$$

and we have (6.1.11).

For the next result, recall from (4.2.11) (with $\mu = \sigma$) that

$$\begin{gathered} W^{\#} = \text{closure in } H^1(\Omega) \oplus L^2(\partial\Omega,\sigma) \text{ of} \\ \{(u, u|_{\partial\Omega}) : u \in \mathrm{Lip}(\overline{\Omega})\}. \end{gathered} \tag{6.1.16}$$

Propositon 6.1.3. *Let $\Omega \subset M$ be a UR domain. If we set*

$$\mathcal{S}^{\#} f = (\mathcal{S}f, Sf), \tag{6.1.17}$$

then

$$\mathcal{S}^{\#} : L^2(\partial\Omega, \sigma) \longrightarrow W^{\#}. \tag{6.1.18}$$

Proof. We know that

$$\mathcal{S}^{\#} : L^2(\partial\Omega, \sigma) \longrightarrow H^1(\Omega) \oplus L^2(\partial\Omega, \sigma). \tag{6.1.19}$$

Since $L^\infty(\partial\Omega)$ is dense in $L^2(\partial\Omega, \sigma)$, it suffices to show that

$$\begin{aligned} &f \in L^\infty(\partial\Omega) \Rightarrow \exists\, w_\nu \in \mathrm{Lip}(\overline{\Omega}) \text{ such that} \\ &\qquad\qquad w_\nu \to \mathcal{S}f \text{ in } H^1(\Omega) \text{ and } w_\nu\big|_{\partial\Omega} \to Sf \text{ in } L^2(\partial\Omega, \sigma). \end{aligned} \tag{6.1.20}$$

We have seen that

$$f \in L^\infty(\partial\Omega) \Longrightarrow \mathcal{S}f \in C^\omega(M) \cap H^{1,p}(M), \quad \forall\, p < \infty, \tag{6.1.21}$$

so setting $w_\nu = \Phi_\nu(\mathcal{S}f)$, where Φ_ν is a standard mollifier on M, yields

$$\begin{aligned} w_\nu &\longrightarrow \mathcal{S}f \text{ in } H^{1,p}(M), \quad \forall\, p < \infty, \text{ and} \\ w_\nu\big|_{\partial\Omega} &\longrightarrow Sf \text{ uniformly on } \partial\Omega, \end{aligned} \tag{6.1.22}$$

which is sufficient.

Recalling that $\overline{H}^1(\Omega)$ is the closure of $\mathrm{Lip}(\overline{\Omega})$ in $H^1(\Omega)$, we have:

Corollary 6.1.4. *Let $\Omega \subset M$ be a UR domain. Then*

$$\mathcal{S} : L^2(\partial\Omega, \sigma) \longrightarrow \overline{H}^1(\Omega). \tag{6.1.23}$$

We note that (6.1.19) can be improved, and this leads to improvements of Proposition 6.1.3 and Corollary 6.1.4. In fact, as shown in §3.2 of [MMT], given that $n = \dim M$,

$$\|u\|_{L^{pn/(n-1)}(\Omega)} \le C\|u^*\|_{L^p(\partial\Omega,\sigma)}, \tag{6.1.24}$$

for $p \in [1, \infty)$, as long as Ω is Ahlfors regular. Also, given the estimate (6.1.7), use of Ahlfors regularity implies that

$$S : L^1(\partial\Omega, \sigma) \longrightarrow L^q(\partial\Omega, \sigma), \quad \forall\, q \in \Big[1, \frac{n-1}{n-2}\Big), \tag{6.1.25}$$

and, by duality,

$$S : L^r(\partial\Omega, \sigma) \longrightarrow L^\infty(\partial\Omega), \quad \forall\, r > n-1. \tag{6.1.26}$$

Interpolation gives

$$S : L^r(\partial\Omega, \sigma) \longrightarrow L^2(\partial\Omega, \sigma), \quad \forall\, r > \frac{2(n-1)}{n+1} \text{ (and } r \geq 1). \tag{6.1.27}$$

It follows that

$$\mathcal{S}^\# : L^{2(n-1)/n}(\partial\Omega, \sigma) \longrightarrow H^1(\Omega) \oplus L^2(\partial\Omega, \sigma), \quad \text{for } \ n \geq 3, \tag{6.1.28}$$

and

$$\mathcal{S}^\# : L^p(\partial\Omega, \sigma) \longrightarrow H^1(\Omega) \oplus L^2(\partial\Omega, \sigma), \quad \forall\, p > 1, \quad \text{if } \ n = 2. \tag{6.1.29}$$

The conclusion (6.1.18) is accordingly improved:

Proposition 6.1.5. *In the setting of Proposition 6.1.3,*

$$\mathcal{S}^\# : L^p(\partial\Omega, \sigma) \longrightarrow W^\#, \tag{6.1.30}$$

for

$$\begin{aligned} p &\geq \frac{2(n-1)}{n} && \textit{if } \ n \geq 3, \\ p &> \quad 1 && \textit{if } \ n = 2. \end{aligned} \tag{6.1.31}$$

Consequently,

$$\mathcal{S} : L^p(\partial\Omega, \sigma) \longrightarrow \overline{H}^1(\Omega), \tag{6.1.32}$$

for such p.

For later use, we record the following result.

Proposition 6.1.6. *If $\Omega \subset M$ is a UR domain, then*

$$S : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial\Omega, \sigma) \ \ \textit{is injective}, \quad \forall\, p \in (1, \infty). \tag{6.1.33}$$

Proof. Assume $f \in L^p(\partial\Omega, \sigma)$ and $Sf = 0$. Set $u = \mathcal{S}f$. Then $u, \nabla u \in \mathfrak{L}^2(\Omega) \oplus \mathfrak{L}^2(\Omega^-)$ and

$$\Delta u = 0 \ \text{ on } \ \Omega, \quad (\Delta - V)u = 0 \ \text{ on } \ \Omega^-, \quad u\big|_{\partial\Omega} = 0. \tag{6.1.34}$$

Hence $u \equiv 0$, so

$$0 = \partial_\nu u\big|_{\partial\Omega^\pm} = \Big(\mp\frac{1}{2}I + A^*\Big)f. \tag{6.1.35}$$

Taking the difference of these two identities yields the desired result, $f = 0$.

The symmetry $E(x, y) = E(y, x)$ implies $S = S^*$, so we have the following:

Corollary 6.1.7. *In the setting of Proposition 6.1.6,*

$$S : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial\Omega, \sigma) \ \ \textit{has dense range}, \quad \forall\, p \in (1, \infty). \tag{6.1.36}$$

6.2. Behavior of N and PI on UR domains

Here $\Omega \subset M$ is a UR domain. The operator N is defined as in §4.2. Recall from (4.2.25) that

$$\begin{aligned} \mathcal{D}(N^{1/2}) &= \tau^{\#}(W^{\#}) \\ &= \{f \in L^2(\partial\Omega, \sigma) : (u, f) \in W^{\#} \text{ for some } u \in \overline{H}^1(\Omega)\}. \end{aligned} \tag{6.2.1}$$

We hence have from Proposition 6.1.5 the following.

Proposition 6.2.1. *If $\Omega \subset M$ is a UR domain of dimension n,*

$$S : L^p(\partial\Omega, \sigma) \longrightarrow \mathcal{D}(N^{1/2}), \tag{6.2.2}$$

for

$$\begin{aligned} p &\geq \frac{2(n-1)}{n} \quad && \text{if } \ n \geq 3, \\ p &> \quad 1 && \text{if } \ n = 2. \end{aligned} \tag{6.2.3}$$

Recall from §4.4 that we defined (in the setting of finite-perimeter domains)

$$\mathrm{PI} : \mathcal{D}(N^{1/2}) \longrightarrow \overline{H}^1(\Omega) \tag{6.2.4}$$

in (4.4.3)–(4.4.5) and showed in Proposition 4.4.1 that $C^\infty(\partial\Omega)$ is dense in $\mathcal{D}(N^{1/2})$. We also brought in

$$\mathrm{PI}_0 : C^\infty(\partial\Omega) \longrightarrow \overline{H}^1(\Omega) \tag{6.2.5}$$

in (4.4.14), as an operator that is positivity-preserving and has a unique continuous linear extension to

$$\mathrm{PI}_0 : C(\partial\Omega) \longrightarrow L^\infty(\Omega), \tag{6.2.6}$$

cf. (4.4.18). In Proposition 4.4.2 we showed that, if

$$\overset{\circ}{H}{}^1(\Omega) = H^1_0(\Omega), \tag{6.2.7}$$

where $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$ and $\overset{\circ}{H}{}^1(\Omega)$ is defined by (4.2.18A), then

$$\mathrm{PI}\, f = \mathrm{PI}_0\, f, \quad \forall\, f \in C^\infty(\partial\Omega). \tag{6.2.8}$$

We will make (6.2.7) a standing hypothesis for the rest of this section.

To take (6.2.8) further, let us set

$$T^{1,p}(\partial\Omega) = \{u|_{\partial\Omega} : u \in H^{1,p}(M)\}, \quad \text{given} \;\; p > n. \tag{6.2.9}$$

Then $T^{1,p}(\partial\Omega)$ is a Banach space, via

$$T^{1,p}(\partial\Omega) \approx H^{1,p}(M)/\{u \in H^{1,p}(M) : u|_{\partial\Omega} = 0\}. \tag{6.2.10}$$

It follows from (6.2.1) that

$$\forall\, p > n, \quad T^{1,p}(\partial\Omega) \subset \mathcal{D}(N^{1/2}), \tag{6.2.11}$$

and since $C^\infty(\partial\Omega)$ is dense in $\mathcal{D}(N^{1/2})$, so is $T^{1,p}(\partial\Omega)$, for $p > n$. The results described in (6.2.4)–(6.2.8) imply that, if (6.2.7) holds, then

$$\mathrm{PI}\, f = \mathrm{PI}_0\, f, \quad \forall\, f \in T^{1,p}(\partial\Omega),\ p > n. \tag{6.2.12}$$

We are now ready to bring in material from §6.1.

Proposition 6.2.2. *Assume $\Omega \subset M$ is a UR domain satisfying (6.2.7). Then*

$$\mathrm{PI}(Sf) = \mathcal{S}f \quad on \;\; \Omega, \quad \forall\, f \in L^\infty(\partial\Omega). \tag{6.2.13}$$

Proof. Given $f \in L^\infty(\partial\Omega)$, it follows from Proposition 6.1.2 that $\mathcal{S}f \in H^{1,p}(M)$ for all $p < \infty$, and hence that $Sf \in T^{1,p}(\partial\Omega)$ for all $p \in (n, \infty)$. We have

$$\mathcal{S}f = \mathrm{PI}_0(Sf), \quad \forall\, f \in L^\infty(\partial\Omega), \tag{6.2.14}$$

since the left side is a harmonic function in $C(\overline{\Omega})$ with boundary value equal to Sf. Then (6.2.13) follows from (6.2.12).

Using Proposition 6.2.1, we can extend the scope of Proposition 6.2.2.

Proposition 6.2.3. *In the setting of Proposition 6.2.2, if $p \ge p(n)$, where $p(n)$ is such that (6.2.3) holds, then*

$$p \ge p(n) \Longrightarrow \mathrm{PI}(Sf) = \mathcal{S}f, \quad \forall\, f \in L^p(\partial\Omega, \sigma). \tag{6.2.15}$$

Proof. Both sides of (6.2.15) are continuous in $f \in L^p(\partial\Omega)$, with values in $\overline{H}^1(\Omega)$, the left side by Proposition 6.2.1, and the right side by Proposition 6.1.5. Proposition 6.2.2 gives identity for f in the dense linear subspace $L^\infty(\partial\Omega)$.

To proceed, take $f, g \in L^p(\partial\Omega, \sigma),\ p \ge p(n)$, and set

$$\varphi = Sf,\ \psi = Sg, \quad \varphi, \psi \in \mathcal{D}(N^{1/2}). \tag{6.2.16}$$

Then

$$\langle N^{1/2}\varphi, N^{1/2}\psi\rangle = \int_{\Omega} \langle \nabla u, \nabla v\rangle \, dV, \tag{6.2.17}$$

with

$$u = \mathrm{PI}\,\varphi = \mathcal{S}f, \quad v = \mathrm{PI}\,\psi = \mathcal{S}g. \tag{6.2.18}$$

Now, by Proposition B.3,

$$f, g \in L^2(\partial\Omega, \sigma) \Longrightarrow \nabla u, \nabla v \in \mathfrak{L}^2(\Omega) \text{ and } v \in \mathfrak{L}^q(\Omega),\ q > 2, \tag{6.2.19}$$

and then Proposition B.2 gives

$$\int_{\Omega} \langle \nabla u, \nabla v\rangle \, dV = \int_{\partial\Omega} (\partial_\nu u) v \, d\sigma. \tag{6.2.20}$$

Taking into account (6.0.15), we get

$$\begin{aligned} \langle N^{1/2}(Sf), N^{1/2}(Sg)\rangle &= \int_{\partial\Omega} (\partial_\nu \mathcal{S}f)(Sg)\, d\sigma \\ &= \Big\langle \Big(-\frac{1}{2}I + A^*\Big) f, Sg \Big\rangle, \end{aligned} \tag{6.2.21}$$

for all $f, g \in L^2(\partial\Omega, \sigma)$, with A^* as in (6.0.16)–(6.0.17). This leads to the following result.

Proposition 6.2.4. *Assume $\Omega \subset M$ is a UR domain satisfying (6.2.7), and assume*

$$S(L^2(\partial\Omega, \sigma)) \ \text{ is dense in } \ \mathcal{D}(N^{1/2}). \tag{6.2.22}$$

Then

$$S : L^2(\partial\Omega, \sigma) \longrightarrow \mathcal{D}(N), \tag{6.2.23}$$

and

$$N(Sf) = \partial_\nu \mathcal{S}f\big|_{\partial\Omega} = \Big(-\frac{1}{2}I + A^*\Big) f, \quad \forall\, f \in L^2(\partial\Omega, \sigma). \tag{6.2.24}$$

Remark. A sufficient condition for (6.2.22) is

$$S(L^2(\partial\Omega, \sigma)) \supset \mathrm{Lip}(\partial\Omega). \tag{6.2.25}$$

We will see in §§6.3–6.4 cases where this holds.

The following result will be useful in the next two sections.

Proposition 6.2.5. *Let $\Omega \subset M$ be a UR domain satisfying (6.2.7) and (6.2.22). Assume*

$$-\frac{1}{2}I + A^* : L^2(\partial\Omega, \sigma) \longrightarrow L^2(\partial\Omega, \sigma) \quad \text{is Fredholm, of index } 0. \tag{6.2.26}$$

Then

$$S : L^2(\partial\Omega, \sigma) \longrightarrow \mathcal{D}(N) \quad \text{is an isomorphism.} \tag{6.2.27}$$

Proof. From (6.2.24), we have

$$(N+1)S = -\frac{1}{2}I + A^* + S \quad \text{on} \quad L^2(\partial\Omega, \sigma). \tag{6.2.28}$$

It is elementary that

$$S : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial\Omega, \sigma) \ \text{ is compact}, \quad \forall\, p \in (1, \infty), \tag{6.2.29}$$

so (6.2.26) implies that the right side of (6.2.28) is Fredholm of index 0 on $L^2(\partial\Omega, \sigma)$. On the other hand, Proposition 6.1.6 implies that the left side of (6.2.28) is injective on $L^2(\partial\Omega, \sigma)$. Thus the right side is *invertible* on $L^2(\partial\Omega, \sigma)$. Since $N+1 : \mathcal{D}(N) \to L^2(\partial\Omega, \sigma)$ is an isomorphism, this gives (6.2.27).

6.3. Lipschitz domains revisited

The special class of UR domains known as Lipschitz domains was described in §3. Our goal here is to show that the operator N, constructed in §4.2 and studied in §6.2, coincides with that constructed in §3.1 when Ω is a Lipschitz domain.

To start, we note that a Lipschitz domain Ω always satisfies the condition (6.2.7), as a special case of Proposition 4.5.3. Also, Proposition 4.5.2 implies $\mathfrak{Z}(\Omega,\sigma)=0$. Going further, we have the following.

Proposition 6.3.1. *If $\Omega\subset M$ is a Lipschitz domain, Proposition 6.2.5 is applicable, hence*

$$S : L^2(\partial\Omega,\sigma) \longrightarrow \mathcal{D}(N) \quad \textit{is an isomorphism.} \tag{6.3.1}$$

Proof. We need to verify (6.2.22) and (6.2.26). First, as mentioned in (3.2.19),

$$S : L^p(\partial\Omega,\sigma) \xrightarrow{\approx} H^{1,p}(\partial\Omega), \quad \forall\, p\in(1,q(\Omega)), \tag{6.3.2}$$

where $q(\Omega)>2$. Taking $p=2$, we get (6.2.25), hence (6.2.22). Next,

$$-\frac{1}{2}I+A^* : L^p(\partial\Omega,\sigma)\longrightarrow L^p(\partial\Omega,\sigma) \ \text{ is Fredholm, of index } 0, \tag{6.3.3}$$

for $1<p<q(\Omega)$. This is shown in [MT3] and [MT4], for metric tensors that are Lipschitz, or satisfy (3.0.1)–(3.0.2), respectively, following earlier work applicable to Lipschitz domains in Euclidean space.

Together, (6.3.1) and (6.3.2) yield

$$\mathcal{D}(N)=H^{1,2}(\partial\Omega). \tag{6.3.4}$$

Also, (6.2.24) applies, so we have

$$Nf=\partial_\nu\mathcal{S}(S^{-1}f)=\Bigl(-\frac{1}{2}I+A^*\Bigr)S^{-1}f, \quad \forall\, f\in H^{1,2}(\partial\Omega). \tag{6.3.5}$$

This coincides with (3.2.15), and hence leads back to the characterization of N in (3.1.3)–(3.1.4).

6.4. Regular SKT domains

The class of regular SKT domains arose in work [Se], [KT] of Semmes, Kenig, and Toro, where they were called chord-arc domains with vanishing constant. The label "regular SKT domains" was proposed in [HMT]. This class can be defined as follows. First, we assume $\Omega \subset M$ is an Ahlfors regular domain. Then Ω is a regular SKT domain provided

$$\begin{aligned} &\Omega \text{ satisfies a 2-sided local John condition, and} \\ &\nu \in \mathrm{vmo}(\partial\Omega). \end{aligned} \tag{6.4.1}$$

To say Ω satisfies a local John condition is to say that there exist $\theta \in (0,1)$ and $R > 0$ with the following properties. For each $p \in \partial\Omega$ and $r \in (0,R)$, there is a point $p_r \in B_r(p) \cap \Omega$ such that

$$B_{\theta r}(p_r) \subset \Omega, \tag{6.4.2}$$

and, for each $x \in \partial\Omega \cap B_r(p)$, there is a path γ_x from x to p_r, of length $\le r/\theta$, such that

$$\operatorname{dist}(\gamma_x(t), \partial\Omega) \ge \theta \operatorname{dist}(\gamma_x(t), x), \quad \forall\, t. \tag{6.4.3}$$

The first hypothesis in (6.4.1) is that both Ω and $M\setminus\overline{\Omega}$ satisfy a local John condition.

This characterization of regular SKT domains is given in §4.2 of [HMT], where it is shown to be equivalent to the definitions given in [Se] and [KT]. It is also shown in [HMT] that an Ahlfors regular domain that satisfies a 2-sided local John condition is a UR domain. Here is another result of [HMT]:

$$\text{If } \Omega \subset M \text{ is a } \mathrm{vmo}_1 \text{ domain, then it is a regular SKT domain.} \tag{6.4.4}$$

One says Ω is a vmo_1 domain if $\partial\Omega$ is locally the graph of a function g, satisfying (4.5.4), but with (4.5.5) replaced by

$$\nabla g \in \mathrm{vmo}\,. \tag{6.4.5}$$

Thus the class of vmo_1 domains is larger than the class of C^1 domains. On the other hand, a vmo_1 domain need not be a Lipschitz domain, and vice-versa.

Parallel to Proposition 6.3.1, we have the following result.

Proposition 6.4.1. *Let $\Omega \subset M$ be a regular SKT domain. Assume that*

$$\overset{\circ}{H}{}^1(\Omega) = H_0^1(\Omega). \tag{6.4.6}$$

Then

$$S : L^2(\partial\Omega, \sigma) \longrightarrow \mathcal{D}(N) \quad \text{is an isomorphism.} \tag{6.4.7}$$

Proof. As in Proposition 6.3.1, we want to show that Proposition 6.2.5 is applicable, so we need to verify (6.2.22) and (6.2.26). One key result, established in §5.2 of [HMT], is that

$$A^* : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial\Omega, \sigma) \ \text{ is compact}, \quad \forall\, p \in (1, \infty), \tag{6.4.8}$$

when Ω is a regular SKT domain. Hence,

$$-\frac{1}{2}I + A^* : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial\Omega, \sigma) \ \text{ is Fredholm, of index } 0, \tag{6.4.9}$$

for $1 < p < \infty$, so (6.2.26) holds. Furthermore, it was shown in §6.4 of [HMT] that

$$S : L^p(\partial\Omega, \sigma) \longrightarrow H^{1,p}(\partial\Omega) \ \text{ is an isomorphism}, \tag{6.4.10}$$

for $1 < p < \infty$. This implies (6.2.22), and completes the proof.

As in §6.3, we can deduce from (6.4.7) and (6.4.10) that

$$\mathcal{D}(N) = H^{1,2}(\partial\Omega), \tag{6.4.11}$$

and that

$$Nf = \partial_\nu \mathcal{S}(S^{-1}f) = \Bigl(-\frac{1}{2}I + A^*\Bigr)S^{-1}f, \quad \forall\, f \in H^{1,2}(\partial\Omega). \tag{6.4.12}$$

Remark. There is a theory of L^p-Sobolev spaces $H^{1,p}(\partial\Omega)$ for Ahlfors regular domains Ω. It is less straightforward than for Lipschitz domains. For this development, see §3.6 and §4.3 of [HMT], and also Appendix A.2 of [MMT]. It is still the case that $\mathrm{Lip}(\partial\Omega) \subset H^{1,p}(\partial\Omega)$, for each $p \in (1, \infty)$.

As in (6.2.28), we can pass from (6.4.12) to

$$N + 1 = \Bigl(-\frac{1}{2}I + A^* + S\Bigr)S^{-1} \tag{6.4.13}$$

on $H^{1,2}(\partial\Omega)$. By (6.4.10) and (6.0.17), the right side extends uniquely to a bounded linear operator from $H^{1,p}(\partial\Omega)$ to $L^p(\partial\Omega,\sigma)$, for each $p \in (1,\infty)$. Furthermore,

$$-\frac{1}{2}I + A^* + S : L^p(\partial\Omega,\sigma) \longrightarrow L^p(\partial\Omega,\sigma) \tag{6.4.14}$$

is Fredholm, of index 0, for each $p \in (1,\infty)$. As we have seen, it is injective on $L^2(\partial\Omega,\sigma)$, hence on $L^p(\partial\Omega,\sigma)$ for $p \in [2,\infty)$, hence invertible on $L^p(\partial\Omega,\sigma)$ for $p \in [2,\infty)$. Thus its adjoint is invertible on $L^p(\partial\Omega,\sigma)$ for $p \in (1,2]$, hence injective on $L^p(\partial\Omega,\sigma)$ for $p \in [2,\infty)$, and also Fredholm of index 0, hence invertible. We deduce that (6.4.14) is invertible on $L^p(\partial\Omega,\sigma)$ for all $p \in (1,\infty)$. Hence

$$N + 1 : H^{1,p}(\partial\Omega) \xrightarrow{\approx} L^p(\partial\Omega,\sigma), \quad \forall\, p \in (1,\infty). \tag{6.4.15}$$

It is also useful to have Sobolev embedding theorems in this setting. As shown in §4.3 of [HMT], if Ω is an n-dimensional Ahlfors regular domain that satisfies a local 2-sided John condition, then

$$\begin{aligned} &H^{1,p}(\partial\Omega) \hookrightarrow L^{p*}(\partial\Omega,\sigma), \quad p^* = \frac{(n-1)p}{n-1-p}, \ \text{ if } 1 < p < n-1, \\ &H^{1,p}(\partial\Omega) \hookrightarrow C^{\alpha}(\partial\Omega), \quad \alpha = 1 - \frac{n}{p}, \ \text{ if } n < p < \infty, \end{aligned} \tag{6.4.16}$$

and

$$H^{1,p}(\partial\Omega) \hookrightarrow L^p(\partial\Omega,\sigma) \ \text{ is compact}, \ \forall\, p \in (1,\infty). \tag{6.4.17}$$

In the current setting of regular SKT domains, (6.4.17) also follows from (6.4.10) and the compactness of S on $L^p(\partial\Omega,\sigma)$. It follows from (6.4.15)–(6.4.16) that, in the setting of Proposition 6.4.1,

$$(N+1)^{-1} : L^p(\partial\Omega,\sigma) \longrightarrow L^{p^*}(\partial\Omega,\sigma), \quad 1 < p < n-1, \tag{6.4.18}$$

with p^* as in (6.4.16). Iteration then yields a natural analogue of Proposition 3.1.2.

It is also of interest to consider the semigroup e^{-tN} in this setting. We know it is a semigrup of positive self-adjoint contractions on $L^2(\partial\Omega,\sigma)$. Also, Proposition 4.3.1 applies, so this is a symmetric Markov semigroup. Therefore a number of results of §3.3 can be established in this setting. Let us focus on Proposition 3.3.8. Results of [Var] imply that an estimate of the form

$$\|e^{-tN} f\|_{L^\infty(\partial\Omega)} \le C_2 t^{-\sigma/2} \|f\|_{L^2(\partial\Omega)}, \quad 0 < t \le 1, \tag{6.4.19}$$

is equivalent to

$$\|f\|^2_{L^{2\sigma/(\sigma-1)}(\partial\Omega)} \le C_1 \Big[\langle Nf, f\rangle + \|f\|^2_{L^2(\partial\Omega)} \Big], \quad f \in \mathcal{D}(N), \tag{6.4.20}$$

as long as $\sigma > 1$. Note that the right side of (6.4.20) is C_1 times

$$\|N^{1/2}f\|^2_{L^2(\partial\Omega)} + \|f\|^2_{L^2(\partial\Omega)} = \|f\|^2_{\mathcal{D}(N^{1/2})}. \tag{6.4.21}$$

Since N is positive and self adjoint, general Hilbert space theory gives

$$\mathcal{D}(N^{1/2}) = [L^2(\partial\Omega,\sigma),\mathcal{D}(N)]_{1/2}, \tag{6.4.21}$$

the right side denoting the complex interpolation space. In view of (6.4.11), this space is $[L^2(\partial\Omega,\sigma),H^{1,2}(\partial\Omega)]_{1/2}$. It is natural to define fractional-order Sobolev spaces by interpolation:

$$H^{s,p}(\partial\Omega) = [L^p(\partial\Omega,\sigma),H^{1,p}(\partial\Omega)]_s, \quad 0 < s < 1,\ 1 < p < \infty. \tag{6.4.22}$$

Then

$$\mathcal{D}(N^{1/2}) = H^{1/2,2}(\partial\Omega). \tag{6.4.23}$$

In the setting of Lipschitz domains, we have the embedding results (3.3.30). Some of these follow from (6.4.16) and interpolation in the setting of Ahlfors regular, 2-sided John domains. For example, a special case of (6.4.16) is

$$H^{1,2}(\partial\Omega) \subset L^{2(n-1)/(n-3)}(\partial\Omega,\sigma), \quad \text{if } \ n > 3. \tag{6.4.24}$$

In such a case, interpolation yields

$$H^{1/2,2}(\partial\Omega) \subset L^{2(n-1)/(n-2)}(\partial\Omega,\sigma), \quad \text{if } \ n > 3. \tag{6.4.25}$$

This result has the same form as its analogue for Lipschitz domains in (3.3.30), except that (6.4.25) does not cover the case $n = 3$. Extending (6.4.25) to $n = 3$ when Ω is an Ahlfors regular, 2-sided John domain, along with further analyses of fractional Sobolev spaces $H^{s,p}(\partial\Omega)$ in this setting, is an interesting topic for future work. At this point, we have the following result, via the sort of argument used in Proposition 3.3.8.

Proposition 6.4.2. *In the setting of Proposition 6.4.1,*

$$\|e^{-tN}f\|_{L^\infty(\partial\Omega)} \le C(t\wedge 1)^{-(n-1)/2}\|f\|_{L^2(\partial\Omega)}, \tag{6.4.26}$$

provided $\dim\Omega = n > 3$.

It is tempting to conjecture that such a result also holds for $n = 2$ and 3, as it does for Lipschitz domains, but we leave this aside.

Having Proposition 6.4.2, we can derive results on the spectrum of N parallel to those done in the setting of Lipschitz domains in §3.4. In fact, parallel to (3.4.5)–(3.4.7), we deduce from (6.4.26) that

$$\|e^{-tN}\|^2_{HS} \le Ct^{-(n-1)}, \quad 0 < t \le 1. \tag{6.4.27}$$

Hence we have (3.4.8)–(3.4.10) in this setting, yielding the following parallel to Proposition 3.4.1.

Proposition 6.4.3. *In the setting of Proposition 6.4.1, the spectrum of N is discrete, and we have*

$$N\varphi_j = \lambda_j \varphi_j, \quad 0 \le \lambda_j \nearrow +\infty, \tag{6.4.28}$$

where $\{\varphi_j : j \ge 0\}$ is an orthonormal basis of $L^2(\partial\Omega, \sigma)$. Then

$$\sum_{j\ge 0} e^{-t\lambda_j} \le C(t \wedge 1)^{-(n-1)}, \tag{6.4.29}$$

hence

$$\sum_{j\ge 0} (\lambda_j + 1)^{-s} \le C + \frac{C}{s-(n-1)}, \quad s > n-1, \tag{6.4.30}$$

at least provided $\dim \Omega = n > 3$.

7. More singular domains

In this chapter we continue to take Ω to be an open subset of a compact, connected, n-dimensional Riemannian manifold M, but we move beyond the class of finite perimeter domains. The finite positive measure μ we place on $\partial\Omega$ will typically have nothing to do with surface area, as given by $(n-1)$-dimensional Hausdorff measure. This use of measures beyond surface area played some role in §5.3, but there we depended very much on σ, surface measure on $\partial_*\Omega$, to guarantee that $\mu \geq \sigma$ satisfied

$$\int_\Omega |u|^2 \, dV \leq C \int_\Omega |\nabla u|^2 \, dV + C \int_{\partial\Omega} |u|^2 \, d\mu, \tag{7.0.1}$$

for all $u \in \mathrm{Lip}(\overline{\Omega})$. This will be the defining condition for the construction of N. We note that the idea of developing this theory for such a class of measures was mentioned on p. 2103 of [AtE]. Here we provide tools to construct such measures and point out some interesting natural examples.

In §7.1 we establish conditions on a measure μ on $\partial\Omega$ that yield (7.0.1). We see that such measures exist as long as

$$\mathrm{Cap}(\partial\Omega) > 0, \tag{7.0.2}$$

under some conditions on Ω, such as the existence of a continuous linear extension map $E : \overline{H}^1(\Omega) \to H^1(M)$. Here we do not assume (4.0.1). On the contrary, we are interested in a number of cases in which

$$\partial\Omega = M \setminus \Omega, \tag{7.0.3}$$

including some cases where $\partial\Omega$ is totally disconnected.

Once we have a postive finite measure μ on $\partial\Omega$ for which (7.0.1) holds, construction of N as a self-adjoint operator on $L^2(\partial\Omega,\mu)$ proceeds much as in §4.2. We record these results in §7.2. We also examine the semigroup $\{e^{-tN} : t \geq 0\}$ as a symmetric Markov semigroup on $L^2(\partial\Omega,\mu)$, and establish conditions under which it is irreducible.

Section 7.3. is devoted to domains Ω in the Riemann sphere $\widehat{\mathbb{C}}$ whose boundaries are Julia sets. We take

$$\partial\Omega = \mathcal{J}_R, \quad \Omega = \widehat{\mathbb{C}} \setminus \mathcal{J}_R = \mathcal{F}_R, \tag{7.0.4}$$

where $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree ≥ 2, $\mathcal{J}_R$ is its Julia set, and $\mathcal{F}_R$ its Fatou set. The set $\mathcal{J}_R$ carries a probability measure λ, known as the maximal entropy measure. We show that (7.0.1) holds for $\mu = \lambda$ (provided $\mathcal{J}_R$ has Lebesgue measure 0), so the results of §7.2 apply. We show that the semigroup $\{e^{-tN} : t \geq 0\}$ is irreducible in all these cases. Julia sets present a variety of fractal behavior, from spikey connected sets known as "dendrites" to totally disconnected sets that are Cantor sets with positive Hausdorff dimension, along with awesomely many other types of behavor.

7.1. Measures that satisfy (7.0.1)

As above, Ω is an open subset of the compact, n-dimensional Riemannian manifold M, and μ is a finite positive measure supported on $\partial\Omega$. We seek conditions that imply the estimate (7.0.1), i.e.,

$$\int_{\Omega} |u|^2 \, dV \le C \int_{\Omega} |\nabla u|^2 \, dV + C \int_{\partial\Omega} |u|^2 \, d\mu, \quad \forall \, u \in \mathrm{Lip}(\overline{\Omega}). \tag{7.1.1}$$

In the current setting, Ω is not necessarily a finite perimeter domain, and μ might have nothing to do with $(n-1)$-dimensional Hausdorff measure. To tackle (7.1.1), we need an approach much different from that used in §4.1, which involved the divergence theorem. We will look for such measures among those that also satisfy

$$\int_{\partial\Omega} |u|^2 \, d\mu \le C \int_{\Omega} |\nabla u|^2 \, dV + C \int_{\Omega} |u|^2 \, dV, \quad \forall \, u \in \mathrm{Lip}(\overline{\Omega}). \tag{7.1.2}$$

This is equivalemt to the condition that $\mathrm{Tr} : \mathrm{Lip}(\overline{\Omega}) \to \mathrm{Lip}(\partial\Omega)$, given by $\mathrm{Tr}\, u = u|_{\partial\Omega}$, has a continuous extension to

$$\mathrm{Tr} : \overline{H}^1(\Omega) \longrightarrow L^2(\partial\Omega, \mu). \tag{7.1.3}$$

We will give specific conditions on μ implying (7.1.2) below. Given (7.1.2), the quadratic form

$$Q(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\Omega)} + \int_{\partial\Omega} uv \, d\mu, \quad u,v \in \mathrm{Lip}(\overline{\Omega}), \tag{7.1.4}$$

is closable, and its closure $\overline{Q}$ has form domain $\mathcal{D}(\overline{Q}) = \overline{H}^1(\Omega)$. Thus we have a positive, self-adjoint operator T such that

$$\mathcal{D}(T^{1/2}) = \overline{H}^1(\Omega), \quad (T^{1/2}u, T^{1/2}v)_{L^2(\Omega)} = \overline{Q}(u,v). \tag{7.1.5}$$

We set $S = T - I$. Then S is positive semi-definite and

$$\|S^{1/2}u\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} + \int_{\partial\Omega} |u|^2 \, d\mu, \quad \forall \, u \in \overline{H}^1(\Omega). \tag{7.1.6}$$

If (7.1.1) is to hold, we certainly want

$$\mathrm{Ker}\, S = 0. \tag{7.1.7}$$

If $u \in \operatorname{Ker} S \subset \overline{H}^1(\Omega)$, then $u \equiv 0$ on Ω, so u is constant on each connected component of Ω. If

$$\mu(\partial\mathcal{O}_j) > 0 \text{ for each component } \mathcal{O}_j \text{ of } \Omega, \tag{7.1.8}$$

then each such constant is 0, and we have (7.1.7). In particular, this holds if Ω is connected. An alternative condition that leads to (7.1.7) (which will be particularly useful in §7.3) involves the following concept.

Definition. We say Ω is *quasi-connected* provided

$$u \in \overline{H}^1(\Omega), \quad \nabla u \equiv 0 \text{ on } \Omega \Longrightarrow u = \text{ const.} \tag{7.1.9}$$

Given this (and the hypothesis (7.1.2)), we clearly have

$$\Omega \text{ quasi-connected } \Longrightarrow \operatorname{Ker} S = 0. \tag{7.1.10}$$

Here is a condition that implies Ω is quasi-connected.

Lemma 7.1.1. *Assume*

$$\Omega = \mathcal{O} \setminus K, \quad K \subset \overline{\mathcal{O}} \textit{ compact}, \quad \textit{and } \mathcal{H}^n(K) = 0. \tag{7.1.11}$$

Then

$$\mathcal{O} \textit{ connected} \Longrightarrow \Omega \textit{ quasi-connected.} \tag{7.1.12}$$

Proof. From the hypotheses, we have that $\mathrm{Lip}(\overline{\Omega}) = \mathrm{Lip}(\overline{\mathcal{O}})$, that $\overline{H}^1(\Omega) = \overline{H}^1(\overline{\mathcal{O}})$, and that $u \in \overline{H}^1(\mathcal{O}), \nabla u \equiv 0$ on $\Omega \Rightarrow \nabla u \equiv 0$ on $\mathcal{O}$.

Our next task is to go from (7.1.7) to (7.1.1). The following result gives a condition under which this can be done.

Proposition 7.1.2. *Assume that (7.1.2) holds, that* $\operatorname{Ker} S = 0$*, and that*

$$\overline{H}^1(\Omega) \hookrightarrow L^2(\Omega) \textit{ is compact.} \tag{7.1.13}$$

Then (7.1.1) holds.

Proof. Under these hypotheses, S is a positive, self-adjoint operator with $\mathcal{D}(S^{1/2}) = \overline{H}^1(\Omega)$, and hence S has compact resolvent. We deduce from $\operatorname{Ker} S = 0$ that $\operatorname{Spec} S^{1/2} \subset [a, \infty)$ for some $a > 0$, so $\|S^{1/2}u\|^2_{L^2(\Omega)} \geq a^2\|u\|^2_{L^2(\Omega)}$, for all $u \in \mathcal{D}(S^{1/2})$, giving (7.1.1), with $C = a^2$.

We turn to a discussion of conditions on Ω and μ that imply (7.1.2). One useful condition to place on Ω is

$$\exists \text{continuous extension operator } E : \overline{H}^1(\Omega) \longrightarrow H^1(M). \tag{7.1.14}$$

Note that (7.1.14) holds provided there exists an open $\mathcal{O} \subset M$ such that (7.1.11) holds and there is an extension operator $E : \overline{H}^1(\mathcal{O}) \to H^1(M)$. In particular, (7.1.14) holds provided

$$\Omega = M \setminus K, \quad K \text{ compact}, \quad \text{and } \mathcal{H}^n(K) = 0, \tag{7.1.15}$$

since then $\overline{H}^1(\Omega) = H^1(M)$. Note, by the way, that (7.1.14) implies (7.1.13).

When (7.1.14) holds and μ is a finite, positive measure supported on $\partial\Omega$, the estimate (7.1.2) holds whenever

$$\int_{\partial\Omega} |u|^2 \, d\mu \le C\|u\|^2_{H^1(M)}, \quad \forall\, u \in \mathrm{Lip}(M). \tag{7.1.16}$$

An equivalent condition is that $M_\mu : \mathrm{Lip}(M) \to \mathcal{D}'(M)$, defined by

$$M_\mu u = u\mu, \tag{7.1.17}$$

has a continuous extension to

$$M_\mu : H^1(M) \longrightarrow H^{-1}(M). \tag{7.1.18}$$

Given that

$$\begin{aligned} u, v \in H^1(M) \Rightarrow uv \in H^{1,p}(M), \ \text{ for } \ p = \frac{n}{n-1}, \text{ if } n > 3, \\ \text{for all } \ p < 2, \quad \text{if } n = 2, \end{aligned} \tag{7.1.19}$$

we see that (7.1.18) holds provided

$$\begin{aligned} &\mu \in H^{-1,n}(M), && \text{if } \ n \ge 3, \\ &\mu \in H^{-1,r}(M), \quad \text{for some } r > 2, && \text{if } \ n = 2. \end{aligned} \tag{7.1.20}$$

We can use this plus Theorem 4.7.4 of [Zie] to deduce that (7.1.18) holds whenever

$$\mu(B_r(x)) \le Ar^\alpha, \quad \forall\, x \in M, \ r > 0, \tag{7.1.21}$$

for some $A < \infty$ and

$$\alpha > n - 1 - \frac{1}{n-1}. \tag{7.1.22}$$

Here, $B_r(x)$ denotes the ball in M is radius r, centered at x. In particular, for $n = 2$, it suffices to have (7.1.21) for some $\alpha > 0$. Let us summarize this result.

Lemma 7.1.3. *If μ is a finite, positive measure on M, we have (7.1.18) (and hence (7.1.16)) provided (7.1.21) holds, with α satisfying (7.1.22).*

Another useful criterion for (7.1.16) is the following result, from Theorem 1.2.2 of [MS].

Lemma 7.1.4. *If μ is a finite, positive measure on M, we have (7.1.18) provided there exists $A < \infty$ such that*

$$\mu(S) \le A\,\mathrm{Cap}(S), \quad \forall\, \textit{compact } S \subset M. \tag{7.1.23}$$

Corollary 7.1.5. *Under the hypotheses of either Lemma 7.1.3 or Lemma 7.1.4, if (7.1.14) holds, then (7.1.2) holds.*

Examples of measures satisfying (7.1.21)–(7.1.22) include the maximal entropy measures on Julia sets, as we will discuss in §7.3. Here we record some interesting metric properties of sets supporting such a measure. Thus, let μ be a positive measure satisfying (7.1.21), and set

$$S = \mathrm{supp}\,\mu, \tag{7.1.24}$$

so S is a compact subset of M, $\mu(M \setminus S) = 0$, and

$$p \in S,\ \mathcal{O} \text{ neighborhood of } p \text{ in } M \Longrightarrow \mu(\mathcal{O} \cap S) > 0. \tag{7.1.25}$$

For the measures relevant for (7.1.2), $S \subset \partial\Omega$. Frequently, but not always, $S = \partial\Omega$. Our first result is a Hausdorff dimension estimate.

Proposition 7.1.6. *If μ is a positive measure satisfying (7.1.21) and $S = \mathrm{supp}\,\mu$, then*

$$\textit{the Hausdorff dimension of } S \textit{ is } \ge \alpha. \tag{7.1.26}$$

In fact, there exists $C < \infty$ such that for each compact $K \subset S$,

$$\mu(K) \le C\mathcal{H}^\alpha(K). \tag{7.1.27}$$

Proof. The α-dimensional Hausdorff measure $\mathcal{H}^\alpha(K)$ is characterized by

$$h_\delta^\alpha(K) \nearrow \mathcal{H}^\alpha(K), \tag{7.1.28}$$

as $\delta \searrow 0$, where

$$h_\delta^\alpha(K) = \inf\Bigl\{\sum_{j\ge1}(\mathrm{diam}\,A_j)^\alpha : K \subset \bigcup_{j\ge1} A_j,\ A_j \text{ compact},\ \mathrm{diam}\,A_j \le \delta\Bigr\}. \tag{7.1.29}$$

Now (7.1.21) implies

$$\mu(A_j) \le C(\operatorname{diam} A_j)^\alpha, \tag{7.1.30}$$

so

$$K \subset \bigcup_{j\ge1} A_j \Rightarrow \mu(K) \le \sum_{j\ge1} \mu(A_j) \le C \sum_{j\ge1} (\operatorname{diam} A_j)^\alpha, \tag{7.1.31}$$

and hence

$$\mu(K) \le C h_\delta^\alpha(K), \quad \forall\, \delta > 0, \tag{7.1.32}$$

which implies (7.1.27).

Taking into account (5.3.8), we have the following.

Corollary 7.1.7. *If μ is a positive measure satisfying (7.1.21)–(7.1.22), and $S = \operatorname{supp} \mu$, then, for each compact $K \subset S$,*

$$\mu(K) > 0 \Longrightarrow \operatorname{Cap}(K) > 0. \tag{7.1.33}$$

We now bring in harmonic measure as a class of measures on $\partial\Omega$ satisfying the hypothesis of Lemma 7.1.4. Its construction uses the solution operator to the generalized Dirichlet problem

$$\mathrm{PI}_0 : C(\partial\Omega) \longrightarrow \{u \in L^\infty(\Omega) : \Delta u = 0\} \tag{7.1.34}$$

described in (4.4.14)–(4.4.18), with the scope extended, in so far as the construction works not merely when (4.0.1) holds, but under the more general hypothesis that

$$\operatorname{Cap}(M \setminus \Omega) > 0. \tag{7.1.35}$$

This still yields the positivity condition (4.4.17). Hence, for each $x \in \Omega$, there is a unique positive Borel measure

$$\omega_x \text{ on } \partial\Omega, \text{ of total mass } 1, \tag{7.1.36}$$

such that

$$\mathrm{PI}_0 f(x) = \int_{\partial\Omega} f\, d\omega_x, \quad \forall\, f \in C(\partial\Omega). \tag{7.1.37}$$

The measure ω_x is harmonic measure on $\partial\Omega$. This depends on x. However, a Harnack inequality argument (cf. [CKL]) shows that, if Ω is connected, given any compact $K \subset \Omega$, there exists $C \in (0,1)$ such that for each positive $f \in C(\partial\Omega)$,

$$C\, \mathrm{PI}_0 f(x) \le \mathrm{PI}_0 f(y) \le C^{-1}\, \mathrm{PI}_0 f(x), \quad \forall\, x, y \in K. \tag{7.1.38}$$

Consequently,

$$C\omega_x(S) \le \omega_y(S) \le C^{-1}\omega_x(S), \tag{3.1.39}$$

for each Borel set $S \subset \partial\Omega$. It is also useful to note that PI_0 has a natural extension to the class of bounded, Borel measurable funcitons f on $\partial\Omega$, and (7.1.37) continues to hold for f in this larger class. For example, if $S \subset \partial\Omega$ is compact

$$\begin{aligned} &f_j \in C(\partial\Omega),\ f_j \searrow \chi_S \\ &\Longrightarrow \mathrm{PI}_0 f_j(x) \searrow \mathrm{PI}_0\, \chi_S(x) = \omega_x(S), \quad \forall\, x \in \Omega. \end{aligned} \tag{7.1.40}$$

Our next goal is to establish the following.

Proposition 7.1.8. *Let $\Omega \subset M$ be open and connected, and satisfy (7.1.35). Fix $x \in \Omega$. Then there exists $C \in (0, \infty)$ such that, for each compact $S \subset \partial\Omega$,*

$$\omega_x(S) \le C \operatorname{Cap}(S). \tag{7.1.41}$$

We define $\operatorname{Cap}(S)$ as follows. Take a smoothly bounded closed set $B \subset \Omega$ such that $M \setminus B$ is connected and $x \notin B$, and set $\mathcal{O} = M \setminus B$. For use below, make sure that $\Omega \setminus B$ is also connected. We will define $\operatorname{Cap}(S)$ for compact $S \subset \mathcal{O}$. First, if S is a smoothly bounded compact subset of $\mathcal{O}$, we define the capacitary potential U_S so that

$$\begin{gathered} U_S \in C(\overline{\mathcal{O}}), \quad \Delta U_S = 0 \text{ on } \mathcal{O} \setminus S, \\ U_S = 1 \text{ on } S, \quad U_S = 0 \text{ on } \partial B. \end{gathered} \tag{7.1.42}$$

Given $\varphi \in C_0^\infty(\mathcal{O})$, Green's formula gives

$$\int\limits_{\mathcal{O}} U_S(x) \Delta\varphi(x)\, dV(x) = -\int\limits_{\partial S} \varphi(y) \partial_\nu U_S(y)\, dS(y), \tag{7.1.43}$$

where ν is the unit normal to ∂S, pointing into S. By Zaremba's principle, $\partial_\nu U_S(y) > 0$ for $y \in \partial S$, so

$$\Delta U_S = -\mu_S \ \text{ on } \ \mathcal{O}, \tag{7.1.44}$$

where μ_S is a positive measure supported on ∂S. We set

$$\operatorname{Cap}(S) = \int\limits_S d\mu_S. \tag{7.1.45}$$

For general compact $S \subset \mathcal{O}$, we can take smoothly bounded $S_j \searrow S$ and (cf. [T1], Chapter 11),

$$U_{S_j} \searrow U_S, \quad \operatorname{Cap}(S_j) \searrow \operatorname{Cap}(S), \tag{7.1.46}$$

and we also have (7.1.44)–(7.1.45).

To begin the proof of Proposition 7.1.8, note that the left side of (7.1.41) is equal to $\mathrm{PI}_0\, \chi_S(x)$, if S is a compact subset of $\partial\Omega$. It will be useful to set $\widetilde{\Omega} = \Omega \setminus B$ and consider

$$\widetilde{\mathrm{PI}}_0 : C(\partial\widetilde{\Omega}) \longrightarrow \{u \in L^\infty(\widetilde{\Omega}) : \Delta u = 0 \text{ on } \widetilde{\Omega}\}, \tag{7.1.47}$$

defined as in (7.1.34) but with Ω replaced by $\widetilde{\Omega}$, and then $\widetilde{\mathrm{PI}}_0 f$ extended to bounded Borel measurable f on $\partial\widetilde{\Omega}$. Clearly, for compact $S \subset \partial\Omega \subset \partial\widetilde{\Omega},\ x \in \widetilde{\Omega}$,

$$\widetilde{\mathrm{PI}}_0 \chi_S(x) \le \mathrm{PI}_0\, \chi_S(x). \tag{7.1.48}$$

An argument using Harnack's inequality and the strong maximum principle yields the following.

Lemma 7.1.9. *Let K be a compact subset of $\widetilde{\Omega}$. There exists $C \in (1,\infty)$ such that for all $f \in C(\partial\widetilde{\Omega})$ such that $f \ge 0$ on $\partial\Omega$, $f = 0$ on ∂B,*

$$\mathrm{PI}_0\, f(x) \le C\widetilde{\mathrm{PI}}_0 f(x), \quad \forall\, x \in K, \tag{7.1.49}$$

and, for each compact $S \subset \partial\Omega$,

$$\mathrm{PI}_0\, \chi_S(x) \le C\widetilde{\mathrm{PI}}_0 \chi_S(x), \quad \forall\, x \in K. \tag{7.1.50}$$

Proof. As a preliminary, since $\widetilde{\Omega}$ is connected, use of Harnack's estimate yields the following. Given $K, K' \subset \widetilde{\Omega}$ compact, there exists $A = A(K,K') \in (1,\infty)$ such that

$$\begin{aligned} A^{-1}u(p) \le u(q) \le Au(p), \quad &\forall\, p \in K, q \in K', u \in C(\widetilde{\Omega}) \\ &\text{such that } u > 0 \text{ and } \Delta u = 0 \text{ on } \widetilde{\Omega}. \end{aligned} \tag{7.1.51}$$

To proceed, let $\widetilde{B} \subset \widetilde{\Omega}$ be a smoothly bounded closed subset containing B in its interior, and set $K = \partial\widetilde{B}$. For $x \in K$, $f \in C(\partial\widetilde{\Omega})$ as in the statement of the lemma, we have

$$\mathrm{PI}_0\, f(x) = \widetilde{\mathrm{PI}}_0 f(x) + \widetilde{\mathrm{PI}}_0 g(x), \tag{7.1.52}$$

where

$$\begin{aligned} g(y) = \quad 0 \quad &\text{for } \ y \in \partial\widetilde{\Omega}, \\ \mathrm{PI}_0 f(y) \quad &\text{for } \ y \in \partial B. \end{aligned} \tag{7.1.53}$$

Now the maximal principle applied to harmonic functions on $\widetilde{B}$ implies

$$\sup_{y\in\partial B} g(y) < \sup_{x\in\partial\tilde{B}} \mathrm{PI}_0\, f(x). \tag{7.1.54}$$

We have, for $x \in K = \partial\widetilde{B}$,

$$0 \le \widetilde{\mathrm{PI}}_0 g(x) \le \gamma\widetilde{\mathrm{PI}}_0 \chi(x), \tag{7.1.55}$$

where

$$\gamma = \sup_{y\in\partial B} g(y), \tag{7.1.56}$$

and

$$\begin{aligned} \chi(y) = 0 \quad &\text{for } \ y \in \partial\widetilde{\Omega}, \\ 1 \quad &\text{for } \ y \in \partial B. \end{aligned} \tag{7.1.57}$$

Another application of the (strong) maximum principle yields that, for $x \in K$,

$$0 < \widetilde{\mathrm{PI}}_0 \chi(x) \le \alpha, \quad \alpha < 1. \tag{7.1.58}$$

We deduce that, for $x \in K$,

$$0 \le \widetilde{\mathrm{PI}}_0 g(x) \le \alpha \sup_{y \in K} \mathrm{PI}_0 f(y). \tag{7.1.59}$$

Consequently, given $f \in C(\partial \widetilde{\Omega})$ such that $f \ge 0$ on $\partial\Omega$, $f = 0$ on ∂B,

$$\exists\, x \in K \text{ such that } \widetilde{\mathrm{PI}}_0 f(x) \ge (1-\alpha)\, \mathrm{PI}_0 f(x). \tag{7.1.60}$$

In light of (7.1.51), this proves (7.1.49), which in turn readily yields (7.1.50).

To proceed with the proof of Proposition 7.1.8, given compact $S \subset \partial\Omega$, we have from the maximum principle that

$$\widetilde{\mathrm{PI}}_0 \chi_S(x) \le U_S(x), \quad \forall\, x \in \widetilde{\Omega}. \tag{7.1.61}$$

Now $\mathrm{Cap}(S)$ is given by (7.1.45), and, by (7.1.42) and (7.1.44), we have

$$U_S(x) = -\int_{\mathcal{O}} G(x,y)\, d\mu_S(y), \quad \forall\, x \in \mathcal{O}, \tag{7.1.62}$$

and in particular for all $x \in \widetilde{\Omega}$. Here $G(x,y)$ is the Green kernel for the Laplace operator on $\mathcal{O} = M \setminus B$, with Dirichlet boundary condition on $\partial\mathcal{O}$, which is bounded on $x \in K$, $y \in \partial\Omega$, with $K \subset \widetilde{\Omega}$ compact. Since μ_S is supported on S, we have for each compact $K \subset \widetilde{\Omega}$ a constant $C \in (0,\infty)$ such that

$$0 \le U_S(x) \le C\mu_S(S) = C\, \mathrm{Cap}(S), \quad \forall\, x \in K. \tag{7.1.63}$$

Putting together (7.1.50), (7.1.61), and (7.1.63), and recalling that $\omega_x(S) = \mathrm{PI}_0\, \chi_S(x)$, we have Proposition 7.1.8.

Combining Proposition 7.1.8 with Corollary 7.1.5, we have the following.

Proposition 7.1.10. *Let $\Omega \subset M$ be open and connected and satisfy (7.1.35). Assume there is an extension operator as in (7.1.14). Then, given $x \in \Omega$, harmonic measure $\mu = \omega_x$ satisfies (7.1.2), and consequently it satisfies (7.1.1).*

Proof. It remains only to treat the last assertion. For this, note that (7.1.14) implies the compactness (7.1.13), and that connectedness of Ω implies $\mathrm{Ker}\, S = 0$. Hence Proposition 7.1.2 applies.

7.2. Construction of N

Recall that Ω is an open subset of a compact, n-dimensional Riemannian manifold M, that μ is a finite positive measure on $\partial\Omega$, and that (7.0.1) holds, i.e.,

$$\int_\Omega |u|^2\,dV \le C\int_\Omega |\nabla u|^2\,dV + C\int_{\partial\Omega} |u|^2\,d\mu, \quad \forall\, u\in \operatorname{Lip}(\overline{\Omega}). \tag{7.2.1}$$

As opposed to the setup in Chapter 4, we do not assume $M\setminus\overline{\Omega}\neq\emptyset$, but explicitly allow the possibility that $\overline{\Omega}=M$. The hypothesis (4.0.1) played a key role in the estimate (4.1.1) (a special case of (7.2.1)), but, as discussed in §7.1, we take different routes here to (7.2.1), which do not appeal to such an hypothesis.

In the current setting, the construction of N in §4.2 goes through smoothly. In fact, since (7.2.1) agrees with (4.2.3), no changes (not even notational) are required in the analysis presented in (4.2.4)–(4.2.36). We summarize the first set of results, parallel with Proposition 4.2.1:

Proposition 7.2.1. *Given (7.2.1), there is a unique positive, self-adjoint operator N on $L^2(\partial\Omega,\mu)$ satisfying*

$$\mathcal{D}(N^{1/2}) = \{f\in L^2(\partial\Omega,\mu) : (u,f)\in V \text{ for some } u\in \overline{H}^1(\Omega)\}, \tag{7.2.2}$$

and

$$\langle N^{1/2}f, N^{1/2}g\rangle = \int_\Omega \langle\nabla u,\nabla v\rangle\,dV, \quad \text{for } \ (u,f),(v,g)\in V. \tag{7.2.3}$$

Here, $\langle\ ,\ \rangle$ on the left side denotes the inner product in the Hilbert space $L^2(\partial\Omega,\mu)$,

$$V = \Big\{(u,f)\in W^\# : \int_\Omega \langle\nabla u,\nabla v\rangle\,dV = 0,\ \forall\, v\in \mathring{H}^1(\Omega)\Big\}, \tag{7.2.4}$$

$$W^\# = \text{ closure of } \{(u,u|_{\partial\Omega}) : u\in\operatorname{Lip}(\overline{\Omega})\} \text{ in } \overline{H}^1(\Omega)\oplus L^2(\partial\Omega,\mu), \tag{7.2.5}$$

and

$$\mathring{H}^1(\Omega) = \{v\in \overline{H}^1(\Omega) : (v,0)\in W^\#\}. \tag{7.2.6}$$

From (7.2.3) we have

$$\langle N^{1/2}f, N^{1/2}f\rangle = \int_{\Omega} |\nabla u|^2 \, dV, \quad \text{for} \ \ (u,f) \in V. \tag{7.2.7}$$

Also, parallel to (4.2.31),

$$\langle N^{1/2}f, N^{1/2}f\rangle = \inf\Bigl\{ \int_{\Omega} |\nabla u|^2 \, dV : (u,f) \in W^{\#} \Bigr\}. \tag{7.2.8}$$

Next we have an extension of Proposition 4.2.2. That result had the hypothesis that Ω was connected, but here we emphasize that more generally Ω can be quasi-connected, as defined in (7.1.9).

Proposition 7.2.2. *In the setting of Proposition 7.2.1, assume that Ω is quasi-connected. Then*

$$\operatorname{Ker} N = \operatorname{Span}(1) + \mathfrak{Z}(\Omega, \mu), \tag{7.2.9}$$

where

$$\mathfrak{Z}(\Omega,\mu) = \{ f \in L^2(\partial\Omega, \mu) : (0, f) \in W^{\#} \}. \tag{7.2.10}$$

Proof. As in (4.2.33), given $f \in L^2(\partial\Omega, \mu)$,

$$f \in \operatorname{Ker} N \Longleftrightarrow (u,f) \in W^{\#} \ \ \text{for some } u \in \overline{H}^1(\Omega) \text{ satisfying } \nabla u = 0. \tag{7.2.11}$$

As noted in §4.2, if Ω is connected, then $u \in \overline{H}^1(\Omega)$, $\nabla u = 0$ implies u is constant on Ω. The content of (7.1.11) is that this implication holds whenever Ω is quasi-connected.

Remark. What makes this extension interesting is Lemma 7.1.1.

Note that $f \in \mathfrak{Z}(\Omega,\mu)$ if and only if there exist $u_k \in \operatorname{Lip}(\overline{\Omega})$ such that

$$u_k \longrightarrow 0 \ \ \text{in} \ \ H^1(\Omega), \quad u_k\bigr|_{\partial\Omega} \longrightarrow f \ \ \text{in} \ \ L^2(\partial\Omega,\mu). \tag{7.2.12}$$

Hence, parallel to the remark following Proposition 4.2.2, we have:

Proposition 7.2.3. *If (7.1.2) holds, then $\mathfrak{Z}(\Omega,\mu) = 0$.*

The results of §4.3 on the semigroup $\{e^{-tN} : t \geq 0\}$ also extend in a straightforward fashion. We move from Proposition 4.3.1 to

Proposition 7.2.4. *Given $f \in L^2(\partial\Omega, \mu)$,*

$$f \geq 0 \Longrightarrow e^{-tN} f \geq 0, \quad \forall\, t > 0. \tag{7.2.13}$$

We can also extend Proposition 4.3.2, and relax the connectivity hypothesis on Ω.

Proposition 7.2.5. *In the setting of Proposition 7.2.1, assume Ω is quasi-connected. If $\mathfrak{Z}(\Omega, \mu) = 0$, then $\{e^{-tN} : t \geq 0\}$ is irreducible.*

We illustrate the conclusion of Proposition 7.2.5 with the following example, involving a quasi-connected, but not connected set Ω (though not at all singular), namely

$$\Omega = (-1, 1) \setminus \{0\}. \tag{7.2.14}$$

This parallels the calculations in (2.3.9)–(2.3.13). We have $L^2(\partial\Omega) \approx \mathbb{R}^3$, via $f \mapsto (f(-1), f(0), f(1))^t$. In this case

$$\mathrm{PI}\begin{pmatrix}1\\1\\1\end{pmatrix} = 1, \quad \mathrm{PI}\begin{pmatrix}0\\0\\1\end{pmatrix} = x_+, \quad \mathrm{PI}\begin{pmatrix}1\\0\\0\end{pmatrix} = (-x)_+, \tag{7.2.15}$$

so

$$N\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad N\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad N\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \tag{7.2.16}$$

hence

$$N = \begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}. \tag{7.2.17}$$

We see that N has eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3, \tag{7.2.18}$$

with associated eigenvectors

$$v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\-2\\1\end{pmatrix}. \tag{7.2.19}$$

Rather than working out e^{-tN} explicitly from (7.2.18)–(7.2.19), let us note that

$$M = N - I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{7.2.20}$$

has square

$$M^2 = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \tag{7.2.21}$$

Now

$$e^{-tN} = e^{-t} e^{-tM} = e^{-t} \Big(I - tM + \frac{t^2}{2} M^2 \Big) + O(t^3), \tag{7.2.22}$$

as $t \to 0$. Plugging in (7.2.20)–(7.2.21) yields that all 9 matrix entries of e^{-tN} are > 0 for sufficiently small $t > 0$. Then the semigroup property $e^{-(s+t)N} = e^{-sN} e^{-tN}$ implies that all the matrix entries of e^{-tN} are > 0, for all $t > 0$.

More complicated (and interesting) cases of Proposition 7.2.4 arise in §7.3.

7.3. Domains bounded by Julia sets

Here we consider a class of domains in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \approx S^2$ that are bounded by Julia sets, and a class of probability measures on their boundary, known as measures of maximal entropy (MME). Let us introduce the basic concepts.

A rational map

$$R(z) = \frac{P(z)}{Q(z)}, \tag{7.3.1}$$

where $P(z)$ and $Q(z)$ are polynomials with no nontrivial common factors, yields a holomorphic map $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Its degree is $\deg R = \max\{\deg P, \deg Q\}$. One has the partition

$$\widehat{\mathbb{C}} = \mathcal{J}_R \cup \mathcal{F}_R, \tag{7.3.2}$$

where $\mathcal{J}_R$ is the *Julia set* of R and $\mathcal{F}_R$ is the *Fatou set*. By definition, a point $\zeta \in \widehat{\mathbb{C}}$ belongs to $\mathcal{F}_R$ if and only if there is a neighborhood $\mathcal{O}$ of ζ such that $\{R^n|_{\mathcal{O}} : n \geq 1\}$ is a normal family of maps from $\mathcal{O}$ to $\widehat{\mathbb{C}}$. The set $\mathcal{F}_R$ is an open subset of $\widehat{\mathbb{C}}$, and $\mathcal{J}_R$ (which is closed) is defined to be its complement.

The map R preserves $\mathcal{J}_R$ and $\mathcal{F}_R$. On $\mathcal{F}_R$ its behavior is regular, and on $\mathcal{J}_R$ it is chaotic. One signature of the chaos is that, given $\zeta \in \mathcal{J}_R$, and given a neighborhood $\mathcal{O}$ of ζ in $\widehat{\mathbb{C}}$, there is an m such that

$$R^m(\mathcal{O} \cap \mathcal{J}_R) \supset \mathcal{J}_R. \tag{7.3.3}$$

Whenever $\deg R \geq 2$, $\mathcal{J}_R \neq \emptyset$. If $\mathcal{J}_R \neq \widehat{\mathbb{C}}$, then its interior is empty. Proofs of these facts, and others used below, can be found in [Mil] and [MNTU]. From here on, we assume $\deg R \geq 2$ and $\mathcal{J}_R \neq \widehat{\mathbb{C}}$.

We next describe the maximal entropy measure λ on $\mathcal{J}_R$, introduced in [Lyu] and [FLM]. It is constructed as follows. Pick $p \in \mathcal{J}_R$ and set

$$\lambda_{p,k} = \frac{1}{d^k} \sum_{q \in R^{-k}(p)} \delta_q, \tag{7.3.4}$$

with $q \in R^{-k}(p)$ counted according to multiplicity. Here $d = \deg R$, so for each k, $\lambda_{p,k}$ is a probability measure supported on $\mathcal{J}_R$. Then (cf. Theorem 5.4.1 of [MNTU]), for each $p \in \mathcal{J}_R$, it is the case that

$$\lambda_{p,k} \longrightarrow \lambda, \quad \text{weak}^*, \quad \text{as} \quad k \to \infty, \tag{7.3.5}$$

and the limit is independent of p. Furthermore,

$$\operatorname{supp} \lambda = \mathcal{J}_R. \tag{7.3.6}$$

Another important property of λ (though not central to the work here) is that

$$R \text{ preserves } \lambda \text{ and acts ergodically on } (\mathcal{J}_R, \lambda). \tag{7.3.7}$$

The ergodicity is another signature of the chaos on $\mathcal{J}_R$, and its proof makes essential use of (7.3.3); cf. [MNTU], p. 189.

Of key importance here is that Lemma 7.1.3 is applicable to the MME λ. That is to say, there exist $a > 0$ and $C < \infty$ such that, for each disk D_ρ of radius ρ,

$$\lambda(D_\rho) \le C\rho^a. \tag{7.3.8}$$

See [MNTU], p. 190. Since $n = 2$ here, we have (7.1.21)–(7.1.22).

Consequently, if we take

$$\Omega = \mathcal{F}_R \subset M = \widehat{\mathbb{C}}, \quad \partial\Omega = \mathcal{J}_R, \tag{7.3.9}$$

where R is a rational map of degree ≥ 2 and $\mathcal{J}_R \ne \emptyset$, (7.1.2) applies to $\mu = \lambda$. The set $\mathcal{F}_R$ might or might not be connected; examples will be discussed below. However, we can apply Lemma 7.1.1 to see that

$$\mathcal{F}_R \text{ is quasi-connected,} \tag{7.3.10}$$

provided

$$\mathcal{H}^2(\mathcal{J}_R) = 0. \tag{7.3.11}$$

Then (7.1.10) applies. Also, whenever (7.3.11) holds, we have (7.1.15) (with $K = \mathcal{J}_R$), and hence

$$\overline{H}^1(\mathcal{F}_R) = H^1(\widehat{\mathbb{C}}), \tag{7.3.12}$$

so Proposition 7.1.2 is applicable. Consequently (7.1.1) holds, with $\mu = \lambda$. Therefore, Proposition 7.2.1 applies. We record the result.

Proposition 7.3.1. *Let R be a rational map of degree ≥ 2, and assume (7.3.11) holds. Then there is a unique positive, self-adjoint operator N on $L^2(\mathcal{J}_R, \lambda)$ satisfying*

$$\mathcal{D}(N^{1/2}) = \{f \in L^2(\mathcal{J}_R, \lambda) : (u, f) \in V \text{ for some } u \in \overline{H}^1(\mathcal{F}_R)\}, \tag{7.3.13}$$

and

$$(7.3.14) \qquad \langle N^{1/2} f, N^{1/2} g\rangle = \int_{\mathcal{F}_R} \langle \nabla u, \nabla v\rangle \, dS, \quad \textit{for} \ \ (u,f),(v,g) \in V.$$

Here dS denotes the standard area element on $\widehat{\mathbb{C}} \approx S^2$.

Remark. There exist Julia sets $\mathcal{J}_R \neq \widehat{\mathbb{C}}$ for which $\mathcal{H}^2(\mathcal{J}_R) > 0$, but (7.3.11) is known to hold under a wide variety of conditions. See [BC].

Continuing with applications of §7.2, we note that, since (7.1.2) applies, with $\mu = \lambda$, Proposition 7.2.3 gives

$$(7.3.15) \qquad \mathfrak{Z}(\mathcal{F}_R, \lambda) = 0,$$

hence, by (7.2.9),

$$(7.3.16) \qquad \operatorname{Ker} N = \operatorname{Span}(1),$$

given (7.3.11). Next, Propositions 7.2.4–7.2.5 apply, giving:

Proposition 7.3.2. *In the setting of Proposition 7.3.1, $\{e^{-tN} : t \geq 0\}$ is a symmetric Markov semigroup on $L^2(\mathcal{J}_R, \lambda)$, and it is irreducible.*

We now describe some types of Julia sets that occur, referring to [Mil] and [MNTU] for details. As a preliminary, we note that $\mathcal{J}_R$ cannot have any isolated points. More precisely, given (7.3.8), we can apply Proposition 7.1.5 and deduce that there exists $C < \infty$ such that for each compact $K \subset \mathcal{J}_R$,

$$(7.3.17) \qquad \lambda(K) \leq C\mathcal{H}^a(K).$$

Together with (7.3.6), this implies that the Hausdorff dimension of $\mathcal{J}_R \cap \mathcal{O}$ is $\geq a$ for each open $\mathcal{O} \subset \widehat{\mathbb{C}}$ that intersects $\mathcal{J}_R$. Here are some cases.

(A) $\mathcal{J}_R$ could be totally disconnected.
We see from (7.3.17) that $\mathcal{J}_R$ must be a Cantor set, of positive Hausdorff dimension. In this case, $\mathcal{F}_R$ is connected. One class of examples for this is

$$(7.3.18) \qquad R(z) = z^2 + c, \quad c \notin \mathcal{M},$$

where $\mathcal{M}$ is the Mandelbrot set.

(B) Both $\mathcal{J}_R$ and $\mathcal{F}_R$ could be connected.

For example, $\mathcal{J}_R$ could be a "dendrite;" see [Mil], pp. 41–42, [Dev], p. 290. A standard example of this is

$$R(z) = z^2 + i. \tag{7.3.19}$$

(C) $\mathcal{F}_R$ could have 2 connected components.
This occurs, for example, for $R(z) = z^2 + c$ when c is in the "main cardioid" of the Mandelbrot set $\mathcal{M}$. For $c = 0$, $\mathcal{J}_R$ is the unit circle. In other cases, it is typically not a smooth curve. In this situation, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are the two components of $\mathcal{F}_R$, one has

$$\partial\mathcal{O}_1 = \partial\mathcal{O}_2 = \mathcal{J}_R. \tag{7.3.20}$$

(D) $\mathcal{F}_R$ could have infinitely many connected components.
In fact, if $\mathcal{F}_R$ has more than 2 components, it must have infinitely many (cf. [B], Theorem 5.6.2). Examples include $R(z) = z^2 + c$ when c is in any component of the interior of $\mathcal{M}$ other than the main cardioid. In such a case, we have

(D1) $\mathcal{J}_R$ is connected.
A popular example is the "Duady rabbit," arising for

$$R(z) = z^2 + (-0.122 + 0.745\, i). \tag{7.3.21}$$

See [Mil], p. 42. Generally, in this situation, $\mathcal{J}_R$ might or might not be locally connected.

For other rational maps, it might be that

(D2) $\mathcal{J}_R$ is not connected.
For an example, see [Mil], p. 50. It is the case (cf. [Mil], Corollary 4.15) that if $\mathcal{J}_R$ is not connected, it must have uncountably many connected components.

In a number of cases, there is a nonempty "residual set,"

$$\mathcal{J}_0 = \mathcal{J}_R \setminus \bigcup_j \partial\mathcal{O}_j, \tag{7.3.22}$$

where $\{\mathcal{O}_j\}$ are the connected components of $\mathcal{F}_R$. In such a case, one has

$$\lambda(\partial\mathcal{O}_j) = 0, \tag{7.3.23}$$

for all j. If $R(z)$ is a polynomial, of degree ≥ 2, then the connected component $\mathcal{O}^\infty$ of $\mathcal{F}_R$ containing ∞ has the property that $\partial\mathcal{O}^\infty = \mathcal{J}_R$, and then $\mathcal{J}_0 = \emptyset$. In many cases, it can be shown that if $\mathcal{J}_0 = \emptyset$, then (7.3.23) holds except for one component of $\mathcal{F}_R$. Further discussion of this matter can be found in [HT].

A. Quadratic forms and self-adjoint operators

Here we discuss a couple of results that associate a positive semi-definite self-adjoint operator to a quadratic form, starting with the classical Friedrichs extension method. All vector spaces will be vector spaces over $\mathbb{R}$.

In the setup for the Friedrichs method, we have a Hilbert space H, with inner product $(\ ,\ )_H$, and another Hilbert space V, with inner product $Q(\ ,\ )$, and a continuous, injective map

$$J : V \longrightarrow H, \tag{A.1}$$

with dense range. Via J, we identify V with a dense linear subspace of H. Composing with $J^t : H \to V'$ (which is also injective, with dense range), we have

$$V \hookrightarrow H \hookrightarrow V'. \tag{A.2}$$

Meanwhile, the inner product $Q(\ ,\ )$ on V induces an isomorphism

$$A_Q : V \xrightarrow{\approx} V', \quad \text{with inverse } \ T_Q : V' \longrightarrow V. \tag{A.3}$$

We restrict T_Q to H, mapping H to $V \subset H$, to get the bounded linear operator

$$T : H \longrightarrow H. \tag{A.4}$$

Given $u, v \in H$, we can pick $f, g \in V$ such that $u = A_Q f, v = A_Q g$. Then

$$\begin{aligned} (u, Tv)_H &= (A_Q f, T_Q A_Q g)_H \\ &= (A_Q f, g)_H \\ &= Q(f, g). \end{aligned}$$

Hence

$$(u, Tv)_H = (v, Tu)_H,$$

so the bounded operator T is self-adjoint. Clearly T is injective, hence has dense range $\mathcal{R}(T) \subset H$. Thus the inverse A of T is densely defined,

$$A : H \longrightarrow H, \quad \mathcal{D}(A) = \mathcal{R}(T), \tag{A.5}$$

and it is a classical result that A is self adjoint (cf. [T1], Appendix A, Proposition 8.2). Clearly

$$u, v \in \mathcal{D}(A) \Longrightarrow (Au, v)_H = Q(u, v). \tag{A.6}$$

One verifies that

$$\mathcal{D}(A^{1/2}) = V. \tag{A.7}$$

Let us also note that

$$\mathcal{D}(A) = \{u \in V : A_Q u \in H\}. \tag{A.8}$$

Given that V is a Hilbert space, we say Q is a closed quadratic form. More generally, we can let V_0 have a positive-definite inner product Q and a continuous injection $J : V_0 \to H$. Then we can form the Hilbert space completion V of V_0, and extend J to a continuous linear map $J : V \to H$. If J is injective on V, we say Q is *closable*, and associate the self-adjoint operator A by the process described above.

Here is a basic example of a non-closable quadratic form. Set

$$H = L^2([0,1]), \quad V_0 = C([0,1]), \quad \alpha(f,g) = f(0)g(0). \tag{A.9}$$

Of course, α is not positive definite, so we fix $\delta > 0$ and set

$$Q : V_0 \times V_0 \to \mathbb{R}, \quad Q(f,g) = \alpha(f,g) + \delta(f,g)_H. \tag{A.10}$$

It is readily seen that the Hilbert-space completion of V_0 is

$$V^\# = \ell^2(\{0\}) \oplus L^2([0,1]), \tag{A.11}$$

and that the extension of J, $J^\# : V^\# \to H$, annihilates the first factor on the right side of (A.11) and maps the second factor isomorphically onto H. On the other hand, we can set

$$V = V^\# / \operatorname{Ker} J^\#, \tag{A.12}$$

and obtain the setting described above. Thus there arises a self-adjoint operator A on H, and we readily check that

$$A = \delta I. \tag{A.13}$$

Consequently the effect of the quadratic form α on A has disappeared.

We move on to an expanded method of assigning a self-adjoint operator to a quadratic form, introduced in [AtE2] and used in [AtE]. In this setting, we have a real Hilbert space H, with inner product $(\ ,\ )_H$, a real vector space W_0, a linear map

$$\tau : W_0 \longrightarrow H, \quad \text{with dense range}, \tag{A.14}$$

and a positive symmetric bilinear form

$$\alpha : W_0 \times W_0 \to \mathbb{R}, \quad \alpha(u,u) \geq 0, \ \forall\, u \in W_0. \tag{A.15}$$

We do not want to assume that α is positive definite. However, we do assume that

$$\operatorname{Ker} \alpha \cap \operatorname{Ker} \tau = 0, \tag{A.16}$$

which, if necessary, can be arranged by passing to a quotient space of W_0. Then

$$\beta(u, v) = \alpha(u, v) + (\tau u, \tau v)_H \tag{A.17}$$

is a positive-definite inner product on W_0. Let $W^\#$ denote its Hilbert space completion, yielding the continuous linear extension of τ,

$$\tau^\# : W^\# \longrightarrow H, \quad \text{with dense range.} \tag{A.18}$$

Now we form

$$V = W^\# / \operatorname{Ker} \tau^\#, \tag{A.19}$$

which has a natural Hilbert space structure, isomorphic to the orthogonal complement of $\operatorname{Ker} \tau^\#$ in $W^\#$. Also $\tau^\#$ in (A.18) induces a natural continuous injection

$$J : V \longrightarrow H, \tag{A.20}$$

with dense range, and we are in the setting described as the beginning of this appendix, yielding a positive-semidefinite self-adjoint operator

$$B : H \longrightarrow H. \tag{A.21}$$

In view of how this arises from (A.17), we have $\operatorname{Spec} B \subset [1, \infty)$, and the self-adjoint operator A associated with α is

$$A = B - I. \tag{A.22}$$

Note that

$$\mathcal{D}(A^{1/2}) = \mathcal{D}(B^{1/2}) = V, \tag{A.23}$$

where we identify V with a dense linear subspce of H, via (A.20). Equivalently,

$$\mathcal{D}(A^{1/2}) = \tau^\#(W^\#). \tag{A.24}$$

B. Green's formula

Here we discuss sufficient conditions for the identity

$$\int_{\Omega} \langle \nabla u, \nabla v\rangle \, dV = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, v \, d\sigma, \tag{B.1}$$

given

$$\Delta u = 0 \quad \text{on} \quad \Omega, \tag{B.2}$$

and given further conditions on the domain Ω and on the functions u and v on Ω. In all cases, we assume $\Omega \subset M$ is a finite perimeter domain, as defined in Chapter 4 (with $\dim M = n$), and σ is surface measure, as in (4.0.2)–(4.0.4).

The identity (B.1) arises in (3.1.5) when $\Omega \subset M$ is a Lipschitz domain and

$$u = \mathrm{PI}\, f, \quad v = \mathrm{PI}\, g, \quad f, g \in H^1(\partial\Omega). \tag{B.3}$$

As stated in (3.1.2), such an hypothesis on u implies

$$\nabla u \in \mathfrak{L}^2(\Omega), \tag{B.4}$$

where we set

$$\begin{aligned} \mathfrak{L}^p(\Omega) = \{ w \in C^0(\Omega) : w^* \in L^p(\partial\Omega) \text{ and } \exists \text{ nontangential} \\ \text{limit } w\big|_{\partial\Omega}, \text{ a.e. on } \partial\Omega \}. \end{aligned} \tag{B.5}$$

(Recall that w^* denotes the nontangential maximal function.) Meanwhile, since $H^1(\partial\Omega) \subset L^q(\partial\Omega)$ with $q = 2(n-1)/(n-3)$ for $n > 3$, any $q < \infty$ if $n = 3$, and $q = \infty$ if $n = 2$, results on $\mathrm{PI}\, g$, for $g \in H^1(\partial\Omega) \subset L^q(\partial\Omega)$ give

$$v \in \mathfrak{L}^q(\Omega), \quad q > 2, \tag{B.6}$$

as well as (B.4) for ∇v. Let us note that $\mathfrak{L}^p(\Omega) \subset L^p(\Omega)$. In fact, a stronger result holds:

$$\mathfrak{L}^p(\Omega) \subset L^{pn/(n-1)}(\Omega). \tag{B.7}$$

This is fairly elementary when Ω is a Lipschitz domain. In [HMT], §3.2, it is established for a broader class of domains, namely the class of *Ahlfors regular domains*, defined as follows.

Let $\Omega \subset M$ be an open subset, and assume

$$\mathcal{H}^{n-1}(\partial\Omega) < \infty, \quad \mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \tag{B.8}$$

where $\partial_*\Omega$ is the measure-theoretic boundary of Ω. Assume also that there exists $\alpha \in (1, \infty)$ such that for all $p \in \partial\Omega,\ r \in (0, 1]$,

$$\alpha^{-1} r^{n-1} \le \sigma(\partial\Omega \cap B_r(p)) \le \alpha r^{n-1}, \tag{B.9}$$

where σ is surface area on $\partial\Omega$ (i.e., $(n-1)$-dimensional Hausdorff measure). It is elementary that each Lipschitz domain is Ahlfors regular.

Returning to (B.1), we will show that it follows from the divergence theorem, in the form

$$\int_\Omega \operatorname{div} X \, dV = \int_{\partial\Omega} \langle \nu, X \rangle \, d\sigma, \tag{B.10}$$

when the vector field X is given by

$$X = v \nabla u, \tag{B.11}$$

in which case

$$\operatorname{div} X = \nabla v \cdot \nabla u + v \Delta u = \nabla v \cdot \nabla u, \tag{B.12}$$

provided $\Delta u = 0$. Note that (B.4) and (B.6) imply

$$X \in \mathfrak{L}^p(\Omega), \quad \text{for some } \ p > 1. \tag{B.13}$$

Meanwhile, given $\nabla u \in \mathfrak{L}^2(\Omega)$ and also $\nabla v \in \mathfrak{L}^2(\Omega)$, we have

$$\nabla u \cdot \nabla v \in \mathfrak{L}^1(\Omega) \subset L^{n/(n-1)}(\Omega) \subset L^1(\Omega). \tag{B.14}$$

The following form of the divergence theorem was established in [HMT], in §2.3 for domains Ω in Euclidean space, and in §5.3 for domains Ω in a compact Riemannian manifold, with a continuous metric tensor.

Proposition B.1. *Let M be a compact Riemannian manifold and let $\Omega \subset M$ be an Ahlfors regular domain. Assume that X is a vector field on Ω satisfying*

$$X \in \mathfrak{L}^p(\Omega), \quad \textit{and} \ \ \operatorname{div} X \in L^1(\Omega), \tag{B.15}$$

for some $p > 1$. Then (B.10) holds.

Remark. This result has been extended in [MMM] to include the case $p = 1$. This extension is useful for the study of real Hardy spaces on $\partial\Omega$, but we will not use this extension here.

We record the general result on (B.1) that follows from Proposition B.1 and the calculations involving (B.10)–(B.14), of use in Chapter 6.

Proposition B.2. *Let $\Omega \subset M$ be an Ahlfors regular domain. Assume u and v satisfy*

$$\nabla u, \nabla v \in \mathfrak{L}^2(\Omega), \quad v \in \mathfrak{L}^q(\Omega),\ q > 2, \tag{B.16}$$

and $\Delta u = 0$ on Ω. Then (B.1) holds.

Here is a case of direct interest in Chapter 6.

Proposition B.3. *Let $\Omega \subset M$ be a UR domain, and let*

$$u = \mathcal{S}f, \quad v = \mathcal{S}g, \quad f, g \in L^2(\partial\Omega), \tag{B.17}$$

where $\mathcal{S}$ is the single layer potential operator, given by (6.0.11). Then $\Delta u = \Delta v = 0$ on Ω, and u and v satisfy (B.16).

Proof. That $\Delta u = \Delta v = 0$ on Ω follows from the construction of $E(x, y)$. That $\nabla u, \nabla v \in \mathfrak{L}^2(\Omega)$ follows from (6.0.12)–(6.0.14). That $v \in \mathfrak{L}^q(\Omega)$ for some $q > 2$ follows from calculations done in §5.1 of [HMT].

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