Equivariant Isometric Embeddings of Homogeneous Spaces Into Hilbert Space

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Let G be a Lie group, $K \subset G$ a compact subgroup. Denote the associated Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$. Let π be a unitary repesentation of G on a Hilbert space H, and let $u \in H$ be a smooth vector. We denote the associated Lie algebra representation of $\mathfrak{g}$ by $d\pi$. Let us assume that, given $X \in \mathfrak{g}$,

$$d\pi(X)u = 0 \Longleftrightarrow X \in \mathfrak{k}, \tag{1}$$

that

$$g \in K \Longrightarrow \pi(g)u = u, \tag{2}$$

and that, given $g \in G$,

$$\pi(g)u = u \Longrightarrow g \in K. \tag{3}$$

Under these hypotheses, define

$$\varphi : G \longrightarrow H, \quad \varphi(g) = \pi(g)u. \tag{4}$$

By (2), this gives rise to a map

$$\psi : G/K \longrightarrow H, \quad \psi([g]) = \varphi(g), \tag{5}$$

and by (3) this map is one-to-one. Note that, given $X \in \mathfrak{g} = T_eG$,

$$D\varphi(e)X = d\pi(X)u, \tag{6}$$

so the hypotheses above imply ψ is an embedding. Let $\widetilde{X}$ denote the left invariant vector field on G associated to X:

$$\widetilde{X}(g) = \frac{d}{dt} g \operatorname{Exp}(tX)\Big|_{t=0} \in T_gG. \tag{7}$$

Note that

$$\begin{aligned} D\varphi(g)\widetilde{X}(g) &= \frac{d}{dt}\varphi(g\operatorname{Exp}(tX))\Big|_{t=0} \\ &= \frac{d}{dt}\pi(g\operatorname{Exp}(tX))u\Big|_{t=0} \\ &= \pi(g)d\pi(X)u. \end{aligned} \tag{8}$$

Hence, if also $Y \in \mathfrak{g}$, we have a (degenerate) inner product on T_gG given by

$$\begin{aligned}\langle \widetilde{X}(g), \widetilde{Y}(g)\rangle &= \mathrm{Re}\big(D\varphi(g)\widetilde{X}(g), D\varphi(g)\widetilde{Y}(g)\big)_H \\ &= \mathrm{Re}\big(\pi(g)d\pi(X)u, \pi(g)d\pi(Y)u\big)_H,\end{aligned} \tag{9}$$

which by unitarity of π is equal to

$$\langle X, Y\rangle = \mathrm{Re}\big(d\pi(X)u, d\pi(Y)u\big)_H. \tag{10}$$

Consequently, the embedding (5) induces a Riemannian metric on $M = G/K$ that is G-invariant, and with respect to which (5) is an isometric embedding, onto a manifold in H that is a G-orbit.

Now $M = G/K$ might come with a G-invariant metric tensor, and the question arises whether it must coincide (up to a constant factor) with the metric produced by (5). Indeed, sometimes the following holds:

$$M = G/K \ \text{ has a unique } G\text{-invariant Riemannian metric, up to a constant factor.} \tag{11}$$

To restate this condition, take $p = [e] \in M$ and note the natural action of K on T_pM. The condition (9) is equivalent to

$$T_pM \ \text{ has a unique } K\text{-invariant inner product, up to a constant factor.} \tag{12}$$

This is equivalent to saying that $\mathfrak{g}/\mathfrak{k}$ has no proper $\mathrm{ad}(\mathfrak{k})$-invariant linear subspace. Such a condition holds, for example, if M is a rank-1 symmetric space. See [CR] for the special case of balls with the hyperbolic metric.

On the other hand, (11)–(12) can fail, for example when $K = \{e\}$ and dim $G > 1$. The set of K-invariant inner products on T_pM is a nonempty, open, convex cone Γ in $S^2_K T^*_p$, the linear space of K-invariant symmetric bilinear forms on T_p. Even when dim $S^2_K T^*_p > 1$, the following is true.

Proposition 1. *If (M, h) is a Riemannian manifold on which G acts transitively, as a group of isometries, with $K \subset G$ the compact subgroup fixing $p \in M$, then M has an equivariant isometric embedding into a Hilbert space.*

To begin the proof, we bring in the regular representation of G on $L^2(M, h)$:

$$L(g)u(x) = u(g^{-1}x), \quad u \in L^2(M, h). \tag{13}$$

Pick $u \in C_0^\infty(M)$ to be a positive, monotonically decreasing function of $d(x, p)$. Then (1)–(3) hold, for $\pi = L$. Thus we have an embedding

$$\psi : M \longrightarrow L^2(M, h), \quad \psi(g \cdot p) = L(g)u, \tag{14}$$

giving as in (8)–(10) a G-invariant metric tensor $\tilde{g}$.

Let us denote by Q the inner product on T_pM given by h at p, and also the associated degenerate inner product on $\mathfrak{g}$. With u as above, set

$$u_\delta(x) = \delta^{1-n/2}u(\delta^{-1}x), \tag{15}$$

in exponential coordinates centered at p, and set

$$Q_\delta(X, Y) = (dL(X)u_\delta, dL(Y)u_\delta)_{L^2}. \tag{16}$$

The following is readily established.

Lemma 2. *There exists* $A \in (0,\infty)$ *such that, for all* $X, Y \in \mathfrak{g}$,

$$\lim_{\delta \to 0} Q_\delta(X,Y) = AQ(X,Y). \tag{17}$$

Replacing u by $A^{-1/2}u$, we have

$$\lim_{\delta \to 0} Q_\delta(X,Y) = Q(X,Y). \tag{18}$$

If $\dim S^2_K T^*_p = \ell = 1$, we are of course done. If $\ell > 1$, we proceed as follows. Let $A_j,\ 1 \le j \le \ell$ be a basis of $S^2_K T^*_p$, and pick $\eta > 0$ so small that

$$Q_{\pm j} = Q \pm \eta A_j \ \text{ all belong to } \ \Gamma. \tag{19}$$

Then, by arguments as above, pick $u_{\pm j}$ such that if $u_{\pm j,\delta}(x) = \delta^{1-n/2} u_{\pm j}(\delta^{-1}x)$,

$$Q_{\delta,\pm,j}(X,Y) = (dL(X)u_{\pm j,\delta}, dL(Y)u_{\pm j,\delta})_{L^2}, \tag{20}$$

then

$$\lim_{\delta \to 0} Q_{\delta,\pm,j}(X,Y) = Q_{\pm j}(X,Y). \tag{21}$$

Picking δ sufficiently small, we have

$$Q \in \ \text{convex hull of} \ \ \{Q_{\delta,\pm,j}\}. \tag{22}$$

Then we can achieve the metric tensor h by embedding M into a finite sum of copies of $L^2(M,h)$, via (13) with u replaced by $u_{\pm j,\delta}$.

Remark. Sometimes the procedure described above yields finite dimensional equivariant isometric embeddings. For example, take $M = \mathbb{RP}^2$, covered by S^2. If we let π denote the natural representation of $SO(3)$ on the space of spherical harmonics on S^2 corresponding to harmonic polynomials on $\mathbb{R}^3$ that are homogeneous of degree 2 (a 5-dimensional space), we get an equivariant isometric embedding of $\mathbb{RP}^2$ into $\mathbb{R}^5$.

Reference

[CR] M. Chuaqui and G. Riera, Euclidean models for the hyperbolic disk and its group of motions, Preprint, 2010.