Remarks on Fractional Diffusion Equations

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1. Introduction

Work on non-Gaussian probability distributions has led people to consider "fractional diffusion equations" of the following sort:

$$ {}^c\partial_t^\beta u = -(-\Delta)^\alpha u, \quad t \ge 0; \qquad u(0,x) = f(x), \tag{1.1} $$

with $\alpha, \beta \in (0,1]$, the case $\alpha = \beta = 1$ being the standard diffusion equation. Here, Δ is the Laplace operator, the fractional power $(-\Delta)^\alpha$ is a positive self-adjoint operator, defined by the spectral theorem, and ${}^c\partial_t^\beta$ is a Caputo fractional derivative (a variant of the Riemann-Liouville fractional derivative, better suited for initial-value problems):

$$ {}^c\partial_t^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\, \partial_s v(s)\, ds, \tag{1.2} $$

if $\beta \in (0,1)$. There have been a number of recent papers on this topic, with emphasis on the case $\Delta = \partial_x^2$, acting on functions on the line $\mathbb{R}$. See, for example, [CCL], [CCL2], [MPG], and references therein.

Here we point out that in a more general context the solution operator $S^t_{\beta,\alpha}$ to (1.1) yields a family of probability distributions, by virtue of being positivity-preserving:

$$ f \ge 0 \Longrightarrow S^t_{\beta,\alpha} f \ge 0, \tag{1.3} $$

and having the property

$$ \int S^t_{\beta,\alpha} f(x)\, dx = \int f(x)\, dx, \tag{1.4} $$

under appropriate hypotheses. This will hold, e.g., when Δ is the Laplace operator on $\mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$, with the Neumann boundary condition. (With the Dirichlet boundary condition, (1.3) will hold, but not (1.4). In such a case one would have a diffusion with absorption.) The key behind this is the demonstration that

$$ S^t_{\beta,\alpha} = \int_0^\infty \Psi^t_{\beta,\alpha}(s) e^{s\Delta}\, ds, \tag{1.5} $$

where

$$ \Psi^t_{\beta,\alpha}(s) \ge 0 \quad \text{for} \quad s, t > 0, \quad \alpha, \beta \in (0,1] \tag{1.6} $$

(and $(\alpha,\beta)\neq(1,1)$), and

$$\int_0^\infty \Psi^t_{\beta,\alpha}(s)\,ds = 1. \tag{1.7}$$

It will be convenient to work in the more general setting of symmetric diffusion semigroups. We also break up the analysis of positivity into two pieces. In §2 we analyze the case $\beta = 1$ of (1.1), generalized to

$$\partial_t u = -L^\alpha u, \quad u(0) = f, \tag{1.8}$$

where L is a positive self-adjoint operator and e^{-tL} a symmetric diffusion semigroup. This analysis is classical and we merely sketch the results, described in more detail in Chapter IX of [Y]. The basic conclusion is that e^{-tL^α} is also a symmetric diffusion semigroup, for $\alpha \in (0,1)$. It will be useful to have this analysis for the next step, tackled in §3:

$$^c\partial_t^\beta u = -Au, \quad u(0) = f, \tag{1.9}$$

where e^{-tA} is a symmetric diffusion semigroup and $\beta \in (0,1)$. A familiar Laplace transform analysis writes the solution operator S^t_β to (1.9) as

$$S^t_\beta = E_\beta(-t^\beta A), \tag{1.10}$$

where $E_\beta(z)$ is a special function (the Mittag-Leffler function) and the right side of (1.10) is defined by the functional calculus for self-adjoint operators. Known Laplace transform identities involving $E_\beta(z)$ (cf. (3.6), (3.11)) serendipitously allow us to deduce (1.5)–(1.7) (in a more general context, with $-\Delta$ replaced by L) from the results of §2.

In §4 we consider an extension of (1.1) to $\beta \in (1,2]$. In such a case (1.5)–(1.7) fails. One still has (1.3) for $\alpha = 1$ and $\Delta = \partial_x^2$ on functions on $\mathbb{R}$ (as shown in [MPG]), but we note that such positivity fails in higher dimension.

In §5 we construct functions $\psi(\xi)$, homogeneous of degree $\alpha \in (0,2)$, such that $e^{-t\psi(D)}$, acting on functions on $\mathbb{R}^n$, satisfies (1.3), including as special cases (with $n = 1$) various fractional derivatives. The probability distributions so obtained are known as α-stable distributions. We mention connections with material in [ST], and also our notes [T3].

In §6 we briefly discuss a class of fractional diffusion-reaction equations. In §7 we present the results of some numerical calculations of solutions to some linear diffusion and fractional diffusion equations and fractional diffusion-reaction equations of Fisher-Kolmogorov type, for functions $u(t,x)$ defined on $[0,\infty)\times S^1$.

In §8 we discuss formulas and estimates for the solution to inhomogeneous fractional diffusion equations, of the form

$$^c\partial_t^\beta u = -Au + q(t), \quad u(0) = f. \tag{1.11}$$

In §9 we apply results of §8 to establish the short time existence to fractional diffusion-reaction equations of the form

$$ {}^c\partial_t^\beta u = -Au + F(u), \quad u(0) = f, \quad A = (-\Delta)^{m/2}, \quad 0 < m \le 2. \tag{1.12} $$

when $\beta \in (0,1)$, the case $\beta = 1$ having been discussed in §6. We consider the cases $f \in C(M)$ and $f \in L^6(M)$, when M is a compact n-dimensional Riemannian manifold. The latter case requires the restriction $n/2 < m \le 2$. Also, for this result, and for the results of §§10-11, we essentially require $F(u)$ to be a cubic polynomial in u, a situation that is popular in the study of reaction-diffusion equations.

In §10 we consider (1.12) for $f \in L^{3q}(M)$, when

$$ q > 1 \ \text{ and } \ \frac{3n}{3q} < m \le 2. \tag{1.13} $$

In §11 we push this a bit, in the case $n = 2,\ m = 2$, and obtain local existence given $f \in L^p(M),\ p > 2$.

In Appendix A we recall some basic material on Riemann-Liouville fractional integrals and the Caputo fractional derivative, used in the main body of this paper. In Appendix B we briefly discuss results on finite linear systems, of the form

$$ {}^c\partial_t^\beta u = Lu, \quad u(0) = f, \tag{1.14} $$

where

$$ f \in V, \quad L \in \operatorname{End}(V), \quad \dim V < \infty. \tag{1.15} $$

In Appendix C we provide several approaches to deriving the formula (1.10), with the power series (3.5) for $E_\beta(z)$.

2. Subordination identities

Let L be a positive self-adjoint operator. By the spectral theorem, one has

$$e^{-tL^\alpha} = \int_0^\infty \Phi_{t,\alpha}(s) e^{-sL}\, ds, \quad 0 < \alpha < 1, \tag{2.1}$$

for $t > 0$, where $\Phi_{t,\alpha}$ has the property

$$e^{-t\lambda^\alpha} = \int_0^\infty \Phi_{t,\alpha}(s) e^{-s\lambda}\, ds, \quad \lambda > 0. \tag{2.2}$$

The fact that

$$(-1)^k \partial_\lambda^k e^{-t\lambda^\alpha} \ge 0 \quad \text{for} \quad \lambda, t > 0, \quad k \in \mathbb{Z}^+ \tag{2.3}$$

implies

$$\Phi_{t,\alpha}(s) \ge 0, \quad \text{for} \quad s \in [0,\infty), \tag{2.4}$$

given $t \in (0,\infty)$, $\alpha \in (0,1)$. One also has

$$\int_0^\infty \Phi_{t,\alpha}(s)\, ds = 1. \tag{2.5}$$

This is discussed in a more general context in §IX.11 of [Y].

We recall that the most familiar case is the case $\alpha = 1/2$, where

$$\Phi_{t,1/2}(s) = \frac{t}{2\pi^{1/2}} e^{-t^2/4s}\, s^{-3/2}. \tag{2.6}$$

This particular subordination identity has numerous applications to analysis; cf. [T], Chapter 3, (5.22)–(5.31), and Chapter 11, (2.24), for some examples.

The positivity in (2.4) has the implication that if e^{-sL} is a diffusion semigroup, so is e^{-tL^α}, for each $\alpha \in (0,1)$.

We record some further useful properties of $\Phi_{t,\alpha}$. First, a change of variable gives

$$\Phi_{t,\alpha}(s) = t^{-1/\alpha} \Phi_{1,\alpha}(t^{-1/\alpha} s). \tag{2.7}$$

Next, up to a constant factor,

$$f_\alpha(\xi) = e^{-(i\xi)^\alpha} \tag{2.8}$$

is the Fourier transform of $\Phi_{1,\alpha}$, extended by 0 on $(-\infty, 0]$. For $\alpha \in (0,1)$, f_α is rapidly decreasing, with all derivatives, as $|\xi| \to \infty$. It follows that $\Phi_{1,\alpha}(s)$, so extended, is C^∞ on $\mathbb{R}$, in particular, vanishing to all orders as $s \to 0$, as illustrated in case $\alpha = 1/2$ by

$$\Phi_{1,1/2}(s) = \frac{1}{2\pi^{1/2}} e^{-1/4s} s^{-3/2}, \quad s > 0. \tag{2.9}$$

On the other hand, the nature of the singularity of f_α at $\xi = 0$ implies that $\Phi_{1,\alpha}(s)$ has the following asymptotic behavior as $s \to +\infty$:

$$\Phi_{1,\alpha}(s) \sim \sum_{k \geq 1} \gamma_{\alpha k} s^{-k\alpha - 1}, \quad s \to +\infty, \tag{2.10}$$

also illustrated by (2.9) in case $\alpha = 1/2$.

3. Fractional diffusion equations

Let A be a positive self-adjoint operator. We analyze the solution to

$$(3.1)\qquad {}^c\partial_t^\beta u = -Au, \quad t > 0; \qquad u(0) = f,$$

given $\beta \in (0, 1)$, and show that if e^{-sA} is a diffusion semigroup the solution to (3.1) is also given by a diffusion, i.e., a family of positivity-preserving operators. As is standard, we use the fact that, with

$$(3.2)\qquad \mathcal{L}u(s) = \int_0^\infty e^{-st} u(t)\, dt,$$

the equation (3.1) becomes

$$(3.3)\qquad (s^\beta + A)\mathcal{L}u(s) = s^{\beta-1} f.$$

Application of Laplace inversion (cf. [MPG], Appendix A) gives

$$(3.4)\qquad u(t) = E_\beta(-t^\beta A) f,$$

where $E_\beta(z)$ is the Mittag-Leffler function

$$(3.5)\qquad E_\beta(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta n + 1)},$$

and the linear operator $E_\beta(-t^\beta A)$ in (3.4) is given by the standard operator calculus for self-adjoint operators. As derived in (A.37) of [MPG], one has

$$(3.6)\qquad E_\beta(-s) = \int_0^\infty M_\beta(r) e^{-rs}\, dr, \quad s > 0,$$

given $\beta \in (0, 1)$, where

$$(3.7)\qquad M_\beta(r) = \frac{1}{2\pi i} \int_\gamma e^{\zeta - r\zeta^\beta} \frac{d\zeta}{\zeta^{1-\beta}},$$

and γ can be taken as a vertical line $\{i\sigma + \varepsilon : \sigma \in \mathbb{R}\}$, with small $\varepsilon > 0$. It follows that

$$(3.8)\qquad E_\beta(-t^\beta A) = \int_0^\infty M_\beta(r) e^{-rt^\beta A}\, dr, \quad t > 0, \quad \beta \in (0, 1).$$

Some particular cases of $M_\beta(r)$, mentioned in (A.34)–(A.35) of [MPG], are

$$(3.9)\qquad M_{1/2}(r) = \pi^{-1/2} e^{-r^2/4}, \quad M_{1/3}(r) = 3^{2/3} \mathrm{Ai}(3^{-1/3} r).$$

These examples illustrate the following important result.

Proposition 3.1. *Given* $0 < \beta < 1, \ r \geq 0$*, we have*

$$M_\beta(r) \geq 0. \tag{3.10}$$

Proof. This can be deduced from the following identity, due to [P], and noted in (A.41) of [MPG]:

$$\beta \int_0^\infty r^{-\beta-1} M_\beta(r^{-\beta}) e^{-rs} \, dr = e^{-s^\beta}, \tag{3.11}$$

given $\beta \in (0,1)$. Comparison with (2.2) gives

$$\beta r^{-\beta-1} M_\beta(r^{-\beta}) = \Phi_{1,\beta}(r). \tag{3.12}$$

Thus the positivity (3.10) follows from (2.4)

We are now able to prove the positivity assertion made in the introduction. We merely plug (2.1) into (3.8) to obtain (1.5)–(1.7).

4. The case $\beta \in (1,2]$

Work in [MPG] also considered (1.1) for $\beta \in (1,2]$. Here the Caputo fractional derivative ${}^c\partial_t^\beta$ is given by

$$ {}^c\partial_t^\beta v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{-\beta+1}\, \partial_s^2 v(s)\, ds, \quad 1 < \beta < 2. $$

One continues to get (3.4), i.e.,

$$ u(t) = S^t_{\beta,\alpha} f = E_\beta(-t^\beta A) f, \quad A = (-\Delta)^\alpha. \tag{4.1} $$

One has in particular

$$ E_2(-s) = \cos s^{1/2}, \tag{4.2} $$

and hence

$$ S^t_{2,\alpha} = \cos t(-\Delta)^{\alpha/2}, \tag{4.3} $$

the solution operator to the Cauchy problem

$$ \bigl(\partial_t^2 + (-\Delta)^\alpha\bigr) u = 0, \quad u(0) = f,\ \partial_t u(0) = 0. \tag{4.4} $$

For $\alpha = 1$ one gets the wave equation:

$$ (\partial_t^2 - \Delta) u = 0, \quad u(0) = f,\ \partial_t u(0) = 0. \tag{4.5} $$

If $A = -\partial_x^2$, acting on functions on the line, then, as shown in [MPG], one has a diffusion. In fact, by (4.6) of [MPG], for $\beta < 2$,

$$ E_\beta(t^\beta \partial_x^2)\delta(x) = \frac{1}{2t^{1/2}} M_{\beta/2}(t^{-\beta/2}|x|), \quad x \in \mathbb{R}, \tag{4.6} $$

for $t > 0$. For $\beta \in (1,2)$ we have $\beta/2 \in (1/2,1)$, and Proposition 3.1 yields positivity of (4.6). As for the endpoint case, $\beta = 2$, one has

$$ \Bigl(\cos t\sqrt{-\partial_x^2}\Bigr)\delta(x) = \frac{1}{2}\bigl[\delta(x+t) + \delta(x-t)\bigr], \quad x \in \mathbb{R}. \tag{4.7} $$

Well known formulas for $\cos t\sqrt{-\Delta}\ \delta(x)$ with $x \in \mathbb{R}^n$ (cf. [T], Chapter 3, §5) involve distributions that are not positive measures. Hence positivity fails for $S^t_{2,1}$ on functions on $\mathbb{R}^n$ with $n \ge 2$. It follows by continuity that positivity fails for $S^t_{\beta,1}$ for β close to 2. One might investigate in more detail just how $S^t_{\beta,\alpha}$ behaves on functions on $\mathbb{R}^n$ for $n \ge 2,\ \beta \in (1,2)$.

5. Diffusion semigroups with homogeneous generators

Here we consider semigroups of the form $e^{-t\psi(D)}$, where $\psi(D)$ acts on functions on $\mathbb{R}^n$ via Fourier multiplication by $\psi(\xi)$. We construct functions homogeneous of degree $\alpha \in (0,2)$ for which $e^{-t\psi(D)}$ is positivity preserving and furthermore satisfies

$$0 \le f \le 1 \Longrightarrow 0 \le e^{-t\psi(D)} f \le 1, \quad \forall\, t > 0. \tag{5.1}$$

Of course

$$\psi(\xi) = |\xi|^\alpha, \quad 0 \le \alpha \le 2, \tag{5.2}$$

works, by the results of §2. We obtain further cases by specializing the Levy-Khinchin formula (cf. [J], §3.7). In this way we obtain the following such homogeneous generators:

$$\begin{aligned} \Phi_{\alpha,g}(\xi) &= -\int_{\mathbb{R}^n} (e^{iy\cdot\xi} - 1) g(y) |y|^{-n-\alpha}\, dy, && 0 < \alpha < 1, \\ \Psi_{\alpha,g}(\xi) &= -\int_{\mathbb{R}^n} (e^{iy\cdot\xi} - 1 - iy\cdot\xi) g(y) |y|^{-n-\alpha}\, dy, && 1 < \alpha < 2. \end{aligned} \tag{5.3}$$

The function g is assumed to be positive, bounded, and homogeneous of degree 0, i.e.,

$$g \ge 0, \quad g \in L^\infty(\mathbb{R}^n), \quad g(ry) = g(y), \quad \forall\, r > 0. \tag{5.4}$$

It is easy to verify that both integrals in (5.3) are absolutely convergent, and, for $r > 0$,

$$\begin{aligned} \Phi_{\alpha,g}(r\xi) &= r^\alpha \Phi_{\alpha,g}(\xi), \quad 0 < \alpha < 1, \\ \Psi_{\alpha,g}(r\xi) &= r^\alpha \Psi_{\alpha,g}(\xi), \quad 1 < \alpha < 2. \end{aligned} \tag{5.5}$$

When $g \equiv 1$ we obtain a positive multiple of (5.2).

We now specialize to $n = 1$ and $g = \chi_{\mathbb{R}^+}$, so we look at

$$\begin{aligned} \varphi_\alpha(\xi) &= -\int_0^\infty (e^{iy\xi} - 1) y^{-1-\alpha}\, dy, && 0 < \alpha < 1, \\ \psi_\alpha(\xi) &= -\int_0^\infty (e^{iy\xi} - 1 - iy\xi) y^{-1-\alpha}\, dy, && 1 < \alpha < 2. \end{aligned} \tag{5.6}$$

Clearly φ_α and ψ_α are holomorphic in $\{\xi \in \mathbb{C} : \operatorname{Im} \xi > 0\}$, and homogeneous of degree α in ξ. Also, for $\eta > 0$,

$$\begin{aligned} \varphi_\alpha(i\eta) &= -\int_0^\infty (e^{-y\eta} - 1) y^{-1-\alpha}\, dy > 0, && 0 < \alpha < 1, \\ \psi_\alpha(i\eta) &= -\int_0^\infty (e^{-y\eta} - 1 + y\eta) y^{-1-\alpha}\, dy < 0, && 1 < \alpha < 2, \end{aligned} \tag{5.7}$$

since, for $r > 0$, $1 - r < e^{-r} < 1$. It follows that $\varphi_\alpha(\xi)$ and $\psi_\alpha(\xi)$ are positive multiples of

$$\begin{aligned} \varphi_\alpha^\#(\xi) &= (-i\xi)^\alpha, && 0 < \alpha < 1, \\ \psi_\alpha^\#(\xi) &= -(-i\xi)^\alpha, && 1 < \alpha < 2, \end{aligned} \tag{5.8}$$

restrictions to $\mathbb{R}$ of functions holomorphic on $\{\xi \in \mathbb{C} : \operatorname{Im} \xi > 0\}$. Taking instead $g = \chi_{\mathbb{R}^-}$, we obtain positive multiples of

$$\begin{aligned} \varphi_\alpha^b(\xi) &= (i\xi)^\alpha, && 0 < \alpha < 1, \\ \psi_\alpha^b(\xi) &= -(i\xi)^\alpha, && 1 < \alpha < 2, \end{aligned} \tag{5.9}$$

restrictions to $\mathbb{R}$ of functions holomorphic on $\{\xi \in \mathbb{C} : \operatorname{Im} \xi < 0\}$, satisfying

$$\varphi_\alpha^b(-i\eta) > 0, \quad \psi_\alpha^b(-i\eta) < 0, \quad \forall\, \eta > 0. \tag{5.10}$$

The functions in (5.8) and (5.9) are well known examples of homogeneous functions $\psi(\xi)$ for which $e^{-t\psi(D)}$ satisfies (5.1). The associated operators $\psi(D)$ are fractional derivatives.

It is also useful to observe the explicit formulas

$$e^{-t\varphi_\alpha^\#(\xi)} = e^{-t(\cos \pi\alpha/2)|\xi|^\alpha} \Big[\cos\Big(t\big(\sin\frac{\pi\alpha}{2}\big)|\xi|^\alpha\Big) + i\sigma(\xi)\sin\Big(t\big(\sin\frac{\pi\alpha}{2}\big)|\xi|^\alpha\Big)\Big]. \tag{5.11}$$

for $t > 0$, $0 < \alpha < 1$, where

$$\sigma(\xi) = \operatorname{sgn} \xi, \tag{5.12}$$

and

$$e^{-t\psi_\alpha^\#(\xi)} = e^{t(\cos \pi\alpha/2)|\xi|^\alpha} \Big[\cos\Big(t\big(\sin\frac{\pi\alpha}{2}\big)|\xi|^\alpha\Big) - i\sigma(\xi)\sin\Big(t\big(\sin\frac{\pi\alpha}{2}\big)|\xi|^\alpha\Big)\Big], \tag{5.13}$$

fior $t > 0$, $1 < \alpha < 2$. Note that

$$0 < \alpha < 1 \Rightarrow \cos\frac{\pi\alpha}{2} > 0, \quad 1 < \alpha < 2 \Rightarrow \cos\frac{\pi\alpha}{2} < 0, \tag{5.14}$$

so of course we have decaying exponentials in both (5.11) and (5.13). We get similar formulas with # replaced by b, since in fact

$$\varphi^b_\alpha(\xi) = \varphi^\#_\alpha(-\xi), \quad \psi^b_\alpha(\xi) = \psi^\#_\alpha(-\xi). \tag{5.15}$$

Returning to the general formulas (5.3), we can switch to polar coordinates and write

$$\begin{aligned} \Phi_{\alpha,g}(\xi) &= -\int_{S^{n-1}} \int_0^\infty (e^{is\omega\cdot\xi} - 1) g(\omega) s^{-1-\alpha}\, ds\, dS(\omega), \\ \Psi_{\alpha,g}(\xi) &= -\int_{S^{n-1}} \int_0^\infty (e^{is\omega\cdot\xi} - 1 - is\omega\cdot\xi) g(\omega) s^{-1-\alpha}\, ds\, dS(\omega), \end{aligned} \tag{5.16}$$

and hence

$$\begin{aligned} \Phi_{\alpha,g}(\xi) &= \int_{S^{n-1}} \varphi_\alpha(\omega\cdot\xi) g(\omega)\, dS(\omega), \\ \Psi_{\alpha,g}(\xi) &= \int_{S^{n-1}} \psi_\alpha(\omega\cdot\xi) g(\omega)\, dS(\omega). \end{aligned} \tag{5.17}$$

We can extend the scope, replacing $g(\omega)\, dS(\omega)$ by a general positive, finite Borel measure on S^{n-1}. Taking into account the calculations yielding (5.8)–(5.9), we obtain homogeneous generators satisfying (5.1), of the form

$$\begin{aligned} \Phi^b_{\alpha,\nu}(\xi) &= \int_{S^{n-1}} (i\omega\cdot\xi)^\alpha\, d\nu(\omega), \qquad 0 < \alpha < 1, \\ \Psi^b_{\alpha,\nu}(\xi) &= -\int_{S^{n-1}} (i\omega\cdot\xi)^\alpha\, d\nu(\omega), \quad 1 < \alpha < 2, \end{aligned} \tag{5.18}$$

where ν is a positive, finite Borel measure on S^{n-1}.

It remains to discuss the case $\alpha = 1$. For $n = 1$ it is seen that positive multiples of

$$|\xi| + ia\xi, \quad a \in \mathbb{R}, \tag{5.19}$$

work. Hence the following functions on $\mathbb{R}^n$ work:

$$|\omega\cdot\xi| + ia\omega\cdot\xi, \quad \omega \in S^{n-1},\ a \in \mathbb{R}.$$

We can take positive superpositions of such functions and, in analogy with (5.18), obtain generators of diffusion semigroups whose negatives are Fourier multiplication by

$$ib\cdot\xi + \Xi_\nu(\xi), \tag{5.20}$$

where $b \in \mathbb{R}^n$ and

$$\Xi_\nu(\xi) = \int_{S^{n-1}} |\omega \cdot \xi| \, d\nu(\omega). \tag{5.21}$$

We now tie in results derived above with material given in Chapters 1–2 of [ST]. For such functions $\psi(\xi)$, homogeneous of degree $\alpha \in (0, 2]$, as constructed above, the probability distributions

$$p_t(x) = e^{-t\psi(D)}\delta(x) \tag{5.22}$$

are known as α-stable distributions. In the notation (1.1.6) of [ST], consider

$$\psi(\xi) = \sigma^\alpha \, |\xi|^\alpha \Big(1 - i\beta(\operatorname{sgn}\xi) \tan \frac{\pi\alpha}{2}\Big), \quad \xi \in \mathbb{R}. \tag{5.23}$$

Here

$$\sigma \in (0, \infty), \quad \beta \in [-1, 1], \tag{5.24}$$

and $\alpha \in (0, 2)$ but $\alpha \neq 1$. Also, take $\mu \in \mathbb{R}$. Then $e^{-\psi(D)+i\mu D}\delta(x)$ is a probability distribution on the line called an α-stable distribution with scale parameter σ, skewness parameter β, and shift parameter μ. It is clear from (5.11)–(5.13) that each function of the form (5.23) is a positive linear combination of $\varphi^\#_\alpha(\xi)$ and $\varphi^b_\alpha(\xi)$ if $\alpha \in (0, 1)$ and a positive linear combination of $\psi^\#_\alpha(\xi)$ and $\psi^b_\alpha(\xi)$ if $\alpha \in (1, 2)$.

In case $\alpha = 1$, one goes beyond $\psi(\xi)$ homogeneous of degree 1 in ξ, to consider

$$\psi(\xi) = \sigma \, |\xi| \Big(1 + i\frac{2\beta}{\pi}(\operatorname{sgn}\xi) \log |\xi|\Big) + i\mu\xi, \quad \xi \in \mathbb{R}, \tag{5.25}$$

again with $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$. Then $e^{-\psi(D)}\delta(x)$ is a probability distribution on $\mathbb{R}$ called a 1-stable distribution, with scale parameter σ, skewness β, and shift μ. The cases arising from (5.19) all have skewness $\beta = 0$.

Similarly, functions $\psi(\xi)$ of the form (5.18) and (5.20)–(5.21) produce probability distributions $e^{-\psi(D)}\delta(x)$ on $\mathbb{R}^n$ that are α-stable. These, plus analogues with a shift incorporated, comprise all of them except when $\alpha = 1$, in which case one generalizes (5.21) to

$$\widetilde{\Xi}_\nu(\xi) = \int_{S^{n-1}} |\omega \cdot \xi| \Big(1 + \frac{2i}{\pi}(\operatorname{sgn}\omega \cdot \xi) \log |\omega \cdot \xi|\Big) d\nu(\omega). \tag{5.26}$$

Compare (2.3.1)–(2.3.2) in [ST].

We return to the case $n = 1$ and make some more comments on the probability distributions

$$\begin{aligned} p^\alpha_t(x) &= e^{-t\varphi^\#_\alpha(D)}\delta(x), \quad 0 < \alpha < 1, \\ p^\alpha_t(x) &= e^{-t\psi^\#_\alpha(D)}\delta(x), \quad 1 < \alpha < 2, \end{aligned} \tag{5.27}$$

and their variants with # replaced by b, which are simply $p_t^\alpha(-x)$. Explicitly, we have

$$p_t^\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\cdot\xi - t\varphi_\alpha^\#(\xi)}\, d\xi, \tag{5.28}$$

for $0 < \alpha < 1$, with $\varphi_\alpha^\#(\xi)$ replaced by $\psi_\alpha^\#(\xi)$ for $1 < \alpha < 2$. Recall that $\varphi_\alpha^\#$ and $\psi_\alpha^\#$ are holomorphic in $\{\xi \in \mathbb{C} : \operatorname{Im}\xi > 0\}$. It follows from the Paley-Wiener theorem that, for each $t > 0$,

$$p_t^\alpha(x) = 0, \quad \text{for} \ \ x \in [0,\infty),\ 0 < \alpha < 1. \tag{5.29}$$

This theorem does not apply when $\alpha \in (1,2)$, but a shift in the contour of integration to $\{\xi + ib : \xi \in \mathbb{R}\}$, with arbitrary $b > 0$ yields

$$p_t^\alpha(x) = e^{-bx} \times \ \text{bounded function of } x, \tag{5.30}$$

for $x \in \mathbb{R}$, whenever $1 < \alpha < 2$, hence

$$p_t^\alpha(x) = o(e^{-bx}), \quad \forall\, b > 0, \quad \text{as} \ \ x \to +\infty, \quad \text{for} \ \ 1 < \alpha < 2. \tag{5.31}$$

A more precise asymptotic behavior is stated in (1.2.11) of [ST].

We also note that, for $\alpha \in (1.2)$, $p_t^\alpha(x)$ is real analytic in $x \in \mathbb{R}$, and in fact extends to an entire holomorphic function in $x \in \mathbb{C}$, for each $t > 0$, due to rapidity with which $\operatorname{Re}\psi_\alpha^\#(\xi) \to +\infty$ as $|\xi| \to \infty$, which of course forbids (5.29) in this case.

6. Fractional diffusion-reaction equations

We consider $\ell \times \ell$ systems of equations

$$\frac{\partial u}{\partial t} = -Lu + X(u), \quad u(0) = f, \tag{6.1}$$

where $u = u(t, x)$ takes values in $\mathbb{R}^\ell$, X is a real vector field on $\mathbb{R}^\ell$, and L is a diagonal operator,

$$L = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_\ell \end{pmatrix}, \tag{6.2}$$

where each operator $-A_j$ generates a diffusion semigroup, satisfying

$$a \le f \le b \Longrightarrow a \le e^{-tA_j} f \le b, \quad \forall\, t > 0. \tag{6.3}$$

In case the operators A_j are second order differential operators satisfying (6.3), the system (6.1) is a reaction-diffusion equation. Recent studies have considered A_j given by fractional derivatives. For example, [CCL3] considers the following scalar equation (a modification of the Fisher-Kolmogorov equation):

$$\frac{\partial u}{\partial t} = -\psi_\alpha^b(D)u + u(1-u), \quad u(0) = f, \tag{6.4}$$

where $\alpha \in (1, 2)$ and ψ_α^b is given by (5.9).

Our next goal is to present an extension of Proposition 4.4 in Chapter 15 of [T], giving a global existence result and some qualitative information on an important class of systems of the form (6.1). Here is the set-up. We assume there is a family $\{K_s : 0 \le s < \infty\}$ of compact subsets of $\mathbb{R}^\ell$ such that each K_s has the invariance property

$$f(x) \in K_s\ \forall\ x \Longrightarrow e^{-tL} f(x) \in K_s\ \forall\ x. \tag{6.5}$$

For example, K_s could be a Cartesian product of intervals, and then (6.3) implies (6.5). Furthermore, we assume that

$$\mathcal{F}_X^t(K_s) \subset K_{s+t}, \quad s, t \in \mathbb{R}^+, \tag{6.6}$$

where $\mathcal{F}_X^t$ is the flow on $\mathbb{R}^\ell$ generated by X. Then we have the following result.

Proposition 6.1. *Under the hypotheses (6.5)–(6.6), if $f(x) \in K_0$ for all x, then (6.1) has a solution for all $t \in [0, \infty)$, and, for each $t > 0$,*

$$u(t, x) \in K_t, \quad \forall\, x. \tag{6.7}$$

The proof is basically the same as the proof of Proposition 4.4 mentioned above. The key behind (6.7) is the nonlinear Trotter product formula:

$$u(t) = \lim_{n \to \infty} \Big(e^{-(t/n)L} \mathcal{F}^{t/n} \Big)^n f, \tag{6.8}$$

where

$$\mathcal{F}^t f(x) = \mathcal{F}_X^t(f(x)). \tag{6.9}$$

As one application, in case $\ell = 1$, we see that if $0 < a < b < \infty$, and if

$$a \le f(x) \le b, \quad \forall\, x \in \mathbb{R}, \tag{6.10}$$

then (6.4) has a solution for all $t \in [0, \infty)$, and

$$\lim_{t \to \infty} u(t, x) \equiv 1. \tag{6.11}$$

With a little more work, we could allow $a = 0$ and obtain (6.11) as long as f is not identially zero. In [CCL3] there is an intriguing discussion of finer qualitative behavior of moving front solutions to (a variant of) (6.4), based on numerical evidence. See §7 for some more on this.

One can consider various other reaction-diffusion equations, such as the Fitzhugh-Nagumo equations, and variants, with ∂_x^2 replaced by fractional derivatives, to which Proposition 6.1 would be applicable. See Chapter 15, §4 of [T] for other examples, which could be similarly generalized.

7. Numerical experiments

Here we discuss numerical results on five linear (fractional) diffusion equations:

$$\frac{\partial u}{\partial t} = -Lu, \quad u(0) = f, \tag{7.1}$$

and five (fractional) diffusion-reaction equations of Fisher-Kolmogorov type:

$$\frac{\partial u}{\partial t} = -Lu + X(u), \quad u(0) = f, \tag{7.2}$$

for $u = u(t, x)$ defined on $[0, \infty) \times S^1$, where $S^1 \approx \mathbb{R}/(2\pi\mathbb{Z})$ is the circle. In (7.2) we take

$$X(u) = 6u(1 - u), \tag{7.3}$$

and the five operators L we consider are, respectively,

$$-\frac{d^2}{dx^2}, \quad \Big(-\frac{d^2}{dx^2}\Big)^{1/2}, \quad \Big(-\frac{d^2}{dx^2}\Big)^{1/4}, \quad \psi^b_{3/2}(D), \quad \varphi^\#_{1/2}(D), \tag{7.4}$$

where $\psi^b_\alpha(\xi)$ and $\varphi^\#_\alpha(\xi)$ are given by (5.8)–(5.9). In all cases we take

$$\begin{aligned} f(x) = 1 \quad &\text{if} \ \ |x| < \frac{2\pi}{10}, \\ 0 \quad &\text{otherwise}, \end{aligned} \tag{7.5}$$

and we picture $S^1 = [-\pi, \pi]$, with the endpoints identified.

To solve (7.1), we represent the solution as a Fourier multiplier, namely Fourier multiplication by $e^{-tL(\xi)}$, where $L(\xi)$ is given, respectively, by

$$\xi^2, \quad |\xi|, \quad |\xi|^{1/2}, \quad \psi^b_{3/2}(\xi), \quad \varphi^\#_{1/2}(\xi). \tag{7.6}$$

In particular, by (5.11)–(5.15), we have

$$e^{-t\psi^b_{3/2}(\xi)} = e^{-(\sqrt{2}/2)t|\xi|^{3/2}} \left[\cos\Big(\frac{\sqrt{2}}{2}t|\xi|^{3/2}\Big) + i\sigma(\xi)\sin\Big(\frac{\sqrt{2}}{2}t|\xi|^{3/2}\Big)\right], \tag{7.7}$$

and

$$e^{-t\varphi^\#_{1/2}(\xi)} = e^{-(\sqrt{2}/2)t|\xi|^{1/2}} \left[\cos\Big(\frac{\sqrt{2}}{2}t|\xi|^{1/2}\Big) + i\sigma(\xi)\sin\Big(\frac{\sqrt{2}}{2}t|\xi|^{1/2}\Big)\right]. \tag{7.8}$$

Our numerical approximation uses a 1024 point discrete Fourier transform, implemented by an FFT.

To solve (7.2) numerically, we use Strang's splitting method, a variant of (6.8) given by

$$u(t) = \lim_{n\to\infty} \left(\mathcal{F}^{t/2n} e^{-(t/n)L} \mathcal{F}^{t/2n}\right)^n f, \tag{7.9}$$

which is formally second order accurate. More precisely, we fix a time step $h = 0.001$ and take

$$u(nh) \approx \left(\mathcal{F}^{h/2} e^{-hL} \mathcal{F}^{h/2}\right)^n f, \tag{7.10}$$

for $0 \le n \le 500$, so $t = nh \in [0, 0.5]$. We evaluate e^{-hL} as above, via Fourier multiplication, and we use a difference scheme to approximate the action of $\mathcal{F}^{h/2}$.

Figures 1A–1E illustrate solutions to the five linear equations of the form (7.1), with L given in (7.4). Each figure presents the graph of $u(t,x)$, for $x \in [-\pi, \pi]$, at times $t = n/1000$, with $n = 0, 100, 200, 300, 400, 500$. In Figure 1A, the equation is the standard diffusion equation $u_t = u_{xx}$, and the graphs beyond $t = 0$ certainly look quite Gaussian. Figures 1B and 1C illustrate some symmetric superdiffusions. Figures 1D and 1E illustrate some asymmetric diffusions. In Figure 1D a fat tail sprouts off to the right, while in Figure 1E the fat tail sprouts off to the left, and wraps completely around S^1 by $t = 0.3$.

Figures 2A–2E illustrate solutions to five Fisher-Kolmogorov type equations of the form (7.2), with L again given in (7.4). In all cases, the solution $u(t,x)$ takes values in the interval $[0,1]$, and the vector field X on this interval generates a flow that pushes points away from the critical point 0 and towards the critical point 1. In Figure 2A, the equation is a standard Fisher-Kolmogorov equation. Figures 2B–2C illustrate variants, where the effects of the superdiffusions lead to $u(t,x)$ approaching 1 more rapidly (as t increases) for x far away from 0 than one sees in Figure 2A. In Figures 2D–2E one also sees how the fat tails for the fractional diffusions lead to an approach of $u(t,x)$ to 1, somewhat more rapid in the right (resp., left) direction, in the two respective cases.

One can particularly compare the results illustrated in Figure 2D with numerical results discussed in [CCL3].

8. Inhomogeneous fractional diffusion equations

Here we consider equations of the form

$$
{}^c\partial_t^\beta u = -Au + q(t), \quad u(0) = f, \tag{8.1}
$$

where A is a positive, self-adjoint operator on a Hilbert space H, $f \in H$, and $q \in C(\mathbb{R}^+, H)$. We assume $0 < \beta \le 1$. The operator ${}^c\partial_t^\beta$ is as in (1.2) if $\beta \in (0,1)$. We have the Laplace transform identity

$$
\mathcal{L}({}^c\partial_t^\beta u)(s) = s^\beta \mathcal{L}u(s) - s^{\beta-1}u(0). \tag{8.2}
$$

Hence (8.1) implies

$$
\mathcal{L}u(s) = (s^\beta + A)^{-1}\mathcal{L}q(s) + s^{\beta-1}(s^\beta + A)^{-1} f. \tag{8.3}
$$

Recall that the Laplace transform of $E_\beta(-t^\beta A)$ is $s^{\beta-1}(s^\beta + A)^{-1}$, with E_β as in (3.5)–(3.8). In fact, if

$$
e_\beta(t) = E_\beta(-t^\beta), \tag{8.4}
$$

we have

$$
\int_0^\infty e_\beta(t) e^{-st}\, dt = \frac{s^{\beta-1}}{s^\beta + 1}. \tag{8.5}
$$

It follows that

$$
\int_0^\infty e_\beta(t\gamma) e^{-st}\, dt = \frac{s^{\beta-1}}{s^\beta + \gamma^\beta}, \tag{8.6}
$$

which gives

$$
\int_0^\infty E_\beta(-t^\beta A) e^{-st}\, dt = s^{\beta-1}(s^\beta + A)^{-1}. \tag{8.7}
$$

We also have

$$
\begin{aligned}
A^{1/\beta} \int_0^\infty & e_\beta'(tA^{1/\beta}) e^{-st}\, dt \\
&= s \int_0^t e_\beta(tA^{1/\beta}) e^{-st}\, dt + e_\beta(tA^{1/\beta}) e^{-st}\Big|_0^\infty \\
&= s^\beta (s^\beta + A)^{-1} - 1 \\
&= -A(s^\beta + A)^{-1}.
\end{aligned} \tag{8.8}
$$

With this we can apply Laplace inversion to (8.3) and obtain

$$u(t)=e_\beta(tA^{1/\beta})f-A^{-1+1/\beta}\int_0^t e_\beta'(\tau A^{1/\beta})q(t-\tau)\,d\tau, \tag{8.9}$$

using the fact that

$$g(t)=\int_0^t h(\tau)q(t-\tau)\,d\tau \Longrightarrow \mathcal{L}g(s)=\mathcal{L}h(s)\,\mathcal{L}q(s). \tag{8.10}$$

A formula equivalent to (8.9) is

$$u(t)=E_\beta(-t^\beta A)f+\beta\int_0^t \tau^{\beta-1}E_\beta'(-\tau^\beta A)q(t-\tau)\,d\tau. \tag{8.11}$$

Compare (A.30) of [MPG] for the case $A=1$, and (7.4) of [D] for the general formula.

Recalling (3.6), we have

$$E_\beta'(-s)=\int_0^\infty M_\beta(r)re^{-rs}\,ds,\quad s>0, \tag{8.12}$$

with $M_\beta(r)$ given by (3.7), and also by (3.11)–(3.12). Hence (3.8), i.e.,

$$E_\beta(-t^\beta A)=\int_0^\infty M_\beta(r)e^{-rt^\beta A}\,dr, \tag{8.13}$$

is complemented by

$$E_\beta'(-t^\beta A)=\int_0^\infty M_\beta(r)re^{-rt^\beta A}\,dr, \tag{8.14}$$

for $t>0,\ \beta\in(0,1)$. Note that $M_\beta(r)$ and $M_\beta(r)r$ are positive and integrable on $\mathbb{R}^+$. Hence, if $\{e^{-sA}:s>0\}$ is positivity preserving, on $H=L^2(M)$, so are the operators (8.13) and (8.14).

We desire to obtain some estimates on $E_\beta(-s)$ for $s\in\mathbb{R}^+$, hence on the operators that appear in (8.11). Of course, the formula (3.5) implies this function is smooth on $[0,\infty)$. We want to examine its asymptotic behavior as $s\nearrow+\infty$. We first tackle the behavior of $e_\beta(t)$ as $t\nearrow\infty$. The key tool for is the identity (8.5), which is valid for $\operatorname{Re}s\ge 0$. The evaluation for $s=i\xi,\ \xi\in\mathbb{R}$ gives the Fourier transform of $e_\beta(t)$ (extended to vanish on $\mathbb{R}^-$):

$$\hat{e}_\beta(\xi)=\frac{(i\xi)^{\beta-1}}{(i\xi)^\beta+1},\quad 0<\beta<1. \tag{8.15}$$

This Fourier transform identity enables us to determine the behavior of $e_\beta(t)$ as $t \nearrow \infty$, due to the (almost) classical conormal singularity of $\hat{e}_\beta$ at $\xi = 0$. We get, as $t \nearrow +\infty$,

$$e_\beta(t) \sim \sum_{k\ge 1} a_{\beta k} t^{-k\beta}, \tag{8.16}$$

$$e'_\beta(t) \sim -\sum_{k\ge 1} k\beta a_{\beta k} t^{-k\beta-1}. \tag{8.17}$$

Equivalently, as $s \nearrow +\infty$,

$$E_\beta(-s) = e_\beta(s^{1/\beta}) \sim \sum_{k\ge 1} a_{\beta k} s^{-k}, \tag{8.18}$$

$$E'_\beta(-s) = \frac{1}{\beta} s^{1/\beta-1} e'_\beta(s^{1/\beta}) \sim -\sum_{k\ge 1} k a_{\beta k} s^{-k-1}, \tag{8.19}$$

assuming $\beta \in (0,1)$. We emphasize the leading terms:

$$E_\beta(-s) \sim a_{\beta 0} s^{-1} + \cdots, \quad E'_\beta(-s) \sim -a_{\beta 0} s^{-2} + \cdots. \tag{8.20}$$

By contrast,

$$E_1(-s) = e^{-s}. \tag{8.21}$$

We now collect some operator estimates on $E_\beta(-t^\beta A)$ and $E'_\beta(-t^\beta A)$. First, suppose B is a Banach space on which e^{-tA} acts as a contraction semigroup:

$$\|e^{-tA} f\|_B \le \|f\|_B, \quad \forall\, t > 0. \tag{8.22}$$

Then (8.13)–(8.14), plus the positivity of $M_\beta(r)$ and the fact that $M_\beta(r)$ and $M_\beta(r)r$ integrate to $E_\beta(0)$ and $E'_\beta(0)$, respectively, give

$$\begin{aligned} \|E_\beta(-t^\beta A) f\|_B &\le \|f\|_B, \\ \|E'_\beta(-t^\beta A) f\|_B &\le \frac{1}{\beta\Gamma(\beta)} \|f\|_B. \end{aligned} \tag{8.23}$$

Next, assume H is a Hilbert space and A is a positive, self-adjoint operator on H. Then (8.20) plus the smoothness of $E_\beta(-s)$ on $[0,\infty)$ imply that $sE_\beta(-s)$, $sE_\beta(-s)$, and $s^2 E'_\beta(-s)$ are bounded on $[0,\infty)$, hence

$$\|t^\beta A E_\beta(-t^\beta A) f\|_H, \ \ \|t^\beta A E'_\beta(-t^\beta A) f\|_H, \ \ \|t^{2\beta} A^2 E'_\beta(-t^\beta A) f\|_H \le C\|f\|_H, \tag{8.24}$$

for $t \in [0,\infty)$. Interpolation with (8.23) (with $B = H$) yields further estimates, such as

$$\|\tau^{\sigma\beta} E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(H,\mathcal{D}(A^\sigma))} \le C, \quad \tau > 0, \tag{8.25}$$

given $\sigma \in (0,1)$, hence

$$\|\tau^{\beta-1}E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(H,\mathcal{D}(A^\sigma))} \le C\tau^{-1+(1-\sigma)\beta}. \tag{8.26}$$

We begin to specialize. For the rest of this section, we assume M is a compact, smooth Riemannian manifold, without boundary, and

$$A = (-\Delta)^{m/2}, \quad 0 < m \le 2, \tag{8.27}$$

where Δ is the Laplace-Beltrami operator on M. Then (8.22)–(8.23) hold for

$$B = L^p(M), \quad 1 \le p \le \infty, \quad B = C(M), \tag{8.28}$$

and (8.24)–(8.26) hold for

$$H = L^2(M), \quad \mathcal{D}(A^\sigma) = H^{\sigma m,2}(M). \tag{8.29}$$

We can go further, noting that

$$E_\beta(-s) \in S^{-1}_{1,0}([0,\infty)), \quad E'_\beta(-s) \in S^{-2}_{1,0}([0,\infty)), \tag{8.30}$$

where to say $F \in S^\mu_{1,0}([0,\infty))$ is to say $F \in C^\infty([0,\infty))$ and

$$|\partial_s^j F(s)| \le C_j \langle s\rangle^{\mu-j}, \quad \forall\, j \in \mathbb{Z}^+,\ s \in [0,\infty). \tag{8.31}$$

Now (8.27) implies

$$A \in OPS^m(M) \tag{8.32}$$

is elliptic, as well as positive and self-adjoint. Results in Chapter 12 of [T2] then imply that, given $T_0 \in (0,\infty)$,

$$\begin{aligned} &E_\beta(-t^\beta A), \quad t^\beta A E_\beta(-t^\beta A), \\ &E'_\beta(-t^\beta A), \quad t^\beta A E'_\beta(-t^\beta A), \quad t^{2\beta} A^2 E'_\beta(-t^\beta A) \\ &\text{are bounded in } \ OPS^0_{1,0}(M), \quad \text{for } \ t \in (0,T_0]. \end{aligned} \tag{8.33}$$

Boundedness of elements of $OPS^0_{1,0}(M)$ on $L^p(M)$ for $1 < p < \infty$ yield the following estimates, for such p:

$$\|E_\beta(-t^\beta A)f\|_{L^p}, \ \ t^\beta\|AE_\beta(-t^\beta A)f\|_{L^p} \le C\|f\|_{L^p}, \tag{8.34}$$

and

$$\|E'_\beta(-t^\beta A)f\|_{L^p}, \ \ t^\beta\|AE'_\beta(-t^\beta A)f\|_{L^p}, \ \ t^{2\beta}\|A^2E'_\beta(-t^\beta A)f\|_{L^p} \le C\|f\|_{L^p}. \tag{8.35}$$

Then elliptic regularity yields

$$\|E_\beta(-t^\beta A)\|_{\mathcal{L}(L^p,H^{m,p})},\quad \|E'_\beta(-t^\beta A)\|_{\mathcal{L}(L^p,H^{m,p})} \le Ct^{-\beta}, \tag{8.36}$$

and

$$\|E'_\beta(t^{-\beta} A)\|_{\mathcal{L}(L^p,H^{2m,p})} \le Ct^{-2\beta}, \tag{8.37}$$

uniformly for $t \in (0,T_0]$. As in (8.25), interpolation of (8.36) with some of the estimates in (8.35) gives, for $\sigma \in (0,1),\ p \in (1,\infty)$,

$$\|\tau^{\sigma\beta} E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(L^p,H^{\sigma m,p})} \le C, \tag{8.38}$$

hence

$$\|\tau^{\beta-1} E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(L^p,H^{\sigma m,p})} \le C\tau^{-1+(1-\sigma)\beta}, \tag{8.39}$$

uniformly for $\tau \in (0,T_0]$. We also get estimates on Zygmund spaces, such as

$$\|\tau^{\beta-1} E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(C^0_*,C^{\sigma m}_*)} \le C\tau^{-1+(1-\sigma)\beta}, \tag{8.40}$$

and similar estimates on other familite of Besov spaces.

We can produce another demonstration of (8.34)–(8.35), and extend the scope of these estimates, using (8.13)–(8.14) in concert with the following result, which for $\beta = 1/2$ and $\beta = 1/3$ is illustrated by (3.9).

Proposition 8.1. *For $\beta \in (0,1)$, the function $M_\beta(r)$ in (8.13)–(8.14) satisfies*

$$M_\beta \in \mathcal{S}([0,\infty)), \tag{8.41}$$

i.e., M_β is smooth on $[0,\infty)$ and rapidly decreasing, with all its derivatives, at infinity.

Proof. We make use of the identity (3.12),

$$\beta r^{-\beta-1} M_\beta(r^{-\beta}) = \Phi_{1,\beta}(r),$$

plus the results about $\Phi_{1,\beta}$ established at the end of §2. The fact that $\Phi_{1,\beta}(s)$ is smooth on $[0,\infty)$ and vanishes to all orders as $s \to 0$ implies M_β is smooth on $(0,\infty)$ and vanishes rapidly, with all derivatives, at ∞.

It remains to show that $M_\beta(r)$ is smooth up to $r = 0$. For this, we use the asymptotic expansion (2.10), which implies

$$M_\beta(r) = \frac{1}{\beta} r^{-1-1/\beta} \Phi_{1,\beta}(r^{-1/\beta}) \sim \frac{1}{\beta} \sum_{k\ge 1} \gamma_{\beta k} r^{k-1},$$

as $r \searrow 0$.

We can exploit Proposition 8.1 as follows. Given (8.41), we can write

$$
\begin{aligned}
sE_\beta(-s) &= -\int_0^\infty M_\beta(r)\frac{\partial}{\partial r}e^{-rs}\,dr \\
&= \int_0^\infty M_\beta'(r)e^{-rs}\,dr + M_\beta(0),
\end{aligned}
\tag{8.42}
$$

and deduce that, whenever B is a Banach space such that (8.22) holds, or more generally

$$
\|e^{-tA}f\|_B \le C\|f\|_B, \quad \forall\, t>0,
\tag{8.43}
$$

then

$$
\|t^\beta AE_\beta(-t^\beta A)f\|_B \le C\|f\|_B.
\tag{8.44}
$$

Similarly,

$$
\|t^\beta AE_\beta'(-t^\beta A)f\|_B,\ \ \|t^{2\beta}A^2E_\beta'(-t^\beta A)f\|_B \le C\|f\|_B.
\tag{8.45}
$$

As advertised, this provides another demonstration of (8.34)–(8.35), and extends the scope of these estimates.

9. Fractional diffusion-reaction equations – local existence

Here we study the initial-value problem

$$ {}^c\partial_t^\beta u = -Au + F(u), \quad u(0) = f, \tag{9.1} $$

on $[0, T_0] \times M$, given a suitable f on M (perhaps with values in $\mathbb{R}^k$). We assume $\beta \in (0, 1)$. As in the end of §8, we assume M is a compact Riemannian manifold, and

$$ A = (-\Delta)^{m/2}, \quad 0 < m \le 2, \tag{9.2} $$

where Δ is the Laplace-Beltrami operator on M.

Using (8.11), we rewrite (9.1) as

$$ u(t) = E_\beta(-t^\beta A)f + \beta \int_0^t \tau^{\beta-1} E'_\beta(-\tau^\beta A) F(u(t-\tau))\, d\tau. \tag{9.3} $$

Hence we desire to solve

$$ \Phi u = u, \tag{9.4} $$

where

$$ \Phi u(t) = E_\beta(-t^\beta A)f + \beta \int_0^t \tau^{\beta-1} E'_\beta(-\tau^\beta A) F(u(t-\tau))\, d\tau. \tag{9.5} $$

Thus we seek a fixed point of

$$ \Phi : \mathfrak{X} \longrightarrow \mathfrak{X}, \tag{9.6} $$

where $\mathfrak{X}$ is a suitably chosen complete metric space.

To begin, we assume $f \in C(M)$. We pick $a \in (0, \infty)$ and set

$$ \mathfrak{X} = \{u \in C(I, C(M)) : u(0) = f,\ \sup_{t \in I} \|u(t) - f\|_{L^\infty} \le a\}, \quad I = [0, \delta], \tag{9.7} $$

where $\delta > 0$ will be specified below. We assume u takes values in $\mathbb{R}^k$, $F : \mathbb{R}^k \to \mathbb{R}^k$, and

$$ u \in \mathbb{R}^k,\ |u| \le A \Longrightarrow |F(u)| \le K,\ |DF(u)| \le L. \tag{9.8} $$

Here $|\cdot|$ denotes some convenient norm on $\mathbb{R}^k$ and also the associated operator norm on $\mathrm{End}(\mathbb{R}^k)$. Now $t \mapsto E_\beta(-t^\beta A)$ is strongly continuous on $C(M)$ (by (8.13)), and $E_\beta(0) = I$, so we can pick $\delta > 0$ so small that

$$t \in (0, \delta] \Longrightarrow \|E_\beta(-t^\beta A)f - f\|_{L^\infty} \le \frac{1}{2}a. \tag{9.9}$$

To get $\Phi : \mathfrak{X} \to \mathfrak{X}$, it suffices to ensure that

$$t \in I,\ u \in \mathfrak{X} \Rightarrow \beta \int_0^t \tau^{\beta-1} \|E'_\beta(-\tau^\beta A)F(u(t-\tau))\|_{L^\infty}\, d\tau \le \frac{1}{2}a. \tag{9.10}$$

By (9.8), $u \in \mathfrak{X} \Rightarrow \|F(u(t-\tau))\|_{L^\infty} \le K$. Then (8.23), with $B = C(M)$, implies $\|E'_\beta(-\tau^\beta A)F(u(t-\tau))\|_{L^\infty} \le K/\beta\Gamma(\beta)$, so (9.10) holds provided

$$t \in I \Longrightarrow \frac{K}{\Gamma(\beta)} \int_0^t \tau^{\beta-1}\, d\tau \le \frac{a}{2}, \tag{9.11}$$

i.e., provided

$$\delta^\beta \le \frac{\beta\Gamma(\beta)}{2} \frac{a}{K}. \tag{9.12}$$

Hence $\Phi : \mathfrak{X} \to \mathfrak{X}$ whenever (9.9) and (9.12) hold.

We next produce a condition that guarantees Φ is a contraction on $\mathfrak{X}$. Given $u, v \in \mathfrak{X},\ t \in I$, we have

$$\|\Phi u(t) - \Phi v(t)\|_{L^\infty} \le \beta \int_0^t \tau^{\beta-1} \big\|E'_\beta(-\tau^\beta A)[F(u(t-\tau) - F(v(t-\tau))]\big\|_{L^\infty}\, d\tau. \tag{9.13}$$

Now

$$F(u) - F(v) = \int_0^1 \frac{d}{ds} F(su + (1-s)v)\, ds = G(u,v)(u-v), \tag{9.14}$$

with

$$G(u,v) = \int_0^1 DF(su + (1-s)v)\, ds. \tag{9.15}$$

Hence $t \in I,\ \tau \in [0,t]$ imply, via (9.8),

$$\|F(u(t-\tau)) - F(v(t-\tau))\|_{L^\infty} \le L\|u(t-\tau) - v(t-\tau)\|_{L^\infty}, \tag{9.16}$$

so, again by (8.23), the right side of (9.13) is bounded by

$$\begin{aligned} &\frac{L}{\Gamma(\beta)} \int_0^t \tau^{\beta-1} \|u(t-\tau) - v(t-\tau)\|_{L^\infty}\, d\tau \\ &\qquad \le \frac{L}{\beta\Gamma(\beta)} \sup_{0\le\tau\le t} \|u(\tau) - v(\tau)\|_{L^\infty}\, t^\beta. \end{aligned} \tag{9.17}$$

Thus we get

$$\sup_{t\in I} \|\Phi u(t) - \Phi v(t)\|_{L^\infty} \le \theta \sup_{t\in I} \|u(t) - v(t)\|_{L^\infty}, \tag{9.18}$$

provided

$$\delta^\beta \le \beta\Gamma(\beta)\,\frac{\theta}{L}. \tag{9.19}$$

Hence, as long as δ satisfies (9.9), (9.12), and (9.18), with $\theta \in (0,1)$, Φ is a contraction on $\mathfrak{X}$, given by (9.7). We record the local existence result.

Proposition 9.1. *Assume M is a compact Riemannian manifold and A is given by (9.2). Assume $F : \mathbb{R}^k \to \mathbb{R}^k$ satisfies (9.8). Take $f \in C(M)$. Then (9.1) has a solution in $C([0,\delta], C(M))$ provided $\delta > 0$ satisfies (9.9), (9.12), and (9.18), with $\theta < 1$.*

We look at situations with more singular initial data. Not to get too general, we assume

$$f \in L^6(M). \tag{9.20}$$

The analysis will be dimension dependent; say

$$\dim M = n. \tag{9.21}$$

We again take $a \in (0,\infty)$ and set

$$\mathfrak{X} = \{u \in C(I, L^6(M)) : u(0) = f,\ \sup_{t\in I} \|u(t) - f\|_{L^6} \le a\}, \quad I = [0,\delta], \tag{9.22}$$

with $\delta > 0$ to be specified below. This time, we assume

$$|F(u)| \le K(1 + |u|^3), \quad |DF(u)| \le L(1 + |u|^2), \tag{9.23}$$

which holds if $F(u)$ is a cubic polynomial in u. Again Φ is given by (9.5). We desire to show that if $\delta > 0$ is small enough, $\Phi : \mathfrak{X} \to \mathfrak{X}$ and is a contraction. We will succeed in case

$$\frac{n}{3} < m \le 2, \tag{9.24}$$

with m as in (9.2). Note that this requires $n \le 5$.

To start, $t \mapsto E_\beta(-t^\beta A)$ is strongly continuous on $L^6(M)$, again by (8.13), so we can pick $\delta > 0$ so small that

$$t \in (0,\delta] \Longrightarrow \|E_\beta(-t^\beta A)f - f\|_{L^6} \le \frac{a}{2}. \tag{9.25}$$

To get $\Phi : \mathfrak{X} \to \mathfrak{X}$, it suffices to show that

$$t \in I,\ u \in \mathfrak{X} \Rightarrow \beta \int_0^t \tau^{\beta-1} \|E'_\beta(-\tau^\beta A)F(u(t-\tau))\|_{L^6}\,d\tau \le \frac{a}{2}. \tag{9.26}$$

By (9.23),

$$u \in \mathfrak{X} \Longrightarrow \|F(u(t-\tau))\|_{L^2} \le C(a,K). \tag{9.27}$$

The estimate (8.26), with $H = L^2(M)$ (or (8.39), with $p = 2$) gives

$$\tau^{\beta-1}\|E'(-\tau^\beta A)F(u(t-\tau))\|_{H^{\sigma m,2}} \le C\tau^{-1+(1-\sigma)\beta}, \tag{9.28}$$

for $\sigma \in (0,1)$. Sobolev embedding theorems give

$$H^{\sigma m,2}(M) \subset L^6(M), \quad \text{for some } \ \sigma < 1, \tag{9.29}$$

provided (9.24) holds. We mention parenthetically that $H^{\sigma m,2}(M) \subset L^\infty(M)$ for some $\sigma < 1$ provided $n/2 < m \le 2$. Consequently, if (9.24) holds, we have the integral in (9.26) bounded by

$$Ct^{(1-\sigma)\beta}, \tag{9.30}$$

which is $\le a/2$ for all $t \in (0,\delta]$ if δ is small enough. This gives $\Phi : \mathfrak{X} \to \mathfrak{X}$.

We next want to show that Φ is a contraction on $\mathfrak{X}$ if $\delta > 0$ is small enough. This would follow if we could show that, for $u, v \in \mathfrak{X}$,

$$\begin{aligned} &\beta \int_0^t \tau^{\beta-1}\|E'_\beta(-\tau^\beta A)[F(u(t-\tau)) - F(v(t-\tau))]\|_{L^6}\,d\tau \\ &\qquad \le Ct^{(1-\sigma)\beta} \sup_{0\le\tau\le t} \|u(\tau) - v(\tau)\|_{L^6}, \end{aligned} \tag{9.31}$$

since this would yield

$$\sup_{t\in I} \|\Phi u(t) - \Phi v(t)\|_{L^6} \le \theta \sup_{t\in I} \|u(t) - v(t)\|_{L^6}, \tag{9.32}$$

for $u, v \in \mathfrak{X}$, for some $\theta < 1$, if $I = [0,\delta]$ and $\delta > 0$ is small enough.

To proceed, with notation as in (9.14)–(9.15), we have, for $t, \tau \in I$, $u = u(t-\tau)$, $v = v(t-\tau)$, elements of $\mathfrak{X}$,

$$\begin{aligned} \|F(u) - F(v)\|_{L^2} &= \|G(u,v)(u-v)\|_{L^2} \\ &\le \|G(u,v)\|_{L^3}\|u-v\|_{L^6} \\ &\le C(a)\|u-v\|_{L^6}, \end{aligned} \tag{9.33}$$

the last inequality by the hypothesis (9.23) on $DF(u)$. Hence the left side of (9.31) is

$$\le C(A)\beta \int_0^t \tau^{\beta-1}\|E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(L^2,L^6)}\|u(t-\tau) - v(t-\tau)\|_{L^6}\,d\tau. \tag{9.34}$$

Via (8.26) or (8.39), plus (9.29), we have

$$\tau^{\beta-1}\|E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(L^2,L^6)} \le C\tau^{-1+(1-\sigma)\beta}, \tag{9.35}$$

provided (9.24) holds. Hence (9.34)–(9.35) yield the desired estimate (9.31), and we have the contraction property. We record the result.

Proposition 9.2. *Assume M is a compact Riemannian manifold of dimension n, A is given by (9.2), and m satisfies (9.24). Let $F : \mathbb{R}^k \to \mathbb{R}^k$ satisfy (9.23). Take $f \in L^6(M)$. Then (9.1) has a solution $u \in C([0,\delta], L^6(M))$ provided $\delta > 0$ is sufficiently small.*

10. More local existence results

Here we seek other complete metric spaces $\mathfrak{X}$ for which $\Phi : \mathfrak{X} \to \mathfrak{X}$ is a contraction, given Φ as in (9.5), i.e.,

$$\Phi u(t) = E\beta(-t^\beta A)f + \beta \int_0^t \tau^{\beta-1} E'_\beta(-\tau^\beta A) F(u(t-\tau))\, d\tau. \tag{10.1}$$

We continue to assume $\beta \in (0,1),\ A = (-\Delta)^{m/2},\ m \in (0,2]$, and $F : \mathbb{R}^k \to \mathbb{R}^k$ satisfies (9.23), i.e.,

$$|F(u)| \le K(1+|u|^3), \quad |DF(u)| \le L(1+|u|^2), \tag{10.2}$$

which holds if F is a cubic polnomial in u. We also continue to assume M is a compact Riemannian manifold of dimension n. Generalizing (9.20)–(9.22), we pick $q \in (1,\infty),\ a \in (0,\infty)$,

$$f \in L^{3q}(M), \tag{10.3}$$

and set

$$\mathfrak{X} = \{u \in C(I, L^{3q}(M)) : u(0) = f,\ \sup_{t\in I} \|u(t) - f\|_{L^{3q}} \le a\}, \quad I = [0,\delta], \tag{10.4}$$

with $\delta > 0$ to be specified.

Parallel to (9.25), since $t \mapsto E_\beta(-t^\beta A)$ is strongly continuous on $L^{3q}(M)$, we can pick $\delta > 0$ so small that

$$t \in (0,\delta] \Longrightarrow \|E_\beta(-t^\beta A)f - f\|_{L^{3q}} \le \frac{a}{2}. \tag{10.5}$$

To get $\Phi : \mathfrak{X} \to \mathfrak{X}$, it suffices to show that

$$t \in I,\ u \in \mathfrak{X} \Rightarrow \beta \int_0^t \tau^{\beta-1} \|E'_\beta(-\tau^\beta A) F(u(t-\tau))\|_{L^{3q}}\, d\tau \le \frac{a}{2}. \tag{10.6}$$

By (10.2),

$$u \in \mathfrak{X} \Longrightarrow \|F(u(t-\tau))\|_{L^q} \le C(a,K). \tag{10.7}$$

The estimate (8.39) gives

$$\tau^{\beta-1} \|E'_\beta(-\tau^\beta A) F(u(t-\tau))\|_{H^{\sigma m,q}} \le C\tau^{-1+(1-\sigma)\beta}, \tag{10.8}$$

for $\sigma \in (0,1)$. We seek a condition implying

$$H^{\sigma m,q}(M) \subset L^{3q}(M). \tag{10.9}$$

for some $\sigma \in (0,1)$. If $n = \dim M$, Sobolev embedding results imply

$$\begin{aligned} H^{\sigma m,q}(M) \subset L^{\infty}(M), &\quad \text{for some } \sigma < 1, \text{ if } \ mq > n, \\ L^{nq/(n-\sigma mq)}(M), &\quad \text{if } \ mq \le n. \end{aligned} \tag{10.10}$$

Thus (10.9) holds provided either $mq > n$ or $mq \le n$ and $nq/(n-\sigma mq) \ge 3q$ for some $\sigma \in (0,1)$. Hence (10.9) holds provided

$$3q > \frac{2n}{m}. \tag{10.11}$$

As for how this constrains m, recalling that $m \le 2$, we require

$$\frac{2n}{3q} < m \le 2. \tag{10.12}$$

This requires $n < 3q$. For $q = 2,\ 3q = 6$, this is (9.24). If (10.9) holds, (10.8) yields

$$\tau^{\beta-1}\|E'_\beta(-\tau^\beta A)F(u(t-\tau))\|_{L^{3q}} \le C\tau^{-1+(1-\sigma)\beta}, \tag{10.13}$$

and we get (10.6), as long as $\delta > 0$ is small enough. Hence $\Phi : \mathfrak{X} \to \mathfrak{X}$.

We next want to show that Φ is a contraction on $\mathfrak{X}$ if $\delta > 0$ is small enough. Parallel to (9.31), this would follow if we could show that, for $u, v \in \mathfrak{X}$,

$$\begin{aligned} \beta \int_0^t \tau^{\beta-1}\|E'_\beta(-\tau^\beta A)[F(u(t-\tau)) - F(v(t-\tau))]\|_{L^{3q}}\, d\tau \\ \le Ct^{(1-\sigma)\beta} \sup_{0\le\tau\le t} \|u(\tau) - v(\tau)\|_{L^{3q}}. \end{aligned} \tag{10.14}$$

To proceed, with notation as in (9.14)–(9.15), and parallel to (9.33), we have, for $t, \tau \in I,\ u = u(t-\tau),\ v = v(t-\tau)$, elements of $\mathfrak{X}$,

$$\begin{aligned} \|F(u) - F(v)\|_{L^q} &= \|G(u,v)(u-v)\|_{L^q} \\ &\le \|G(u,v)\|_{L^{3q/2}}\|u-v\|_{L^{3q}} \\ &\le C(a)\|u-v\|_{L^{3q}}, \end{aligned} \tag{10.15}$$

the last inequality by the hypothesis (10.2) on $DF(u)$. Hence the left side of (10.14) is

$$\le C(a)\beta \int_0^t \tau^{\beta-1}\|E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(L^q,L^{3q})}\|u(t-\tau) - v(t-\tau)\|_{L^{3q}}\, d\tau. \tag{10.16}$$

Via (10.8)–(10.10), we have

$$\tau^{\beta-1}\|E'_\beta(-\tau^\beta A)\|_{\mathcal{L}(L^q,L^{3q})} \le C\tau^{-1+(1-\sigma)\beta}, \tag{10.17}$$

provided (10.12) holds. Hence (10.16)–(10.17) yield the desired estimate (10.14), and we have the contraction property. We record the result.

Proposition 10.1. *Assume M is a compact, n-dimensional, Riemannian manifold, $A = (-\Delta)^{m/2}$, $f \in L^{3q}(M)$, and m and q satisfy $q > 1$ and (10.12). Let $F : \mathbb{R}^k \to \mathbb{R}^k$ satisfy (10.2). Then (9.1) has a solution $u \in C([0,\delta], L^{3q}(M))$ provided $\delta > 0$ is small enough.*

11. Further variants

Let us write the putative solution of (9.1) as

$$u(t) = u_0(t) + v(t), \quad u_0(t) = E_\beta(-t^\beta A)f. \tag{11.1}$$

Then the integral equation (9.3) is equivalent to

$$v(t) = \beta \int_0^t \tau^{\beta-1} E'_\beta(-\tau^\beta A) F(u_0(t-\tau) + v(t-\tau))\, d\tau, \tag{11.2}$$

or

$$\Psi v = v, \tag{11.3}$$

where

$$\Psi v(t) = \beta \int_0^t \tau^{\beta-1} E'_\beta(-\tau^\beta A) F(u_0(t-\tau) + v(t-\tau))\, d\tau. \tag{11.4}$$

Thus we seek a complete metric space $\mathfrak{Z}$ for which

$$\Psi : \mathfrak{Z} \longrightarrow \mathfrak{Z} \tag{11.5}$$

is a contraction.

For example, picking $q \in (1, \infty)$, $a \in (0, \infty)$, we can take

$$\mathfrak{Z} = \{v \in C(I, L^{3q}(M)) : v(0) = 0, \sup_{t\in I} \|v(t)\|_{L^{3q}} \le a\}, \quad I = [0, \delta]. \tag{11.6}$$

We assume F satisfies (10.2). We assume $f \in L^{3q}(M)$, so $u_0 \in C(I, L^{3q}(M))$. Estimates parallel to those given in §10 show that if (10.12) holds, then, for $\delta > 0$ small enough, (11.5) holds and Ψ is a contraction.

In a search for other candidates for the space $\mathfrak{Z}$, we investigate the behavior of $v_1 = \Psi 0$, i.e., of

$$v_1(t) = \beta \int_0^t \tau^{\beta-1} E'_\beta(-\tau^\beta A) F(u_0(t-\tau))\, d\tau. \tag{11.7}$$

To start, let us take

$$n = \dim M = 2, \quad f \in L^2(M). \tag{11.8}$$

Then, for $\sigma \in (0,1]$,

$$\|u_0(t-\tau)\|_{H^{\sigma m,2}} \le C(t-\tau)^{-\sigma\beta}. \tag{11.9}$$

We have

$$\begin{aligned} H^{\sigma m,2}(M) \subset L^\infty(M), &\quad \text{if } \ \sigma m > 1, \\ L^{4/(2-2\sigma m)}, &\quad \text{if } \ \sigma m < 1. \end{aligned} \tag{11.10}$$

In particular, $4/(2-2\sigma m) = 6$ if $\sigma m = 2/3$, so

$$\|u_0(t-\tau)\|_{L^6} \le C(t-\tau)^{-\sigma\beta} \ \text{ if } \ \sigma m \ge \frac{2}{3}, \tag{11.11}$$

hence

$$\|F(u_0(t-\tau))\|_{L^2} \le C(t-\tau)^{-3\sigma\beta}, \tag{11.12}$$

for $0 < \tau < t \le T_0$, if $\sigma m \ge 2/3$, while also $\sigma \le 1$, i.e., if

$$\frac{2}{3m} \le \sigma \le 1, \tag{11.13}$$

which is possible provided

$$\frac{2}{3} \le m \le 2. \tag{11.14}$$

In such a case,

$$\|v_1(t)\|_{L^2} \le C \int_0^t \tau^{\beta-1}(t-\tau)^{-3\sigma\beta}\, d\tau, \tag{11.15}$$

which is finite provided

$$3\sigma\beta < 1. \tag{11.16}$$

This is consistent with (11.13) if

$$\frac{2}{m}\beta < 1, \quad \text{i.e.,} \ \ 2\beta < m, \quad \text{or} \ \ \beta < \frac{m}{2}. \tag{11.17}$$

In such a case we can take

$$\sigma = \frac{2}{3m}, \quad \text{so} \ \ 3\sigma\beta = \frac{2\beta}{m}, \tag{11.18}$$

and (11.15) yields

$$\|v_1(t)\|_{L^2} \le C \int_0^t \tau^{\beta-1}(t-\tau)^{-2\beta/m}\, d\tau = \widetilde{C} t^{-(2-m)\beta/m}. \tag{11.19}$$

In particular,

$$\|v_1(t)\|_{L^2} \le \widetilde{C} \ \text{ if } \ m=2. \tag{11.20}$$

So let's assume

$$n=2, \quad f \in L^2(M), \quad m=2, \quad \sigma = \frac{1}{3}, \quad \beta \in (0,1). \tag{11.21}$$

In such a case, we have the conclusion (11.20). Under the hypotheses of (11.21), let us pick $a, b \in (0,\infty)$ and set

$$\begin{aligned} \mathfrak{Z} = \{v \in C(I, L^2(M)) : v(0)=0,\ \|v(t)\|_{L^2} \le a, \\ \|v(t)\|_{L^6} \le bt^{-\sigma\beta},\ \forall\, t \in I\}, \end{aligned} \tag{11.22}$$

with $I=[0,\delta]$. Then

$$\begin{aligned} v \in \mathfrak{Z} &\Rightarrow \|u_0(t-\tau)+v(t-\tau)\|_{L^6} \le C(t-\tau)^{-\sigma\beta} \\ &\Rightarrow \|F(u_0(t-\tau)+v(t-\tau))\|_{L^2} \le C(t-\tau)^{-3\sigma\beta} \\ &\Rightarrow \|\Psi v(t)||_{L^2} \le C\int_0^t \tau^{\beta-1}(t-\tau)^{-3\sigma\beta}\, d\tau = \widetilde{C}. \end{aligned} \tag{11.23}$$

However, we cannot guarantee that $\widetilde{C} \le a$, even if we shrink I.

Nevertheless, we proceed to estimate $\|\Psi v(t)\|_{L^6}$. We have

$$\|\tau^{\beta-1}E'_\beta(-\tau^\beta A)F\|_{H^{\sigma m,2}} \le C\tau^{-1+(1-\sigma)\beta}\|F\|_{L^2}. \tag{11.24}$$

Hence, from the L^2 estimate of F in (11.23), if $v \in \mathfrak{Z}$,

$$\begin{aligned} \tau^{\beta-1}\|E'_\beta(-\tau^\beta A)F(u_0(t-\tau)+v(t-\tau))\|_{H^{\sigma m,2}} \\ \le C\tau^{-1+(1-\sigma)\beta}(t-\tau)^{-3\sigma\beta}, \end{aligned} \tag{11.25}$$

under hypothesis (11.21), hence

$$\begin{aligned} \|\Psi v(t)\|_{L^6} &\le C\|\Psi v(t)\|_{H^{\sigma m,2}} \\ &\le C\int_0^t \tau^{-1+(1-\sigma)\beta}(t-\tau)^{-3\sigma\beta}\, d\tau \\ &= \widetilde{C}t^{-\beta/3}. \end{aligned} \tag{11.26}$$

Again we get an estimate of $\widetilde{C}t^{-\sigma\beta}$, since $\sigma = 1/3$, but we cannot establish that $\widetilde{C} \le b$. In other words, the hypothesis (11.21) seems to be of "critical" type.

We will try again, with the hypothesis $f \in L^2(M)$ replaced by

$$f \in L^p(M), \quad \text{for some} \quad p > 2. \tag{11.27}$$

We already know that things work out if

$$p = 3q > \frac{2n}{m} = 2, \text{ when } \ n = m = 2, \text{ provided also } \ q > 1, \text{ i.e., } \ p > 3. \tag{11.27A}$$

Now we want to take p closer to 2, when $n = m = 2$. We need further estimates on $v_1(t)$, in order to set up a replacement for the space (11.22).

To start, we need an estimate on

$$\|u_0(t-\tau)\|_{L^{3p}}, \tag{11.28}$$

parallel to that in (11.11). Parallel to (11.9), we have

$$\|u_0(t-\tau)\|_{H^{\sigma m,p}} \le C(t-\tau)^{-\sigma\beta}, \tag{11.29}$$

and, parallel to (11.10), we have (when $n = 2$)

$$\begin{aligned} H^{\sigma m,p}(M) \subset L^\infty(M), &\quad \text{if } \ \sigma m > \frac{2}{p}, \\ L^{2p/(2-\sigma m p)}, &\quad \text{if } \ \sigma m < \frac{2}{p}. \end{aligned} \tag{11.30}$$

In particular, $2p/(2-\sigma m p) = 3p$ if $\sigma m = 4/3p$, so

$$\|u_0(t-\tau)\|_{L^{3p}} \le C(t-\tau)^{-\sigma\beta} \ \text{ if } \ \sigma m \ge \frac{4}{3p}, \tag{11.31}$$

hence

$$\|F(u_0(t-\tau))\|_{L^p} \le C(t-\tau)^{-3\sigma\beta}, \tag{11.32}$$

for $0 < \tau < t \le T_0$, if $\sigma m \ge 4/3p$, while also $\sigma \le 1$, i.e., if

$$\frac{4}{3pm} \le \sigma \le 1, \tag{11.33}$$

or, assuming $m = 2$, if

$$\frac{2}{3p} \le \sigma \le 1, \tag{11.34}$$

which of course is true if $p > 2$, so we can take

$$\sigma = \frac{2}{3p}, \quad \text{so} \;\; 3\sigma\beta = \frac{2\beta}{p}, \tag{11.35}$$

and we have

$$\begin{aligned} \|v_1(t)\|_{L^p} &\le C \int_0^t \tau^{\beta-1}(t-\tau)^{-2\beta/p}\, d\tau \\ &= \widetilde{C} t^{\beta(1-2/p)}. \end{aligned} \tag{11.36}$$

Also (11.32) and the analogue of (11.25) with $H^{\sigma m,2}$ replaced by $H^{\sigma m,p}$, give

$$\begin{aligned} \|v_1(t)\|_{H^{\sigma m,p}} &\le C \int_0^t \tau^{-1+(1-\sigma)\beta}(t-\tau)^{-2\beta/p}\, d\tau \\ &= \widetilde{C} t^{-\beta(8/3p-1)} \\ &= \widetilde{C} t^{-\beta(4\sigma-1)}. \end{aligned} \tag{11.37}$$

Compare (11.29). Note that $4\sigma - 1 < \sigma \Leftrightarrow \sigma < 1/3$, which by (11.35) holds if $p > 2$. Hence $\|v_1(t)\|_{H^{\sigma m,p}}$ has a gentler blow-up as $t \searrow 0$ than $\|u_0(t)\|_{H^{\sigma m,p}}$ does (given $m = 2$).

In light of these observations, under hypothesis (11.27), plus

$$n = m = 2, \tag{11.38}$$

and with σ as in (11.35), it is natural to take $a, b \in (0, \infty)$, and set

$$\begin{aligned} \mathfrak{Z} = \{v \in C(I, L^p(M)) : v(0) = 0,\ \|v(t)\|_{L^p} \le a, \\ \|v(t)\|_{L^{3p}} \le b t^{-\sigma\beta},\ \forall\, t \in I\}, \end{aligned} \tag{11.39}$$

with $I = [0, \delta]$. We desire to show that, for $\delta > 0$ small enough, Ψ, given by (11.4), maps $\mathfrak{Z}$ to itself, as a contraction.

To start, under the hypotheses (11.27) and (11.38), and taking σ as in (11.35), we have

$$\begin{aligned} v \in \mathfrak{Z} &\Rightarrow \|u_0(t-\tau) + v(t-\tau)\|_{L^{3p}} \le C(t-\tau)^{-\sigma\beta} \\ &\Rightarrow \|F(u_0(t-\tau) + v(t-\tau))\|_{L^p} \le C(t-\tau)^{-3\sigma\beta} \\ &\Rightarrow \|\Psi v(t)\|_{L^p} \le C \int_0^t \tau^{\beta-1}(t-\tau)^{-2\beta/p}\, d\tau = \widetilde{C} t^{\beta(1-2/p)}. \end{aligned} \tag{11.40}$$

We require of δ that

$$\widetilde{C} \delta^{\beta(1-2/p)} \le a, \tag{11.41}$$

which is possible since $p > 2$.

Next we estimate $\|\Psi v(t)\|_{H^{\sigma m,p}}$, which leads to an estimate of $\|\Psi v(t)\|_{L^{3p}}$. We have

$$\tau^{\beta-1}\|E'_\beta(-\tau^\beta A)F\|_{H^{\sigma m,p}} \le C\tau^{-1+(1-\sigma)\beta}\|F\|_{L^p}, \tag{11.42}$$

hence, from the L^p estimates of F in (11.40), if $v \in \mathfrak{Z}$,

$$\begin{aligned}\tau^{\beta-1}\|E'_\beta(-\tau^\beta A)F(u_0(t-\tau)+v(t-\tau))\|_{H^{\sigma m,p}} \\ \le C\tau^{-1+(1-\sigma)\beta}(t-\tau)^{-3\sigma\beta},\end{aligned} \tag{11.43}$$

under hypotheses (11.27) and (11.38). Hence, bringing in (11.35),

$$\begin{aligned}\|\Psi v(t)\|_{L^{3p}} &\le C\|\Psi v(t)\|_{H^{\sigma m,p}} \\ &\le C\int_0^t \tau^{-1+(1-\sigma)\beta}(t-\tau)^{-2\beta/p}\,d\tau \\ &= \widetilde{C}t^{-\beta(4\sigma-1)},\end{aligned} \tag{11.44}$$

parallel to (11.37). We require of δ that

$$\widetilde{C}\delta^{-\beta(4\sigma-1)} \le b\delta^{-\beta\sigma}, \tag{11.45}$$

which is possible since $4\sigma - 1 < \sigma$. Then $\Psi : \mathfrak{Z} \to \mathfrak{Z}$.

Similar estimates show that, with δ perhaps further shrunk, Ψ is a contraction on $\mathfrak{Z}$. We omit the details. We record the resulting existence theorem.

Proposition 11.1. *Let M be a compact, 2-dimensional Riemannian manifold, $A = -\Delta$, and $\beta \in (0,1)$. Assume F satisfies (10.2). Assume $f \in L^p(M)$ for some $p > 2$. Then, for some $\delta > 0$, the initial value problem (9.1) has a unique solution $u \in C(I, L^p(M))$ of the form $u = u_0 + v$, as in (11.1), such that v belongs to $\mathfrak{Z}$, given by (11.39), with $\sigma = 2/3p$. Furthermore,*

$$\|v(t)\|_{H^{2\sigma,p}} \le Ct^{-\beta(4\sigma-1)}. \tag{11.46}$$

Note. For $n = 2$, $m = 2$, Proposition 10.1 requires $p = 3q > 3$, so Proposition 11.1 is an improvement.

A. Riemann-Liouville fractional integrals and Caputo fractional derivatives

For $\beta > 0$, the Riemann-Liouville fractional integral J^β is defined by

$$J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau, \tag{A.1}$$

for $t \geq 0$, where f is a suitable function on $[0,\infty)$, say continuous on $[0,\infty)$ and polynomially bounded. We mention that

$$J^\beta 1(t) = \frac{1}{\Gamma(\beta+1)} t_+^\beta. \tag{A.2}$$

With the Laplace transform given by

$$\mathcal{L}f(s) = \int_0^\infty f(t) e^{-st}\, dt, \quad \operatorname{Re} s > 0, \tag{A.3}$$

we have

$$u(s) = \int_0^t g(t-\tau) f(\tau)\, d\tau \Longrightarrow \mathcal{L}u(s) = \mathcal{L}g(s)\, \mathcal{L}f(s), \tag{A.4}$$

and

$$g_\beta(t) = t_+^{\beta-1},\ \beta > 0 \Longrightarrow \mathcal{L}g_\beta(s) = \Gamma(\beta) s^{-\beta}. \tag{A.5}$$

Hence

$$\mathcal{L}J^\beta f(s) = s^{-\beta} \mathcal{L}f(s). \tag{A.6}$$

For $\beta \in (0,1)$, the Riemann-Liouville fractional derivative is given by

$${}^r\partial_t^\beta f = \partial_t J^{1-\beta} f, \tag{A.7}$$

and the Caputo fractional derivative is given by

$${}^c\partial_t^\beta f = J^{1-\beta} \partial_t f. \tag{A.8}$$

One has

$${}^r\partial_t^\beta J^\beta f = f \quad \text{and} \quad {}^c\partial_t^\beta J^\beta f = f. \tag{A.9}$$

However, ${}^r\partial_t^\beta$ and ${}^c\partial_t^\beta$ are not identical. For example, given $\beta \in (0,1)$,

$$\text{(A.10)} \qquad {}^c\partial_t^\beta 1 \equiv 0, \quad {}^r\partial_t^\beta 1 = \frac{1}{\Gamma(\beta)} t_+^{\beta-1}.$$

We next consider how the Laplace transform interacts with these two fractional derivatives. Note that

$$\text{(A.11)} \qquad \begin{aligned} \mathcal{L}\partial_t f(s) &= \int_0^\infty f'(t) e^{-s}\, dt \\ &= s\mathcal{L}f(s) - f(0), \end{aligned}$$

the last identity by integration by parts. It follows that, for $\beta \subset (0,1)$,

$$\text{(A.12)} \qquad \mathcal{L}\,{}^r\partial_t^\beta f(s) = s^\beta \mathcal{L}f(s) - J^{1-\beta} f(0),$$

and

$$\text{(A.13)} \qquad \mathcal{L}\,{}^c\partial_t^\beta f(s) = s^\beta \mathcal{L}f(s) - s^{\beta-1} f(0).$$

Consequently, one can apply Laplace transform techniques conveniently to initial value problems for fractional differential equations involving the Caputo fractional derivative ${}^c\partial_t^\beta$, but not so well for those involving the Riemann-Liouville fractional derivative ${}^r\partial_t^\beta$.

For application in Appendix C, we compute ${}^c\partial_t^\beta t^\gamma$, for $\beta \in (0,1)$, $\gamma \geq \beta$. We have

$$\text{(A.14)} \qquad \begin{aligned} {}^c\partial_t^\beta t^\gamma &= J^{1-\beta} \partial_t t^\gamma \\ &= \gamma J^{1-\beta} t^{\gamma-1} \\ &= \gamma \Gamma(\gamma) J^{1-\beta} J^{\gamma-1} 1(t), \end{aligned}$$

the last identity by (A.2). Now (A.6) implies $J^{1-\beta} J^{\gamma-1} = J^{\gamma-\beta}$, so

$$\text{(A.15)} \qquad \begin{aligned} {}^c\partial_t^\beta t^\gamma &= \gamma \Gamma(\gamma) J^{\gamma-\beta} 1(t) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} t^{\gamma-\beta}, \end{aligned}$$

invoking (A.2) again. In particular, for $k \in \mathbb{N}$,

$$\text{(A.16)} \qquad {}^c\partial_t^\beta t^{k\beta} = \frac{\Gamma(k\beta+1)}{\Gamma(k\beta-\beta+1)} t^{(k-1)\beta}.$$

(Recall (A.10) for the case $k = 0$.)

Remark. One can extend the conclusion of (A.15) to $\gamma > 0$ by a direct computation of $J^{1-\beta} t^{\gamma-1}$, using (A.1).

B. Finite-dimensional linear fractional differential systems

Here we briefly discuss linear systems

$$ {}^c\partial_t^\beta u = Lu, \quad u(0) = f, \tag{B.1} $$

when L is not necessarily a negative self adjoint operator on a Hilbert space, but rather

$$ f \in V, \quad L \in \operatorname{End}(V), \tag{B.2} $$

and V is a complex vector space of dimension $k < \infty$. For more details, see [D].

Parallel to (3.4), the solution to (B.1) is given by

$$ u(t) = E_\beta(t^\beta L) f. \tag{B.3} $$

Now we can write

$$ V = \bigoplus_j V_j, \tag{B.4} $$

where, for λ_j in the spectrum of L,

$$ L\Big|_{V_j} = \lambda_j I + N_j, \tag{B.5} $$

with N_j nilpotent on V_j. Then, in the obvious sense,

$$ E_\beta(t^\beta L) = \bigoplus_j E_\beta(t^\beta(\lambda_j I + N_j)). \tag{B.6} $$

Furthermore, standard holomorphic functional calculus gives, for nilpotent N and $\lambda \in \mathbb{C}$,

$$ E_\beta(\lambda I + N) = \sum_{k \ge 0} \frac{1}{k!} E_\beta^{(k)}(\lambda) N^k, \tag{B.7} $$

the sum being finite if N is nilpotent. Hence

$$ E_\beta(t^\beta(\lambda_j I + N_j)) = \sum_{k \ge 0} \frac{1}{k!} E_\beta^{(k)}(t^\beta \lambda_j) t^{k\beta} N_j^k. \tag{B.8} $$

Note that (8.20) extends to

$$E_\beta^{(k)}(-s) \sim a_\beta^k s^{-k-1} + \cdots, \quad s \nearrow +\infty. \tag{B.9}$$

This implies decay of (B.8) as $t \to +\infty$, when $\lambda_j < 0$, though only at a rate $O(t^{-\beta})$, when $\beta \in (0,1)$, not at an exponential rate, as for $\beta = 1$.

To go further, one can extend the scope of (B.9), by extending that of (8.15)–(8.19). With

$$\eta_\beta(\xi) = \frac{(i\xi)^{\beta-1}}{(i\xi)^\beta + 1}, \tag{B.10}$$

as in (8.15), we have, up to a constant factor,

$$\hat{\eta}_\beta(t) = e_\beta(t). \tag{B.11}$$

Analytic continuation arguments give

$$E_\beta^{(k)}(z) \sim a_\beta^k (-z)^{-k-1} + \cdots, \quad \text{as } |z| \to \infty, \quad \text{for } |\text{Arg } z| > \frac{\pi\beta}{2}. \tag{B.12}$$

See [D]. Hence

$$E_\beta(t^\beta(\lambda_j I + N_j)) \longrightarrow 0 \quad \text{as} \quad t \nearrow +\infty, \tag{B.13}$$

provided

$$|\text{Arg}\, \lambda_j| > \frac{\pi\beta}{2}. \tag{B.14}$$

C. Derivation of power series for $E_\beta(t)$

We approach the solution to

$$ {}^c\partial_t^\beta u = au, \quad u(0) = 1, \tag{C.1}$$

given $\beta \in (0,1)$, $a \in \mathbb{C}$, taking a cue from (A.16), which suggests trying

$$ u(t) = \sum_{k\ge 0} c_k t^{k\beta}. \tag{C.2}$$

In fact, granted appropriate convergence, applying (A.16) to (C.2) yields

$$ \begin{aligned} {}^c\partial_t^\beta u &= \sum_{k\ge 1} \frac{\Gamma(k\beta+1)}{\Gamma(k\beta-\beta+1)} c_k t^{(k-1)\beta} \\ &= \sum_{\ell\ge 0} \frac{\Gamma(\ell\beta+\beta+1)}{\Gamma(\ell\beta+1)} c_{\ell+1} t^{\ell\beta}. \end{aligned} \tag{C.3}$$

Comparison with the series for au, given by multiplying (C.2) by a, yields

$$ c_{\ell+1} = a\frac{\Gamma(\ell\beta+1)}{\Gamma(\ell\beta+\beta+1)} c_\ell. \tag{C.4}$$

Given $c_0 = 1$, we have

$$ c_1 = \frac{a}{\Gamma(\beta+1)}, \quad c_2 = \frac{a^2}{\Gamma(\beta+1)} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}, \dots, \tag{C.5}$$

and inductively,

$$ c_k = \frac{a^k}{\Gamma(k\beta+1)}. \tag{C.6}$$

Hence we arrive at

$$ u(t) = E_\beta(t^\beta a), \tag{C.7}$$

where

$$ E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}, \tag{C.8}$$

as the solution to (C.1).

To go backwards, note that, for $\beta \in (0,1)$,

$$J^{\beta}\, {}^c\partial_t^{\beta} u(t) = J\partial_t u(t) \\ = u(t) - u(0), \tag{C.9}$$

so (C.1) implies

$$u(t) = 1 + aJ^{\beta} u(t), \tag{C.10}$$

and in fact, by (A.9)–(A.10), (C.1) and (C.10) are equivalent. This suggests another approach. Write (C.10) as

$$(I - aJ^{\beta})u(t) = 1, \tag{C.11}$$

and then

$$u(t) = \sum_{k \geq 0} a^k J^{k\beta} 1(t), \tag{C.12}$$

which via (A.2) again leads to (C.7)–(C.8).

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