Fourier Analysis and the FFT

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Abstract. These notes introduce three types of Fourier transforms, one for periodic functions of one or more real variables (also called Fourier series), one for integrable (or square-integrable) functions of one or more real variables, and one for functions defined on a set of n points, identified with the integers mod n (also called the discrete Fourier transform, or DFT). We discuss in each case the important Fourier inversion formula. We also discuss relations of these various Fourier transforms with each other. Finally, we discuss 'fast' algorithms for computing the DFT, when n is a power of 2.

Contents

1. Fourier series

Let f be an integrable function on the torus $\mathbf{T}^n$, naturally isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$ and to the Cartesian product of n copies $S^1 \times \cdots \times S^1$ of the circle. Its Fourier series is by definition a function on $\mathbb{Z}^n$ given by

$$\hat{f}(k) = \frac{1}{(2\pi)^n} \int\limits_{\mathbf{T}^n} f(\theta) e^{-ik\cdot\theta} d\theta \tag{1.1}$$

where $k = (k_1, \dots, k_n),\ k \cdot \theta = k_1\theta_1 + \cdots + k_n\theta_n$. We use the notation

$$\mathcal{F}f(k) = \hat{f}(k). \tag{1.2}$$

Clearly we have a continuous linear map

$$\mathcal{F} : L^1(\mathbf{T}^1) \longrightarrow \ell^\infty(\mathbb{Z}^n), \tag{1.3}$$

where $\ell^\infty(\mathbb{Z}^n)$ denotes the space of bounded functions on $\mathbb{Z}^n$, with the sup norm. If $f \in C^\infty(\mathbf{T}^n)$, then we can integrate by parts to get

$$k^\alpha \hat{f}(k) = \frac{1}{(2\pi)^n} \int\limits_{\mathbf{T}^n} (D^\alpha f)(\theta)\ e^{-ik\cdot\theta}\ d\theta \tag{1.4}$$

where $k^\alpha = k_1^{\alpha_1} \cdots k_n^{\alpha_n}$, and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_j = i \frac{\partial}{\partial \theta_j}. \tag{1.5}$$

It follows easily that

$$\mathcal{F} : C^\infty(\mathbf{T}^n) \longrightarrow s(\mathbb{Z}^n), \tag{1.6}$$

where $s(\mathbb{Z}^n)$ consists of functions u on $\mathbb{Z}^n$ which are rapidly decreasing, in the sense that, for each N,

$$p_N(u) = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^N \ |u(k)| < \infty. \tag{1.7}$$

If we use the inner product

$$(f, g) = (f, g)_{L^2} = \frac{1}{(2\pi)^n} \int\limits_{\mathbf{T}^n} f(\theta) \overline{g(\theta)} d\theta, \tag{1.8}$$

for $f, g \in C^\infty(\mathbf{T}^n)$, or more generally for $f, g \in L^2(\mathbf{T}^n)$, and if on $s(\mathbb{Z}^n)$, or more generally on $\ell^2(\mathbb{Z}^n)$, the space of square summable functions on $\mathbb{Z}^n$, we use the inner product

$$(u, v) = (u, v)_{\ell^2} = \sum_{k \in \mathbb{Z}^n} u(k) \overline{v(k)}, \tag{1.9}$$

we have the formula

$$(\mathcal{F} f, u)_{\ell^2} = (f, \mathcal{F}^* u)_{L^2}, \tag{1.10}$$

valid for $f \in C^\infty(\mathbf{T}^n), \ u \in s(\mathbb{Z}^n)$, where

$$\mathcal{F}^* : s(\mathbb{Z}^n) \longrightarrow C^\infty(\mathbf{T}^n) \tag{1.11}$$

is given by

$$\big(\mathcal{F}^* u\big)(\theta) = \sum_{k \in \mathbb{Z}^n} u(k) \ e^{ik \cdot \theta}. \tag{1.12}$$

Indeed, taking (1.12) as a definition, the identity (1.10) follows easily from the identity

$$\frac{1}{(2\pi)^n} \int\limits_{\mathbf{T}^n} e^{ik \cdot \theta} \ e^{-i\ell \cdot \theta} \ d\theta = \delta_{k\ell}, \tag{1.13}$$

where $\delta_{k\ell} = 1$ if $k = \ell$, $\delta_{k\ell} = 0$ otherwise.

Our main goal here is to establish the Fourier inversion formula

$$f(\theta) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, e^{ik\cdot\theta}, \tag{1.14}$$

the sum on the right in (1.14) converging in the appropriate function space, depending on the nature of f. Let us single out another space of functions on $\mathbf{T}^n$, the trigonometric polynomials:

$$\mathcal{TP} = \Big\{ \sum_{k\in\mathbb{Z}^n} a(k) e^{ik\cdot\theta} : a(k) = 0 \text{ except for finitely many } k \Big\}. \tag{1.15}$$

Clearly

$$\mathcal{F} : \mathcal{TP} \longrightarrow c_{00}(\mathbb{Z}^n) \tag{1.16}$$

where $c_{00}(\mathbb{Z}^n)$ consists of functions on $\mathbb{Z}^n$ which vanish except at a finite number of points; this follows from (1.13). The formula (1.12) gives

$$\mathcal{F}^* : c_{00}(\mathbb{Z}^n) \longrightarrow \mathcal{TP}, \tag{1.17}$$

and the formula (1.13) easily yields

$$\mathcal{F}\mathcal{F}^* = I \text{ on } c_{00}(\mathbb{Z}^n), \tag{1.18}$$

and even

$$\mathcal{F}\mathcal{F}^* = I \text{ on } s(\mathbb{Z}^n). \tag{1.19}$$

By comparison, the inversion formula (1.14) states

$$\mathcal{F}^*\mathcal{F} = I, \tag{1.20}$$

on $C^\infty(\mathbf{T}^n)$, or some other space of functions on $\mathbf{T}^n$, as specified below. Before getting to this, let us note one other implication of (1.13), namely, if

$$f_j(\theta) = \sum_k \varphi_j(k) e^{ik\cdot\theta} \tag{1.21}$$

are elements of $\mathcal{TP}$, or more generally, if $\varphi_j \in s(\mathbb{Z}^n)$, then we have the Parseval identity

$$(f_1, f_2)_{L^2} = \sum_{k\in\mathbb{Z}^n} \varphi_1(k)\overline{\varphi_2(k)}; \tag{1.22}$$

in particular the Plancherel identity

$$\|f_j\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^n} |\varphi_j(k)|^2, \tag{1.23}$$

for $f_j \in \mathcal{TP}$, or more generally for any f_j of the form (1.21) with $\varphi_j \in s(\mathbb{Z}^n)$. In particular, the map $\mathcal{F}^*$ given by (1.12), and satisfying (1.11) and (1.17), has a unique continuous extension to $\ell^2(\mathbb{Z}^n)$, and

$$\mathcal{F}^* : \ell^2(\mathbb{Z}^n) \longrightarrow L^2(\mathbf{T}^n) \tag{1.24}$$

is an isometry of $\ell^2(\mathbb{Z}^n)$ onto its range. Part of the inversion formula will be that the map (1.24) is also surjective.

Let us note that, if $f_j \in \mathcal{TP}$, satisfying (1.21), then (1.13) implies $\hat{f}_j(k) = \varphi_j(k)$, so we have directly in this case:

$$\mathcal{F}^*\mathcal{F} = I \text{ on } \mathcal{TP}. \tag{1.25}$$

One approach to more general inversion formulas would be to establish that $\mathcal{TP}$ is dense in various function spaces, on which $\mathcal{F}^*\mathcal{F}$ can be shown to act continuously. Here, we will take a superficially different approach. We will make use of such basic results from real analysis as the denseness of $C(\mathbf{T}^n)$ in $L^p(\mathbf{T}^n)$, for $1 \le p < \infty$.

Our approach to (1.14) will be to establish the following Abel summability result. Consider

$$J_r f(\theta) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, r^{|k|}\, e^{ik\cdot\theta} \tag{1.26}$$

where $|k| = |k_1| + \cdots + |k_n|,\ r \in [0,1)$. We will show that

$$J_r f \to f \text{ as } r \nearrow 1, \tag{1.27}$$

in the appropriate spaces. The operator J_r in (1.26) is defined for any $f \in L^1(\mathbf{T}^n)$, if $r < 1$, and we have the formula

$$J_r f(\theta) = (2\pi)^{-n} \int_{\mathbf{T}^n} f(\theta') \sum_{k\in\mathbb{Z}^n} r^{|k|}\, e^{ik\cdot(\theta-\theta')}\, d\theta'. \tag{1.28}$$

The sum over $\mathbb{Z}^n$ inside the integral can be written

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n} r^{|k|}\, e^{ik\cdot(\theta-\theta')} &= P_n(r,\theta-\theta') \\ &= p(r,\theta_1-\theta_1')\cdots p(r,\theta_n-\theta_n'), \end{aligned} \tag{1.29}$$

where

$$
\begin{aligned}
p(r,\theta) &= \sum_{k=-\infty}^{\infty} r^{|k|}\ e^{ik\theta} \\
&= 1 + \sum_{k=1}^{\infty} \bigl(r^k e^{ik\theta} + r^k e^{-ik\theta}\bigr) \\
&= (1-r^2)/(1-2r\cos\theta + r^2).
\end{aligned} \tag{1.30}
$$

Then we have the explicit integral formula

$$
\begin{aligned}
J_r f(\theta) &= (2\pi)^{-n} \int\limits_{\mathbf{T}^n} f(\theta') P_n(r,\theta-\theta') d\theta' \\
&= (2\pi)^{-n} \int\limits_{\mathbf{T}^n} f(\theta-\theta') P_n(r,\theta') d\theta'.
\end{aligned} \tag{1.31}
$$

Let us examine $p(r,\theta)$. It is clear that the numerator and denominator on the right side of (1.30) are positive, so $p(r,\theta) > 0$ for each $r \in [0,1),\ \theta \in S^1$. Of course, as $r \nearrow 1$, the numerator tends to 0; as $r \nearrow 1$, the denominator tends to a nonzero limit, except at $\theta = 0$. Since it is clear that

$$
(2\pi)^{-1} \int\limits_{S^1} p(r,\theta) d\theta = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum r^{|k|} e^{ik\theta} d\theta = 1, \tag{1.32}
$$

we see that, for r close to 1, $p(r,\theta)$ as a function of θ is highly peaked near $\theta = 0$ and small elsewhere, as in figure 1.1.

We are now prepared to prove the following result giving Abel summability (1.27).

Proposition 1.1**.** *If $f \in C(\mathbf{T}^n)$, then*

$$
J_r f \to f \ \textit{ uniformly on } \mathbf{T}^n \ \textit{ as } r \nearrow 1. \tag{1.33}
$$

Furthermore, for any $p \in [1,\infty)$, if $f \in L^p(\mathbf{T}^n)$, then

$$
J_r f \to f \ \textit{ in } L^p(\mathbf{T}^n) \ \textit{ as } r \nearrow 1. \tag{1.34}
$$

The proof of (1.33) is an immediate consequence of (1.31) and the peaked nature of $P_n(\theta')$ near $\theta' = 0$ discussed above, together with the observation that, if f is continuous at θ, then it does not vary very much near θ. The convergence in (1.34) is in the L^p-norm, defined by

$$
\|g\|_{L^p} = \Bigl[(2\pi)^{-n} \int\limits_{\mathbf{T}^n} |g(\theta)|^p d\theta\Bigr]^{1/p}. \tag{1.35}
$$

We have the well known triangle inequality in such a norm:

$$\|g_1 + g_2\|_{L^p} \le \|g_1\|_{L^p} + \|g_2\|_{L^p}, \tag{1.36}$$

and this implies, via (1.31)-(1.32),

$$\begin{aligned} \|J_r f\|_{L^p} &= (2\pi)^{-n} \Big\| \int_{\mathbf{T}^n} P_n(\theta')\tau_{\theta'} f \, d\theta' \Big\|_{L^p} \\ &\le (2\pi)^{-n} \int P_n(\theta') \|\tau_{\theta'} f\|_{L^p} d\theta' \\ &= \|f\|_{L^p}, \end{aligned} \tag{1.37}$$

where

$$\tau_{\theta'} f(\theta) = f(\theta - \theta'), \tag{1.38}$$

which implies $\|\tau_{\theta'} f\|_{L^p} = \|f\|_{L^p}$. In other words,

$$\|J_r\|_{\mathcal{L}(L^p)} \le 1, \quad 1 \le p < \infty, \tag{1.39}$$

where we are using the operator norm on L^p :

$$\|T\|_{\mathcal{L}(L^p)} = \sup\{\|Tf\|_{L^p} : \|f\|_{L^p} \le 1\}. \tag{1.40}$$

Using this, we can deduce (1.34) from (1.33), and the denseness of $C(\mathbf{T}^n)$ in each space $L^p(\mathbf{T}^n)$, for $1 \le p < \infty$. Indeed, given $f \in L^p(\mathbf{T}^n)$, and given $\varepsilon > 0$, find $g \in C(\mathbf{T}^n)$ such that $\|f - g\|_{L^p} < \varepsilon$. Note that, generally, $\|g\|_{L^p} \le \|g\|_{\sup}$. Now we have

$$\begin{aligned} \|J_r f - f\|_{L^p} &\le \|J_r(f-g)\|_{L^p} + \|J_r g - g\|_{L^p} + \|g - f\|_{L^p} \\ &< \varepsilon + \|J_r g - g\|_{L^\infty} + \varepsilon, \end{aligned} \tag{1.41}$$

making use of (1.39). By (1.33), the middle term is $< \varepsilon$ if r is close enough to 1, so this proves (1.34).

Corollary 1.2. *If $f \in C^\infty(\mathbf{T}^n)$, then the Fourier inversion formula (1.14) holds.*

Proof. In such a case, as noted, we have $\hat{f} \in s(\mathbb{Z}^n)$, so certainly the right side of (1.14) is absolutely convergent to some $f^\# \in C(\mathbf{T}^n)$. In such a case, one a fortiori has

$$\lim_{r \nearrow 1} \sum_{k \in \mathbb{Z}^n} \hat{f}(k) r^{|k|} e^{ik\cdot\theta} = f^\#(\theta). \tag{1.42}$$

But now Proposition 1.1 implies (1.42) is equal to $f(\theta)$, i.e., $f^\# = f$, so the inversion formula is proved for $f \in C^\infty(\mathbf{T}^n)$.

As a result, we see that

$$\mathcal{F}^* : s(\mathbb{Z}^n) \longrightarrow C^\infty(\mathbf{T}^n) \tag{1.43}$$

is surjective, as well as injective, with two sided inverse $\mathcal{F} : C^\infty(\mathbf{T}^n) \to s(\mathbb{Z}^n)$. This of course implies that the map (1.24) has dense range in $L^2(\mathbf{T}^n)$; hence

$$\mathcal{F}^* : \ell^2(\mathbb{Z}^n) \longrightarrow L^2(\mathbf{T}^n) \text{ is unitary.} \tag{1.44}$$

Another way of stating this is

$$\{e^{ik\cdot\theta} : k \in \mathbb{Z}^n\} \text{ is an orthonormal basis of } L^2(\mathbf{T}^n), \tag{1.45}$$

with inner product given by (1.8). Also, the inversion formula

$$\mathcal{F}^*\mathcal{F} = I \text{ on } C^\infty(\mathbf{T}^n) \tag{1.46}$$

implies

$$\|\mathcal{F}f\|_{\ell^2} = \|f\|_{L^2}, \tag{1.47}$$

so therefore $\mathcal{F}$ extends by continuity from $C^\infty(\mathbf{T}^n)$ to a map

$$\mathcal{F} : L^2(\mathbf{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n), \text{ unitary,} \tag{1.48}$$

inverting (1.44). The denseness $C^\infty(\mathbf{T}^n) \subset L^2(\mathbf{T}^n) \subset L^1(\mathbf{T}^n)$ implies that this $\mathcal{F}$ coincides with the restriction to $L^2(\mathbf{T}^n)$ of the map (1.3). Note that the fact that (1.44) and (1.48) are inverses of each other extends the inversion result of Corollary 1.2.

We devote a little space to conditions implying that the Fourier series (1.14) is absolutely convergent, weaker than the hypothesis that $f \in C^\infty(\mathbf{T}^n)$. Note that since $|e^{ik\cdot\theta}| = 1$, absolute convergence of (1.14) implies uniform convergence. By (1.4), we see that

$$f \in C^\ell(\mathbf{T}^n) \Longrightarrow |\hat{f}(k)| \le C\langle k\rangle^{-\ell}, \tag{1.49}$$

which in turn clearly gives absolute convergence provided

$$\ell \ge n+1. \tag{1.50}$$

Using Plancherel's identity and Cauchy's inequality, we can do somewhat better:

Proposition 1.3. *If $f \in C^\ell(\mathbf{T}^n)$, then the Fourier series for f is absolutely convergent provided*

$$\ell > n/2. \tag{1.51}$$

Proof. We have

$$\begin{aligned} \sum_k |\hat{f}(k)| &= \sum_k \langle k\rangle^{-\ell} \langle k\rangle^{\ell} |\hat{f}(k)| \\ &\le \Bigl[\sum_k \langle k\rangle^{-2\ell}\Bigr]^{\frac{1}{2}} \cdot \Bigl[\sum_k \langle k\rangle^{2\ell} |\hat{f}(k)|^2\Bigr]^{\frac{1}{2}} \\ &\le C\Bigl[\sum_k \langle k\rangle^{2\ell} |\hat{f}(k)|^2\Bigr]^{\frac{1}{2}}, \end{aligned} \tag{1.52}$$

as long as (1.51) holds. The square of the right side is dominated by

$$\begin{aligned} C' \sum_k \sum_{|\gamma|\le\ell} \bigl|k^\gamma \hat{f}(k)\bigr|^2 &= C' \sum_{|\gamma|\le\ell} \|D^\gamma f\|_{L^2}^2 \\ &\le C''\|f\|_{C^\ell}^2, \end{aligned} \tag{1.53}$$

so the proposition is proved.

See some of the exercises below for more on convergence in the case $n = 1$.

Exercises.

1. Given $f, g \in L^1(\mathbf{T}^n)$, show that

$$\hat{f}(k)\hat{g}(k) = \hat{u}(k)$$

with

$$u(\theta) = (2\pi)^{-n} \int_{\mathbf{T}^n} f(\varphi) g(\theta - \varphi)\, d\varphi.$$

2. Given $f, g \in C(\mathbf{T}^n)$, show that

$$\widehat{(fg)}(k) = \sum_m \hat{f}(k-m)\hat{g}(m).$$

3. Using the proof of Proposition 1.3, show that every $f \in \operatorname{Lip}(S^1)$ has an absolutely convergent Fourier series.

4. Show that if $f \in L^\infty(S^1)$ has bounded variation, then $|\hat{f}(k)| \le C/\langle k \rangle$.

5. For $f \in L^1(S^1)$, set

$$S_N f(\theta) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\theta}.$$

Show that $S_N f(\theta) = (1/2\pi) \int_{-\pi}^{\pi} f(\theta - \varphi) D_N(\varphi)\, d\varphi$ where

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin(N + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}. \tag{1.55}$$

<u>Hint</u>. To evaluate the sum, recall how to sum a finite geometrical series.
$D_N(\theta)$ is called the Dirichlet kernel.

6. Let $f \in L^\infty(S^1)$ have bounded variation and also have the following property of 'vanishing' at $\theta = 0$:

$$f(\theta) \Big/ \sin \frac{1}{2}\theta \text{ is of bounded variation on } [-\pi, \pi].$$

Show that $\left|\int\limits_{S^1} f(\theta)D_N(\theta)d\theta\right| \le C/N$, and hence $S_N f(0) \to 0$ as $N \to \infty$.
Hint. Use the formula (1.55) for $D_N(\theta)$ and perform an integration by parts.

7. Deduce that if $f \in L^\infty(S^1)$ has bounded variation, and is Lipschitz continuous at θ_0, then $S_N f(\theta_0) \to f(\theta_0)$ as $N \to \infty$. If furthermore f is Lipschitz on an open interval $J \subset S^1$, then $S_N f \to f$ uniformly on compact subsets of J.

8. Let $f \in L^\infty(S^1)$ be piecewise C^1, with a finite number of simple jumps. Show that $S_N f(\theta) \to f(\theta)$ at points of continuity. If f has a jump at θ_j, with limiting values $f_\pm(\theta_j)$, show that

$$S_N f(\theta_j) \to \frac{1}{2}\big[f_+(\theta_j) + f_-(\theta_j)\big], \tag{1.56}$$

as $N \to \infty$.
Hint. By Problem 7, it remains only to establish (1.56). Show that this can be reduced to the case $\theta_j = \pi,\ f(\theta) = \theta$, for $-\pi \le \theta < \pi$. Verify that this function has Fourier series

$$2\sum_{k=1}^\infty \frac{(-1)^k}{k} \sin k\theta.$$

9. Work out the Fourier series of the function $f \in \mathrm{Lip}(S^1)$ given by

$$f(\theta) = |\theta|, \quad -\pi \le \theta \le \pi.$$

Examining this at $\theta = 0$, establish that

$$\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}.$$

10. One can obtain Fourier coefficients of functions θ^k and $|\theta|^k$ on $[-\pi,\pi]$ in terms of the Fourier coefficients of

$$q_k(\theta) = \theta^k \text{ on } [0,\pi], \quad = 0 \text{ on } [-\pi, 0].$$

Show that, for $n \ne 0$,

$$\sum_{k=0}^\infty \frac{1}{k!}\hat{q}_k(n)\ (is)^k = -\frac{1}{2\pi i n}\Big[(-1)^n \cdot e^{\pi i s} - 1\Big]\Big(1 - \frac{s}{n}\Big)^{-1},$$

and use this to work out the Fourier series for these functions. Apply this to Problem 9, and to the calculation at the end of Problem 8.

2. The Fourier transform

The Fourier transform is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\xi} dx \tag{2.1}$$

when $f \in L^1(\mathbb{R}^n)$. It is clear that

$$\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^n). \tag{2.2}$$

This is analogous to (1.3). The analogue for $C^\infty(\mathbf{T}^n)$, and simultaneously for $s(\mathbb{Z}^n)$, of section 1, in this case is the Schwartz space of rapidly decreasing functions:

$$\mathcal{S}(\mathbb{R}^n) = \big\{u \in C^\infty(\mathbb{R}^n) : x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n) \text{ for all } \alpha, \beta \geq 0\big\}, \tag{2.3}$$

where $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, with $D_j = i\partial/\partial x_j$. It is easy to verify that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \tag{2.4}$$

and

$$\xi^\alpha D_\xi^\beta \mathcal{F}f(\xi) = \mathcal{F}(D^\alpha x^\beta f)(\xi). \tag{2.5}$$

We define $\mathcal{F}^*$ by

$$\mathcal{F}^* f(\xi) = \tilde{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{ix\cdot\xi} dx, \tag{2.6}$$

which differs from (2.1) only in the sign of the exponent. It is clear that $\mathcal{F}^*$ satisfies the mapping properties (2.2), (2.4), and

$$(\mathcal{F}u, v) = (u, \mathcal{F}^* v) \tag{2.7}$$

for $u, v \in \mathcal{S}(\mathbb{R}^n)$, where (u, v) denotes the usual L^2-inner product, $(u,v) = \int_{\mathbb{R}^n} u(x)\, \overline{v(x)}\, dx$.

As in the theory of Fourier series, the first major result is the Fourier inversion formula. The following is our first version.

Proposition 2.1. *We have the inversion formula*

$$\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = I \text{ on } \mathcal{S}(\mathbb{R}^n). \tag{2.8}$$

Our proof will have in common with the proof of the inversion formula for Fourier series, via Proposition 1.1, that we will sneak up on the inversion formula by throwing in a convergence factor which will allow interchange of orders of integration (in the proof of Proposition 1.1, the orders of an integral and an infinite series were interchanged). Thus, let us write, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \mathcal{F}^*\mathcal{F}f(x) &= (2\pi)^{-n} \int \Big[\int f(y) e^{-iy\cdot\xi} dy\Big] e^{ix\cdot\xi} d\xi \\ &= (2\pi)^{-n} \lim_{\varepsilon \searrow 0} \iint f(y)\ e^{-\varepsilon|\xi|^2}\ e^{i(x-y)\cdot\xi}\ dy\ d\xi. \end{aligned} \tag{2.9}$$

We can interchange the order of integration on the right for any $\varepsilon > 0$, to obtain

$$\mathcal{F}^*\mathcal{F}f(x) = \lim_{\varepsilon \searrow 0} \int f(y) p(\varepsilon, x-y) dy, \tag{2.10}$$

where

$$p(\varepsilon, x) = (2\pi)^{-n} \int e^{-\varepsilon|\xi|^2 + ix\cdot\xi} d\xi. \tag{2.11}$$

Note that

$$p(\varepsilon, x) = \varepsilon^{-n/2}\ q(x/\sqrt{\varepsilon}) \tag{2.12}$$

where $q(x) = p(1, x)$. In a moment we will show that

$$p(\varepsilon, x) = (4\pi\varepsilon)^{-n/2}\ e^{-|x|^2/4\varepsilon}. \tag{2.13}$$

The derivation of this identity will also show that

$$\int_{\mathbb{R}^n} q(x)\ dx = 1. \tag{2.14}$$

From this, it follows as in the proof of Proposition 1.1 that

$$\lim_{\varepsilon \searrow 0} \int f(y) p(\varepsilon, x-y) dy = f(x) \tag{2.15}$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$, even for f bounded and continuous, so we have proved $\mathcal{F}^*\mathcal{F} = I$ on $\mathcal{S}(\mathbb{R}^n)$; the proof that $\mathcal{F}\mathcal{F}^* = I$ on $\mathcal{S}(\mathbb{R}^n)$ is identical.

It remains to verify (2.13). We observe that $p(\varepsilon, x)$, defined by (2.11), is an entire analytic function of $x \in \mathbb{C}^n$, for any $\varepsilon > 0$. It is convenient to verify that

$$p(\varepsilon, ix) = (4\pi\varepsilon)^{-n/2}\ e^{|x|^2/4\varepsilon}, \quad x \in \mathbb{R}^n, \tag{2.16}$$

from which (2.13) follows by analytic continuation. Now

$$\begin{aligned} p(\varepsilon, ix) &= (2\pi)^{-n} \int e^{-x\cdot\xi - \varepsilon|\xi|^2}\, d\xi \\ &= (2\pi)^{-n} e^{|x|^2/4\varepsilon} \int e^{-|x/2\sqrt{\varepsilon} + \sqrt{\varepsilon}\xi|^2}\, d\xi \\ &= (2\pi)^{-n} \varepsilon^{-n/2} e^{|x|^2/4\varepsilon} \int\limits_{\mathbb{R}^n} e^{-|\xi|^2}\, d\xi. \end{aligned} \tag{2.17}$$

To prove (2.16), it remains to show that

$$\int\limits_{\mathbb{R}^n} e^{-|\xi|^2}\, d\xi = \pi^{n/2}. \tag{2.18}$$

Indeed, if

$$A = \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi, \tag{2.19}$$

then the left side of (2.18) is equal to A^n. But for $n = 2$ we can use polar coordinates:

$$A^2 = \int\limits_{\mathbb{R}^2} e^{-|\xi|^2}\, d\xi = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\ dr\ d\theta = \pi. \tag{2.20}$$

This completes the proof of the identity (2.16), and hence of (2.13).

In light of (2.7) and the Fourier inversion formula (2.8), we see that, for $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathcal{F}u, \mathcal{F}v) = (u, v) = (\mathcal{F}^* u, \mathcal{F}^* v). \tag{2.21}$$

Thus $\mathcal{F}$ and $\mathcal{F}^*$ extend uniquely from $\mathcal{S}(\mathbb{R}^n)$ to isometries on $L^2(\mathbb{R}^n)$, and are inverses to each other. Thus we have the Plancherel theorem:

Proposition 2.2. *The Fourier transform*

$$\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \tag{2.22}$$

is unitary, with inverse $\mathcal{F}^*$.

We make a remark about the computation of the Fourier integral (2.11), done above via analytic continuation. The following derivation does not make any direct use of complex analysis. It suffices to handle the case $\varepsilon = \frac{1}{2}$, i.e., to show

$$\hat{G}(\xi) = e^{-|\xi|^2/2} \text{ if } G(x) = e^{-|x|^2/2}, \text{ on } \mathbb{R}^n. \tag{2.23}$$

We have interchanged the roles of x and ξ compared to those in (2.11) and (2.13). It suffices to get (2.23) in the case $n = 1$, by the obvious multiplicativity. Now $G(x) = e^{-x^2/2}$ satisfies the differential equation

$$(d/dx + x)G(x) = 0. \tag{2.24}$$

By the intertwining property (2.5), it follows that $(d/d\xi + \xi)\hat{G}(\xi) = 0$, and uniqueness of solutions to this ODE yields $\hat{G}(\xi) = Ce^{-\xi^2/2}$. The constant C is evaluated via the identity (2.20); $C = 1$; and we are done.

As for the necessity of computing the Fourier integral (2.11) to prove the Fourier inversion formula, let us note the following. For <u>any</u> $g \in \mathcal{S}(\mathbb{R}^n)$ with $g(0) = 1$, ($g(\xi) = e^{-|\xi|^2}$ being an example), we have (replacing ε by δ^2), just as in (2.9),

$$\begin{aligned} \mathcal{F}^*\mathcal{F}f(x) &= (2\pi)^{-n} \lim_{\delta \searrow 0} \int f(y) g(\delta\xi) e^{i(x-y)\cdot\xi} d\xi \\ &= \lim_{\delta \searrow 0} \int f(y) h_\delta(x-y)\,dy \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} h_\delta(x) &= (2\pi)^{-n} \int g(\delta\xi) e^{ix\cdot\xi} d\xi \\ &= (2\pi)^{-n/2} \delta^{-n} \tilde{g}(x/\delta). \end{aligned} \tag{2.26}$$

By the peaked nature of h_δ as $\delta \to 0$, we see that the limit in (2.25) is equal to

$$C\ f(x) \tag{2.27}$$

where

$$C = \int h_1(x)\,dx = (2\pi)^{-n/2} \int \tilde{g}(x)\,dx. \tag{2.28}$$

The argument (2.25)-(2.27) shows C is independent of the choice of $g \in \mathcal{S}(\mathbb{R}^n)$, and we only have to find a single example g such that $\tilde{g}(x)$ can be evaluated explicitly and then the integral on the right in (2.28) can be evaluated explicitly. In most natural examples one picks g to be even, so $\tilde{g} = \hat{g}$.

We remark that one does not need to have $g \in \mathcal{S}(\mathbb{R}^n)$ in the argument above; it suffices to have $g \in L^1(\mathbb{R}^n)$, bounded and continuous, and such that $\hat{g} \in L^1(\mathbb{R}^n)$. An example, in the case $n = 1$, is

$$g(\xi) = e^{-|\xi|}. \tag{2.29}$$

In this case, elementary calculations give

$$\hat{g}(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \frac{1}{x^2+1}. \tag{2.30}$$

In this case, (2.28) can be evaluated in terms of the arc tangent. Another example, in the case $n = 1$, is

$$\begin{aligned} g(\xi) = 1 - |\xi| &\text{ if } |\xi| \le 1 \\ 0 \quad &\text{ if } |\xi| \ge 1. \end{aligned} \tag{2.31}$$

In this case,

$$\hat{g}(x) = (2\pi)^{-\frac{1}{2}} \Big(\frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\Big)^2, \tag{2.32}$$

and (2.28) can be evaluated by the method of residues. The calculation of (2.32) can be achieved by evaluating

$$\int_0^1 (1-\xi) \cos x\xi \ d\xi$$

via an integration by parts, though there is a more painless way, mentioned below.

We now make some comments on the relation between the Fourier transform and convolutions. The convolution $u * v$ of two functions on $\mathbb{R}^n$ is defined by

$$\begin{aligned} u * v(x) &= \int u(y)v(x-y)dy \\ &= \int u(x-y)v(y)dy. \end{aligned} \tag{2.33}$$

Note that $u * v = v * u$. If $u, v \in \mathcal{S}(\mathbb{R}^n)$, so is $u * v$. Also

$$\|u * v\|_{L^p(\mathbb{R}^n)} \le \|u\|_{L^1} \|v\|_{L^p},$$

so the convolution has a unique continuous extension to a bilinear map

$$L^1(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \tag{2.34}$$

for $1 \leq p < \infty$; one can directly perceive this also works for $p = \infty$. Note that the right side of (2.10), for any $\varepsilon > 0$, is an example of a convolution. Computing the Fourier transform of (2.33) leads immediately to the formula

$$\mathcal{F}(u * v)(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{v}(\xi). \tag{2.35}$$

We also note that, if

$$P = \sum_{|\alpha| \leq k} a_\alpha D^\alpha \tag{2.36}$$

is a constant coefficient differential operator, we have

$$P(u * v) = (Pu) * v = u * (Pv) \tag{2.37}$$

if $u, v \in \mathcal{S}(\mathbb{R}^n)$. This also generalizes; if $u \in \mathcal{S}(\mathbb{R}^n), v \in L^p(\mathbb{R}^n)$, the first identity continues to hold.

We mention the following simple application of (2.35), to a short calculation of (2.32). With g given by (2.31), we have $g = g_1 * g_1$, where

$$g_1(\xi) = 1 \text{ for } \xi \in [-\frac{1}{2}, \frac{1}{2}], \quad 0 \text{ otherwise.} \tag{2.38}$$

Thus

$$\begin{aligned} \hat{g}_1(x) &= (2\pi)^{-1/2} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-ix\xi}\, d\xi \\ &= (2\pi)^{-1/2} \big(\sin \frac{1}{2}x\big) \Big/ \big(\frac{1}{2}x\big), \end{aligned} \tag{2.39}$$

and then (2.32) follows immediately from (2.35).

Exercises.

1. Show that $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ denotes the space of functions v, continuous on $\mathbb{R}^n$, such that $v(\xi) \to 0$ as $|\xi| \to \infty$.
Hint. Use the denseness of $\mathcal{S}(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$.
This result is called the Riemann-Lebesgue Lemma.

2. Show that the Fourier transforms (2.1) and (2.22) coincide on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

3. For $f \in L^1(\mathbb{R}^n)$, set $S_R f(x) = (2\pi)^{-1/2} \int_{-R}^{R} \hat{f}(\xi) e^{ix\xi} d\xi$. Show that

$$S_R f(x) = D_R * f(x) = \int_{-\infty}^{\infty} D_R(x-y) f(y) dy$$

where

$$D_R(x) = (2\pi)^{-1} \int_{-R}^{R} e^{ix\xi} d\xi = \frac{\sin Rx}{\pi x}.$$

Compare exercise 5 of §1.

4. Show that $f \in L^2(\mathbb{R}) \Rightarrow S_R f \to f$ in L^2-norm as $R \to \infty$.

5. Show that there exist $f \in L^1(\mathbb{R})$ such that $S_R f \notin L^1(\mathbb{R})$ for any $R \in (0, \infty)$.
Hint. Note that $D_R \notin L^1(\mathbb{R})$.

6. For $f \in L^1(\mathbb{R})$, set

$$C_R f(x) = (2\pi)^{-1/2} \int_{-R}^{R} \Bigl(1 - \frac{|\xi|}{R}\Bigr) \hat{f}(\xi) e^{ix\xi} d\xi.$$

Show that $C_R f(x) = E_R * f(x)$ where

$$E_R(x) = (2\pi)^{-1} \int_{-R}^{R} \Bigl(1 - \frac{|\xi|}{R}\Bigr) e^{ix\xi} d\xi = \frac{2}{\pi R} \Bigl[\frac{\sin \frac{1}{2} R x}{x}\Bigr]^2.$$

Note that $E_R \in L^1(\mathbb{R})$. Show that, for $1 \le p < \infty$,

$$f \in L^p(\mathbb{R}) \Longrightarrow C_R f \to f \text{ in } L^p\text{-norm, as } R \to \infty.$$

We say the Fourier transform of f is Cesaro-summable if $C_R f \to f$ as $R \to \infty$.

In exercises 7-13, suppose $f \in \mathcal{S}(\mathbb{R}),\ f \geq 0,\ \int_{-\infty}^{\infty} f(x)dx = 1$, and $\int_{-\infty}^{\infty} xf(x)dx = 0$. Set $F(\xi) = (2\pi)^{1/2}\hat{f}(\xi)$. The point of the exercises is to obtain a version of the Central Limit Theorem.

7. Show that $F(0) = 1,\ F'(0) = 0,\ F''(0) = -2a < 0$.

8. Set $F_n(\xi) = F(\xi/\sqrt{n})^n$. Relate $(2\pi)^{-1/2}\tilde{F}_n(x)$ to the convolution of n copies of f.

9. Show that there exist $A > 0$ and $G \in C^\infty([-A, A])$ such that $f(\xi) = e^{-a\xi^2}G(\xi)$ for $|\xi| \leq A$, and $G(0) = 1,\ G'(0) = G''(0) = 0$. Hence

$$F_n(\xi) = e^{-a\xi^2}G(\xi/\sqrt{n})^n \text{ for } |\xi| \leq A\sqrt{n}.$$

10. Show that $\left|G(\xi/\sqrt{n})^n - 1\right| \leq Cn^{-\alpha}$ if $|\xi| \leq n^{(\frac{1}{2}-\alpha)/3}$, for n large.
Fix $\alpha \in (0, \frac{1}{2})$ and set $\gamma = (\frac{1}{2} - \alpha)/3 \in (0, \frac{1}{6})$.

11. Show that, for $|\xi| \geq n^\gamma$, $\left|F(\xi/\sqrt{n})\right| \leq 1 - \frac{1}{2}an^{-(1-2\gamma)}$, for n large, so $|F_n(\xi)| \leq e^{-an^{2\gamma}/4} = \delta_n$. Deduce that

$$\int\limits_{|\xi|\geq n^\gamma} |F_n(\xi)|d\xi \leq C\ \delta_n^{(n-1)/n}\sqrt{n} \to 0 \text{ as } n \to \infty.$$

12. From problems 9-11, deduce that $F_n \to e^{-a\xi^2}$ in $L^1(\mathbb{R})$ as $n \to \infty$.

13. Deduce now that $(2\pi)^{-1/2}\tilde{F}_n \to (4\pi a)^{-1/2}e^{-x^2/4a}$ in both $C_0(\mathbb{R})$ and $L^1(\mathbb{R})$, as $n \to \infty$. Relate this to the Central Limit Theorem of probability theory. Weaken the hypotheses on f as much as you can.
Hint. In passing from the C_0 result to the L^1 result, positivity of $\tilde{F}_n$ will be useful.

14. With $p_\varepsilon(x) = (4\pi\varepsilon)^{-1/2} e^{-x^2/4\varepsilon}$, as in (3.13) for $n = 1$, show that, for any $u(x)$, continuous and compactly supported on $\mathbb{R}$, $p_\varepsilon * u \to u$ uniformly as $\varepsilon \to 0$.
Show that for each $\varepsilon > 0$, $p_\varepsilon * u(x)$ is the restriction to $\mathbb{R}$ of an entire holomorphic function of $x \in \mathbb{C}$.

15. Using exercise 14, prove the <u>Weierstrass approximation theorem</u>:
Any $f \in C([a, b])$ is a uniform limit of polynomials.
<u>Hint</u>. Extend f to u as above, approximate u by $p_\varepsilon * u$, and expand this in a power series.

3. The discrete Fourier transform

When doing numerical work involving Fourier series, it is convenient to discretize, and replace S^1, pictured as the group of complex numbers of modulus 1, by the group Γ_n generated by $\omega = e^{2\pi i/n}$. One can also approximate $\mathbf{T}^d$ by $(\Gamma_n)^d$, a product of d copies of Γ_n. We will restrict attention to the case $d = 1$ here; results for general d are obtained similarly.

The cyclic group Γ_n is isomorphic to the group $\mathbb{Z}_n = \mathbb{Z}/(n)$, but we will observe a distinction between these two groups; an element of Γ_n is a certain complex number of modulus 1 and an element of $\mathbb{Z}_n$ is an equivalence class of integers. For n large, we think of Γ_n as an approximation to S^1 and $\mathbb{Z}_n$ as an approximation to $\mathbb{Z}$. We note the natural dual pairing $\Gamma_n \times \mathbb{Z}_n \to \mathbb{C}$ given by $(\omega^j, \ell) \mapsto \omega^{j\ell}$, which is well defined since $\omega^{jn} = 1$.

Now, given a function $f : \Gamma_n \to \mathbb{C}$, its discrete Fourier transform $f^\# = \Phi_n f$, mapping $\mathbb{Z}_n$ to $\mathbb{C}$, is defined by

$$f^\#(\ell) = \frac{1}{n} \sum_{\omega^j \in \Gamma_n} f(\omega^j) \omega^{-j\ell}. \tag{3.1}$$

Similarly, given a function $g : \mathbb{Z}_n \to \mathbb{C}$, its 'inverse Fourier transform' $g^b : \Gamma_n \to \mathbb{C}$ is defined by

$$g^b(\omega^j) = \sum_{\ell \in \mathbb{Z}_n} g(\ell) \omega^{j\ell}. \tag{3.2}$$

The following is the Fourier inversion formula in this context.

Proposition 3.1**.** *The map*

$$\Phi_n : L^2(\Gamma_n) \longrightarrow L^2(\mathbb{Z}_n) \tag{3.3}$$

is a unitary isomorphism, with inverse defined by (3.2), so

$$f(\omega^j) = \sum_{\ell \in \mathbb{Z}_n} f^\#(\ell) \omega^{j\ell}. \tag{3.4}$$

Here the space $L^2(\mathbb{Z}_n)$ is defined by counting measure and $L^2(\Gamma_n)$ by $1/n$ times counting measure, i.e.,

$$(u, v)_{L^2(\Gamma_n)} = \frac{1}{n} \sum_{\omega^j \in \Gamma_n} u(\omega^j) \overline{v(\omega^j)}. \tag{3.5}$$

Note that, if we define functions e_j on Γ_n by

$$e_j(\omega^k) = \omega^{jk}, \tag{3.6}$$

then Proposition 3.1 is equivalent to:

Proposition 3.2. *The functions e_j, $1 \leq j \leq n$, form an orthonormal basis of $L^2(\Gamma_n)$.*

Proof. Since $L^2(\Gamma_n)$ has dimension n, we need only check that the e_js are mutually orthogonal. Note that

$$(e_k, e_\ell) = \frac{1}{n} \sum_{\omega^j \in \Gamma_n} \omega^{mj}, \quad m = k - \ell.$$

Denote the sum by S_m. If we multiply by ω^m, we have a sum of the same set of powers of ω, so $S_m = \omega^m S_m$. Thus $S_m = 0$ whenever $\omega^m \neq 1$, which completes the proof. Alternatively, the series is easily summed as a finite geometrical series.

Note that the functions e_j in (3.6) are the restrictions to Γ_n of $e^{ij\theta}$ (i.e., values at $\theta = 2\pi k/n$). These restrictions depend only on the residue class of j mod n, which leads to the following simple but fundamental connection between Fourier series on S^1 and on Γ_n.

Proposition 3.3. *If $f \in C(S^1)$ has absolutely convergent Fourier series, then*

$$f^{\#}(\ell) = \sum_{j=-\infty}^{\infty} \hat{f}(\ell + jn). \tag{3.7}$$

We will use (3.7) as a tool to see how well a function on S^1 is approximated by discretization, involving restriction to Γ_n. Precisely, we consider the operators

$$R_n : C(S^1) \longrightarrow L^2(\Gamma_n), \quad E_n : L^2(\Gamma_n) \longrightarrow C^\infty(S^1) \tag{3.8}$$

given by

$$(R_n f)(\omega^j) = f(2\pi j/n) \tag{3.9}$$

for $f = f(\theta),\ 0 \leq \theta \leq 2\pi$, and

$$E_n\Bigl(\sum_{\ell \in \mathbb{Z}_n} g(\ell)\omega^{j\ell}\Bigr) = \sum_{\ell=-\nu}^{\nu-1} g(\ell)e^{i\ell\theta},\ n = 2\nu. \tag{3.10}$$

We assume $n = 2\nu$ is even; one can also treat $n = 2\nu - 1$, changing the upper limit in the last sum from $\nu - 1$ to ν. Clearly $R_n E_n$ is the identity operator on $L^2(\Gamma_n)$. The question of interest to us is: how close is $E_n R_n f$ to f, a function on S^1? The answer depends on smoothness properties of f, and is expressed in terms involving (typically) negative powers of n.

We compare $E_n R_n$ and the partial summing operator

$$P_n f = \sum_{\ell=-\nu}^{\nu-1} \hat{f}(\ell)e^{i\ell\theta} \tag{3.11}$$

for Fourier series. Note that

$$E_n R_n f(\theta) = \sum_{\ell=-\nu}^{\nu-1} f^{\#}(\ell) e^{i\ell\theta}. \tag{3.12}$$

Consequently,

$$E_n R_n f = P_n f + Q_n f \tag{3.13}$$

with

$$Q_n f(\theta) = \sum_{\ell=-\nu}^{\nu-1} \bigl[f^{\#}(\ell) - \hat{f}(\ell)\bigr] e^{i\ell\theta}. \tag{3.14}$$

By (3.7), we have, for $-\nu \leq \ell \leq \nu - 1$,

$$f^{\#}(\ell) - \hat{f}(\ell) = \sum_{j \in \mathbb{Z} \setminus 0} \hat{f}(\ell + jn). \tag{3.15}$$

Consequently, the sup norm of $Q_n f$ is bounded by

$$\sum_{\ell=-\nu}^{\nu-1} \bigl|f^{\#}(\ell) - \hat{f}(\ell)\bigr| \leq \sum_{|k| \geq \nu} |\hat{f}(k)|. \tag{3.16}$$

The right side also dominates the sup norm of $f - P_n f$, proving:

Proposition 3.4. *If $f \in C(S^1)$ has absolutely convergent Fourier series, then*

$$\|f - E_n R_n f\|_{L^\infty} \leq 2 \sum_{|k| \geq \nu} |\hat{f}(k)|. \tag{3.17}$$

The estimates of various norms of $f - P_n f$ is an exercise in Fourier analysis on S^1. Here we note one simple estimate, for $m \geq 1$:

$$\begin{aligned} \|f - P_n f\|_{C^\ell(S^1)} &\leq \sum_{|k| \geq \nu} |k|^\ell |\hat{f}(k)| \\ &\leq C_{m\ell} \|f\|_{C^{\ell+m+1}(S^1)} \cdot n^{-m}, \end{aligned} \tag{3.18}$$

the last inequality following from (1.49). As the reader can verify, use of the proof of Proposition 1.3 can lead to a sharper estimate. As for an estimate of the contribution of Q_n to the discretization error, from (3.14)-(3.16) we easily obtain

$$\begin{aligned} \|Q_n f\|_{C^\ell(S^1)} &\leq (n/2)^\ell \sum_{|k| \geq \nu} |\hat{f}(k)| \\ &\leq C_{\ell m} \|f\|_{C^{\ell+m+1}(S^1)} \cdot n^{-m}. \end{aligned} \tag{3.19}$$

We reiterate that sharper estimates are possible.

We know that solutions to a number of partial differential equations are given by Fourier multipliers on $L^2(S^1)$, of the form

$$F(D)u(\theta) = \sum_{\ell=-\infty}^{\infty} F(\ell)\hat{u}(\ell)e^{i\ell\theta}. \tag{3.20}$$

We want to compare such an operator with its discretized version on $L^2(\Gamma_n)$:

$$F(D_n)\Big[\sum_{\ell\in\mathbb{Z}_n} g(\ell)\omega^{j\ell}\Big] = \sum_{\ell=-\nu}^{\nu-1} F(\ell)g(\ell)\omega^{j\ell}. \tag{3.21}$$

In fact, a simple calculation yields

$$E_nF(D_n)R_nu(\theta) = \sum_{\ell=-\nu}^{\nu-1} F(\ell)u^{\#}(\ell)e^{i\ell\theta} \tag{3.22}$$

and hence

$$E_nF(D_n)R_nu = P_nF(D)u + \Psi_nu \tag{3.23}$$

where

$$\Psi_nu(\theta) = \sum_{\ell=-\nu}^{\nu-1} F(\ell)\Big[\sum_{j\in\mathbb{Z}\setminus 0} \hat{u}(\ell+jn)\Big]e^{i\ell\theta}. \tag{3.24}$$

This implies the estimate

$$\|\Psi_nu\|_{L^\infty} \le \Big[\sup_{|\ell|\le\nu} |F(\ell)|\Big] \sum_{|k|\ge\nu} |\hat{u}(k)|. \tag{3.25}$$

Also, as in (3.18), we have, for $m \ge 1$,

$$\|\Psi_nu\|_{C^\ell(S^1)} \le C_{\ell m}\Big[\sup_{|\ell|\le\nu} |F(\ell)|\Big]\ \|u\|_{C^{\ell+m+1}(S^1)} \cdot n^{-m}. \tag{3.26}$$

The significance of these statements is that, for u smooth, and n large, the discretized $F(D_n)$ provides a very accurate approximation to $F(D)$. This is of practical importance for a number of numerical problems.

Note the distinction between D_n and the centered difference operator Δ_n, defined by

$$(\Delta_nf)(\omega^j) = \frac{n}{4\pi i}\big[f(\omega^{j+1}) - f(\omega^{j-1})\big].$$

We have, in place of (3.21),

$$F(\Delta_n)\Big[\sum_{\ell\in\mathbb{Z}_n} g(\ell)\omega^{j\ell}\Big] = \sum_{\ell=-\nu}^{\nu-1} F\Big(\frac{n}{2\pi}\sin\big(\frac{2\pi\ell}{n}\big)\Big)g(\ell)\omega^{j\ell}, \tag{3.27}$$

so, for $g^b \in L^2(\Gamma_n)$ given by (3.2),

$$F(\Delta_n)g^b(\omega^j) - F(D_n)g^b(\omega^j) = \sum_{\ell=-\nu}^{\nu-1}\Big[F\Big(\frac{n}{2\pi}\sin\big(\frac{2\pi\ell}{n}\big)\Big) - F(\ell)\Big]g(\ell)\omega^{j\ell}. \tag{3.28}$$

This identity leads to a variety of estimates, of which the following is a simple example. If $|F'(\lambda)| \le K$ for $-\nu \le \lambda \le \nu$, then

$$\|F(\Delta_n)u - F(D_n)u\|_{L^\infty} \le (2/3)\pi^2 K\Big[\sum_{\ell=-\nu}^{\nu-1} |\ell|^3\ |u^\#(\ell)|\Big]\cdot n^{-2}, \tag{3.29}$$

since, for $-\pi \le x \le \pi$, $|\sin x - x| \le (1/6)|x|^3$. The basic content of this is that $F(\Delta_n)$ furnishes a <u>second order</u> accurate approximation to $F(D)$ (as $n\to\infty$). This is an improvement over the first order accuracy one would get by using a one sided difference operator, e.g.,

$$(\Delta_n^+ f)(\omega^j) = \frac{n}{2\pi i}\big[f(\omega^{j+1}) - f(\omega^j)\big],$$

but not as good as the 'infinite order accuracy' one gets for $F(D_n)$, as a consequence of (3.23)-(3.26).

Similarly to the case of functions on S^1, we have, for $u \in L^2(\Gamma_n)$,

$$\begin{aligned} F(D_n)u(\omega^j) &= (k_F * u)(\omega^j) \\ &= \frac{1}{n}\sum_{\ell\in\mathbb{Z}_n} k_F(\omega^{j-\ell})u(\omega^\ell) \end{aligned} \tag{3.30}$$

where

$$k_F(\omega^j) = \sum_{\ell=-\nu}^{\nu-1} F(\ell)\omega^{j\ell}. \tag{3.31}$$

For example, with $F(\lambda) = e^{-y|\lambda|}$, we get the discrete version of the Poisson kernel:

$$k_F(\omega^j) = p_y(\omega^j) = \sum_{\ell=-\nu}^{\nu-1} e^{-y|\ell|}\omega^{j\ell} \tag{3.32}$$

which we can write as a sum of two finite geometrical series to get

$$p_y(\omega^j) = \frac{1 - r^2 - 2r^{\nu+1}(-1)^j \cos(2\pi j/n)}{1 + r^2 - 2r\cos(2\pi j/n)} + r^\nu \omega^{-j\nu} \tag{3.33}$$

with $r = e^{-y}$, and, as usual, $\omega = e^{2\pi i/n}$, $n = 2\nu$. Compare (1.30). The reader can produce a similar formula for n odd.

As in the case of S^1, the sum (3.31) for the (discretized) heat kernel, with $F(\ell) = e^{-t\ell^2}$, cannot generally be simplified to an expression whose size is independent of n. However, when t is an imaginary integer, such an evaluation can be performed. Such expressions are called Gauss sums, and their evaluation is regarded as one of the pearls of early nineteenth century mathematics. We present one such result here.

Proposition 3.5. *For any $n \geq 1$, even or odd,*

$$\sum_{k=0}^{n-1} e^{2\pi i k^2/n} e^{2\pi i \ell k/n} = \frac{1}{2}(1+i)e^{-\pi i \ell^2/2n}\left[1 + (-1)^\ell i^{-n}\right]\, n^{\frac{1}{2}}. \tag{3.34}$$

Proof. The sum on the left is $n \cdot f^{\#}(-\ell)$, where $f \in C(S^1)$ is given by

$$f(\theta) = e^{in\theta^2/2\pi}, \quad 0 \leq \theta \leq 2\pi.$$

Note that f is Lipschitz on S^1, with a simple jump in its derivative, so $\hat{f}(k) = O(|k|^{-2})$. Hence Proposition 3.3 applies, and (3.7) yields

$$f^{\#}(-\ell) = \sum_{j=-\infty}^{\infty} \int_0^1 e^{2\pi i n[y^2 + (j+\ell/n)y]}\, dy. \tag{3.35}$$

To evaluate his, we use the 'Gaussian integral,' (convergent though not absolutely convergent):

$$\int_{-\infty}^{\infty} e^{2\pi i n y^2}\, dy = n^{-\frac{1}{2}}\gamma, \quad \gamma = \frac{1}{2}(1+i), \tag{3.36}$$

obtained from (2.20) by a change of variable and analytic continuation. We will break up the real line as a union of intervals $\bigcup_k [k+a, k+a+1]$, in two different ways, and then evaluate (3.35). Note that

$$\int_{k+a}^{k+a+1} e^{2\pi i n y^2} dy = \int_0^1 e^{2\pi i n[y^2 + 2(k+a)y]} dy \cdot e^{2\pi i n(k+a)^2}. \tag{3.37}$$

If we pick $a = \ell/2n$, then $2(k+a) = 2k + \ell/n$, and as k runs over $\mathbb{Z}$, we get those integrands in (3.35) for which j is even. If we pick $a = -\frac{1}{2} + \ell/2n$, then $2(k+a) = 2k - 1 + \ell/n$. Furthermore, we have

$$e^{2\pi i n(k+a)^2} = e^{\pi i \ell^2/2n} \text{ and } e^{\pi i(\ell-n)^2/2n}, \tag{3.38}$$

respectively, for these two choices of a. Thus the sum in (3.35) is equal to $n^{-\frac{1}{2}}\gamma$ times $e^{-\pi i \ell^2/2n} + e^{-\pi i(\ell-n)^2/2n}$, which gives the desired formula (3.34).

The basic case of this sum is the $\ell = 0$ case:

$$\sum_{k=0}^{n-1} e^{2\pi i k^2/n} = \frac{1}{2}(1+i)(1+i^{-n})n^{\frac{1}{2}} = \sigma_n \cdot n^{\frac{1}{2}}, \tag{3.39}$$

where σ_n is periodic of period 4 in n, with

$$\sigma_0 = 1+i,\ \sigma_1 = 1,\ \sigma_2 = 0,\ \sigma_3 = i. \tag{3.40}$$

This result, particularly when $n = p$ is a prime, is used as a tool to obtain fascinating number theoretical results. For more on this, see the exercises and references given there.

Exercises.

1. Generalize the Gauss sum identity (3.34) to

$$\sum_{k=0}^{n-1} e^{2\pi i k^2 m/n} e^{2\pi i \ell k/n} = \frac{1+i}{2}\Big(\frac{n}{m}\Big)^{\frac{1}{2}} e^{-\pi i \ell^2/2mn} \sum_{\nu=0}^{2m-1} e^{-\pi i n\nu^2/2m} e^{-\pi i \nu\ell/m}. \tag{3.41}$$

<u>Hint</u>. The left side is $n \cdot f^{\#}(-\ell)$ with

$$f(\theta) = e^{inm\theta^2/2\pi}, \quad 0 \le \theta \le 2\pi.$$

For this, one has a formula like (3.35):

$$f^{\#}(-\ell) = \sum_{j=-\infty}^{\infty} \int_0^1 e^{2\pi i nm[y^2+(1/m)(j+\ell/n)y]}\, dy.$$

Write $j = 2m\mu + \nu$, so $\sum_{j=-\infty}^{\infty} = \sum_{\nu=0}^{2m-1} \sum_{j=\nu \bmod 2m}$. For fixed ν, the sum becomes a multiple of the Gaussian integral (3.36), with n replaced by nm, and the formula (3.41) arises. Note the $\ell = 0$ case of this:

$$\sum_{k=0}^{n-1} e^{2\pi i k^2 m/n} = \frac{1+i}{2}\Big(\frac{n}{m}\Big)^{\frac{1}{2}} \sum_{\nu=0}^{2m-1} e^{-\pi i n\nu^2/2m}.$$

2. Let $\#(\ell,n)$ denote the number of solutions $k \in \mathbb{Z}_n$ to

$$\ell = k^2 \pmod{n}.$$

Show that, with $\omega = e^{2\pi i/n}$,

$$\sum_{k=0}^{n-1} \omega^{jk^2} = \sum_{\ell=0}^{n-1} \#(\ell,n)\omega^{j\ell}.$$

3. Show that, more generally,

$$\Big(\sum_{k=0}^{n-1} \omega^{jk^2}\Big)^{\nu} = \sum_{\ell=0}^{n-1} \#(\ell,n;\nu)\ \omega^{j\ell}$$

where $\#(\ell, n; \nu)$ denotes the number of solutions $(k_1, \ldots, k_\nu) \in (\mathbb{Z}_n)^\nu$ to

$$\ell = k_1^2 + \cdots + k_\nu^2 \pmod{n}.$$

4. Let p be a prime. The Legendre symbol $(\ell|p)$ is defined to be $+1$ if $\ell = k^2$ mod p for some k, and $\ell \neq 0$, 0 if $\ell = 0$, and -1 otherwise. If p is an odd prime, $\#(\ell, p) = (\ell|p) + 1$. The Legendre symbol has the useful multiplicative property: $(\ell_1\ell_2|p) = (\ell_1|p)(\ell_2|p)$. Check this. Show that, with $\omega = e^{2\pi i/p}$, if p is an odd prime,

$$\sum_{k=0}^{p-1} \omega^{k^2} = \sum_{\ell=0}^{p-1} (\ell|p)\omega^\ell$$

and, more generally,

$$\sum_{k=0}^{p-1} \omega^{jk^2} = \sum_{\ell=0}^{p-1} (\ell|p)\omega^{j\ell} + p\delta_{j0}$$

where $\delta_{j0} = 1$ if $j = 0 \pmod{p}$, 0 otherwise.
Hint. Use exercise 2.

5. Denoting $\sum\limits_{k=0}^{p-1} \omega^{k^2}$ by G_p, p an odd prime, show that

$$\sum_{k=0}^{p-1} \omega^{jk^2} = (j|p) \cdot G_p + p \cdot \delta_{j0}.$$

Hint. If $1 \leq j \leq p-1$, use $\sum\limits_{\ell=0}^{p-1} (\ell|p)\omega^\ell = \sum\limits_{\ell=0}^{p-1} (j\ell|p)\omega^{j\ell}$ and $(j\ell|p) = (j|p)(\ell|p)$.

Denote by $S(m, n)$ the Gauss sum

$$S(m, n) = \sum_{k=0}^{n-1} e^{2\pi i k^2 m/n}.$$

Then the content of problem 5 is that $S(j, p) = (j|p)S(1, p)$ for $1 \leq j \leq p-1$, when p is an odd prime.

6. Assume p and q are distinct odd primes. Show that

$$S(1, pq) = S(q, p)S(p, q).$$

Hint. To re-sum $\sum_{k=0}^{pq-1} e^{2\pi i k^2/pq}$, use the fact that, as μ runs over $\{0,1,\dots,p-1\}$ and ν runs over $\{0,1,\dots,q-1\}$, then $k=\mu q+\nu p$ runs once over $\mathbb{Z}$ mod pq.

7. From problems 5-6, it follows that, when p and q are distinct odd primes,

$$(p|q)(q|p)=\frac{S(1,pq)}{S(1,p)S(1,q)}.$$

Use the evaluation (3.39) of $S(1,n)$ to deduce the quadratic reciprocity law:

$$(p|q)(q|p)=(-1)^{(p-1)(q-1)/4}.$$

This law, together with the complementary results

$$(-1|p)=(-1)^{(p-1)/2},\quad (2|p)=(-1)^{(p^2-1)/8},$$

allow for an effective computation of $(\ell|p)$, as one application, but the significance of quadratic reciprocity goes beyond this. It and other implications of Gauss sums are absolutely fundamental in number theory. For material on this, see [Hua], [Land], [Rad].

4. The fast Fourier transform

In the last section we discussed some properties of the discrete Fourier transform

$$f^{\#}(\ell) = \frac{1}{n} \sum_{\omega^j \in \Gamma_n} f(\omega^j) \omega^{-j\ell}, \tag{4.1}$$

where $\ell \in \mathbb{Z}_n = \mathbb{Z}/(n)$ and Γ_n is the multiplicative group of unit complex numbers generated by $\omega = e^{2\pi i/n}$. We now turn to a discussion of efficient numerical computation of the discrete Fourier transform. Note that, for any fixed ℓ, computing the right side of (4.1) involves $n-1$ additions and n multiplications of complex numbers, plus n integer products $j\ell = m$ and looking up ω^m and $f(\omega^j)$. If the computations for varying ℓ are done independently, the total effort to compute $f^{\#}$ involves n^2 multiplications and $n(n-1)$ additions of complex numbers, plus some further chores. The Fast Fourier Transform (denoted FFT) is a method for computing $f^{\#}$ in $Cn(\log n)$ steps, in case n is a power of 2.

The possibility of doing this arises from observing redundancies in the calculation of the Fourier coefficients $f^{\#}(\ell)$. Let us illustrate this in the case of Γ_4. We can write

$$\begin{aligned} 4f^{\#}(0) &= \big[f(1) + f(i^2)\big] + \big[f(i) + f(i^3)\big] \\ 4f^{\#}(2) &= \big[f(1) + f(i^2)\big] - \big[f(i) + f(i^3)\big] \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} 4f^{\#}(1) &= \big[f(1) - f(i^2)\big] - i\big[f(i) - f(i^3)\big] \\ 4f^{\#}(3) &= \big[f(1) - f(i^2)\big] + i\big[f(i) - f(i^3)\big] \end{aligned} \tag{4.3}$$

Note that each term in square brackets appears twice. Note also that (4.2) gives the Fourier coefficients of a function on Γ_2; namely, if

$${}^0f(1) = f(1) + f(-1), \quad {}^0f(-1) = f(i) + f(i^3), \tag{4.4}$$

then

$$2f^{\#}(2\ell) = {}^0f^{\#}(\ell) \text{ for } \ell = 0 \text{ or } 1. \tag{4.5}$$

Similarly, if we set

$${}^1f(1) = f(1) - f(-1), \quad {}^1f(-1) = -i\big[f(i) - f(i^3)\big] \tag{4.6}$$

then

$$2f^{\#}(2\ell+1) = {}^1f^{\#}(\ell) \text{ for } \ell = 0 \text{ or } 1. \tag{4.7}$$

This phenomenon is a special case of a more general result which leads to a fast inductive procedure for evaluating the Fourier transform $f^{\#}$.

Suppose $n = 2^k$; let us use the notation $G_k = \Gamma_n$. Note that G_{k-1} is a subgroup of G_k. Furthermore, there is a homomorphism of G_k onto G_{k-1}, given by $\omega^j \mapsto \omega^{2j}$. Given $f : G_k \to \mathbb{C}$, define the following functions ${}^0f$ and ${}^1f$ on G_{k-1}, with $\omega_1 = \omega^2$, generating G_{k-1} :

$${}^0f(\omega_1^j) = f(\omega^j) + f(\omega^{j+\frac{1}{2}n}), \tag{4.8}$$

$${}^1f(\omega_1^j) = \overline{\omega}^j \big[f(\omega^j) - f(\omega^{j+\frac{1}{2}n})\big]. \tag{4.9}$$

Note that the factor $\overline{\omega}^j$ in (12.9) makes ${}^1f(\omega_1^j)$ well defined for $j \in \mathbb{Z}_{n/2}$, i.e., the right side of (12.9) is unchanged if j is replaced by $j + \frac{1}{2}n$. Then ${}^0f^{\#}$ and ${}^1f^{\#}$, the discrete Fourier transforms of the functions ${}^0f$ and ${}^1f$, are functions on $\mathbb{Z}_{n/2} = \mathbb{Z}/(2^{k-1})$.

Proposition 4.1. *We have the following identities relating the Fourier transforms of* ${}^0f$, ${}^1f$, *and* f :

$$2f^{\#}(2\ell) = {}^0f^{\#}(\ell), \tag{4.10}$$

and

$$2f^{\#}(2\ell+1) = {}^1f^{\#}(\ell), \tag{4.11}$$

for $\ell \in \{0, 1, \dots, \frac{1}{2}n - 1\}$.

Proof. Recall that we set $\omega_1 = \omega^2$. Since $\omega^n = 1$ and $\omega_1^{n/2} = 1$, we have

$$\begin{aligned} nf^{\#}(2\ell) &= \sum_{\omega^j \in G_k} f(\omega^j)\overline{\omega}^{2j\ell} \\ &= \sum_{\omega_1^j = \omega^{2j} \in G_{k-1}} \big[f(\omega^j) + f(\omega^{j+\frac{1}{2}n})\big]\overline{\omega}_1^{j\ell}, \end{aligned} \tag{4.12}$$

proving (4.10), and, since $\omega^{n/2} = -1$,

$$\begin{aligned} nf^{\#}(2\ell+1) &= \sum_{\omega^j \in G_k} f(\omega^j)\overline{\omega}^j \, \overline{\omega}^{2j\ell} \\ &= \sum_{\omega_1^j = \omega^{2j} \in G_{k-1}} \overline{\omega}^j \big[f(\omega^j) - f(\omega^{j+\frac{1}{2}n})\big]\overline{\omega}_1^{j\ell}, \end{aligned} \tag{4.13}$$

proving (4.11).

Thus the problem of computing $f^{\#}$, given $f \in L^2(G_k)$, is transformed, after $\frac{1}{2}n$ multiplications and n additions of complex numbers in (4.8)-(4.9), to the problem of computing the Fourier transforms of <u>two</u> functions on G_{k-1}. After $\frac{1}{4}n$ new multiplications and $\frac{1}{2}n$ new additions for each of these functions ${}^0f$ and ${}^1f$, i.e., after an additional total of $\frac{1}{2}n$ new multiplications and n additions, this is reduced to the problem of computing <u>four</u> Fourier transforms of functions on G_{k-2}. After k iterations, we obtain 2^k functions on $G_0 = \{1\}$, which precisely give the Fourier coefficients of f. Doing this hence takes $kn = (\log_2 n)n$ additions and $\frac{1}{2}kn = \frac{1}{2}(\log_2 n)n$ multplications of complex numbers, plus a comparable number of integer operations and fetching from memory values of given or previously computed functions.

To describe an explicit implementation of Proposition 4.1 for a computation of $f^{\#}$, let us identify an element $\ell \in \mathbb{Z}_n$ $(n = 2^k)$ with a k-tuple $L = (L_{k-1}, \dots, L_1, L_0)$ of elements of $\{0, 1\}$ giving the binary expansion of the integer in $\{0, \dots, n-1\}$ representing ℓ, i.e., $L_0 + L_1 \cdot 2 + \cdots + L_{k-1} \cdot 2^{k-1} = \ell \bmod n$. To be a little fussy, we use the notation

$$f^{\#}(\ell) = f^{\#\#}(L). \tag{4.14}$$

Then the formulas (4.10)-(4.11) state that

$$2f^{\#\#}(L_{k-1}, \dots, L_1, 0) = {}^0f^{\#\#}(L_{k-1}, \dots, L_1) \tag{4.15}$$

and

$$2f^{\#\#}(L_{k-1}, \dots, L_1, 1) = {}^1f^{\#\#}(L_{k-1}, \dots, L_1). \tag{4.16}$$

The inductive procedure described above gives, from ${}^0f$ and ${}^1f$ defined on G_{k-1}, the functions

$${}^{00}f = {}^0({}^0f), \quad {}^{10}f = {}^1({}^0f), \quad {}^{01}f = {}^0({}^1f), \quad {}^{11}f = {}^1({}^1f) \tag{4.17}$$

defined on G_{k-2}, and so forth, and we see from (4.15)-(4.16) that

$$f^{\#}(\ell) = (1/n)\ {}^Lf, \tag{4.18}$$

where ${}^Lf = {}^Lf(1)$ is defined on $G_0 = \{1\}$. From (4.8)-(4.9) we have the following inductive formula for ${}^{L_{m+1}L_m\cdots L_1}f$ on G_{k-m-1} :

$$\begin{aligned} {}^{0L_m\cdots L_1}f(\omega_{m+1}^j) &= {}^{L_m\cdots L_1}f(\omega_m^j) + {}^{L_m\cdots L_1}f(\omega_m^{j+2^{k-m-1}}), \\ {}^{1L_m\cdots L_1}f(\omega_{m+1}^j) &= \overline{\omega}_m^j\big[{}^{L_m\cdots L_1}f(\omega_m^j) - {}^{L_m\cdots L_1}f(\omega_m^{j+2^{k-m-1}})\big], \end{aligned} \tag{4.19}$$

where ω_m is the generator of G_{k-m}, defined by $\omega_0 = \omega = e^{2\pi i/n}$ $(n = 2^k)$, $\omega_{m+1} = \omega_m^2$, i.e., $\omega_m = \omega^{2^m}$.

When doing computations, particularly in a higher level language, it may be easier to work with integers ℓ than with m-tuples $(L_1, \ldots, L_1)$. Therefore, let us set

$$ {}^{L_m \cdots L_1} f(\omega_m^j) = F_m(2^m \cdot j + \ell), \tag{4.20} $$

where

$$ \ell = L_1 + L_2 \cdot 2 + \cdots + L_m \cdot 2^{m-1} \in \{0, 1, \ldots, 2^m - 1\} $$

and

$$ j \in \{0, 1, \ldots, 2^{k-m} - 1\}. $$

Note that this precisely defines F_m on $\{0, 1, \ldots, 2^k - 1\}$. For $m = 0$ we have

$$ F_0(j) = f(\omega^j), \quad 0 \le j \le 2^k - 1. \tag{4.21} $$

The iterative formulas (4.19) give

$$ \begin{aligned} &F_{m+1}(2^{m+1}j + \ell) = F_m(2^m j + \ell) + F_m(2^m j + 2^{k-1} + \ell), \\ &F_{m+1}(2^m + 2^{m+1}j + \ell) = \overline{\omega}_m^j \big[F_m(2^m j + \ell) - F_m(2^m j + 2^{k-1} + \ell)\big], \end{aligned} \tag{4.22} $$

for $0 \le j \le 2^{k-m-1} - 1,\ 0 \le \ell \le 2^m - 1$. It is easy to write a computer program to implement such an iteration. The formula (4.18) for the Fourier transform of f becomes

$$ f^{\#}(\ell) = (1/n) F_k(\ell), \quad 0 \le \ell \le 2^k - 1. \tag{4.23} $$

While (4.21)-(4.23) provides an easily implementable FFT algorithm, it is not necessarily the best. One drawback is the following. In passing from F_m to F_{m+1} via (4.22), you need two different arrays of n complex numbers. A variant of (12.19), where ${}^{L_{m+1} L_m \cdots L_1} f$ is replaced by $f^{L_1 \cdots L_m L_{m+1}}$, leads to an iterative procedure where a transformation of the type (4.19) is performed 'in place,' and only one such array needs to be used. If memory is expensive and one needs to make best use of it, this savings can be important. At the end of such an iteration, one needs to perform a 'bit reversal' to produce $f^{\#}$. Details, including sample programs, can be found in [Pnr].

On any given computer, a number of factors would influence the choice of the best FFT algorithm. These include such things as relative speed of memory access and floating point performance, efficiency of computing trigonometric functions (e.g., whether this is implemented in hardware), degree of accuracy required, and other factors. Also special features, such as computing the Fourier transform of a real valued function, or of a function whose Fourier transform is known to be real valued, would affect specific computer programs designed for maximum efficiency. Working out how best to implement FFTs on various computers presents many interesting problems.

Exercises.

1. Using the FFT, write a computer program to solve numerically the initial value problem for the heat equation $\partial u/\partial t - u_{xx} = 0$ on $\mathbb{R}^+ \times S^1$.

2. Consider multidimensional generalizations of the discrete Fourier transform, and in particular the FFT. What size 3-dimensional FFT could be handled by a computer with 4 megabytes of RAM? With 256 megs?

3. Generalize the FFT algorithm to a cyclic group Γ_n with $n = 3^k$. To $n = p_1 \cdots p_k$ where p_j are 'small' primes.

5. An FFT procedure in Pascal

The procedure **fft** below implements the FFT algorithm described in §4, via the recursive formula (4.22). A related procedure **invfft** calculates the inverse DFT, defined by (3.2). A programmer using these procedures needs to declare a type,

```
fctn=array[0..255] of double;
```

A pair of elements of type ‘fctn’ represent the real and imaginary parts of a complex-valued function on Γ_n.

```
(* fft.pas *)

var
   romega,iomega:array[0..255] of double;
   twopow:array[0..8] of integer;
   rfx,ifx:array[0..1,0..255] of double;


procedure initfft;
var
   jj:integer;
   pp:double;
begin
   pp:=pi/128;
   for jj:=0 to 255 do romega[jj]:=cos(jj*pp);
   for jj:=0 to 255 do iomega[jj]:=-sin(jj*pp);
   twopow[0]:=1;
   for jj:=1 to 8 do twopow[jj]:=2*twopow[jj-1];
end;


procedure fft(rff:fctn;iff:fctn;var rgg:fctn; var igg:fctn);
var
   mm,ma,mb,jj,el,tm,tmj,tmm:integer;
   tmjl,ttmjl,tmjl8,ttmjlt:integer;
   rxx,ixx,rom,iom:double;
begin
   for jj:=0 to 255 do rfx[0,jj]:=rff[jj];
   for jj:=0 to 255 do ifx[0,jj]:=iff[jj];
   for mm:=0 to 7 do
```

```
   begin
      ma:=mm mod 2;
      mb:=(ma+1) mod 2;
      tm:=twopow[mm];
      for jj:=0 to twopow[7-mm]-1 do
      begin
         tmj:=tm*jj;
         tmm:=tmj mod 256;
         rom:=romega[tmm];
         iom:=iomega[tmm];
         for el:=0 to tm-1 do
         begin
            tmjl:=tmj+el;      (* (2^m)j+l *)
            ttmjl:=tmj+tmjl;      (* (2^(m+1)j+l *)
            tmjl8:=tmjl+128;      (* (2^m)j+l+128 *)
            ttmjlt:=ttmjl+tm;      (* (2^m)+(2^(m+1))j+l *)
            rfx[mb,ttmjl]:=rfx[ma,tmjl]+rfx[ma,tmjl8];
            ifx[mb,ttmjl]:=ifx[ma,tmjl]+ifx[ma,tmjl8];
            rxx:=rfx[ma,tmjl]-rfx[ma,tmjl8];
            ixx:=ifx[ma,tmjl]-ifx[ma,tmjl8];
            rfx[mb,ttmjlt]:=rom*rxx-iom*ixx;
            ifx[mb,ttmjlt]:=rom*ixx+iom*rxx;
         end;
      end;
   end;
   for jj:=0 to 255 do rgg[jj]:=rfx[0,jj]/256;
   for jj:=0 to 255 do igg[jj]:=ifx[0,jj]/256;
end;


procedure invfft(rff:fctn;iff:fctn;var rgg:fctn; var igg:fctn);
var
   mm,ma,mb,jj,el,tm,tmj,tmm:integer;
   tmjl,ttmjl,tmjl8,ttmjlt:integer;
   rxx,ixx,rom,iom:double;
begin
   for jj:=0 to 255 do rfx[0,jj]:=rff[jj];
   for jj:=0 to 255 do ifx[0,jj]:=iff[jj];
   for mm:=0 to 7 do
   begin
      ma:=mm mod 2;
      mb:=(ma+1) mod 2;
      tm:=twopow[mm];
      for jj:=0 to twopow[7-mm]-1 do
```

```
      begin
         tmj:=tm*jj;
         tmm:=tmj mod 256;
         rom:=romega[tmm];
         iom:=iomega[tmm];
         for el:=0 to tm-1 do
         begin
            tmjl:=tmj+el;      (* (2^m)j+l *)
            ttmjl:=tmj+tmjl;      (* (2^(m+1)j+l *)
            tmjl8:=tmjl+128;      (* (2^m)j+l+128 *)
            ttmjlt:=ttmjl+tm;      (* (2^m)+(2^(m+1))j+l *)
            rfx[mb,ttmjl]:=rfx[ma,tmjl]+rfx[ma,tmjl8];
            ifx[mb,ttmjl]:=ifx[ma,tmjl]+ifx[ma,tmjl8];
            rxx:=rfx[ma,tmjl]-rfx[ma,tmjl8];
            ixx:=ifx[ma,tmjl]-ifx[ma,tmjl8];
            rfx[mb,ttmjlt]:=rom*rxx+iom*ixx;
            ifx[mb,ttmjlt]:=rom*ixx-iom*rxx;
            (* sign change on iom *)
         end;
      end;
   end;
   for jj:=0 to 255 do rgg[jj]:=rfx[0,jj];
   for jj:=0 to 255 do igg[jj]:=ifx[0,jj];
   (* no division by 256, unlike direct fft *)
end;
```

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