Differential Forms and Applications to Complex Analysis, Harmonic Functions, and Degree Theory

Michael Taylor

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Introduction

Here we introduce differential forms on surfaces, establish basic properties, and explore some applications. These notes are excerpted from my text *Introduction to Analysis in Several Variables*, available online at http://www.unc.edu/math/Faculty/met/math521.html.

The concept of a differential form in introduced in §6. It is seen that a k-form can be integrated over a k-dimensional surface, endowed with an extra piece of structure called an "orientation." In §7, we define exterior products of forms, and interior products of forms with vector fields, and we define the exterior derivative of a k-form, which is a $(k+1)$-form. Section 8 is devoted to a general Stokes formula, an important integral identity on a surface with boundary that contains as special cases the classical identities of Green, Gauss, and Stokes. These special cases are discussed in more detail in §9.

In §10 we use Green's theorem to derive fundamental properties of holomorphic functions of a complex variable. In this section we also produce some results on the closely related study of harmonic functions. One result is Liouville's theorem, stating that a bounded harmonic function on all of $\mathbb{R}^n$ must be constant. When specialized to holomorphic functions on $\mathbb{C} = \mathbb{R}^2$, this yields a proof of the fundamental theorem of algebra.

Sections 11–12 apply the Stokes formula for differential forms to obtain some basic results of degree theory. This is done in §12, after the notion of smooth homotopy is introduced in §11. As a preliminary, we obtain the Brouwer no-retraction theorem and fixed point theorem, and a result that smooth vector fields tangent to an even dimensional sphere must vanish somewhere. It is noted that these results follow from the existence of top-dimensional forms that are not exact. We obtain a general sufficient condition for exactness, and this leads to a definition of the degree of a smooth map between smooth, compact, oriented surfaces of the same dimension. Applications are given to a smooth version of the Jordan-Brouwer separation theorem, to the index of a vector field, and to the Euler characteristic of a smooth, compact n-dimensional surface.

Appendix C presents a proof of the change of variable formula for multiple integrals, following an idea presented by Lax in [L], but using the calculus of differential forms to carry out the details.

6. Differential forms

It is very desirable to be able to make constructions that depend as little as possible on a particular choice of coordinate system. The calculus of differential forms, whose study we now take up, is one convenient set of tools for this purpose.

We start with the notion of a 1-form. It is an object that gets integrated over a curve; formally, a 1-form on $\Omega \subset \mathbb{R}^n$ is written

$$\alpha = \sum_j a_j(x)\, dx_j. \tag{6.1}$$

If $\gamma : [a,b] \to \Omega$ is a smooth curve, we set

$$\int_\gamma \alpha = \int_a^b \sum a_j\big(\gamma(t)\big)\gamma_j'(t)\, dt. \tag{6.2}$$

In other words,

$$\int_\gamma \alpha = \int_I \gamma^*\alpha \tag{6.3}$$

where $I = [a,b]$ and

$$\gamma^*\alpha = \sum_j a_j(\gamma(t))\gamma_j'(t)\, dt$$

is the *pull-back* of α under the map γ. More generally, if $F : \mathcal{O} \to \Omega$ is a smooth map ($\mathcal{O} \subset \mathbb{R}^m$ open), the pull-back $F^*\alpha$ is a 1-form on $\mathcal{O}$ defined by

$$F^*\alpha = \sum_{j,k} a_j(F(y))\, \frac{\partial F_j}{\partial y_k}\, dy_k. \tag{6.4}$$

The usual change of variable for integrals gives

$$\int_\gamma \alpha = \int_\sigma F^*\alpha \tag{6.5}$$

if γ is the curve $F \circ \sigma$.

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism, and

$$X = \sum b^j(x) \frac{\partial}{\partial x_j} \tag{6.6}$$

is a vector field on Ω, recall from (3.40) that we have the vector field on $\mathcal{O}$:

$$F_\# X(y) = \big(DF^{-1}(p)\big) X(p), \quad p = F(y). \tag{6.7}$$

If we define a pairing between 1-forms and vector fields on Ω by

$$\langle X, \alpha \rangle = \sum_j b^j(x) a_j(x) = b \cdot a, \tag{6.8}$$

a simple calculation gives

$$\langle F_\# X, F^* \alpha \rangle = \langle X, \alpha \rangle \circ F. \tag{6.9}$$

Thus, a 1-form on Ω is characterized at each point $p \in \Omega$ as a linear transformation of the space of *vectors* at p to $\mathbb{R}$.

More generally, we can regard a k-form α on Ω as a k-multilinear map on vector fields:

$$\alpha(X_1, \dots, X_k) \in C^\infty(\Omega); \tag{6.10}$$

we impose the further condition of anti-symmetry when $k \geq 2$:

$$\alpha(X_1, \dots, X_j, \dots, X_\ell, \dots, X_k) = -\alpha(X_1, \dots, X_\ell, \dots, X_j, \dots, X_k). \tag{6.11}$$

Let us note that a 0-form is simply a function.

There is a special notation we use for k-forms. If $1 \leq j_1 < \cdots < j_k \leq n,\ j = (j_1, \dots, j_k)$, we set

$$\alpha = \frac{1}{k!} \sum_j a_j(x)\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} \tag{6.12}$$

where

$$a_j(x) = \alpha(D_{j_1}, \dots, D_{j_k}), \quad D_j = \partial/\partial x_j. \tag{6.13}$$

More generally, we assign meaning to (6.12) summed over all k-indices $(j_1, \dots, j_k)$, where we identify

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = (\operatorname{sgn} \sigma)\ dx_{j_{\sigma(1)}} \wedge \cdots \wedge dx_{j_{\sigma(k)}}, \tag{6.14}$$

σ being a permutation of $\{1,\dots,k\}$. If any $j_m = j_\ell$ $(m \neq \ell)$, then (6.14) vanishes. A common notation for the statement that α is a k-form on Ω is

$$\alpha \in \Lambda^k(\Omega). \tag{6.15}$$

In particular, we can write a 2-form β as

$$\beta = \frac{1}{2} \sum b_{jk}(x)\ dx_j \wedge dx_k \tag{6.16}$$

and pick coefficients satisfying $b_{jk}(x) = -b_{kj}(x)$. According to (6.12)–(6.13), if we set $U = \sum u_j(x)\partial/\partial x_j$ and $V = \sum v_j(x)\partial/\partial x_j$, then

$$\beta(U,V) = \sum b_{jk}(x) u^j(x) v^k(x). \tag{6.17}$$

If b_{jk} is not required to be antisymmetric, one gets $\beta(U,V) = (1/2)\sum(b_{jk} - b_{kj})u^j v^k$.

If $F : \mathcal{O} \to \Omega$ is a smooth map as above, we define the pull-back $F^*\alpha$ of a k-form α, given by (6.12), to be

$$F^*\alpha = \sum_j a_j\big(F(y)\big)(F^* dx_{j_1}) \wedge \cdots \wedge (F^* dx_{j_k}) \tag{6.18}$$

where

$$F^* dx_j = \sum_\ell \frac{\partial F_j}{\partial y_\ell}\, dy_\ell, \tag{6.19}$$

the algebraic computation in (6.18) being performed using the rule (6.14). Extending (6.9), if F is a diffeomorphism, we have

$$(F^*\alpha)(F_\# X_1, \dots, F_\# X_k) = \alpha(X_1, \dots, X_k) \circ F. \tag{6.20}$$

If $B = (b_{jk})$ is an $n \times n$ matrix, then, by (6.14),

$$\begin{aligned} \Big(\sum_k b_{1k} dx_k\Big) \wedge \Big(\sum_k b_{2k} dx_k\Big) &\wedge \cdots \wedge \Big(\sum_k b_{nk} dx_k\Big) \\ &= \sum_{k_1,\dots,k_n} b_{1k_1} \cdots b_{nk_n}\, dx_{k_1} \wedge \cdots \wedge dx_{k_n} \\ &= \Big(\sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}\Big) dx_1 \wedge \cdots \wedge dx_n \\ &= \big(\det B\big)\, dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{6.21}$$

Here S_n denotes the set of permutations of $\{1,\dots,n\}$, and the last identity is the formula for the determinant presented in (1.101). It follows that if $F : \mathcal{O} \to \Omega$ is a C^1 map between two domains of dimension n, and

$$\alpha = A(x)\, dx_1 \wedge \cdots \wedge dx_n \tag{6.22}$$

is an n-form on Ω, then

$$F^*\alpha = \det DF(y)\ A(F(y))\ dy_1 \wedge \cdots \wedge dy_n. \tag{6.23}$$

Comparison with the change of variable formula for multiple integrals suggests that one has an intrinsic definition of $\int_\Omega \alpha$ when α is an n-form on Ω, $n = \dim\,\Omega$. To implement this, we need to take into account that det $DF(y)$ rather than $|\det DF(y)|$ appears in (6.21). We say a smooth map $F : \mathcal{O} \to \Omega$ between two open subsets of $\mathbb{R}^n$ *preserves orientation* if $\det DF(y)$ is everywhere positive. The object called an "orientation" on Ω can be identified as an equivalence class of nowhere vanishing n-forms on Ω, two such forms being equivalent if one is a multiple of another by a positive function in $C^\infty(\Omega)$; the standard orientation on $\mathbb{R}^n$ is determined by $dx_1 \wedge \cdots \wedge dx_n$. If S is an n-dimensional surface in $\mathbb{R}^{n+k}$, an orientation on S can also be specified by a nowhere vanishing form $\omega \in \Lambda^n(S)$. If such a form exists, S is said to be orientable. The equivalence class of positive multiples $a(x)\omega$ is said to consist of "positive" forms. A smooth map $\psi : S \to M$ between oriented n-dimensional surfaces preserves orientation provided $\psi^*\sigma$ is positive on S whenever $\sigma \in \Lambda^n(M)$ is positive. If S is oriented, one can choose coordinate charts which are all orientation preserving. We mention that there exist surfaces that cannot be oriented, such as the famous "Möbius strip," and also the projective space $\mathbb{P}^2$, discussed in §5.

We define the integral of an n-form over an oriented n-dimensional surface as follows. First, if α is an n-form supported on an open set $\Omega \subset \mathbb{R}^n$, given by (6.22), then we set

$$\int_\Omega \alpha = \int_\Omega A(x)\, dV(x), \tag{6.24}$$

the right side defined as in §4. If $\mathcal{O}$ is also open in $\mathbb{R}^n$ and $F : \mathcal{O} \to \Omega$ is an orientation preserving diffeomorphism, we have

$$\int_{\mathcal{O}} F^*\alpha = \int_\Omega \alpha, \tag{6.25}$$

as a consequence of (6.23) and the change of variable formula (4.47). More generally, if S is an n-dimensional surface with an orientation, say the image of an open set $\mathcal{O} \subset \mathbb{R}^n$ by $\varphi : \mathcal{O} \to S$, carrying the natural orientation of $\mathcal{O}$, we can set

$$\int_S \alpha = \int_{\mathcal{O}} \varphi^*\alpha \tag{6.26}$$

for an n-form α on S. If it takes several coordinate patches to cover S, define $\int_S \alpha$ by writing α as a sum of forms, each supported on one patch.

We need to show that this definition of $\int_S \alpha$ is independent of the choice of coordinate system on S (as long as the orientation of S is respected). Thus, suppose $\varphi : \mathcal{O} \to U \subset S$

and $\psi : \Omega \to U \subset S$ are both coordinate patches, so that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is an orientation-preserving diffeomorphism, as in Fig. 5.1 of the last section. We need to check that, if α is an n-form on S, supported on U, then

$$\int\limits_{\mathcal{O}} \varphi^* \alpha = \int\limits_{\Omega} \psi^* \alpha. \tag{6.27}$$

To establish this, we first show that, for any form α of any degree,

$$\psi \circ F = \varphi \Longrightarrow \varphi^* \alpha = F^* \psi^* \alpha. \tag{6.28}$$

It suffices to check (6.28) for $\alpha = dx_j$. Then (6.19) gives $\psi^*\, dx_j = \sum (\partial \psi_j / \partial x_\ell)\, dx_\ell$, so

$$F^* \psi^*\, dx_j = \sum_{\ell, m} \frac{\partial F_\ell}{\partial x_m} \frac{\partial \psi_j}{\partial x_\ell}\, dx_m, \quad \varphi^*\, dx_j = \sum_m \frac{\partial \varphi_j}{\partial x_m}\, dx_m; \tag{6.29}$$

but the identity of these forms follows from the chain rule:

$$D\varphi = (D\psi)(DF) \Longrightarrow \frac{\partial \varphi_j}{\partial x_m} = \sum_\ell \frac{\partial \psi_j}{\partial x_\ell} \frac{\partial F_\ell}{\partial x_m}. \tag{6.30}$$

Now that we have (6.28), we see that the left side of (6.27) is equal to

$$\int\limits_{\mathcal{O}} F^*(\psi^* \alpha), \tag{6.31}$$

which is equal to the right side of (6.27), by (6.25). Thus the integral of an n-form over an oriented n-dimensional surface is well defined.

Exercises

1. If $F : U_0 \to U_1$ and $G : U_1 \to U_2$ are smooth maps and $\alpha \in \Lambda^k(U_2)$, then (6.26) implies

$$(G \circ F)^* \alpha = F^*(G^* \alpha) \text{ in } \Lambda^k(U_0). \tag{6.32}$$

In the special case that $U_j = \mathbb{R}^n$ and F and G are linear maps, and $k = n$, show that this identity implies

$$\det(GF) = (\det F)(\det G). \tag{6.33}$$

Compare this with the derivation of (1.87).

2. Let $\Lambda^k\mathbb{R}^n$ denote the space of k-forms (6.12) with constant coefficients. Show that

$$\dim_{\mathbb{R}} \Lambda^k\mathbb{R}^n = \binom{n}{k}. \tag{6.34}$$

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then T^* preserves this class of spaces; we denote the map

$$\Lambda^k T^* : \Lambda^k\mathbb{R}^n \longrightarrow \Lambda^k\mathbb{R}^m. \tag{6.35}$$

Similarly, replacing T by T^* yields

$$\Lambda^k T : \Lambda^k\mathbb{R}^m \longrightarrow \Lambda^k\mathbb{R}^n. \tag{6.36}$$

3. Show that $\Lambda^k T$ is uniquely characterized as a linear map from $\Lambda^k\mathbb{R}^m$ to $\Lambda^k\mathbb{R}^n$ which satisfies

$$(\Lambda^k T)(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k), \quad v_j \in \mathbb{R}^m.$$

4. Show that if $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps, then

$$\Lambda^k(ST) = (\Lambda^k S) \circ (\Lambda^k T). \tag{6.37}$$

Relate this to (6.28).

If $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$, define an inner product on $\Lambda^k\mathbb{R}^n$ by declaring an orthonormal basis to be

$$\{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}. \tag{6.38}$$

If $A : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ is a linear map, define $A^t : \Lambda^k\mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ by

$$\langle A\alpha, \beta\rangle = \langle \alpha, A^t\beta\rangle, \quad \alpha, \beta \in \Lambda^k\mathbb{R}^n, \tag{6.39}$$

where $\langle\ ,\ \rangle$ is the inner product on $\Lambda^k\mathbb{R}^n$ defined above.

5. Show that, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, with transpose T^t, then

$$(\Lambda^k T)^t = \Lambda^k(T^t). \tag{6.40}$$

Hint. Check the identity $\langle (\Lambda^k T)\alpha, \beta\rangle = \langle \alpha, (\Lambda^k T^t)\beta\rangle$ when α and β run over the orthonormal basis (6.38). That is, show that if $\alpha = e_{j_1} \wedge \cdots \wedge e_{j_k}$, $\beta = e_{i_1} \wedge \cdots \wedge e_{i_k}$, then

$$\langle Te_{j_1} \wedge \cdots \wedge Te_{j_k}, e_{i_1} \wedge \cdots \wedge e_{i_k}\rangle = \langle e_{j_1} \wedge \cdots \wedge e_{j_k}, T^t e_{i_1} \wedge \cdots \wedge T^t e_{i_k}\rangle. \tag{6.41}$$

Hint. Say $T = (t_{ij})$. In the spirit of (6.21), expand $Te_{j_1} \wedge \cdots \wedge Te_{j_k}$, and show that the left side of (6.41) is equal to

$$\sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) t_{i_{\sigma(1)} j_1} \cdots t_{i_{\sigma(k)} j_k}, \tag{6.42}$$

where S_k denotes the set of permutations of $\{1, \dots, k\}$. Similarly, show that the right side of (6.41) is equal to

$$\sum_{\tau \in S_k} (\operatorname{sgn} \tau) t_{i_1 j_{\tau(1)}} \cdots t_{i_k j_{\tau(k)}}. \tag{6.43}$$

To compare these two formulas, see the hint for (1.106) in §1.

6. Show that if $\{u_1, \dots, u_n\}$ is any orthonormal basis of $\mathbb{R}^n$, then the set $\{u_{j_1} \wedge \cdots \wedge u_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}$ is an orthonormal basis of $\Lambda^k \mathbb{R}^n$.
Hint. Use Exercises 4 and 5 to show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation on $\mathbb{R}^n$ (i.e., preserves the inner product) then $\Lambda^k T$ is an orthogonal transformation on $\Lambda^k \mathbb{R}^n$.

7. Let $v_j, w_j \in \mathbb{R}^n$, $1 \le j \le k$ $(k < n)$. Form the matrices V, whose k columns are the column vectors $v_1, \dots, v_k$, and W, whose k columns are the column vectors $w_1, \dots, w_k$. Show that

$$\begin{aligned} \langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle &= \det W^t V \\ &= \det V^t W. \end{aligned} \tag{6.44}$$

Hint. Show that both sides are linear in each v_j and in each w_j. (To treat the right side, see the exercises on determinants, in §1.) Use this to reduce the problem to verifying (6.44) when each v_j and each w_j is chosen from among the set of basis vectors $\{e_1, \dots, e_n\}$. Use anti-symmetries to reduce the problem further.

8. Deduce from Exercise 7 that if $v_j, w_j \in \mathbb{R}^n$, then

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \sum_{\pi} (\operatorname{sgn} \pi) \langle v_1, w_{\pi(1)} \rangle \cdots \langle v_k, w_{\pi(k)} \rangle, \tag{6.45}$$

where π ranges over the set of permutations of $\{1, \dots, k\}$.

9. Show that the conclusion of Exercise 6 also follows from (6.45).

10. Let $A, B : \mathbb{R}^k \to \mathbb{R}^n$ be linear maps and set $\omega = e_1 \wedge \cdots \wedge e_k \in \Lambda^k \mathbb{R}^k$. We have $\Lambda^k A\omega, \Lambda^k B\omega \in \Lambda^k \mathbb{R}^n$. Deduce from (6.44) that

$$\langle \Lambda^k A\omega, \Lambda^k B\omega \rangle = \det B^t A. \tag{6.46}$$

11. Let $\varphi : \mathcal{O} \to \mathbb{R}^n$ be smooth, with $\mathcal{O} \subset \mathbb{R}^m$ open. Deduce from Exercise 10 that, for each $x \in \mathcal{O}$,

$$\|\Lambda^m D\varphi(x)\omega\|^2 = \det D\varphi(x)^t D\varphi(x), \tag{6.47}$$

where $\omega = e_1 \wedge \cdots \wedge e_m$. Deduce that if $\varphi : \mathcal{O} \to U \subset M$ is a coordinate patch on a smooth m-dimensional surface $M \subset \mathbb{R}^n$ and $f \in C(M)$ is supported on U, then

$$\int_M f\, dS = \int_{\mathcal{O}} f(\varphi(x)) \|\Lambda^m D\varphi(x)\omega\|\, dx. \tag{6.48}$$

12. Show that the result of Exercise 5 in §5 follows from (6.48), via (6.41)–(6.42).

13. Recall the projective spaces $\mathbb{P}^n$, constructed in §5. Show that $\mathbb{P}^n$ is orientable if and only if n is odd.

7. Products and exterior derivatives of forms

Having discussed the notion of a differential form as something to be integrated, we now consider some operations on forms. There is a *wedge product*, or exterior product, characterized as follows. If $\alpha \in \Lambda^k(\Omega)$ has the form (6.12) and if

$$\beta = \sum_i b_i(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Lambda^\ell(\Omega), \tag{7.1}$$

define

$$\alpha \wedge \beta = \sum_{j,i} a_j(x) b_i(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \tag{7.2}$$

in $\Lambda^{k+\ell}(\Omega)$. A special case of this arose in (6.18)–(6.21). We retain the equivalence (6.14). It follows easily that

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{7.3}$$

In addition, there is an *interior product* if $\alpha \in \Lambda^k(\Omega)$ with a vector field X on Ω, producing $\iota_X \alpha = \alpha \rfloor X \in \Lambda^{k-1}(\Omega)$, defined by

$$(\alpha \rfloor X)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}). \tag{7.4}$$

Consequently, if $\alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$, $D_i = \partial/\partial x_i$, then

$$\alpha \rfloor D_{j_\ell} = (-1)^{\ell-1} dx_{j_1} \wedge \cdots \wedge \widehat{dx}_{j_\ell} \wedge \cdots \wedge dx_{j_k} \tag{7.5}$$

where $\widehat{dx}_{j_\ell}$ denotes removing the factor dx_{j_ℓ}. Furthermore,

$$i \notin \{j_1, \dots, j_k\} \Longrightarrow \alpha \rfloor D_i = 0.$$

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism and α, β are forms and X a vector field on Ω, it is readily verified that

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta), \quad F^*(\alpha \rfloor X) = (F^*\alpha) \rfloor (F_\# X). \tag{7.6}$$

We make use of the operators $\wedge_k$ and ι_k on forms:

$$\wedge_k \alpha = dx_k \wedge \alpha, \quad \iota_k \alpha = \alpha \rfloor D_k. \tag{7.7}$$

There is the following useful *anticommutation relation*:

$$\wedge_k \iota_\ell + \iota_\ell \wedge_k = \delta_{k\ell}, \tag{7.8}$$

where $\delta_{k\ell}$ is 1 if $k=\ell$, 0 otherwise. This is a fairly straightforward consequence of (7.5). We also have

$$\wedge_j \wedge_k + \wedge_k \wedge_j = 0, \quad \iota_j \iota_k + \iota_k \iota_j = 0. \tag{7.9}$$

From (7.8)–(7.9) one says that the operators $\{\iota_j, \wedge_j : 1 \le j \le n\}$ generate a "Clifford algebra."

Another important operator on forms is the *exterior derivative*:

$$d : \Lambda^k(\Omega) \longrightarrow \Lambda^{k+1}(\Omega), \tag{7.10}$$

defined as follows. If $\alpha \in \Lambda^k(\Omega)$ is given by (6.12), then

$$d\alpha = \sum_{j,\ell} \frac{\partial a_j}{\partial x_\ell}\, dx_\ell \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{7.11}$$

Equivalently,

$$d\alpha = \sum_{\ell=1}^{n} \partial_\ell \wedge_\ell \alpha \tag{7.12}$$

where $\partial_\ell = \partial/\partial x_\ell$ and $\wedge_\ell$ is given by (7.7). The antisymmetry $dx_m \wedge dx_\ell = -dx_\ell \wedge dx_m$, together with the identity $\partial^2 a_j/\partial x_\ell \partial x_m = \partial^2 a_j/\partial x_m \partial x_\ell$, implies

$$d(d\alpha) = 0, \tag{7.13}$$

for any differential form α. We also have a product rule:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \quad \alpha \in \Lambda^k(\Omega),\ \beta \in \Lambda^j(\Omega). \tag{7.14}$$

The exterior derivative has the following important property under pull-backs:

$$F^*(d\alpha) = dF^*\alpha, \tag{7.15}$$

if $\alpha \in \Lambda^k(\Omega)$ and $F : \mathcal{O} \to \Omega$ is a smooth map. To see this, extending (7.14) to a formula for $d(\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_\ell)$ and using this to apply d to $F^*\alpha$, we have

$$\begin{aligned} dF^*\alpha = &\sum_{j,\ell} \frac{\partial}{\partial x_\ell}\big(a_j \circ F(x)\big)\, dx_\ell \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big) \\ &+ \sum_{j,\nu} (\pm) a_j\big(F(x)\big)\big(F^* dx_{j_1}\big) \wedge \cdots \wedge d\big(F^* dx_{j_\nu}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big). \end{aligned} \tag{7.16}$$

Now the definition (6.18)–(6.19) of pull-back gives directly that

$$F^* dx_i = \sum_\ell \frac{\partial F_i}{\partial x_\ell}\, dx_\ell = dF_i, \tag{7.17}$$

and hence $d(F^* dx_i) = ddf_i = 0$, so only the first sum in (7.16) contributes to $dF^*\alpha$. Meanwhile,

$$F^* d\alpha = \sum_{j,m} \frac{\partial a_j}{\partial x_m}\big(F(x)\big)\ (F^* dx_m) \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big), \tag{7.18}$$

so (7.15) follows from the identity

$$\sum_{\ell} \frac{\partial}{\partial x_\ell}\big(a_j \circ F(x)\big)\, dx_\ell = \sum_m \frac{\partial a_j}{\partial x_m}\big(F(x)\big)\, F^* dx_m, \tag{7.19}$$

which in turn follows from the chain rule.

If $d\alpha = 0$, we say α is *closed*; if $\alpha = d\beta$ for some $\beta \in \Lambda^{k-1}(\Omega)$, we say α is *exact*. Formula (7.13) implies that every exact form is closed. The converse is not always true globally. Consider the multi-valued angular coordinate θ on $\mathbb{R}^2 \setminus (0,0)$; $d\theta$ is a single valued closed form on $\mathbb{R}^2 \setminus (0,0)$ which is not globally exact. An important result of H. Poincaré, is that every closed form is locally exact. A proof is given in §11. (A special case is established in §8.)

Exercises

1. If α is a k-form, verify the formula (7.14), i.e., $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$. If α is closed and β is exact, show that $\alpha \wedge \beta$ is exact.

2. Let F be a vector field on U, open in $\mathbb{R}^3$, $F = \sum_1^3 f_j(x)\partial/\partial x_j$. The vector field $G = \operatorname{curl} F$ is classically defined as a formal determinant

$$\operatorname{curl} F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{pmatrix}, \tag{7.20}$$

where $\{e_j\}$ is the standard basis of $\mathbb{R}^3$. Consider the 1-form $\varphi = \sum_1^3 f_j(x) dx_j$. Show that $d\varphi$ and curl F are related in the following way:

$$\begin{aligned} \operatorname{curl} F &= \sum_1^3 g_j(x)\ \partial/\partial x_j, \\ d\varphi &= g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2. \end{aligned} \tag{7.21}$$

See (9.30)–(9.37) for more on this connection.

3. If F and φ are related as in Exercise 2, show that curl F is uniquely specified by the relation

$$d\varphi \wedge \alpha = \langle \text{curl } F, \alpha \rangle\, \omega \tag{7.22}$$

for all 1-forms α on $U \subset \mathbb{R}^3$, where $\omega = dx_1 \wedge dx_2 \wedge dx_3$ is the volume form.

4. Let B be a ball in $\mathbb{R}^3$, F a smooth vector field on B. Show that

$$\exists\, u \in C^\infty(B) \text{ s.t. } F = \text{ grad } u \Longrightarrow \text{ curl } F = 0. \tag{7.23}$$

Hint. Compare $F = \text{ grad } u$ with $\varphi = du$.

5. Let B be a ball in $\mathbb{R}^3$ and G a smooth vector field on B. Show that

$$\exists \text{ vector field } F \text{ s.t. } G = \text{ curl } F \Longrightarrow \text{ div } G = 0. \tag{7.24}$$

Hint. If $G = \sum_1^3 g_j(x)\partial/\partial x_j$, consider

$$\psi = g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2. \tag{7.25}$$

Show that

$$d\psi = (\operatorname{div} G)\, dx_1 \wedge dx_2 \wedge dx_3. \tag{7.26}$$

6. Show that the 1-form $d\theta$ mentioned below (7.19) is given by

$$d\theta = \frac{x\, dy - y\, dx}{x^2 + y^2}.$$

For the next set of exercises, let Ω be a planar domain, $X = f(x, y)\, \partial/\partial x + g(x, y)\, \partial/\partial y$ a nonvanishing vector field on Ω. Consider the 1-form $\alpha = g(x, y)\, dx - f(x, y)\, dy$.

7. Let $\gamma : I \to \Omega$ be a smooth curve, $I = (a, b)$. Show that the image $C = \gamma(I)$ is the image of an integral curve of X if and only if $\gamma^*\alpha = 0$. Consequently, with slight abuse of notation, one describes the integral curves by $g\, dx - f\, dy = 0$.
If α is exact, i.e., $\alpha = du$, conclude the level curves of u are the integral curves of X.

8. A function φ is called an integrating factor if $\tilde{\alpha} = \varphi\alpha$ is exact, i.e., if $d(\varphi\alpha) = 0$, provided Ω is simply connected. Show that an integrating factor always exists, at least locally. Show that $\varphi = e^v$ is an integrating factor if and only if $Xv = -\text{ div } X$.
Find an integrating factor for $\alpha = (x^2 + y^2 - 1)\, dx - 2xy\, dy$.

9. Define the radial vector field $R = x_1\partial/\partial x_1 + \cdots + x_n\partial/\partial x_n$, on $\mathbb{R}^n$. Show that

$$\omega = dx_1 \wedge \cdots \wedge dx_n \Longrightarrow$$

$$\omega \rfloor R = \sum_{j=1}^{n} (-1)^{j-1} x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Show that

$$d(\omega \rfloor R) = n\, \omega.$$

10. Show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a linear rotation (i.e., $F \in SO(n)$) than $\beta = \omega \rfloor R$ in Exercise 9 has the property that $F^*\beta = \beta$.

8. The general Stokes formula

The Stokes formula involves integrating a k-form over a k-dimensional surface with boundary. We first define that concept. Let S be a smooth k-dimensional surface (say in $\mathbb{R}^N$), and let M be an open subset of S, such that its closure $\overline{M}$ (in $\mathbb{R}^N$) is contained in S. Its boundary is $\partial M = \overline{M} \setminus M$. We say $\overline{M}$ is a smooth surface with boundary if also ∂M is a smooth $(k-1)$-dimensional surface. In such a case, any $p \in \partial M$ has a neighborhood $U \subset S$ with a coordinate chart $\varphi : \mathcal{O} \to U$, where $\mathcal{O}$ is an open neighborhood of 0 in $\mathbb{R}^k$, such that $\varphi(0) = p$ and φ maps $\{x \in \mathcal{O} : x_1 = 0\}$ onto $U \cap \partial M$.

If S is oriented, then $\overline{M}$ is oriented, and ∂M inherits an orientation, uniquely determined by the following requirement: if

$$\overline{M} = \mathbb{R}^k_- = \{x \in \mathbb{R}^k : x_1 \le 0\}, \tag{8.1}$$

then $\partial M = \{(x_2, \dots, x_k)\}$ has the orientation determined by $dx_2 \wedge \cdots \wedge dx_k$.

We can now state the Stokes formula.

Proposition 8.1. *Given a compactly supported $(k-1)$-form β of class C^1 on an oriented k-dimensional surface $\overline{M}$ (of class C^2) with boundary ∂M, with its natural orientation,*

$$\int_M d\beta = \int_{\partial M} \beta. \tag{8.2}$$

Proof. Using a partition of unity and invariance of the integral and the exterior derivative under coordinate transformations, it suffices to prove this when $\overline{M}$ has the form (8.1). In that case, we will be able to deduce (8.2) from the Fundamental Theorem of Calculus. Indeed, if

$$\beta = b_j(x)\ dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k, \tag{8.3}$$

with $b_j(x)$ of bounded support, we have

$$d\beta = (-1)^{j-1} \frac{\partial b_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_k. \tag{8.4}$$

If $j > 1$, we have

$$\int_M d\beta = (-1)^{j-1} \int \Big\{ \int_{-\infty}^{\infty} \frac{\partial b_j}{\partial x_j}\ dx_j \Big\}\, dx' = 0, \tag{8.5}$$

and also $\kappa^*\beta = 0$, where $\kappa : \partial M \to \overline{M}$ is the inclusion. On the other hand, for $j = 1$, we have

$$
\begin{aligned}
\int\limits_M d\beta &= \int \Big\{ \int_{-\infty}^0 \frac{\partial b_1}{\partial x_1}\, dx_1 \Big\}\, dx_2 \cdots dx_k \\
&= \int b_1(0, x')\, dx' \\
&= \int\limits_{\partial M} \beta.
\end{aligned} \tag{8.6}
$$

This proves Stokes' formula (8.2).

It is useful to allow singularities in ∂M. We say a point $p \in \overline{M}$ is a *corner* of dimension ν if there is a neighborhood $\overline{U}$ of p in $\overline{M}$ and a C^2 diffeomorphism of $\overline{U}$ onto a neighborhood of 0 in

$$
K = \{x \in \mathbb{R}^k : x_j \le 0, \ \text{for } 1 \le j \le k - \nu\}, \tag{8.7}
$$

where k is the dimension of M. If M is a C^2 surface and every point $p \in \partial M$ is a corner (of some dimension), we say $\overline{M}$ is a C^2 surface with corners. In such a case, ∂M is a locally finite union of C^2 surfaces with corners. The following result extends Proposition 8.1.

Proposition 8.2. *If $\overline{M}$ is a C^2 surface of dimension k, with corners, and β is a compactly supported $(k-1)$-form of class C^1 on $\overline{M}$, then (8.2) holds.*

Proof. It suffices to establish this when β is supported on a small neighborhood of a corner $p \in \partial M$, of the form $\overline{U}$ described above. Hence it suffices to show that (8.2) holds whenever β is a $(k-1)$-form of class C^1, with compact support on K in (8.7); and we can take β to have the form (8.3). Then, for $j > k - \nu$, (8.5) still holds, while, for $j \le k - \nu$, we have, as in (8.6),

$$
\begin{aligned}
\int\limits_K d\beta &= (-1)^{j-1} \int \Big\{ \int_{-\infty}^0 \frac{\partial b_j}{\partial x_j}\, dx_j \Big\}\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\
&= (-1)^{j-1} \int b_j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_k)\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\
&= \int\limits_{\partial K} \beta.
\end{aligned} \tag{8.8}
$$

This completes the proof.

The reason we required $\overline{M}$ to be a surface of class C^2 (with corners) in Propositions 8.1 and 8.2 is the following. Due to the formulas (6.18)–(6.19) for a pull-back, if β is of class C^j and F is of class C^ℓ, then $F^*\beta$ is generally of class C^μ, with $\mu = \min(j, \ell - 1)$.

Thus, if $j = \ell = 1$, $F^*\beta$ might be only of class C^0, so there is not a well-defined notion of a differential form of class C^1 on a C^1 surface, though such a notion is well defined on a C^2 surface. This problem can be overcome, and one can extend Propositions 8.1–8.2 to the case where $\overline{M}$ is a C^1 surface (with corners), and β is a $(k-1)$-form with the property that both β and $d\beta$ are continuous. We will not go into the details. Substantially more sophisticated generalizations are given in [Fed].

We will mention one useful extension of the scope of Proposition 8.2, to surfaces with piecewise smooth boundary that do not satisfy the corner condition. An example is illustrated in Fig. 8.1. There the point p is a singular point of ∂M that is not a corner, according to the definition using (8.7). However, in many cases M can be divided into pieces (M_1 amd M_2 for the example presented in Fig. 8.1) and each piece M_j is a surface with corners. Then Proposition 8.2 applies to each piece separately:

$$\int\limits_{M_j} d\beta = \int\limits_{\partial M_j} \beta,$$

and one can sum over j to get (8.2) in this more general setting.

We next apply Proposition 8.2 to prove the following special case of the Poincaré lemma, which will be used in §10.

Proposition 8.3. *If α is a 1-form on $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $d\alpha = 0$, then there exists a real valued $u \in C^\infty(B)$ such that $\alpha = du$.*

In fact, let us set

$$u_j(x) = \int\limits_{\gamma_j(x)} \alpha, \tag{8.9}$$

where $\gamma_j(x)$ is a path from 0 to $x = (x_1, x_2)$ which consists of two line segments. The path first goes from 0 to $(0, x_2)$ and then from $(0, x_2)$ to x, if $j = 1$, while if $j = 2$ it first goes from 0 to $(x_1, 0)$ and then from $(x_1, 0)$ to x. See Fig. 8.2. It is easy to verify that $\partial u_j / \partial x_j = \alpha_j(x)$. We claim that $u_1 = u_2$, or equivalently that

$$\int\limits_{\sigma(x)} \alpha = 0, \tag{8.10}$$

where $\sigma(x)$ is a closed path consisting of $\gamma_2(x)$ followed by $\gamma_1(x)$, in reverse. In fact, Stokes' formula, Proposition 8.2, implies that

$$\int\limits_{\sigma(x)} \alpha = \int\limits_{R(x)} d\alpha, \tag{8.11}$$

where $R(x)$ is the rectangle whose boundary is $\sigma(x)$. If $d\alpha = 0$, then (8.11) vanishes, and we have the desired function $u: \ u = u_1 = u_2$.

Exercises

1. In the setting of Proposition 8.1, show that

$$\partial M = \emptyset \Longrightarrow \int_M d\beta = 0.$$

2. Consider the region $\overline{\Omega} \subset \mathbb{R}^2$ defined by

$$\overline{\Omega} = \{(x, y) : 0 \le y \le x^2, 0 \le x \le 1\}.$$

Show that the boundary points $(1, 0)$ and $(1, 1)$ are corners, but $(0, 0)$ is *not* a corner. The boundary of $\overline{\Omega}$ is too sharp at $(0, 0)$ to be a corner; it is called a "cusp." Extend Proposition 8.2. to treat this region.

3. Suppose $U \subset \mathbb{R}^n$ is an open set with smooth boundary $M = \partial U$, and U has the standard orientation, determined by $dx_1 \wedge \cdots \wedge dx_n$. (See the paragraph above (6.23).) Let $\varphi \in C^1(\mathbb{R}^n)$ satisfy $\varphi(x) < 0$ for $x \in U$, $\varphi(x) > 0$ for $x \in \mathbb{R}^n \setminus \overline{U}$, and grad $\varphi(x) \neq 0$ for $x \in \partial U$, so grad φ points out of U. Show that the natural orientation on ∂U, as defined just before Proposition 8.1, is the same as the following. The equivalence class of forms $\beta \in \Lambda^{n-1}(\partial U)$ defining the orientation on ∂U satisfies the property that $d\varphi \wedge \beta$ is a *positive* multiple of $dx_1 \wedge \cdots \wedge dx_n$, on ∂U.

4. Suppose $U = \{x \in \mathbb{R}^n : x_n < 0\}$. Show that the orientation on ∂U described above is that of $(-1)^{n-1}\, dx_1 \wedge \cdots \wedge dx_{n-1}$. If $V = \{x \in \mathbb{R}^n : x_n > 0\}$, what orientation does ∂V inherit?

5. Extend the special case of Poincaré's Lemma given in Proposition 8.3 to the case where α is a closed 1-form on $B = \{x \in \mathbb{R}^n : |x| < 1\}$, i.e., from the case dim $B = 2$ to higher dimensions.

6. Define $\beta \in \Lambda^{n-1}(\mathbb{R}^n)$ by

$$\beta = \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Let $\overline{\Omega} \subset \mathbb{R}^n$ be a smoothly bounded compact subset. Show that

$$\frac{1}{n} \int_{\partial\Omega} \beta = \mathrm{Vol}(\Omega).$$

7. In the setting of Exercise 6, show that if $f \in C^1(\overline{\Omega})$, then

$$\int\limits_{\partial\Omega} f\beta = \int\limits_{\Omega} (Rf + nf)\, dx,$$

where

$$Rf = \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}.$$

8. In the setting of Exercises 6–7, and with $S^{n-1} \subset \mathbb{R}^n$ the unit sphere, show that

$$\int\limits_{S^{n-1}} f\beta = \int\limits_{S^{n-1}} f\, dS.$$

Hint. Let $B \subset \mathbb{R}^{n-1}$ be the unit ball, and define $\varphi : B \to S^{n-1}$ by $\varphi(x') = (x', \sqrt{1 - |x'|^2})$. Compute $\varphi^*\beta$. Compare surface area formulas derived in §5.
Another approach. The unit sphere $S^{n-1} \overset{j}{\hookrightarrow} \mathbb{R}^n$ has a volume form (cf. (9.13)), it must be a scalar multiple $g(j^*\beta)$, and, by Exercise 10 of §7, g must be constant. Then Exercise 6 identifies this constant, in light of results from §5.
See the exercises in §9 for more on this.

9. Given β as in Exercise 6. show that the $(n-1)$-form

$$\omega = |x|^{-n}\, \beta$$

on $\mathbb{R}^n \setminus 0$ is closed. Use Exercise 6 to show that $\int_{S^{n-1}} \omega \neq 0$, and hence ω is not exact.

10. Let $\overline{\Omega} \subset \mathbb{R}^n$ be a compact, smoothly bounded subset. Take ω as in Exercise 9. Show that

$$\int\limits_{\partial\Omega} \omega = \begin{cases} A_{n-1} & \text{if } 0 \in \Omega, \\ 0 & \text{if } 0 \notin \overline{\Omega}. \end{cases}$$

9. The classical Gauss, Green, and Stokes formulas

The case of (8.1) where $S = \overline{\Omega}$ is a region in $\mathbb{R}^2$ with smooth boundary yields the classical Green Theorem. In this case, we have

$$\beta = f\,dx + g\,dy, \quad d\beta = \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\,dx \wedge dy, \tag{9.1}$$

and hence (8.1) becomes the following

Proposition 9.1. *If $\overline{\Omega}$ is a region in $\mathbb{R}^2$ with smooth boundary, and f and g are smooth functions on $\overline{\Omega}$, which vanish outside some compact set in $\overline{\Omega}$, then*

$$\iint\limits_{\Omega} \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Bigr)\,dx\,dy = \int\limits_{\partial\Omega} (f\,dx + g\,dy). \tag{9.2}$$

Note that, if we have a vector field $X = X_1\partial/\partial x + X_2\partial/\partial y$ on $\overline{\Omega}$, then the integrand on the left side of (9.2) is

$$\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} = \operatorname{div} X, \tag{9.3}$$

provided $g = X_1,\ f = -X_2$. We obtain

$$\iint\limits_{\Omega} (\operatorname{div} X)\,dx\,dy = \int\limits_{\partial\Omega} \bigl(-X_2\,dx + X_1\,dy\bigr). \tag{9.4}$$

If $\partial\Omega$ is parametrized by arc-length, as $\gamma(s) = \bigl(x(s), y(s)\bigr)$, with orientation as defined for Proposition 8.1, then the unit normal ν, to $\partial\Omega$, pointing *out* of Ω, is given by $\nu(s) = \bigl(y'(s), -x'(s)\bigr)$, and (9.4) is equivalent to

$$\iint\limits_{\Omega} (\operatorname{div} X)\,dx\,dy = \int\limits_{\partial\Omega} \langle X, \nu\rangle\,ds. \tag{9.5}$$

This is a special case of Gauss' Divergence Theorem. We now derive a more general form of the Divergence Theorem. We begin with a definition of the divergence of a vector field on a surface M.

Let M be a region in $\mathbb{R}^n$, or an n-dimensional surface in $\mathbb{R}^{n+k}$, provided with a volume form

$$\omega_M \in \Lambda^n M. \tag{9.6}$$

Let X be a vector field on M. Then the divergence of X, denoted div X, is a function on M given by

$$(\operatorname{div} X)\,\omega_M = d(\omega_M \rfloor X). \tag{9.7}$$

If $M = \mathbb{R}^n$, with the standard volume element

$$\omega = dx_1 \wedge \cdots \wedge dx_n, \tag{9.8}$$

and if

$$X = \sum X^j(x)\frac{\partial}{\partial x_j}, \tag{9.9}$$

then

$$\omega \rfloor X = \sum_{j=1}^{n} (-1)^{j-1} X^j(x) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{9.10}$$

Hence, in this case, (9.7) yields the familiar formula

$$\operatorname{div} X = \sum_{j=1}^{n} \partial_j X^j, \tag{9.11}$$

where we use the notation

$$\partial_j f = \frac{\partial f}{\partial x_j}. \tag{9.12}$$

Suppose now that M is endowed with both an orientation and a metric tensor $g_{jk}(x)$. Then M carries a natural volume element ω_M, determined by the condition that, if one has an orientation-preserving coordinate system in which $g_{jk}(p_0) = \delta_{jk}$, then $\omega_M(p_0) = dx_1 \wedge \cdots \wedge dx_n$. This condition produces the following formula, in any orientation-preserving coordinate system:

$$\omega_M = \sqrt{g}\; dx_1 \wedge \cdots \wedge dx_n, \quad g = \det(g_{jk}), \tag{9.13}$$

by the same sort of calculations as done in (5.10)–(5.15).

We now compute div X when the volume element on M is given by (9.13). We have

$$\omega_M \rfloor X = \sum_j (-1)^{j-1} X^j \sqrt{g}\; dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \tag{9.14}$$

and hence

$$d(\omega_M \rfloor X) = \partial_j(\sqrt{g} X^j)\; dx_1 \wedge \cdots \wedge dx_n. \tag{9.15}$$

Here, as below, we use the summation convention. Hence the formula (9.7) gives

$$\operatorname{div} X = g^{-1/2}\partial_j(g^{1/2}X^j). \tag{9.16}$$

Compare (5.55).

We now derive the Divergence Theorem, as a consequence of Stokes' formula, which we recall is

$$\int_M d\alpha = \int_{\partial M} \alpha, \tag{9.17}$$

for an $(n-1)$-form on $\overline{M}$, assumed to be a smooth compact oriented surface with boundary. If $\alpha = \omega_M \rfloor X$, formula (9.7) gives

$$\int_M (\operatorname{div} X)\, \omega_M = \int_{\partial M} \omega_M \rfloor X. \tag{9.18}$$

This is one form of the Divergence Theorem. We will produce an alternative expression for the integrand on the right before stating the result formally.

Given that ω_M is the volume form for M determined by a Riemannian metric, we can write the interior product $\omega_M \rfloor X$ in terms of the volume element $\omega_{\partial M}$ on ∂M, with its induced orientation and Riemannian metric, as follows. Pick coordinates on M, centered at $p_0 \in \partial M$, such that ∂M is tangent to the hyperplane $\{x_1 = 0\}$ at $p_0 = 0$ (with M to the left of ∂M), and such that $g_{jk}(p_0) = \delta_{jk}$, so $\omega_M(p_0) = dx_1 \wedge \cdots \wedge dx_n$. Consequently, $\omega_{\partial M}(p_0) = dx_2 \wedge \cdots \wedge dx_2$. It follows that, at p_0,

$$j^*(\omega_M \rfloor X) = \langle X, \nu\rangle \omega_{\partial M}, \tag{9.19}$$

where ν is the unit vector normal to ∂M, pointing out of M and $j : \partial M \hookrightarrow M$ the natural inclusion. The two sides of (9.19), which are both defined in a coordinate independent fashion, are hence equal on ∂M, and the identity (9.18) becomes

$$\int_M (\operatorname{div} X)\, \omega_M = \int_{\partial M} \langle X, \nu\rangle\, \omega_{\partial M}. \tag{9.20}$$

Finally, we adopt the notation of the sort used in §§4–5. We denote the volume element on M by dV and that on ∂M by dS, obtaining the *Divergence Theorem*:

Theorem 9.2. *If $\overline{M}$ is a compact surface with boundary, X a smooth vector field on $\overline{M}$, then*

$$\int_M (\operatorname{div} X)\, dV = \int_{\partial M} \langle X, \nu\rangle\, dS, \tag{9.21}$$

where ν is the unit outward-pointing normal to ∂M.

The only point left to mention here is that M need not be orientable. Indeed, we can treat the integrals in (9.21) as surface integrals, as in §5, and note that all objects in (9.21) are independent of a choice of orientation. To prove the general case, just use a partition of unity supported on orientable pieces.

We obtain some further integral identities. First, we apply (9.21) with X replaced by uX. We have the following "derivation" identity:

$$\text{div}\; uX = u\;\text{div}\, X + \langle du, X\rangle = u\;\text{div}\, X + Xu, \tag{9.22}$$

which follows easily from the formula (9.16). The Divergence Theorem immediately gives

$$\int_M (\text{div}\, X)u\, dV + \int_M Xu\, dV = \int_{\partial M} \langle X, \nu\rangle u\, dS. \tag{9.23}$$

Replacing u by uv and using the derivation identity $X(uv) = (Xu)v + u(Xv)$, we have

$$\int_M \big[(Xu)v + u(Xv)\big]\, dV = -\int_M (\text{div}\, X)uv\, dV + \int_{\partial M} \langle X, \nu\rangle uv\, dS. \tag{9.24}$$

It is very useful to apply (9.23) to a gradient vector field X. If v is a smooth function on M, grad v is a vector field satisfying

$$\langle \text{grad}\; v, Y\rangle = \langle Y, dv\rangle, \tag{9.25}$$

for any vector field Y, where the brackets on the left are given by the metric tensor on M and those on the right by the natural pairing of vector fields and 1-forms. Hence grad $v = X$ has components $X^j = g^{jk}\partial_k v$, where (g^{jk}) is the matrix inverse of (g_{jk}).

Applying div to grad v, we have the *Laplace operator*:

$$\Delta v = \;\text{div grad}\; v = g^{-1/2}\partial_j\big(g^{jk} g^{1/2}\partial_k v\big). \tag{9.26}$$

When M is a region in $\mathbb{R}^n$ and we use the standard Euclidean metric, so div X is given by (9.11), we have the Laplace operator on Euclidean space:

$$\Delta v = \frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_n^2}. \tag{9.27}$$

Now, setting $X = \;\text{grad}\, v$ in (9.23), we have $Xu = \langle \text{grad}\, u, \text{grad}\, v\rangle$, and $\langle X, \nu\rangle = \langle \nu, \text{grad}\; v\rangle$, which we call the normal derivative of v, and denote $\partial v/\partial\nu$. Hence

$$\int_M u(\Delta v)\; dV = -\int_M \langle \text{grad}\; u, \text{grad}\; v\rangle\; dV + \int_{\partial M} u\frac{\partial v}{\partial \nu}\; dS. \tag{9.28}$$

If we interchange the roles of u and v and subtract, we have

$$\int_M u(\Delta v)\, dV = \int_M (\Delta u) v\, dV + \int_{\partial M} \Big[u\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} v \Big]\, dS. \tag{9.29}$$

Formulas (9.28)–(9.29) are also called Green formulas. We will make further use of them in §10.

We return to the Green formula (9.2), and give it another formulation. Consider a vector field $Z = (f, g, h)$ on a region in $\mathbb{R}^3$ containing the planar surface $U = \{(x, y, 0) : (x, y) \in \Omega\}$. If we form

$$\text{curl } Z = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{pmatrix} \tag{9.30}$$

we see that the integrand on the left side of (9.2) is the k-component of curl Z, so (9.2) can be written

$$\iint_U (\text{curl } Z) \cdot k\, dA = \int_{\partial U} (Z \cdot T)\, ds, \tag{9.31}$$

where T is the unit tangent vector to ∂U. To see how to extend this result, note that k is a unit *normal* field to the planar surface U.

To formulate and prove the extension of (9.31) to any compact oriented surface with boundary in $\mathbb{R}^3$, we use the relation between curl and exterior derivative discussed in Exercises 2–3 of §7. In particular, if we set

$$F = \sum_{j=1}^3 f_j(x) \frac{\partial}{\partial x_j}, \quad \varphi = \sum_{j=1}^3 f_j(x)\, dx_j, \tag{9.32}$$

then curl $F = \sum_1^3 g_j(x)\ \partial/\partial x_j$ where

$$d\varphi = g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2. \tag{9.33}$$

Now Suppose $\overline{M}$ is a smooth oriented $(n-1)$-dimensional surface with boundary in $\mathbb{R}^n$. Using the orientation of M, we pick a unit normal field N to M as follows. Take a smooth function v which vanishes on M but such that $\nabla v(x) \neq 0$ on M. Thus ∇v is normal to M. Let $\sigma \in \Lambda^{n-1}(M)$ define the orientation of M. Then $dv \wedge \sigma = a(x)\, dx_1 \wedge \cdots \wedge dx_n$, where $a(x)$ is nonvanishing on M. For $x \in M$, we take $N(x) = \nabla v(x)/|\nabla v(x)|$ if $a(x) > 0$, and $N(x) = -\nabla v(x)/|\nabla v(x)|$ if $a(x) < 0$. We call N the "positive" unit normal field to the oriented surface M, in this case. Part of the motivation for this characterization of N is that, if $\Omega \subset \mathbb{R}^n$ is an open set with smooth boundary $M = \partial\Omega$, and we give M the induced orientation, as described in §8, then the positive normal field N just defined coincides with the unit normal field pointing out of Ω. Compare Exercises 2–3 of §8.

Now, if $G = (g_1, \dots, g_n)$ is a vector field defined near M, then

$$\int_M (N \cdot G)\, dS = \int_M \Big(\sum_{j=1}^n (-1)^{j-1} g_j(x)\, dx_1 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_n \Big). \tag{9.34}$$

This result follows from (9.19). When $n = 3$ and $G = \text{ curl } F$, we deduce from (9.32)–(9.33) that

$$\iint_M d\varphi = \iint_M (N \cdot \text{curl } F)\, dS. \tag{9.35}$$

Furthermore, in this case we have

$$\int_{\partial M} \varphi = \int_{\partial M} (F \cdot T)\ ds, \tag{9.36}$$

where T is the unit tangent vector to ∂M, specied as follows by the orientation of ∂M; if $\tau \in \Lambda^1(\partial M)$ defines the orientation of ∂M, then $\langle T, \tau \rangle > 0$ on ∂M. We call T the "forward" unit tangent vector field to the oriented curve ∂M. By the calculations above, we have the classical Stokes formula:

Proposition 9.3. *If $\overline{M}$ is a compact oriented surface with boundary in $\mathbb{R}^3$, and F is a C^1 vector field on a neighborhood of $\overline{M}$, then*

$$\iint_M (N \cdot \text{curl} F)\ dS = \int_{\partial M} (F \cdot T)\ ds, \tag{9.37}$$

where N is the positive unit normal field on M and T the forward unit tangent field to ∂M.

REMARK. The right side of (9.37) is called the *circulation* of F about ∂M. Proposition 9.3 shows how curl F arises to measure this circulation.

Direct proof of the Divergence Theorem

Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 smooth boundary $\partial\Omega$. Hence, for each $p \in \partial\Omega$, there is a neighborhood U of p in $\mathbb{R}^n$, a rotation of coordinate axes, and a C^1 function $u : \mathcal{O} \to \mathbb{R}$, defined on an open set $\mathcal{O} \subset \mathbb{R}^{n-1}$, such that

$$\Omega \cap U = \{x \in \mathbb{R}^n : x_n \le u(x'), x' \in \mathcal{O}\} \cap U,$$

where $x = (x', x_n),\ x' = (x_1, \dots, x_{n-1})$.

We aim to prove that, given $f \in C^1(\overline{\Omega})$, and any constant vector $e \in \mathbb{R}^n$,

$$\int_{\Omega} e \cdot \nabla f(x)\, dx = \int_{\partial\Omega} (e \cdot N) f \, dS, \tag{9.38}$$

where dS is surface measure on $\partial\Omega$ and $N(x)$ is the unit normal to $\partial\Omega$, pointing out of Ω. At $x = (x', u(x')) \in \partial\Omega$, we have

$$N = (1 + |\nabla u|^2)^{-1/2}(-\nabla u, 1). \tag{9.39}$$

To prove (9.38), we may as well suppose f is supported in such a neighborhood U. Then we have

$$\begin{aligned} \int_{\Omega} \frac{\partial f}{\partial x_n}\, dx &= \int_{\mathcal{O}} \Big(\int_{x_n \le u(x')} \partial_n f(x', x_n)\, dx_n \Big)\, dx' \\ &= \int_{\mathcal{O}} f\big(x', u(x')\big)\, dx' \\ &= \int_{\partial\Omega} (e_n \cdot N) f \, dS. \end{aligned} \tag{9.40}$$

The first identity in (9.40) follows from Theorem 4.9, the second identity from the Fundamental Theorem of Calculus, and the third identity from the identification

$$dS = \big(1 + |\nabla u|^2\big)^{1/2}\, dx',$$

established in (5.21). We use the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$.

Such an argument works when e_n is replaced by any constant vector e with the property that we can represent $\partial\Omega \cap U$ as the graph of a function $y_n = \tilde{u}(y')$, with the y_n-axis parallel to e. In particular, it works for $e = e_n + a e_j$, for $1 \le j \le n-1$ and for $|a|$ sufficiently small. Thus, we have

$$\int_{\Omega} (e_n + a e_j) \cdot \nabla f(x)\, dx = \int_{\partial\Omega} (e_n + a e_j) \cdot N \, f \, dS. \tag{9.41}$$

If we subtract (9.40) from this and divide the result by a, we obtain (9.38) for $e = e_j$, for all j, and hence (9.38) holds in general.

Note that replacing e by e_j and f by f_j in (9.38), and summing over $1 \le j \le n$, yields

$$\int_{\Omega} (\mathrm{div}\ F)\, dx = \int_{\partial\Omega} N \cdot F \, dS, \tag{9.42}$$

for the vector field $F = (f_1, \dots, f_n)$. This is the usual statement of Gauss' Divergence Theorem, as given in Theorem 9.2 (specialized to domains in $\mathbb{R}^n$).

Reversing the argument leading from (9.2) to (9.5), we also have another proof of Green's Theorem, in the form (9.2).

Exercises

1. Newton's equation $md^2x/dt^2 = -\nabla V(x)$ for the motion in $\mathbb{R}^n$ of a body of mass m, in a potential force field $F = -\nabla V$, can be converted to a first-order system for (x, ξ), with $\xi = mx$. One gets

$$\frac{d}{dt}(x, \xi) = H_f(x, \xi),$$

where H_f is a "Hamiltonian vector field" on $\mathbb{R}^{2n}$, given by

$$H_f = \sum_{j=1}^{n} \Big[\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big].$$

In the case described above,

$$f(x, \xi) = \frac{1}{2m} |\xi|^2 + V(x).$$

Calculate div H_f from (9.11).

2. Let X be a smooth vector field on a smooth surface M, generating a flow $\mathcal{F}_X^t$. Let $\overline{\mathcal{O}} \subset M$ be a compact, smoothly bounded subset, and set $\overline{\mathcal{O}}_t = \mathcal{F}_X^t(\overline{\mathcal{O}})$. As seen in Proposition 5.7,

$$\frac{d}{dt} \operatorname{Vol}(\overline{\mathcal{O}}_t) = \int_{\mathcal{O}_t} (\operatorname{div} X) \, dV. \tag{9.43}$$

Use the Divergence Theorem to deduce from this that

$$\frac{d}{dt} \operatorname{Vol}(\overline{\mathcal{O}}_t) = \int_{\partial \mathcal{O}_t} \langle X, \nu \rangle \, dS. \tag{9.44}$$

Remark. Conversely, a direct proof of (9.44), together with the Divergence Theorem, would lead to another proof of (9.43).

3. Show that, if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear *rotation*, then, for a C^1 vector field Z on $\mathbb{R}^3$,

$$F_\#(\text{curl } Z) = \text{curl}(F_\# Z). \tag{9.45}$$

4. Let $\overline{M}$ be the graph in $\mathbb{R}^3$ of a smooth function, $z = u(x,y)$, $(x,y) \in \mathcal{O} \subset \mathbb{R}^2$, a bounded region with smooth boundary (maybe with corners). Show that

$$\int_M (\text{curl } F \cdot N)\, dS = \iint_{\mathcal{O}} \Big[\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\Big(-\frac{\partial u}{\partial x}\Big) + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\Big(-\frac{\partial u}{\partial y}\Big) + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\Big]\, dx\, dy, \tag{9.46}$$

where $\partial F_j/\partial x$ and $\partial F_j/\partial y$ are evaluated at $\big(x, y, u(x,y)\big)$. Show that

$$\int_{\partial M} (F \cdot T)\, ds = \int_{\partial \mathcal{O}} \Big(\widetilde{F}_1 + \widetilde{F}_3 \frac{\partial u}{\partial x}\Big)\, dx + \Big(\widetilde{F}_2 + \widetilde{F}_3 \frac{\partial u}{\partial y}\Big)\, dy, \tag{9.47}$$

where $\widetilde{F}_j(x,y) = F_j\big(x, y, u(x,y)\big)$. Apply Green's Theorem, with $f = \widetilde{F}_1 + \widetilde{F}_3(\partial u/\partial x)$, $g = \widetilde{F}_2 + \widetilde{F}_3(\partial u/\partial y)$, to show that the right sides of (9.46) and (9.47) are equal, hence proving Stokes' Theorem in this case.

5. Let $M \subset \mathbb{R}^n$ be the graph of a function $x_n = u(x')$, $x' = (x_1, \dots, x_{n-1})$. If

$$\beta = \sum_{j=1}^{n} (-1)^{j-1} g_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

as in (9.34), and $\varphi(x') = \big(x', u(x')\big)$, show that

$$\begin{aligned} \varphi^*\beta &= (-1)^n \left[\sum_{j=1}^{n-1} g_j\big(x', u(x')\big) \frac{\partial u}{\partial x_j} - g_n\big(x', u(x')\big)\right] dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= (-1)^{n-1} G \cdot (-\nabla u, 1)\ dx_1 \wedge \cdots \wedge dx_{n-1}, \end{aligned}$$

where $G = (g_1, \dots, g_n)$, and verify the identity (9.34) in this case.
Hint. For the last part, recall Exercises 2–3 of §8, regarding the orientation of M.

6. Let S be a smooth oriented 2-dimensional surface in $\mathbb{R}^3$, and M an open subset of S, with smooth boundary; see Fig. 9.1. Let N be the positive unit normal field to S, defined by its orientation. For $x \in \partial M$, let $\nu(x)$ be the unit vector, tangent to M, normal to ∂M, and pointing out of M, and let T be the forward unit tangent vector field to ∂M. Show that, on ∂M,

$$N \times \nu = T, \quad \nu \times T = N.$$

7. If M is an oriented $(n-1)$-dimensional surface in $\mathbb{R}^n$, with positive unit normal field N, show that the volume element ω_M on M is given by

$$\omega_M = \omega \rfloor N,$$

where $\omega = dx_1 \wedge \cdots \wedge dx_n$ is the standard volume form on $\mathbb{R}^n$. Deduce that the volume element on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is given by

$$\omega_{S^{n-1}} = \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

if S^{n-1} inherits the orientation as the boundary of the unit ball.

8. Let M be a C^k surface, $k \geq 2$. Suppose $\varphi : M \to M$ is a C^k isometry, i.e., it preserves the metric tensor. Taking $\varphi^* u(x) = u(\varphi(x))$ for $u \in C^2(M)$, show that

$$\Delta \varphi^* u = \varphi^* \Delta u.$$

Hint. The Laplace operator is uniquely specified by the metric tensor on M, via (9.26).

9. Let X and Y be smooth vector fields on an open set $\Omega \subset \mathbb{R}^3$. Show that

$$Y \cdot \operatorname{curl} X - X \cdot \operatorname{curl} Y = \operatorname{div}(X \times Y).$$

10. In the setting of Exercise 9, assume $\overline{\Omega}$ is compact and smoothly bounded, and that X and Y are C^1 on $\overline{\Omega}$. Show that

$$\int_{\Omega} X \cdot \operatorname{curl} Y \, dx = \int_{\Omega} Y \cdot \operatorname{curl} X \, dx,$$

provided either
(a) X is normal to $\partial\Omega$,
or
(b) X is parallel to Y on $\partial\Omega$.

11. Recall the formula (5.25) for the metric tensor of $\mathbb{R}^n$ in spherical polar coordinates $R : (0, \infty) \times S^{n-1} \to \mathbb{R}^n$, $R(r, \omega) = r\omega$. Using (9.26), show that if $u \in C^2(\mathbb{R}^n)$, then

$$\Delta u(r\omega) = \frac{\partial^2}{\partial r^2} u(r\omega) + \frac{n-1}{r} \frac{\partial}{\partial r} u(r\omega) + \frac{1}{r^2} \Delta_S u(r\omega),$$

where Δ_S is the Laplace operator on S^{n-1}. Deduce that

$$u(x) = f(|x|) \Longrightarrow \Delta u(r\omega) = f''(r) + \frac{n-1}{r} f'(r).$$

12. Show that

$$|x|^{-(n-2)} \ \text{ is harmonic on } \ \mathbb{R}^n \setminus 0.$$

In case $n = 2$, show that

$$\log|x| \text{ is harmonic on } \mathbb{R}^2 \setminus 0.$$

In Exercise 13, we take $n \geq 3$ and consider

$$\begin{aligned} Gf(x) &= \frac{1}{C_n} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy \\ &= \frac{1}{C_n} \int\limits_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-2}}\, dy, \end{aligned}$$

with $C_n = -(n-2)A_{n-1}$.

13. Assume $f \in C_0^2(\mathbb{R}^n)$. Let $\Omega_\varepsilon = \mathbb{R}^n \setminus B_\varepsilon$, where $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$. Verify that

$$\begin{aligned} C_n\, \Delta Gf(0) &= \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} \Delta f(x) \cdot |x|^{2-n}\, dx \\ &= \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} [\Delta f(x) \cdot |x|^{2-n} - f(x) \Delta |x|^{2-n}]\, dx \\ &= -\lim_{\varepsilon \to 0} \int\limits_{\partial B_\varepsilon} \Big[\varepsilon^{2-n} \frac{\partial f}{\partial r} - (2-n)\varepsilon^{1-n} f\Big]\, dS \\ &= -(n-2)A_{n-1} f(0), \end{aligned}$$

using (9.29) for the third identity. Use this to show that

$$\Delta Gf(x) = f(x).$$

14. Work out the analogue of Exercise 13 in case $n = 2$ and

$$Gf(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} f(y) \log|x-y|\, dy.$$

10. Holomorphic functions and harmonic functions

Let f be a *complex-valued* C^1 function on a region $\Omega \subset \mathbb{R}^2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$, via $z = x + iy$, and write $f(z) = f(x, y)$. We say f is *holomorphic* on Ω provided it is complex differentiable, in the sense that

$$\lim_{h \to 0} \frac{1}{h}(f(z+h) - f(z)) \ \text{ exists}, \tag{10.1}$$

for each $z \in \Omega$. When this limit exists, we denote it $f'(z)$, or df/dz. An equivalent condition (given $f \in C^1$) is that f satisfies the Cauchy-Riemann equation:

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}. \tag{10.1A}$$

In such a case,

$$f'(z) = \frac{\partial f}{\partial x}(z) = \frac{1}{i}\,\frac{\partial f}{\partial y}(z). \tag{10.1B}$$

Note that $f(z) = z$ has this property, but $f(z) = \overline{z}$ does not. The following is a convenient tool for producing more holomorphic functions.

Lemma 10.1. *If f and g are holomorphic on Ω, so is fg.*

Proof. We have

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}, \quad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}, \tag{10.2}$$

so if f and g satisfy the Cauchy-Riemann equation, so does fg. Note that

$$\frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z). \tag{10.2A}$$

Using Lemma 10.1, one can show inductively that if $k \in \mathbb{N}$, z^k is holomorphic on $\mathbb{C}$, and

$$\frac{d}{dz}z^k = kz^{k-1}. \tag{10.2B}$$

Also, a direct analysis of (10.1) gives this for $k = -1$, on $\mathbb{C} \setminus 0$, and then an inductive argument gives it for each negative integer k, on $\mathbb{C} \setminus 0$. The exercises explore various other important examples of holomorphic functions.

Our goal in this section is to show how Green's theorem can be used to establish basic results about holomorphic functions on domains in $\mathbb{C}$ (and also develop a study of harmonic functions on domains in $\mathbb{R}^n$). In Theorems 10.2–10.4, Ω will be a bounded domain with piecewise smooth boundary, and we assume Ω can be partitioned into a finite number of C^2 domains with corners, as defined in §8.

To begin, we apply Green's theorem to the line integral

$$\int_{\partial\Omega} f\,dz = \int_{\partial\Omega} f(dx + i\,dy).$$

Clearly (9.2) applies to complex-valued functions, and if we set $g = if$, we get

$$\int_{\partial\Omega} f\,dz = \iint_{\Omega} \Big(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Big)\,dx\,dy. \tag{10.3}$$

Whenever f is holomorphic, the integrand on the right side of (10.3) vanishes, so we have the following result, known as Cauchy's Integral Theorem:

Theorem 10.2. *If $f \in C^1(\overline{\Omega})$ is holomorphic, then*

$$\int_{\partial\Omega} f(z)\,dz = 0. \tag{10.4}$$

Using (10.4), we can establish Cauchy's Integral Formula:

Theorem 10.3. *If $f \in C^1(\overline{\Omega})$ is holomorphic and $z_0 \in \Omega$, then*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z - z_0}\,dz. \tag{10.5}$$

Proof. Note that $g(z) = f(z)/(z - z_0)$ is holomorphic on $\Omega \setminus \{z_0\}$. Let D_r be the disk of radius r centered at z_0. Pick r so small that $D_r \subset \Omega$. Then (10.4) implies

$$\int_{\partial\Omega} \frac{f(z)}{z - z_0}\,dz = \int_{\partial D_r} \frac{f(z)}{z - z_0}\,dz. \tag{10.6}$$

To evaluate the integral on the right, parametrize the curve ∂D_r by $\gamma(\theta) = z_0 + re^{i\theta}$. Hence $dz = ire^{i\theta}\,d\theta$, so the integral on the right is equal to

$$\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\,ire^{i\theta}\,d\theta = i\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta. \tag{10.7}$$

As $r \to 0$, this tends in the limit to $2\pi i f(z_0)$, so (10.5) is established.

Suppose $f \in C^1(\overline{\Omega})$ is holomorphic, $z_0 \in D_r \subset \Omega$, where D_r is the disk of radius r centered at z_0, and suppose $z \in D_r$. Then Theorem 10.3 implies

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta. \tag{10.8}$$

We have the infinite series expansion

$$\frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n, \tag{10.9}$$

valid as long as $|z - z_0| < |\zeta - z_0|$. Hence, given $|z - z_0| < r$, this series is uniformly convergent for $\zeta \in \partial\Omega$, and we have

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int\limits_{\partial\Omega} \frac{f(\zeta)}{\zeta - z_0} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n\, d\zeta. \tag{10.10}$$

We summarize what has been established.

Theorem 10.4. *Given $f \in C^1(\overline{\Omega})$, holomorphic on Ω and a disk $D_r \subset \Omega$ as above, for $z \in D_r$, $f(z)$ has the convergent power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta. \tag{10.11}$$

Note that, when (10.5) is applied to $\Omega = D_r$, the disk of radius r centered at z_0, the computation (10.7) yields

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = \frac{1}{\ell(\partial D_r)} \int\limits_{\partial D_r} f(z)\, ds(z), \tag{10.12}$$

when f is holomorphic and C^1 on D_r, and $\ell(\partial D_r) = 2\pi r$ is the length of the circle ∂D_r. This is a *mean value property*, which extends to harmonic functions on domains in $\mathbb{R}^n$, as we will see below.

Note that we can write (10.1) as $(\partial_x + i\partial_y) f = 0$; applying the operator $\partial_x - i\partial_y$ to this gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{10.13}$$

for any holomorphic function. A general C^2 solution to (10.13) on a region $\Omega \subset \mathbb{R}^2$ is called a *harmonic* function. More generally, if $\mathcal{O}$ is an open set in $\mathbb{R}^n$, a function $f \in C^2(\Omega)$ is called harmonic if $\Delta f = 0$ on $\mathcal{O}$, where, as in (9.27),

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}. \tag{10.14}$$

Generalizing (10.12), we have the following, known as the mean value property of harmonic functions:

Proposition 10.5. *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $u \in C^2(\Omega)$ *be harmonic,* $p \in \Omega$*, and* $B_R(p) = \{x \in \Omega : |x-p| \le R\} \subset \Omega$*. Then*

$$u(p) = \frac{1}{A\big(\partial B_R(p)\big)} \int\limits_{\partial B_R(p)} u(x)\ dS(x). \tag{10.15}$$

For the proof, set

$$\psi(r) = \frac{1}{A(S^{n-1})} \int\limits_{S^{n-1}} u(p + r\omega)\ dS(\omega), \tag{10.16}$$

for $0 < r \le R$. We have $\psi(R)$ equal to the right side of (10.15), while clearly $\psi(r) \to u(p)$ as $r \to 0$. Now

$$\psi'(r) = \frac{1}{A(S^{n-1})} \int\limits_{S^{n-1}} \omega \cdot \nabla u(p + r\omega)\ dS(\omega) = \frac{1}{A\big(\partial B_r(p)\big)} \int\limits_{\partial B_r(p)} \frac{\partial u}{\partial \nu}\ dS(x). \tag{10.17}$$

At this point, we establish:

Lemma 10.6. *If* $\mathcal{O} \subset \mathbb{R}^n$ *is a bounded domain with smooth boundary and* $u \in C^2(\overline{\mathcal{O}})$ *is harmonic in* $\mathcal{O}$*, then*

$$\int\limits_{\partial \mathcal{O}} \frac{\partial u}{\partial \nu}(x)\ dS(x) = 0. \tag{10.18}$$

Proof. Apply the Green formula (9.29), with $M = \mathcal{O}$ and $v = 1$. If $\Delta u = 0$, every integrand in (9.29) vanishes, except the one appearing in (10.18), so this integrates to zero.

It follows from this lemma that (10.17) vanishes, so $\psi(r)$ is constant. This completes the proof of (10.15).

We can integrate the identity (10.15), to obtain

$$u(p) = \frac{1}{V\big(B_R(p)\big)} \int\limits_{B_R(p)} u(x)\ dV(x), \tag{10.19}$$

where $u \in C^2\big(\overline{B_R(p)}\big)$ is harmonic. This is another expression of the mean value property.

The mean value property of harmonic functions has a number of important consequences. Here we mention one result, known as Liouville's Theorem.

Proposition 10.7. *If* $u \in C^2(\mathbb{R}^n)$ *is harmonic on all of* $\mathbb{R}^n$ *and bounded, then* u *is constant.*

Proof. Pick any two points $p, q \in \mathbb{R}^n$. We have, for any $r > 0$,

$$u(p) - u(q) = \frac{1}{V\big(B_r(0)\big)} \left[\int\limits_{B_r(p)} u(x)\,dx - \int\limits_{B_r(q)} u(x)\,dx \right]. \tag{10.20}$$

Note that $V\big(B_r(0)\big) = C_n r^n$, where C_n is evaluated in problem 2 of §5. Thus

$$|u(p) - u(q)| \le \frac{C_n}{r^n} \int\limits_{\Delta(p,q,r)} |u(x)|\,dx, \tag{10.21}$$

where

$$\Delta(p,q,r) = B_r(p) \triangle B_r(q) = \Big(B_r(p) \setminus B_r(q)\Big) \cup \Big(B_r(q) \setminus B_r(p)\Big). \tag{10.22}$$

Note that, if $a = |p - q|$, then $\Delta(p,q,r) \subset B_{r+a}(p) \setminus B_{r-a}(p)$; hence

$$V\big(\Delta(p,q,r)\big) \le C(p,q)\, r^{n-1}, \quad r \ge 1. \tag{10.23}$$

It follows that, if $|u(x)| \le M$ for all $x \in \mathbb{R}^n$, then

$$|u(p) - u(q)| \le M C_n C(p,q)\, r^{-1}, \quad \forall\, r \ge 1. \tag{10.24}$$

Taking $r \to \infty$, we obtain $u(p) - u(q) = 0$, so u is constant.

We will now use Liouville's Theorem to prove the Fundamental Theorem of Algebra:

Theorem 10.8. *If* $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ *is a polynomial of degree* $n \ge 1$ $(a_n \ne 0)$*, then* $p(z)$ *must vanish somewhere in* $\mathbb{C}$.

Proof. Consider

$$f(z) = \frac{1}{p(z)}. \tag{10.25}$$

If $p(z)$ does not vanish anywhere in $\mathbb{C}$, then $f(z)$ is holomorphic on all of $\mathbb{C}$. (See Exercise 9 below.) On the other hand,

$$f(z) = \frac{1}{z^n} \, \frac{1}{a_n + a_{n-1} z^{-1} + \cdots + a_0 z^{-n}}, \tag{10.26}$$

so

$$|f(z)| \longrightarrow 0, \ \text{as } |z| \to \infty. \tag{10.27}$$

Thus f is bounded on $\mathbb{C}$, if $p(z)$ has no roots. By Proposition 10.7, $f(z)$ must be constant, which is impossible, so $p(z)$ must have a complex root.

From the fact that every holomorphic function f on $\mathcal{O} \subset \mathbb{R}^2$ is harmonic, it follows that its real and imaginary parts are harmonic. This result has a converse. Let $u \in C^2(\mathcal{O})$ be harmonic. Consider the 1-form

$$\alpha = -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy. \tag{10.28}$$

We have $d\alpha = -(\Delta u) dx \wedge dy$, so α is closed if and only if u is harmonic. Now, if $\mathcal{O}$ is diffeomorphic to a disk, it follows from Proposition 8.3 that α is exact on $\mathcal{O}$, whenever it is closed, so, in such a case,

$$\Delta u = 0 \text{ on } \mathcal{O} \Longrightarrow \exists\, v \in C^1(\mathcal{O}) \text{ s.t. } \alpha = dv. \tag{10.29}$$

In other words,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{10.30}$$

This is precisely the Cauchy-Riemann equation (10.1) for $f = u + iv$, so we have:

Proposition 10.9. *If $\mathcal{O} \subset \mathbb{R}^2$ is diffeomorphic to a disk and $u \in C^2(\mathcal{O})$ is harmonic, then u is the real part of a holomorphic function on $\mathcal{O}$.*

The function v (which is unique up to an additive constant) is called the *harmonic conjugate* of u.

We close this section with a brief mention of holomorphic functions on a domain $\mathcal{O} \subset \mathbb{C}^n$. We say $f \in C^1(\mathcal{O})$ is holomorphic provided it satisfies

$$\frac{\partial f}{\partial x_j} = \frac{1}{i}\frac{\partial f}{\partial y_j}, \quad 1 \le j \le n. \tag{10.31}$$

Suppose $z \in \mathcal{O},\ z = (z_1, \dots, z_n)$. Suppose $\zeta \in \mathcal{O}$ whenever $|z - \zeta| < r$. Then, by successively applying Cauchy's integral formula (10.5) to each complex variable z_j, we have that

$$f(z) = (2\pi i)^{-n} \int_{\gamma_n} \cdots \int_{\gamma_1} f(\zeta)(\zeta_1 - z_1)^{-1} \cdots (\zeta_n - z_n)^{-1}\, d\zeta_1 \cdots d\zeta_n, \tag{10.32}$$

where γ_j is any simple counterclockwise curve about z_j in $\mathbb{C}$ with the property that $|\zeta_j - z_j| < r/\sqrt{n}$ for all $\zeta_j \in \gamma_j$.

Consequently, if $p \in \mathbb{C}^n$ and $\mathcal{O}$ contains the "polydisc"

$$\overline{D} = \{z \in \mathbb{C}^n : |z_j - p_j| \le \delta,\ \forall\, j\},$$

then, for $z \in D$, the interior of $\overline{D}$, we have

$$f(z) = (2\pi i)^{-n} \int_{C_n} \cdots \int_{C_1} f(\zeta)\big[(\zeta_1 - p_1) - (z_1 - p_1)\big]^{-1} \cdots \big[(\zeta_n - p_n) - (z_n - p_n)\big]^{-1}\, d\zeta_1 \cdots d\zeta_n, \tag{10.33}$$

where $C_j = \{\zeta \in \mathbb{C} : |\zeta - p_j| = \delta\}$. Then, parallel to (10.8)–(10.11), we have

$$f(z) = \sum_{\alpha \ge 0} c_\alpha (z - p)^\alpha, \tag{10.34}$$

for $z \in D$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $(z-p)^\alpha = (z_1 - p_1)^{\alpha_1} \cdots (z_n - p_n)^{\alpha_n}$, as in (1.13), and

$$c_\alpha = (2\pi i)^{-n} \int_{C_n} \cdots \int_{C_1} f(\zeta)(\zeta_1 - p_1)^{-\alpha_1 - 1} \cdots (\zeta_n - p_n)^{-\alpha_n - 1}\, d\zeta_1 \cdots d\zeta_n. \tag{10.35}$$

Thus holomorphic functions on open domains in $\mathbb{C}^n$ have convergent power series expansions.

We refer to [Ahl], [Hil], and [T6] for more material on holomorphic functions of one complex variable, and to [Kr] for material on holomorphic functions of several complex variables. A source of much information on harmonic functions is [Kel]. Also further material on these subjects can be found in [T].

Exercises

1. Let $f_k : \Omega \to \mathbb{C}$ be holomorphic on an open set $\Omega \subset \mathbb{C}$. Assume $f_k \to f$ and $\nabla f_k \to \nabla f$ locally uniformly in Ω. Show that $f : \Omega \to \mathbb{C}$ is holomorphic.

2. Assume

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{10.36}$$

is absolutely convergent for $|z| < R$. Deduce from Proposition 1.10 and Exercise 1 above that f is holomorphic on $|z| < R$, and that

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}, \quad \text{for} \quad |z| < R. \tag{10.37}$$

3. As in (3.89), the exponential function e^z is defined by

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k. \tag{10.38}$$

Deduce from Exercise 2 that e^z is holomorphic in z.

4. By (3.90), we have

$$e^{z+h} = e^z e^h, \quad \forall\, z, h \in \mathbb{C}.$$

Use this to show directly from (10.1) that e^z is complex differentiable and $(d/dz)e^z = e^z$ on $\mathbb{C}$, giving another proof that e^z is holomorphic on $\mathbb{C}$.
Hint. Use the power series for e^h to show that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

5. For another approach to the fact that e^z is holomorphic, use

$$e^z = e^x e^{iy}$$

and (3.89) to verify that e^z satisfies the Cauchy-Riemann equation.

6. For $z \in \mathbb{C}$, set

$$\cos z = \frac{1}{2}\big(e^{iz} + e^{-iz}\big), \quad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big). \tag{10.39}$$

Show that these functions agree with the definitions of $\cos t$ and $\sin t$ given in (3.91)–(3.92), for $z = t \in \mathbb{R}$. Show that $\cos z$ and $\sin z$ are holomorphic in $z \in \mathbb{C}$. Show that

$$\frac{d}{dz} \cos z = -\sin z, \quad \frac{d}{dz} \sin z = \cos z, \tag{10.40}$$

and

$$\cos^2 z + \sin^2 z = 1, \tag{10.41}$$

for all $z \in \mathbb{C}$.

7. Let $\mathcal{O}, \Omega$ be open in $\mathbb{C}$. If f is holomorphic on $\mathcal{O}$, with range in Ω, and g is holomorphic on Ω, show that $h = g \circ f$ is holomorphic on $\mathcal{O}$, and $h'(z) = g'(f(z))f'(z)$.
Hint. See the proof of the chain rule in §1.

8. Let $\Omega \subset \mathbb{C}$ be a connected open set and let f be holomorphic on Ω.
(a) Show that if $f(z_j) = 0$ for distinct $z_j \in \Omega$ and $z_j \to z_0 \in \Omega$, then $f(z) = 0$ for z in a neighborhood of z_0.
Hint. Use the power series expansion (10.1).
(b) Show that if $f = 0$ on a nonempty open set $\mathcal{O} \subset \Omega$, then $f \equiv 0$ on Ω.
Hint. Let $U \subset \Omega$ denote the interior of the set of points where f vanishes. Use part (a) to show that $\overline{U} \cap \Omega$ is open.

9. Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and define $\log : \Omega \to \mathbb{C}$ by

$$\log z = \int_{\gamma_z} \frac{1}{\zeta}\, d\zeta, \tag{10.42}$$

where γ_z is a path from 1 to z in Ω. Use Theorem 10.2 to show that this is independent of the choice of such path. Show that it yields a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$, satisfying

$$\frac{d}{dz} \log z = \frac{1}{z}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

10. Taking $\log z$ as in Exercise 9, show that

$$e^{\log z} = z, \quad \forall\, z \in \mathbb{C} \setminus (-\infty, 0]. \tag{10.43}$$

Hint. If $\varphi(z)$ denotes the left side, show that $\varphi(1) = 1$ and $\varphi'(z) = \varphi(z)/z$. Use uniqueness results from §3 to deduce that $\varphi(x) = x$ for $x \in (0, \infty)$, and from there deduce that $\varphi(z) \equiv z$, using Exercise 8.
Alternative. Apply d/dz to show that

$$\log e^z = z,$$

for z in some neighborhood of 0. Deduce from this (and Exercise 3 of §2) that (10.43) holds for z in some neighborhood of 1. Then get it for all $z \in \mathbb{C} \setminus (-\infty, 0]$ using Exercise 8.

11. With $\Omega = \mathbb{C} \setminus (-\infty, 0]$ as in Exercise 9, and $a \in \mathbb{C}$, define z^a for $z \in \Omega$ by

$$z^a = e^{a \log z}. \tag{10.44}$$

Show that this is holomorphic on Ω and

$$\frac{d}{dz} z^a = a z^{a-1}, \quad z^a z^b = z^{a+b}, \quad \forall\, z \in \Omega. \tag{10.45}$$

12. Let $\mathcal{O} = \mathbb{C} \setminus \{[1,\infty) \cup (-\infty,-1]\}$, and define $As : \mathcal{O} \to \mathbb{C}$ by

$$As(z) = \int\limits_{\sigma_z} (1-\zeta^2)^{-1/2}\, d\zeta,$$

where σ_z is a path from 0 to z in $\mathcal{O}$. Show that this is independent of the choice of such a path, and that it yields a holomorphic function on $\mathcal{O}$.

13. With As as in Exercise 12, show that

$$As(\sin z) = z,$$

for z in some neighborhood of 0. (*Hint.* Apply d/dz.) From here, show that

$$\sin(As(z)) = z, \quad \forall\, z \in \mathcal{O}.$$

Thus we write

$$\arcsin z = \int_0^z (1-\zeta^2)^{-1/2}\, d\zeta. \tag{10.46}$$

Compare (3.97).

14. Look again at Exercise 4 in §1.

15. Look again at Exercises 3–5 in §2. Write the result as an inverse function theorem for holomorphic maps.

16. Differentiate (10.5) to show that, in the setting of Theorem 10.3, for $k \in \mathbb{N}$,

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int\limits_{\partial\Omega} \frac{f(z)}{(z-z_0)^{k+1}}\, dz. \tag{10.47}$$

Show that this also follows from (10.11).

17. Assume f is holomorphic on $\mathbb{C}$, and set

$$M(z_0,R) = \sup_{|z-z_0|\le R} |f(z)|.$$

Use the $k=1$ case of Exercise 16 to show that

$$|f'(z_0)| \le \frac{M(z_0,R)}{R}, \quad \forall\, R \in (0,\infty).$$

18. In the setting of Exercise 17, assume f is bounded, say $|f(z)| \le M$ for all $z \in \mathbb{C}$. Deduce that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$, and in that way obtain another proof of Liouville's theorem, in the setting of holomorphic functions on $\mathbb{C}$. (Note that Proposition 10.7 is more general.)

The next four exercises deal with the function

$$G(z) = \int_{-\infty}^{\infty} e^{-t^2+tz}\,dt, \quad z \in \mathbb{C}. \tag{10.48}$$

19. Show that G is continuous on $\mathbb{C}$.

20. Show that G is holomorphic on $\mathbb{C}$, with

$$G'(z) = \int_{-\infty}^{\infty} te^{-t^2+tz}\,dt.$$

Hint. Write

$$\frac{1}{h}[G(z+h) - G(z)] = \int_{-\infty}^{\infty} e^{-t^2+tz}\frac{1}{h}(e^{th} - 1)\,dt,$$

and

$$\frac{1}{h}(e^{th} - 1) = t + \frac{1}{h}R(th),$$

where

$$e^w = 1 + w + R(w), \quad |R(w)| \le C|w|^2 e^{|w|},$$

so

$$\Bigl|\frac{1}{h}R(th)\Bigr| \le Ct^2|h|e^{|th|}.$$

21. Show that, for $x \in \mathbb{R}$,

$$G(x) = \sqrt{\pi}e^{x^2/4}.$$

Hint. Write

$$G(x) = e^{x^2/4}\int_{-\infty}^{\infty} e^{-(t-x/2)^2}\,dt,$$

and make a change of variable in the integral.

22. Deduce from Exercises 21 and 8 that

$$G(z) = \sqrt{\pi}\,e^{z^2/4}, \quad \forall\, z \in \mathbb{C}. \tag{10.49}$$

The next exercises deal with the Gamma function,

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\, ds, \tag{10.50}$$

defined for $z > 0$ in (5.32).

23. Show that the integral is absolutely convergent for $\operatorname{Re} z > 0$ and defines $\Gamma(z)$ as a holomorphic function on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

24. Extend the identity (5.35), i.e.,

$$\Gamma(z+1) = z\Gamma(z), \tag{10.51}$$

to $\operatorname{Re} z > 0$.

25. Use (10.51) to extend Γ to be holomorphic on $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.

26. Use the result of Exercise 16 to show that if f_ν are holomorphic on an open set $\Omega \subset \mathbb{C}$ and $f_\nu \to f$ uniformly on compact subsets of Ω, then f is holomorphic on Ω and $f_\nu' \to f'$ uniformly on compact subsets.

27. The Riemann zeta function $\zeta(z)$ is defined for $\operatorname{Re} z > 1$ by

$$\zeta(z) = \sum_{k=1}^\infty \frac{1}{k^z}. \tag{10.52}$$

Show that $\zeta(z)$ is holomorphic on $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$.

The following exercises deal with harmonic functions on domains in $\mathbb{R}^n$.

28. Using the formula (9.26) for the Laplace operator together with the formula (5.25) for the metric tensor on $\mathbb{R}^n$ in spherical polar coordinates $x = r\omega,\ x \in \mathbb{R}^n,\ r = |x|,\ \omega \in S^{n-1}$, show that of $u \in C^2(\Omega),\ \Omega \subset \mathbb{R}^n$,

$$\Delta u(r\omega) = \frac{\partial^2}{\partial r^2} u(r\omega) + \frac{n-1}{r} \frac{\partial}{\partial r} u(r\omega) + \frac{1}{r^2} \Delta_S u(r\omega), \tag{10.53}$$

where Δ_S is the Laplace operator on S^{n-1}.

29. If $f(x) = \varphi(|x|)$ on $\mathbb{R}^n$, show that

$$\Delta f(x) = \varphi''(|x|) + \frac{n-1}{|x|} \varphi'(|x|). \tag{10.54}$$

In particular, show that

$$|x|^{-(n-2)} \text{ is harmonic on } \mathbb{R}^n \setminus 0, \tag{10.55}$$

if $n \geq 3$, and

$$\log |x| \text{ is harmonic on } \mathbb{R}^2 \setminus 0. \tag{10.56}$$

If $\mathcal{O}, \Omega$ are open in $\mathbb{R}^n$, a smooth map $\varphi : \mathcal{O} \to \Omega$ is said to be *conformal* provided the matrix function $G(x) = D\varphi(x)^t D\varphi(x)$ is a multiple of the identity, $G(x) = \gamma(x)I$. Recall formula (5.2).

30. Suppose $n = 2$ and φ preserves orientation. Show that φ (pictured as a function $\varphi : \mathcal{O} \to \mathbb{C}$) is conformal *if and only if* it is holomorphic. If φ reverses orientation, φ is conformal $\Leftrightarrow$ $\overline{\varphi}$ is holomorphic (we say φ is anti-holomorphic).

31. If $\mathcal{O}$ and Ω are open in $\mathbb{R}^2$ and u is harmonic on Ω, show that $u \circ \varphi$ is harmonic on $\mathcal{O}$, whenever $\varphi : \mathcal{O} \to \Omega$ is a smooth conformal map.
Hint. Use Exercise 7 and Proposition 10.9.

The following exercises will present an alternative approach to the proof of Proposition 10.5 (the mean value property of harmonic functions). For this, let $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$. Assume u is continuous on B_R and C^2 and harmonic on the interior $\overset{\circ}{B}_R$. We assume $n \geq 2$.

32. Given $g \in SO(n)$, show that $u_g(x) = u(gx)$ is harmonic on $\overset{\circ}{B}_R$.
Hint. See Exercise 7 of §9.

33. As in Exercise 24 of §5, define $\mathcal{A}u \in C(B_R)$ by

$$\mathcal{A}u(x) = \int_{SO(n)} u(gx)\, dg.$$

Thus $\mathcal{A}u(x)$ is a radial function:

$$\mathcal{A}u(x) = \mathcal{S}u(|x|), \quad \mathcal{S}u(r) = \frac{1}{A_{n-1}} \int_{S^{n-1}} u(r\omega)\, dS(\omega).$$

Deduce from Exercise 32 above that $\mathcal{A}u$ is harmonic on $\overset{\circ}{B}_R$.

34. Use Exercise 29 to show that $\varphi(r) = \mathcal{S}u(r)$ satisfies

$$\varphi''(r) + \frac{n-1}{r}\varphi'(r) = 0,$$

for $r \in (0, R)$. Deduce from this differential equation that there exist constants C_0 and C_1 such that

$$\begin{aligned} \varphi(r) = C_0 + C_1 r^{-(n-2)}, &\quad \text{if} \;\; n \geq 3, \\ C_0 + C_1 \log r, &\quad \text{if} \;\; n = 2. \end{aligned}$$

Then show that, since $\mathcal{A}u(x)$ does not blow up at $x = 0$, $C_1 = 0$. Hence

$$\mathcal{A}u(x) = C_0, \quad \forall\, x \in B_R.$$

35. Note that $\mathcal{A}u(0) = u(0)$. Deduce that for each $r \in (0, R]$,

(10.57) $$u(0) = \mathcal{S}u(r) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} u(r\omega)\, dS(\omega).$$

11. Homotopies of maps and actions on forms

Let X and Y be smooth surfaces. Two smooth maps $f_0, f_1 : X \to Y$ are said to be smoothly homotopic provided there is a smooth $F : [0,1] \times X \to Y$ such that $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$. The following result illustrates the significance of maps being homotopic.

Proposition 11.1. *Assume X is a compact, oriented, k-dimensional surface and $\alpha \in \Lambda^k(Y)$ is closed, i.e., $d\alpha = 0$. If $f_0, f_1 : X \to Y$ are smoothly homotopic, then*

$$\int_X f_0^* \alpha = \int_X f_1^* \alpha. \tag{11.1}$$

In fact, with $[0,1] \times X = \overline{\Omega}$, this is a special case of the following.

Proposition 11.2. *Assume $\overline{\Omega}$ is a smoothly bounded, compact, oriented $(k+1)$-dimensional surface, and $\alpha \in \Lambda^k(Y)$ is closed. If $F : \overline{\Omega} \to Y$ is a smooth map, then*

$$\int_{\partial\Omega} F^* \alpha = 0. \tag{11.2}$$

Proof. Stokes' theorem gives

$$\int_{\partial\Omega} F^* \alpha = \int_{\Omega} dF^* \alpha = 0, \tag{11.3}$$

since $dF^*\alpha = F^* d\alpha$ and, by hypothesis, $d\alpha = 0$.

Proposition 11.2 is one generalization of Proposition 11.1. Here is another.

Proposition 11.3. *Assume X is a k-dimensional surface and $\alpha \in \Lambda^\ell(Y)$ is closed. If $f_0, f_1 : X \to Y$ are smoothly homotopic, then $f_0^*\alpha - f_1^*\alpha$ is exact, i.e.,*

$$f_0^*\alpha - f_1^*\alpha = d\beta, \tag{11.4}$$

for some $\beta \in \Lambda^{\ell-1}(X)$.

Proof. Take a smooth $F : \mathbb{R} \times X \to Y$ such that $F(j,x) = f_j(x)$. Consider

$$\tilde{\alpha} = F^*\alpha \in \Lambda^\ell(\mathbb{R} \times X). \tag{11.5}$$

Note that $d\tilde{\alpha} = F^* d\alpha = 0$. Now consider

$$\Phi_s : \mathbb{R} \times X \longrightarrow \mathbb{R} \times X, \quad \Phi_s(t,x) = (s+t, x). \tag{11.6}$$

We claim that

$$\tilde{\alpha} - \Phi_1^* \tilde{\alpha} = d\tilde{\beta}, \tag{11.7}$$

for some $\tilde{\beta} \in \Lambda^{\ell-1}(\mathbb{R} \times X)$. Now take

$$\beta = j^* \tilde{\beta}, \quad j : X \to \mathbb{R} \times X, \quad j(x) = (0, x). \tag{11.8}$$

We have $F \circ j = f_0,\ F \circ \Phi_1 \circ j = f_1$, so it follows that

$$\begin{aligned} f_0^* \alpha - f_1^* \alpha &= j^* \tilde{\alpha} - j^* \Phi_1^* \tilde{\alpha} \\ &= j^* d\tilde{\beta}, \end{aligned} \tag{11.9}$$

given (11.7), which yields (11.4) with β as in (11.8).

It remains to prove (11.7), under the hypothesis that $d\tilde{\alpha} = 0$. The following result gives this. The formula (11.10) uses the interior product, defined by (7.4)–(7.5).

Lemma 11.4. *If $\tilde{\alpha} \in \Lambda^\ell(\mathbb{R} \times X)$ and Φ_s is as in (11.6), then*

$$\frac{d}{ds} \Phi_s^* \tilde{\alpha} = \Phi_s^* \Big(d(\tilde{\alpha} \rfloor \partial_t) + (d\tilde{\alpha}) \rfloor \partial_t \Big). \tag{11.10}$$

Hence, if $d\tilde{\alpha} = 0$, (11.7) holds with

$$\tilde{\beta} = - \int_0^1 (\Phi_s^* \tilde{\alpha}) \rfloor \partial_t \, ds. \tag{11.11}$$

Proof. Since $\Phi_{s+\sigma}^* = \Phi_s^* \Phi_\sigma^* = \Phi_\sigma^* \Phi_s^*$, it suffices to show that (11.10) holds at $s = 0$. It also suffices to work in local coordinates on X. Say

$$\begin{aligned} \tilde{\alpha} = &\sum_i \alpha_i^\#(t, x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \\ &+ \sum_j \alpha_j^b(t, x) \, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{\ell-1}}. \end{aligned} \tag{11.12}$$

We have $\Phi_s^* \tilde{\alpha}$ given by a similar formula, with coefficients replaced by $\alpha_i^\#(t + s, x)$ and $\alpha_j^b(t + s, x)$, hence

$$\begin{aligned} \frac{d}{ds} \Phi_s^* \tilde{\alpha} \big|_{s=0} = &\sum_i \partial_t \alpha_i^\#(t, x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \\ &+ \sum_j \partial_t \alpha_j^b(t, x) \, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{\ell-1}}. \end{aligned} \tag{11.13}$$

Meanwhile

$$\tilde{\alpha}\rfloor\partial_t = \sum_j \alpha_j^b(t,x)\,dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}}, \tag{11.14}$$

so

$$\begin{aligned} d(\tilde{\alpha}\rfloor\partial_t) = &\sum_j \partial_t\alpha_j^b(t,x)\,dt\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}} \\ &+\sum_{j,\nu}\partial_{x_\nu}\alpha_j^b(t,x)\,dx_\nu\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}}. \end{aligned} \tag{11.15}$$

A similar calculation yields

$$\begin{aligned} (d\tilde{\alpha})\rfloor\partial_t = &\sum_i \partial_t\alpha_i^\#(t,x)\,dx_{i_1}\wedge\cdots\wedge dx_{i_\ell} \\ &-\sum_{j,\nu}\partial_{x_\nu}\alpha_j^b(t,x)\,dx_\nu\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{\ell-1}}. \end{aligned} \tag{11.16}$$

Comparison of (11.15)–(11.16) with (11.13) yields (11.10) at $s=0$, proving Lemma 11.4.

The following consequence of Proposition 11.3 contains the Poincaré lemma.

Proposition 11.5. *Let X be a smooth k-dimensional surface. Assume the identity map $I : X \to X$ is smoothly homotopic to a constant map $K : X \to X$, satisfying $K(x)\equiv p$. Then, for all $\ell\in\{1,\dots,k\}$,*

$$\alpha\in\Lambda^\ell(X),\ d\alpha=0 \Longrightarrow \alpha \quad \textit{is exact.} \tag{11.17}$$

Proof. By Proposition 11.3, $\alpha - K^*\alpha$ is exact. However, $K^*\alpha = 0$.

Proposition 11.5 applies to any open $X\subset\mathbb{R}^k$ that is star-shaped, so

$$D_s : X \longrightarrow X \quad \text{for} \quad s\in[0,1], \qquad D_s(x) = sx. \tag{11.18}$$

Thus, for any open star-shaped $X\subset\mathbb{R}^k$, each closed $\alpha\in\Lambda^\ell(X)$ is exact.

We next present an important generalization of Lemma 11.4. Let Ω be a smooth n-dimensional surface. If $\alpha\in\Lambda^k(\Omega)$ and X is a vector field on Ω, generating a flow $\mathcal{F}_X^t$, the *Lie derivative* $\mathcal{L}_X\alpha$ is defined to be

$$\mathcal{L}_X\alpha = \frac{d}{dt}(\mathcal{F}_X^t)^*\alpha\Big|_{t=0}. \tag{11.19}$$

Note the similarity to the definition (3.77) of $\mathcal{L}_XY$ for a vector field Y, for which there was the alternative formula (3.80). The following useful result is known as Cartan's formula for the Lie derivative.

Proposition 11.6. *We have*

$$\mathcal{L}_X\alpha = d(\alpha\rfloor X) + (d\alpha)\rfloor X. \tag{11.20}$$

Proof. We can assume Ω is an open subset of $\mathbb{R}^n$. First we compare both sides in the special case $X = \partial/\partial x_\ell = \partial_\ell$. Note that

$$(\mathcal{F}^t_{\partial_\ell})^*\alpha = \sum_j a_j(x + te_\ell)\, dx_{j_1}\wedge\cdots\wedge dx_{j_k}, \tag{11.21}$$

so

$$\mathcal{L}_{\partial_\ell}\alpha = \sum_j \partial_{x_\ell} a_j(x)\, dx_{j_1}\wedge\cdots\wedge dx_{j_k} = \partial_\ell\alpha. \tag{11.22}$$

To evaluate the right side of (11.21), with $X = \partial_\ell$, we could parallel the calculation (11.14)–(11.16). Alternatively, we can use (7.12) to write this as

$$d(\iota_\ell\alpha) + \iota_\ell d\alpha = \sum_{j=1}^n (\partial_j \wedge_j \iota_\ell + \iota_\ell\partial_j\wedge_j)\alpha. \tag{11.23}$$

Using the commutativity of ∂_j with $\wedge_j$ and with ι_ℓ, and the anticommutativity relations (7.8), we see that the right side of (11.23) is $\partial_\ell\alpha$, which coincides with (11.22). Thus the proposition holds for $X = \partial/\partial x_\ell$.

Now we prove the proposition in general, for a smooth vector field X on Ω. It is to be verified at each point $x_0 \in \Omega$. If $X(x_0) \neq 0$, we can apply Theorem 3.7 to choose a coordinate system about x_0 so $X = \partial/\partial x_1$ and use the calculation above. This shows that the desired identity holds on the set $\{x_0 \in \Omega : X(x_0) \neq 0\}$, and by continuity it holds on the closure of this set. However, if x_0 has a neighborhood on which X vanishes, it is clear that $\mathcal{L}_X\alpha = 0$ near x_0 and also $\alpha\rfloor X$ and $d\alpha\rfloor X$ vanish near x_0. This completes the proof.

From (11.19) and the identity $\mathcal{F}^{s+t}_X = \mathcal{F}^s_X\mathcal{F}^t_X$, it follows that

$$\frac{d}{dt}(\mathcal{F}^t_X)^*\alpha = \mathcal{L}_X(\mathcal{F}^t_X)^*\alpha = (\mathcal{F}^t_X)^*\mathcal{L}_X\alpha. \tag{11.24}$$

It is useful to generalize this. Let F_t be a smooth family of diffeomorphisms of M into M. Define vector fields X_t on $F_t(M)$ by

$$\frac{d}{dt}F_t(x) = X_t(F_t(x)). \tag{11.25}$$

Then, given $\alpha \in \Lambda^k(M)$,

$$\begin{aligned}\frac{d}{dt}F_t^*\alpha &= F_t^*\mathcal{L}_{X_t}\alpha \\ &= F_t^*\Big[d(\alpha\rfloor X_t) + (d\alpha)\rfloor X_t\Big].\end{aligned} \tag{11.26}$$

In particular, if α is closed, then, if F_t are diffeomorphisms for $0 \le t \le 1$,

$$F_1^*\alpha - F_0^*\alpha = d\beta, \quad \beta = \int_0^1 F_t^*(\alpha \rfloor X_t)\, dt. \tag{11.27}$$

The fact that the left side of (11.27) is exact is a special case of Proposition 11.3, but the explicit formula given in (11.27) can be useful.

More on the divergence of a vector field

Let $M \subset \mathbb{R}^n$ be an m-dimensional, oriented surface, with volume form ω. Then $d\omega = 0$ on M, so, if X is a vector field on M,

$$\mathcal{L}_X \omega = d(\omega \rfloor X). \tag{11.28}$$

Comparison with (9.7) gives

$$(\operatorname{div} X)\omega = \mathcal{L}_X \omega. \tag{11.29}$$

This is sometimes taken as the definition of $\operatorname{div} X$. It readily leads to a formula for how the flow $\mathcal{F}_X^t$ affects volumes.

To get this, we start with

$$\begin{aligned} \frac{d}{dt}(\mathcal{F}_X^t)^*\omega &= (\mathcal{F}_X^t)^*\mathcal{L}_X\omega \\ &= (\mathcal{F}_X^t)^*((\operatorname{div} X)\omega). \end{aligned} \tag{11.30}$$

Hence, if $\Omega \subset M$ is a smoothly bounded domain on which the flow $\mathcal{F}_X^t$ is defined for $t \in I$, then, for such t,

$$\begin{aligned} \frac{d}{dt}\operatorname{Vol}\mathcal{F}_X^t(\Omega) &= \frac{d}{dt}\int_\Omega (\mathcal{F}_X^t)^*\omega \\ &= \int_\Omega (\mathcal{F}_X^t)^*((\operatorname{div} X)\,\omega) \\ &= \int_{\mathcal{F}_X^t(\Omega)} (\operatorname{div} X)\,\omega. \end{aligned} \tag{11.31}$$

In other words,

$$\frac{d}{dt}\operatorname{Vol}\mathcal{F}_X^t(\Omega) = \int_{\mathcal{F}_X^t(\Omega)} (\operatorname{div} X)\, dV. \tag{11.32}$$

This result is equivalent to Proposition 5.7, but the derivation here is substantially different. Compare also the discussion in Exercise 2 of §9.

Exercises

1. Show that if α is a k-form and X, X_j are vector fields,

$$(\mathcal{L}_X\alpha)(x_1,\dots,X_k) = X\cdot\alpha(X_1,\dots,X_k) - \sum_j \alpha(X_1,\dots,\mathcal{L}_X X_j,\dots,X_k). \tag{11.33}$$

Recall from (3.80) that $\mathcal{L}_X X_j = [X, X_j]$, and rewrite (11.33) accordingly.

2. Writing (11.20) as

$$\iota_X d\alpha = \mathcal{L}_X\alpha - d\iota_X\alpha,$$

deduce that

$$(d\alpha)(X_0,X_1,\dots,X_k) = (\mathcal{L}_{X_0}\alpha)(X_1,\dots,X_k) - (d\iota_{X_0}\alpha)(X_1,\dots,X_k). \tag{11.34}$$

3. In case α is a one-form, deduce from (11.33)–(11.34) that

$$(d\alpha)(X_0,X_1) = X_0\cdot\alpha(X_1) - X_1\cdot\alpha(X_0) - \alpha([X_0,X_1]). \tag{11.35}$$

4. Using (11.33)–(11.34) and induction on k, show that, if α is a k-form,

$$\begin{aligned}(d\alpha)(X_0,\dots,X_k) = &\sum_{\ell=0}^{k}(-1)^\ell X_\ell\cdot\alpha(X_0,\dots,\widehat{X}_\ell,\dots,X_k)\\ &+\sum_{0\le\ell<j\le k}(-1)^{j+\ell}\alpha([X_\ell,X_j],X_0,\dots,\widehat{X}_\ell,\dots,\widehat{X}_j,\dots,X_k).\end{aligned} \tag{11.36}$$

Here, $\widehat{X}_\ell$ indicates that X_ℓ has been omitted.

5. Show that if X is a vector field, β a 1-form, and α a k-form, then

$$(\wedge_\beta\iota_X + \iota_X\wedge_\beta)\alpha = \langle X,\beta\rangle\alpha. \tag{11.37}$$

Deduce that

$$(df)\wedge(\alpha\rfloor X) + (df\wedge\alpha)\rfloor X = (Xf)\alpha. \tag{11.38}$$

6. Show that the definition (11.19) implies

$$\mathcal{L}_X(f\alpha) = f\mathcal{L}_X\alpha + (XF)\alpha. \tag{11.39}$$

7. Show that the definition (11.19) implies

$$d\mathcal{L}_X\alpha = \mathcal{L}_X(d\alpha). \tag{11.40}$$

8. Denote the right side of (11.20) by $L_X\alpha$, i.e., set

$$L_X\alpha = d(\alpha\rfloor X) + (d\alpha)\rfloor X. \tag{11.41}$$

Show that this definition directly implies

$$L_X(d\alpha) = d(L_X\alpha). \tag{11.42}$$

9. With L_X defined by (11.41), show that

$$L_X(f\alpha) = fL_X\alpha + (Xf)\alpha. \tag{11.43}$$

Hint. Use (11.38).

10. Use the results of Exercises 6–9 to give another proof of Proposition 11.6, i.e., $\mathcal{L}_X\alpha = L_X\alpha$.
Hint. Start with $\mathcal{L}_Xf = Xf = \langle X, df\rangle = L_Xf$.

In Exercises 11–12, let X and Y be smooth vector fields on M and $\alpha \in \Lambda^k(M)$.

11. Show that $\mathcal{L}_{[X,Y]}\alpha = \mathcal{L}_X\mathcal{L}_Y\alpha - \mathcal{L}_Y\mathcal{L}_X\alpha$.

12. Using Exercise 11 and (11.29), show that

$$\operatorname{div}[X,Y] = X(\operatorname{div} Y) - Y(\operatorname{div} X).$$

12. Differential forms and degree theory

Degree theory assigns an integer, $\text{Deg}(f)$, to a smooth map $f : X \to Y$, when X and Y are smooth, compact, oriented surfaces of the same dimension, and Y is connected. This has many uses, as we will see. Results of §11 provide tools for this study. A major ingredient is Stokes' theorem.

As a prelude to our development of degree theory, we use the calculus of differential forms to provide simple proofs of some important topological results of Brouwer. The first two results concern *retractions*. If Y is a subset of X, by definition a retraction of X onto Y is a map $\varphi : X \to Y$ such that $\varphi(x) = x$ for all $x \in Y$.

Proposition 12.1. *There is no smooth retraction $\varphi : B \to S^{n-1}$ of the closed unit ball B in $\mathbb{R}^n$ onto its boundary S^{n-1}.*

In fact, it is just as easy to prove the following more general result. The approach we use is adapted from [Kan].

Proposition 12.2. *If $\overline{M}$ is a compact oriented n-dimensional surface with nonempty boundary ∂M, there is no smooth retraction $\varphi : \overline{M} \to \partial M$.*

Proof. Pick $\omega \in \Lambda^{n-1}(\partial M)$ to be the volume form on ∂M, so $\int_{\partial M} \omega > 0$. Now apply Stokes' theorem to $\beta = \varphi^*\omega$. If φ is a retraction, then $\varphi \circ j(x) = x$, where $j : \partial M \hookrightarrow \overline{M}$ is the natural inclusion. Hence $j^*\varphi^*\omega = \omega$, so we have

$$\int_{\partial M} \omega = \int_{M} d\varphi^*\omega. \tag{12.1}$$

But $d\varphi^*\omega = \varphi^* d\omega = 0$, so the integral (12.1) is zero. This is a contradiction, so there can be no retraction.

A simple consequence of this is the famous Brouwer Fixed-Point Theorem. We first present the smooth case.

Theorem 12.3. *If $F : B \to B$ is a smooth map on the closed unit ball in $\mathbb{R}^n$, then F has a fixed point.*

Proof. We are claiming that $F(x) = x$ for some $x \in B$. If not, define $\varphi(x)$ to be the endpoint of the ray from $F(x)$ to x, continued until it hits $\partial B = S^{n-1}$. An explicit formula is

$$\begin{aligned} \varphi(x) &= x + t(x - F(x)), \quad t = \frac{\sqrt{b^2 + 4ac} - b}{2a}, \\ a &= \|x - F(x)\|^2, \quad b = 2x \cdot (x - F(x)), \quad c = 1 - \|x\|^2. \end{aligned}$$

Here t is picked to solve the equation $\|x + t(x - F(x))\|^2 = 1$. Note that $ac \geq 0$, so $t \geq 0$. It is clear that φ would be a smooth retraction, contradicting Proposition 11.1.

Now we give the general case, using the Stone-Weierstrass theorem (discussed in Appendix E) to reduce it to Theorem 12.3.

Theorem 12.4. *If $G : B \to B$ is a continuous map on the closed unit ball in $\mathbb{R}^n$, then G has a fixed point.*

Proof. If not, then

$$\inf_{x \in B} |G(x) - x| = \delta > 0.$$

The Stone-Weierstrass theorem (Appendix E) implies there exists a polynomial P such that $|P(x) - G(x)| < \delta/8$ for all $x \in B$. Set

$$F(x) = \Bigl(1 - \frac{\delta}{8}\Bigr) P(x).$$

Then $F : B \to B$ and $|F(x) - G(x)| < \delta/2$ for all $x \in B$, so

$$\inf_{x \in B} |F(x) - x| > \frac{\delta}{2}.$$

This contradicts Theorem 12.3.

As a second precursor to degree theory, we next show that an even dimensional sphere cannot have a smooth nonvanishing vector field.

Proposition 12.5. *There is no smooth nonvanishing vector field on S^n if $n = 2k$ is even.*

Proof. If X were such a vector field, we could arrange it to have unit length, so we would have $X : S^n \to S^n$ with $X(v) \perp v$ for $v \in S^n \subset \mathbb{R}^{n+1}$. Thus there would be a unique unit speed curve γ_v along the great circle from v to $X(v)$, of length $\pi/2$. Define a smooth family of maps $F_t : S^n \to S^n$ by $F_t(v) = \gamma_v(t)$. Thus $F_0(v) = v$, $F_{\pi/2}(v) = X(v)$, and $F_\pi = A$ would be the *antipodal map*, $A(v) = -v$. By Proposition 11.3, we deduce that $A^*\omega - \omega = d\beta$ is exact, where ω is the volume form on S^n. Hence, by Stokes' theorem,

$$\int_{S^n} A^*\omega = \int_{S^n} \omega. \tag{12.2}$$

Alternatively, (12.2) follows directly from Proposition 11.1. On the other hand, it is straightforward that $A^*\omega = (-1)^{n+1}\omega$, so (12.2) is possible only when n is odd.

Note that an important ingredient in the proof of both Proposition 12.2 and Proposition 12.5 is the existence of n-forms on a compact oriented n-dimensional surface M that are not exact (though of course they are closed). We next establish the following counterpoint to the Poincaré lemma.

Proposition 12.6. *If M is a compact, connected, oriented surface of dimension n and $\alpha \in \Lambda^n M$, then $\alpha = d\beta$ for some $\beta \in \Lambda^{n-1}(M)$ if and only if*

$$\int_M \alpha = 0. \tag{12.3}$$

We have already discussed the necessity of (12.3). To prove the sufficiency, we first look at the case $M = S^n$.

In that case, any n-form α is of the form $a(x)\omega$, $a \in C^\infty(S^n)$, ω the volume form on S^n, with its standard metric. The group $G = SO(n+1)$ of rotations of $\mathbb{R}^{n+1}$ acts as a transitive group of isometries on S^n. In §5 we constructed the integral of functions over $SO(n+1)$, with respect to Haar measure.

As seen in §5, we have the surjective map

$$\text{Exp} : \text{Skew}(n+1) \longrightarrow SO(n+1),$$

giving a diffeomorphism from a ball $\mathcal{O}$ about 0 in Skew$(n+1)$ onto an open set $U \subset SO(n+1) = G$, a neighborhood of the identity. Since G is compact, we can pick a finite number of elements $\xi_j \in G$ such that the open sets $U_j = \{\xi_j g : g \in U\}$ cover G. Pick $\eta_j \in \text{Skew}(n+1)$ such that Exp $\eta_j = \xi_j$. Define $\Phi_{jt} : U_j \to G$ for $0 \le t \le 1$ by

$$\Phi_{jt}\big(\xi_j \ \text{Exp}(A)\big) = (\text{Exp}\ t\eta_j)(\text{Exp}\ tA), \quad A \in \mathcal{O}. \tag{12.4}$$

Now partition G into subsets Ω_j, each of whose boundaries has content zero, such that $\Omega_j \subset U_j$. If $g \in \Omega_j$, set $g(t) = \Phi_{jt}(g)$. This family of elements of $SO(n+1)$ defines a family of maps $F_{gt} : S^n \to S^n$. Now by (11.27) we have

$$\alpha = g^*\alpha - d\kappa_g(\alpha), \quad \kappa_g(\alpha) = \int_0^1 F_{gt}^*(\alpha \rfloor X_{gt})\, dt, \tag{12.5}$$

for each $g \in SO(n+1)$, where X_{gt} is the family of vector fields on S^n associated to F_{gt}, as in (11.25). Therefore,

$$\alpha = \int_G g^*\alpha \ dg - d\int_G \kappa_g(\alpha)\ dg. \tag{12.6}$$

Now the first term on the right is equal to $\overline{\alpha}\omega$, where $\overline{\alpha} = \int a(g \cdot x)dg$ is a constant; in fact, the constant is

$$\overline{\alpha} = \frac{1}{\text{Vol}\ S^n} \int_{S^n} \alpha. \tag{12.7}$$

Thus in this case (12.3) is precisely what serves to make (12.6) a representation of α as an exact form. This takes care of the case $M = S^n$.

For a general compact, oriented, connected M, proceed as follows. Cover M with open sets $\mathcal{O}_1, \dots, \mathcal{O}_K$ such that each $\overline{\mathcal{O}}_j$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$. Set $U_1 = \mathcal{O}_1$, and inductively enlarge each $\mathcal{O}_j$ to U_j, so that $\overline{U}_j$ is also diffeomorphic to the closed ball, and such that $U_{j+1} \cap U_j \neq \emptyset$, $1 \le j < K$. You can do this by drawing a simple curve from $\overline{\mathcal{O}}_{j+1}$ to a point in U_j and thickening it. Pick a smooth partition of unity φ_j, subordinate to this cover. (See Appendix B.)

Given $\alpha \in \Lambda^n M$, satisfying (12.3), take $\tilde{\alpha}_j = \varphi_j \alpha$. Most likely $\int \tilde{\alpha}_1 = c_1 \neq 0$, so take $\sigma_1 \in \Lambda^n M$, with compact support in $U_1 \cap U_2$, such that $\int \sigma_1 = c_1$. Set $\alpha_1 = \tilde{\alpha}_1 - \sigma_1$, and redefine $\tilde{\alpha}_2$ to be the old $\tilde{\alpha}_2$ plus σ_1. Make a similar construction using $\int \tilde{\alpha}_2 = c_2$, and continue. When you are done, you have

$$\alpha = \alpha_1 + \cdots + \alpha_K, \tag{12.8}$$

with α_j compactly supported in U_j. By construction,

$$\int \alpha_j = 0 \tag{12.9}$$

for $1 \leq j < K$. But then (12.3) implies $\int \alpha_K = 0$ too.

Now pick $p \in S^n$ and define smooth maps

$$\psi_j : M \longrightarrow S^n \tag{12.10}$$

which map U_j diffeomorphically onto $S^n \setminus p$, and map $M \setminus U_j$ to p. There is a unique $v_j \in \Lambda^n S^n$, with compact support in $S^n \setminus p$, such that $\psi^* v_j = \alpha_j$. Clearly

$$\int_{S^n} v_j = 0,$$

so by the case $M = S^n$ of Proposition 12.6 already established, we know that $v_j = dw_j$ for some $w_j \in \Lambda^{n-1} S^n$, and then

$$\alpha_j = d\beta_j, \quad \beta_j = \psi_j^* w_j. \tag{12.11}$$

This concludes the proof of Proposition 12.6.

We are now ready to introduce the notion of the degree of a map between compact oriented surfaces. Let X and Y be compact oriented n-dimensional surfaces. We want to define the degree of a smooth map $F : X \to Y$. To do this, assume Y is connected. We pick $\omega \in \Lambda^n Y$ such that

$$\int_Y \omega = 1. \tag{12.12}$$

We propose to define

$$\operatorname{Deg}(F) = \int_X F^* \omega. \tag{12.13}$$

The following result shows that $\operatorname{Deg}(F)$ is indeed well defined by this formula. The key argument is an application of Proposition 12.6.

Lemma 12.7. *The quantity (12.13) is independent of the choice of* ω, *as long as (12.12) holds.*

Proof. Pick $\omega_1 \in \Lambda^n Y$ satisfying $\int_Y \omega_1 = 1$, so $\int_Y (\omega - \omega_1) = 0$. By Proposition 12.6, this implies

$$\omega - \omega_1 = d\alpha, \text{ for some } \alpha \in \Lambda^{n-1} Y. \tag{12.14}$$

Thus

$$\int_X F^*\omega - \int_X F^*\omega_1 = \int_X dF^*\alpha = 0, \tag{12.15}$$

and the lemma is proved.

The following is a most basic property.

Proposition 12.8. *If* F_0 *and* F_1 *are smoothly homotopic, then* $\operatorname{Deg}(F_0) = \operatorname{Deg}(F_1)$.

Proof. By Proposition 11.1, if F_0 and F_1 are smoothly homotopic, then $\int_X F_0^*\omega = \int_X F_1^*\omega$.

The following result is a simple but powerful extension of Proposition 12.8. Compare the relation between Propositions 11.1 and 11.2.

Proposition 12.9. *Let* $\overline{M}$ *be a compact oriented surface with boundary,* $\dim M = n + 1$. *Take* Y *as above,* $n = \dim Y$. *Given a smooth map* $F : \overline{M} \to Y$, *let* $f = F\big|_{\partial M} : \partial M \to Y$. *Then*

$$\operatorname{Deg}(f) = 0.$$

Proof. Applying Stokes' Theorem to $\alpha = F^*\omega$, we have

$$\int_{\partial M} f^*\omega = \int_M dF^*\omega.$$

But $dF^*\omega = F^*d\omega$, and $d\omega = 0$ if dim $Y = n$, so we are done.

Brouwer's no-retraction theorem is an easy corollary of Proposition 12.9. Compare the proof of Proposition 12.2.

Corollary 12.10. *If* $\overline{M}$ *is a compact oriented surface with nonempty boundary* ∂M, *then there is no smooth retraction* $\varphi : \overline{M} \to \partial M$.

Proof. Without loss of generality, we can assume $\overline{M}$ is connected. If there were a retraction, then $\partial M = \varphi(\overline{M})$ must also be connected, so Proposition 12.9 applies. But then we would have, for the map $id. = \varphi\big|_{\partial M}$, the contradiction that its degree is both zero and 1.

We next give an alternative formula for the degree of a map, which is very useful in many applications. In particular, it implies that the degree is always an *integer*.

A point $y_0 \in Y$ is called a regular value of F, provided that, for each $x \in X$ satisfying $F(x) = y_0$, $DF(x) : T_xX \to T_{y_0}Y$ is an isomorphism. The easy case of Sard's Theorem, discussed in Appendix F, implies that *most* points in Y are regular. Endow X with a volume element ω_X, and similarly endow Y with ω_Y. If $DF(x)$ is invertible, define $JF(x) \in \mathbb{R} \setminus 0$ by $F^*(\omega_Y) = JF(x)\omega_X$. Clearly the *sign* of $JF(x)$, i.e., sgn $JF(x) = \pm 1$, is independent of choices of ω_X and ω_Y, as long as they determine the given orientations of X and Y.

Proposition 12.11. *If y_0 is a regular value of F, then*

$$\operatorname{Deg}(F) = \sum \{\operatorname{sgn} JF(x_j) : F(x_j) = y_0\}. \tag{12.16}$$

Proof. Pick $\omega \in \Lambda^n Y$, satisfying (12.12), with support in a small neighborhood of y_0. Then $F^*\omega$ will be a sum $\sum \omega_j$, with ω_j supported in a small neighborhood of x_j, and $\int \omega_j = \pm 1$ as sgn $JF(x_j) = \pm 1$.

For an application of Proposition 12.11, let X be a compact smooth oriented hypersurface in $\mathbb{R}^{n+1}$, and set $\Omega = \mathbb{R}^{n+1} \setminus X$. Given $p \in \Omega$, define

$$F_p : X \longrightarrow S^n, \quad F_p(x) = \frac{x-p}{|x-p|}. \tag{12.17}$$

It is clear that $\operatorname{Deg}(F_p)$ is constant on each connected component of Ω. It is also easy to see that, when p crosses X, $\operatorname{Deg}(F_p)$ jumps by ± 1. Thus Ω has at least two connected components. This is most of the smooth case of the Jordan-Brouwer separation theorem:

Theorem 12.12. *If X is a smooth compact oriented hypersurface of $\mathbb{R}^{n+1}$, which is connected, then $\Omega = \mathbb{R}^{n+1} \setminus X$ has exactly 2 connected components.*

Proof. X being oriented, it has a smooth global normal vector field. Use this to separate a small collar neighborhood $\mathcal{C}$ of X into 2 pieces; $\mathcal{C} \setminus X = \mathcal{C}_0 \cup \mathcal{C}_1$. The collar $\mathcal{C}$ is diffeomorphic to $[-1,1] \times X$, and each $\mathcal{C}_j$ is clearly connected. It suffices to show that any connected component $\mathcal{O}$ of Ω intersects either $\mathcal{C}_0$ or $\mathcal{C}_1$. Take $p \in \partial\mathcal{O}$. If $p \notin X$, then $p \in \Omega$, which is open, so p cannot be a boundary point of any component of Ω. Thus $\partial\mathcal{O} \subset X$, so $\mathcal{O}$ must intersect a $\mathcal{C}_j$. This completes the proof.

Let us note that, of the two components of Ω, exactly one is unbounded, say Ω_0, and the other is bounded; call it Ω_1. Then we claim

$$p \in \Omega_j \Longrightarrow \operatorname{Deg}(F_p) = j. \tag{12.18}$$

Indeed, for p very far from X, $F_p : X \to S^n$ is not onto, so its degree is 0. And when p crosses X, from Ω_0 to Ω_1, the degree jumps by $+1$.

For a simple closed curve in $\mathbb{R}^2$, Theorem 12.12 is the smooth case of the Jordan curve theorem. That special case of the argument given above can be found in [Sto]. The existence of a smooth normal field simplifies the use of basic degree theory to prove such a result. For a general continuous, simple closed curve in $\mathbb{R}^2$, such a normal field is not available, and the proof of the Jordan curve theorem in this more general context requires a different argument, which can be found in [GrH].

We apply results just established on degree theory to properties of vector fields, particularly of their critical points. A critical point of a vector field V is a point where V

vanishes. Let V be a vector field defined on a neighborhood $\mathcal{O}$ of $p \in \mathbb{R}^n$, with a single critical point, at p. Then, for any small ball B_r about p, $B_r \subset \mathcal{O}$, we have a map

$$V_r : \partial B_r \to S^{n-1}, \quad V_r(x) = \frac{V(x)}{|V(x)|}. \tag{12.19}$$

The degree of this map is called the *index* of V at p, denoted $\operatorname{ind}_p(V)$; it is clearly independent of r. If V has a finite number of critical points, then the index of V is defined to be

$$\operatorname{Index}(V) = \sum \operatorname{ind}_{p_j}(V). \tag{12.20}$$

If $\psi : \mathcal{O} \to \mathcal{O}'$ is an orientation preserving diffeomorphism, taking p to p and V to W, then we claim

$$\operatorname{ind}_p(V) = \operatorname{ind}_p(W). \tag{12.21}$$

In fact, $D\psi(p)$ is an element of $GL(n, \mathbb{R})$ with positive determinant, so it is homotopic to the identity, and from this it readily follows that V_r and W_r are homotopic maps of $\partial B_r \to S^{n-1}$. Thus one has a well defined notion of the index of a vector field with a finite number of critical points on any oriented surface M.

There is one more wrinkle. Suppose X is a smooth vector field on M and p an isolated critical point. If you change the orientation of a small coordinate neighborhood $\mathcal{O}$ of p, then the orientations of both ∂B_r and S^{n-1} in (12.19) get changed, so the associated degree is not changed. Hence one has a well defined notion of the index of a vector field with a finite number of critical points on any smooth surface M, oriented or not.

A vector field V on $\mathcal{O} \subset \mathbb{R}^n$ is said to have a non-degenerate critical point at p provided $DV(p)$ is a nonsingular $n \times n$ matrix. The following formula is convenient.

Proposition 12.13. *If V has a nondegenerate critical point at p, then*

$$\operatorname{ind}_p(V) = \operatorname{sgn} \det DV(p). \tag{12.22}$$

Proof. If p is a nondegenerate critical point, and we set $\psi(x) = DV(p)x$, $\psi_r(x) = \psi(x)/|\psi(x)|$, for $x \in \partial B_r$, it is readily verified that ψ_r and V_r are homotopic, for r small. The fact that $\operatorname{Deg}(\psi_r)$ is given by the right side of (12.22) is an easy consequence of Proposition 12.11

The following is an important global relation between index and degree.

Proposition 12.14. *Let $\overline{\Omega}$ be a smooth bounded region in $\mathbb{R}^{n+1}$. Let V be a vector field on $\overline{\Omega}$, with a finite number of critical points p_j, all in the interior Ω. Define $F : \partial\Omega \to S^n$ by $F(x) = V(x)/|V(x)|$. Then*

$$\operatorname{Index}(V) = \operatorname{Deg}(F). \tag{12.23}$$

Proof. If we apply Proposition 12.9 to $\overline{M} = \overline{\Omega} \setminus \bigcup_j B_\varepsilon(p_j)$, we see that $\operatorname{Deg}(F)$ is equal to the sum of degrees of the maps of $\partial B_\varepsilon(p_j)$ to S^n, which gives (12.23).

Next we look at a process of producing vector fields in higher dimensional spaces from vector fields in lower dimensional spaces.

Proposition 12.15. *Let W be a vector field on $\mathbb{R}^n$, vanishing only at* 0*. Define a vector field V on $\mathbb{R}^{n+k}$ by $V(x,y) = \big(W(x), y\big)$. Then V vanishes only at* $(0,0)$*. Then we have*

$$\mathrm{ind}_0 W = \mathrm{ind}_{(0,0)} V. \tag{12.24}$$

Proof. If we use Proposition 12.11 to compute degrees of maps, and choose $y_0 \in S^{n-1} \subset S^{n+k-1}$, a regular value of W_r, and hence also for V_r, this identity follows.

We turn to a more sophisticated variation. Let X be a compact n dimensional surface in $\mathbb{R}^{n+k}$, W a (tangent) vector field on X with a finite number of critical points p_j. Let $\overline{\Omega}$ be a small tubular neighborhood of X, $\pi : \overline{\Omega} \to X$ mapping $z \in \overline{\Omega}$ to the nearest point in X. Let $\varphi(z) = \operatorname{dist}(z,X)^2$. Now define a vector field V on $\overline{\Omega}$ by

$$V(z) = W(\pi(z)) + \nabla\varphi(z). \tag{12.25}$$

Proposition 12.16. *If $F : \partial\Omega \to S^{n+k-1}$ is given by $F(z) = V(z)/|V(z)|$, then*

$$\mathrm{Deg}(F) = \mathrm{Index}(W). \tag{12.26}$$

Proof. We see that all the critical points of V are points in X that are critical for W, and, as in Proposition 12.15, $\mathrm{Index}(W) = \mathrm{Index}(V)$. Then Proposition 12.14 implies $\mathrm{Index}(V) = \mathrm{Deg}(F)$.

Since $\varphi(z)$ is increasing as one goes away from X, it is clear that, for $z \in \partial\Omega$, $V(z)$ points *out of* $\overline{\Omega}$, provided it is a sufficiently small tubular neighborhood of X. Thus $F : \partial\Omega \to S^{n+k-1}$ is homotopic to the *Gauss map*

$$N : \partial\Omega \longrightarrow S^{n+k-1}, \tag{12.27}$$

given by the outward pointing normal. This immediately gives:

Corollary 12.17. *Let X be a compact n-dimensional surface in $\mathbb{R}^{n+k}$, $\overline{\Omega}$ a small tubular neighborhood of X, and $N : \partial\Omega \to S^{n+k-1}$ the Gauss map. If W is a vector field on X with a finite number of critical points, then*

$$\mathrm{Index}(W) = \mathrm{Deg}(N). \tag{12.28}$$

Clearly the right side of (12.28) is independent of the choice of W. Thus any two vector fields on X with a finite number of critical points have the same index, i.e., $\mathrm{Index}(W)$ is an invariant of X. This invariant is denoted

$$\mathrm{Index}(W) = \chi(X), \tag{12.29}$$

and is called the Euler characteristic of X.

Remark. The existence of smooth vector fields with only nondegenerate critical points (hence only finitely many critical points) on a given compact surface X follows from results presented in Appendix G.

Exercises

1. Let X be a compact, oriented, connected surface. Show that the identity map $I : X \to X$ has degree 1.

2. Suppose Y is also a compact, oriented, connected surface. Show that if $F : X \to Y$ is not onto, then $\operatorname{Deg}(F) = 0$.

3. If $A : S^n \to S^n$ is the antipodal map, show that $\operatorname{Deg}(A) = (-1)^{n-1}$.

4. Show that the homotopy invariance property given in Proposition 12.8 can be deduced as a corollary of Proposition 12.9.
Hint. Take $\overline{M} = X \times [0, 1]$.

5. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. The fundamental theorem of algebra, proved in §10, states that $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$. We aim for another proof, using degree theory. To get this, by contradiction, assume $p : \mathbb{C} \to \mathbb{C} \setminus 0$. For $r \geq 0$, define

$$F_r : S^1 \longrightarrow S^1, \quad F_r(e^{i\theta}) = \frac{p(re^{i\theta})}{|p(re^{i\theta})|}.$$

Show that each F_r is smoothly homotopic to F_0, and note that $\operatorname{Deg}(F_0) = 0$. Then show that there exists r_0 such that

$$r \geq r_0 \Rightarrow F_r \text{ is homotopic to } \Phi,$$

where $\Phi(e^{i\theta}) = e^{in\theta}$. Show that $\operatorname{Deg}(\Phi) = n$, and obtain a contradiction.
Note. Regarding the use of degree theory here, show how the proof of Proposition 12.6 vastly simplifies when $M = S^1$.

6. Show that each odd-dimensional sphere S^{2k-1} has a smooth, nowhere vanishing tangent vector field.
Hint. Regard $S^{2k-1} \subset \mathbb{C}^k$, and multiply the unit normal by i.

7. Let V be a planar vector field. Assume it has a nondegenerate critical point at p. Show

that

$$\begin{aligned} p \text{ saddle} &\Longrightarrow \operatorname{ind}_p(V) = -1 \\ p \text{ source} &\Longrightarrow \operatorname{ind}_p(V) = 1 \\ p \text{ sink} &\Longrightarrow \operatorname{ind}_p(V) = 1 \\ p \text{ center} &\Longrightarrow \operatorname{ind}_p(V) = 1 \end{aligned}$$

8. Let M be a compact oriented 2-dimensional surface. Given a triangulation of M, within each triangle construct a vector field, vanishing at 7 points as illustrated in Fig. 12.1, with the vertices as attractors, the center as a repeller, and the midpoints of each edge as saddle points. Fit these together to produce a smooth vector field X on M. Show directly that

$$\operatorname{Index}(X) = V - E + F,$$

where

$$V = \#\text{ vertices},\ \ E = \#\text{ edges},\ \ F = \#\text{ faces},$$

in the triangulation.

9. With $X = S^n \subset \mathbb{R}^{n+1}$, note that the manifold $\partial\Omega$ in (12.27) consists of two copies of S^n, with opposite orientations. Compute the degree of the map N in (12.27)–(12.28), and use this to show that

$$\chi(S^n) = 2 \text{ if } n \text{ even}, \quad 0 \text{ if } n \text{ odd}, \tag{12.30}$$

granted (12.28)–(12.29).

10. Consider the vector field R on S^2 generating rotation about an axis. Show that R has two critical points, at the "poles." Classify the critical points, compute $\operatorname{Index}(R)$, and compare the $n = 2$ case of (12.30).

11. Generalizing Exercise 9, Let $X \subset \mathbb{R}^{n+1}$ be a smooth, compact, oriented, n-dimensional surface, so again the neighborhood Ω of X as in (12.27) has boundary $\partial\Omega$ consisting essentially of two copies of X, with opposite orientations. Let $\widetilde{N} : X \to S^n$ be the outward pointing unit normal. Show that

$$\operatorname{Deg} \widetilde{N} = \frac{1}{2}\chi(X), \quad \text{if } n \text{ is even}. \tag{12.31}$$

Remark. If ω_S is the volume form on S^n and ω_X that on X, then $\widetilde{N}^*\omega_S = K\,\omega_X$, and $K : X \to \mathbb{R}$ is called the *Gauss curvature* of X. Then (12.31) implies

$$\int_X K(x)\,dS(x) = \frac{1}{2}A_n\,\chi(X),$$

if n is even. This is a basic case of the *Gauss-Bonnet formula.*

12. In the setting of Exercise 11, assume n is *odd*. Show that

$$\chi(X) = 0. \tag{12.32}$$

Give examples where the identity in (12.31) fails.

13. Retain the setting of Exercise 12, especially that n is odd. Let $X = \partial\mathcal{O}$, with $\mathcal{O} \subset \mathbb{R}^{n+1}$ bounded and open. Take a smooth function

$$\varphi : \overline{\mathcal{O}} \longrightarrow [0, \infty), \quad \varphi(x) = \operatorname{dist}(x, X) \ \text{ near } \ X, \quad \varphi(x) > 0 \ \text{ on } \ \mathcal{O}.$$

Let $\Sigma \subset \mathbb{R}^{n+2}$ be the surface

$$\Sigma = \{(x, y) : x \in \overline{\mathcal{O}},\ y^2 = \varphi(x)\},$$

and let $\nu : \Sigma \to S^{n+1}$ be the outward pointing unit normal. Show that

$$\operatorname{Deg} \nu = \operatorname{Deg} \widetilde{N}, \tag{12.33}$$

and deduce that

$$\operatorname{Deg} \widetilde{N} = \frac{1}{2}\chi(\Sigma). \tag{12.34}$$

Hint. Taking $\widetilde{N} : X \to S^n \subset S^{n+1}$, show that each regular value of $\widetilde{N}$ is also a regular value of ν, with the same preimage in $X \subset \Sigma$. Then show that Proposition 12.11 applies.

14. Actually, (12.33) holds whether n is odd or even. Can you get anything else from this?

15. In the setting of Exercise 12 (n is *odd*), generalize the construction of Exercise 6 to show directly that there is a smooth, nowhere vanishing vector field tangent to X.

16. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and $f : \mathcal{O} \to \mathbb{R}$ smooth of class C^2. Let $V = \nabla f$. Assume $p \in \mathcal{O}$ is a nondegenerate critical point of f, so its Hessian $D^2 f(p)$ is a nondegenerate $n \times n$ symmetric matrix. Say

$$D^2 f(p) \text{ has } \ell \text{ positive eigenvalues and } n - \ell \text{ negative eigenvalues.} \tag{12.35}$$

Show that Proposition 12.13 implies

$$\operatorname{ind}_p(V) = (-1)^{n-\ell}. \tag{12.36}$$

17. Let $X \subset \mathbb{R}^{n+k}$ be a smooth, compact, n-dimensional surface. Assume there exists

$f \in C^2(X)$, with just two critical points, a max and a min, both nondegenerate. Use Exercise 16 to show that

$$\chi(X) = 2 \ \text{ if } \ n \ \text{ is even}, \ \ 0 \ \text{ if } n \ \text{ is odd}. \tag{12.37}$$

Considering $S^n \subset \mathbb{R}^{n+1}$, use this to give another demonstration of (12.30).

18. Let $\mathcal{T} \subset \mathbb{R}^3$ be the "inner tube" surface described in Exercise 15 of §5.

(a) Show that rotation about the z-axis is generated by a vector field that is tangent to $\mathcal{T}$ and nowhere vanishing of $\mathcal{T}$.

(b) Define $f : \mathcal{T} \to \mathbb{R}$ by $f(x, y, z) = x, \ (x, y, z) \in \mathcal{T}$. Show that f has four critical points, a max, a min, and two saddles. Deduce from Exercise 16 that ∇f is a vector field on $\mathcal{T}$ of index 0.

(c) Show that both part (a) and part (b) imply $\chi(\mathcal{T}) = 0$.

Let $X \subset \mathbb{R}^n$ be an m-dimensional surface, and let $Y \subset \mathbb{R}^\nu$ be a μ-dimensional surface, both smooth of class C^k. Then

$$X \times Y = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^\nu : x \in X, y \in Y\} \subset \mathbb{R}^n \times \mathbb{R}^\nu$$

has a natural structure of an $(m + \mu)$-dimensional C^k surface.

19. Let X and Y be as above, and assume they are both compact. Let V_1 be a smooth vector field tangent to X and V_2 a smooth vector field tangent to Y, both with only nondegenerate critical points. Say $\{p_i\}$ are the critical points of V_1 and $\{q_j\}$ those of V_2.

(a) Show that $W(x, y) = V_1(x) + V_2(y)$ is a smooth vector field tangent to $X \times Y$. Show that its critical points are precisely the points $\{(p_i, q_j)\}$, each nondegenerate. Show that Proposition 12.13 gives

$$\operatorname{ind}_{(p_i,q_j)} W = (\operatorname{ind}_{p_i} V_1)(\operatorname{ind}_{q_j} V_2). \tag{12.38}$$

(b) Show that

$$\text{Index } W = (\text{Index } V_1)(\text{Index } V_2). \tag{12.39}$$

(c) Deduce that

$$\chi(X \times Y) = \chi(X)\chi(Y). \tag{12.40}$$

Let X and Y be smooth, compact, oriented surfaces in $\mathbb{R}^n$. Assume $k = \dim X, \ \ell = \dim Y$, and $k + \ell = n - 1$. Assume $X \cap Y = \emptyset$. Set

$$\varphi : X \times Y \longrightarrow S^{n-1}, \quad \varphi(x, y) = \frac{x - y}{|x - y|}. \tag{12.41}$$

We define the *linking number*

$$\lambda(X, Y, \mathbb{R}^n) = \operatorname{Deg} \varphi. \tag{12.42}$$

20. Let γ and $\sigma \subset \mathbb{R}^3$ be the following simple closed curves, parametrized by $s, t \in \mathbb{R}/(2\pi\mathbb{Z})$:

$$\gamma(s) = (\cos s, \sin s, 0), \quad \sigma(t) = (0, 1 + \cos t, \sin t).$$

Thus γ is a circle in the (x, y)-plane centered at $(0, 0, 0)$ and σ is a circle in the (y, z)-plane, centered at $(0, 1, 0)$, both of unit radius. Show that

$$\lambda(\gamma, \sigma, \mathbb{R}^3) = 1.$$

Hint. With φ as above, show that $(0, 1, 0) \in S^2$ has exactly one preimage point, under $\varphi : \gamma \times \sigma \to S^2$.

21. Let M be a smooth, compact, oriented, $(n-1)$-dimensional surface, and assume $\varphi : M \to \mathbb{R}^n \setminus 0$ is a smooth map. Set

$$F(x) = \frac{\varphi(x)}{|\varphi(x)|}, \quad F : M \longrightarrow S^{n-1}.$$

Take $\omega \in \Lambda^{n-1}(\mathbb{R}^n \setminus 0)$ to be the form considered in Exercises 9–10 of §8, i.e.,

$$\omega = |x|^{-n} \sum_{j=1}^{n} x_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Show that

$$\operatorname{Deg}(F) = \frac{1}{A_{n-1}} \int_M \varphi^* \omega.$$

Hint. Use Proposition 11.1 (with $Y = \mathbb{R}^n \setminus 0$), plus Exercise 9 of §8, to show that

$$\int_M \varphi^* \omega = \int_M F^* \omega,$$

and show that, under $S^{n-1} \overset{j}{\hookrightarrow} \mathbb{R}^n$, $j^*\omega$ is the area form on S^{n-1}.

22. In Exercise 21, take $n = 3$ and $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$, parametrized by $(s, t) \in \mathbb{R}^2$. Show that

$$\begin{aligned} \varphi^* \omega &= |\varphi|^{-3} \det \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \partial_s \varphi_1 & \partial_s \varphi_2 & \partial_s \varphi_3 \\ \partial_t \varphi_1 & \partial_t \varphi_2 & \partial_t \varphi_3 \end{pmatrix} (ds \wedge dt) \\ &= |\varphi|^{-3} \varphi \cdot \Bigl(\frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t} \Bigr) (ds \wedge dt). \end{aligned}$$

In case $\varphi(s, t) = \gamma(s) - \sigma(t)$, deduce the Gauss linking number formula:

$$\lambda(\gamma, \sigma, \mathbb{R}^3) = -\frac{1}{4\pi} \int_{\mathbb{T}^2} \frac{\gamma(s) - \sigma(t)}{|\gamma(s) - \sigma(t)|^3} \cdot (\gamma'(s) \times \sigma'(t)) \, ds \, dt.$$

C. Differential forms and the change of variable formula

The change of variable formula for one-variable integrals,

$$\int_a^t f\big(\varphi(x)\big)\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(t)} f(x)\,dx, \tag{C.1}$$

given f continuous and φ of class C^1, is easily established, via the fundamental theorem of calculus and the chain rule. We recall how this was done in §0. If we denote the left side of (C.1) by $A(t)$ and the right by $B(t)$, we apply these results to get

$$A'(t) = f\big(\varphi(t)\big)\varphi'(t) = B'(t), \tag{C.2}$$

and since $A(a) = B(a) = 0$, another application of the fundamental theorem of calculus (or simply the mean value theorem) gives $A(t) = B(t)$.

For multiple integrals, the change of variable formula takes the following form, given in Proposition 4.13:

Theorem C.1. *Let $\mathcal{O}, \Omega$ be open sets on $\mathbb{R}^n$ and $\varphi : \mathcal{O} \to \Omega$ be a C^1 diffeomorphism. Given f continuous on Ω, with compact support, we have*

$$\int_{\mathcal{O}} f\big(\varphi(x)\big)\big|\det D\varphi(x)\big|\,dx = \int_{\Omega} f(x)\,dx. \tag{C.3}$$

There are many variants of Theorem C.1. In particular one wants to extend the class of functions f for which (C.3) holds, but once one has Theorem C.1 as stated, such extensions are relatively painless. See the derivation of Theorem 4.15.

Let's face it; the proof of Theorem C.1 given in §4 was a grim affair, involving careful estimates of volumes of images of small cubes under the map φ and numerous pesky details. Recently, P. Lax [L] found a fresh approach to the proof of the multidimensional change of variable formula. More precisely, [L] established the following result, from which Theorem C.1 is an easy consequence.

Theorem C.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Assume $\varphi(x) = x$ for $|x| \ge R$. Let f be a continuous function on $\mathbb{R}^n$ with compact support. Then*

$$\int f\big(\varphi(x)\big)\ \det D\varphi(x)\,dx = \int f(x)\,dx. \tag{C.4}$$

We will give a variant of the proof of [L]. One difference between this proof and that of [L] is that we use the language of differential forms.

Proof of Theorem C.2. Via standard approximation arguments, it suffices to prove this when φ is C^2 and $f \in C_0^1(\mathbb{R}^n)$, which we will assume from here on.

To begin, pick $A > 0$ such that $f(x - Ae_1)$ is supported in $\{x : |x| > R\}$, where $e_1 = (1, 0, \dots, 0)$. Also take A large enough that the image of $\{x : |x| \le R\}$ under φ does not intersect the support of $f(x - Ae_1)$. We can set

$$F(x) = f(x) - f(x - Ae_1) = \frac{\partial \psi}{\partial x_1}(x), \tag{C.5}$$

where

$$\psi(x) = \int_0^A f(x - se_1)\, ds, \quad \psi \in C_0^1(\mathbb{R}^n). \tag{C.6}$$

Then we have the following identities involving n-forms:

$$\begin{aligned} \alpha = F(x)\, dx_1 \wedge \cdots \wedge dx_n &= \frac{\partial \psi}{\partial x_1}\, dx_1 \wedge \cdots \wedge dx_n \\ &= d\psi \wedge dx_2 \wedge \cdots \wedge dx_n \\ &= d(\psi\, dx_2 \wedge \cdots \wedge dx_n), \end{aligned} \tag{C.7}$$

i.e., $\alpha = d\beta$, with $\beta = \psi\, dx_2 \wedge \cdots \wedge dx_n$ a compactly supported $(n-1)$-form of class C^1. Now the pull-back of α under φ is given by

$$\varphi^* \alpha = F\big(\varphi(x)\big) \det D\varphi(x)\, dx_1 \wedge \cdots \wedge dx_n. \tag{C.8}$$

Furthermore, the right side of (C.8) is equal to

$$f\big(\varphi(x)\big) \det D\varphi(x)\, dx_1 \wedge \cdots \wedge dx_n - f(x - Ae_1)\, dx_1 \wedge \cdots \wedge dx_n. \tag{C.9}$$

Hence we have

$$\begin{aligned} &\int f\big(\varphi(x)\big) \det D\varphi(x)\, dx_1 \cdots dx_n - \int f(x)\, dx_1 \cdots dx_n \\ &\qquad = \int \varphi^* \alpha = \int \varphi^* d\beta = \int d(\varphi^* \beta), \end{aligned} \tag{C.10}$$

where we use the general identity

$$\varphi^* d\beta = d(\varphi^* \beta), \tag{C.11}$$

a consequence of the chain rule. On the other hand, a very special case of Stokes' theorem applies to

$$\varphi^* \beta = \gamma = \sum_j \gamma_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n, \tag{C.12}$$

with $\gamma_j \in C_0^1(\mathbb{R}^n)$. Namely

$$d\gamma = \sum_j (-1)^{j-1} \frac{\partial \gamma_j}{\partial x_j} \, dx_1 \wedge \cdots \wedge dx_n, \tag{C.13}$$

and hence, by the fundamental theorem of calculus,

$$\int d\gamma = 0. \tag{C.14}$$

This gives the desired identity (C.4), from (C.10).

We make some remarks on Theorem C.2. Note that φ is not assumed to be one-to-one or onto. In fact, as noted in [L], the identity (C.4) implies that such φ must be onto, and this has important topological implications. We mention that, if one puts absolute values around det $D\varphi(x)$ in (C.4), the appropriate formula is

$$\int f\big(\varphi(x)\big) \, \big|\det D\varphi(x)\big| \, dx = \int f(x) \, n(x) \, dx, \tag{C.15}$$

where $n(x) = \#\{y : \varphi(y) = x\}$. A proof of (C.15) can be found in texts on geometrical measure theory.

As noted in [L], Theorem C.2 was proven in [B-D]. The proof there makes use of differential forms and Stokes' theorem, but it is quite different from the proof given here. A crucial difference is that the proof in [B-D] requires that one knows the change of variable formula as formulated in Theorem C.1.

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