Multivariate Gauss sums

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Let $A \in Gl(N, \mathbb{Z})$ (so det $A = \pm 1$) and assume A is symmetric. Let $Q(\xi) = \xi \cdot A\xi$ and form the second-order differential operator $L = Q(D)$. Consider the Schrödinger equation

$$\frac{\partial u}{\partial t} = iLu, \tag{1}$$

with solution operator e^{itL}. We define $u(t, x)$ on $\mathbb{R} \times \mathbb{T}^N$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We will be able to write out

$$S_A(t, x) = e^{itL}\delta(x) \tag{2}$$

as a finite linear combination of delta functions, when t is a rational multiple of 2π. There are two ways to make such a calculation, and comparing the results gives a reciprocity formula for multivariate Gauss sums, as we will see below. Assume m and n are positive integers.

Our first calculation uses Fourier series on $\mathbb{T}^N$. We have

$$S_A(2\pi m/n, x) = \Bigl(\frac{1}{2\pi}\Bigr)^N \sum_{k\in\mathbb{Z}^N} e^{2\pi i(m/n)k\cdot Ak}\, e^{ik\cdot x}, \tag{3}$$

with convergence in $\mathcal{D}'(\mathbb{T}^N)$. Setting $k = nj + \ell,\ \ell \in [0, n-1]^N$, we obtain a double sum

$$S_A(2\pi m/n, x) = \Bigl(\frac{1}{2\pi}\Bigr)^N \sum_{\ell\in[0,n-1]^N} e^{2\pi i(m/n)\ell\cdot A\ell}\, e^{i\ell\cdot x} \sum_{j\in\mathbb{Z}^N} e^{inj\cdot x}. \tag{4}$$

Now

$$\sum_{j\in\mathbb{Z}^N} e^{inj\cdot x} = \Bigl(\frac{2\pi}{n}\Bigr)^N \sum_{j\in[0,n-1]^N} \delta_{2\pi j/n}(x), \tag{5}$$

so we have

$$S_A(2\pi m/n, x) = \Bigl(\frac{1}{n}\Bigr)^N \sum_{j\in[0,n-1]^N} \Bigl(\sum_{\ell\in[0,n-1]^N} e^{2\pi i(m/n)\ell\cdot A\ell}\, e^{2\pi i\ell\cdot j/n}\Bigr)\delta_{2\pi j/n}(x). \tag{6}$$

Our second calculation starts with the solution to (1) with initial data defined on Euclidean space $\mathbb{R}^N$. There we have the relatively simple formula

$$e^{itL}\delta(x) = \det(-iA)^{-1/2}(4\pi t)^{-N/2}\, e^{-ix\cdot B/4t}, \quad B = A^{-1}. \tag{7}$$

Note that $B \in Gl(N,\mathbb{Z})$. Now, if $S_A(t,x)$ is regarded as defined for $x \in \mathbb{R}^N$, invariant under the translation action of $\mathbb{Z}^N$, then we have

$$S_A(t,x) = \sum_{\nu\in\mathbb{Z}^N} e^{itL}\delta(x-2\pi\nu),$$

with convergence in $\mathcal{S}'(\mathbb{R}^N)$. Hence, setting $d_A = \det(-iA)^{-1/2}$, we have

$$\begin{aligned} &S_A(2\pi m/n,x) \\ &= d_A(4\pi)^{-N}\Bigl(\frac{2n}{m}\Bigr)^{N/2} \sum_{\nu\in\mathbb{Z}^N} e^{-i(x-2\pi\nu)\cdot B(x-2\pi\nu)(n/8\pi m)} \\ &= d_A(4\pi)^{-N}\Bigl(\frac{2n}{m}\Bigr)^{N/2} e^{-(x\cdot Bx)n/8\pi m} \sum_{\nu\in\mathbb{Z}^N} e^{-\pi i(\nu\cdot B\nu)n/2m}\, e^{in(\nu\cdot Bx)/2m}. \end{aligned} \tag{8}$$

Setting $\nu = 2mj+\ell,\ \ell\in[0,2m-1]^N$, we obtain a double sum

$$\begin{aligned} S_A(2\pi m/n,x) = d_A(4\pi)^{-N}\Bigl(\frac{2n}{m}\Bigr)^{N/2} e^{-i(x\cdot Bx)n/8\pi m} \\ \times \sum_{\ell\in[0,2m-1]^N} e^{-\pi in\ell\cdot B\ell/2m}\, e^{in\ell\cdot Bx/2m} \sum_{j\in\mathbb{Z}^N} e^{inj\cdot Bx}. \end{aligned} \tag{9}$$

Now, since $B \in Gl(N,\mathbb{Z})$, we have

$$\sum_{j\in\mathbb{Z}^N} e^{inj\cdot Bx} = \Bigl(\frac{2\pi}{n}\Bigr)^N \sum_{j\in\mathbb{Z}^N} \delta_{2\pi j/n}(x),$$

as an identity on $\mathbb{R}^N$. Descending to $\mathbb{T}^N$, we have

$$\begin{aligned} S_A(2\pi m/n,x) = d_A\Bigl(\frac{1}{2n}\Bigr)^N\Bigl(\frac{2n}{m}\Bigr)^{N/2} \sum_{j\in[0,n-1]^N} e^{-\pi i(j\cdot Bj)/2mn} \\ \times\Bigl(\sum_{\ell\in[0,2m-1]^N} e^{-\pi in\ell\cdot B\ell/2m}\, e^{\pi i\ell\cdot Bj/m}\Bigr)\delta_{2\pi j/n}(x). \end{aligned} \tag{10}$$

Now, comparing (6) and (10), we have the identity

$$\begin{aligned} &\sum_{\ell\in[0,n-1]^N} e^{2\pi i(m/n)\ell\cdot A\ell}\, e^{2\pi\ell\cdot j/n} \\ &\quad = d_A\Bigl(\frac{n}{2m}\Bigr)^{N/2} e^{-\pi i(j\cdot Bj)/2mn} \sum_{\ell\in[0,2m-1]^N} e^{-\pi i(n/2m)\ell\cdot B\ell}\, e^{\pi i\ell\cdot Bj/m}, \end{aligned} \tag{11}$$

for each $j \in \mathbb{Z}^N$. In particular, taking $j=0$, we have

$$\sum_{\ell\in[0,n-1]^N} e^{2\pi i(m/n)\ell\cdot A\ell} = d_A\Bigl(\frac{n}{2m}\Bigr)^{N/2} \sum_{\ell\in[0,2m-1]^N} e^{-\pi i(n/2m)\ell\cdot B\ell}. \tag{12}$$

The case $m = 1$ of (12) gives

$$\sum_{\ell \in [0,n-1]^N} e^{2\pi i (A\ell \cdot \ell)/n} = d_A \Big(\frac{n}{2}\Big)^{N/2} \sum_{\ell \in [0,1]^N} i^{-(\ell \cdot B\ell) n}. \tag{13}$$

Specializing in (13) to $N = 1$ and $A = B = 1$ gives the classical formula of Gauss:

$$\sum_{\ell=0}^{n-1} e^{2\pi i \ell^2/n} = \frac{1+i}{2}\, n^{1/2}\, (1 + i^{-n}). \tag{14}$$

Implicit in (14) is a computation of d_A, which we now discuss. We have

$$e^{itL} = \lim_{\varepsilon \searrow 0} e^{it(L - i\varepsilon\Delta)}, \quad \text{for} \quad t > 0,$$

where $L - i\varepsilon\Delta = Q_\varepsilon(D)$ with $Q_\varepsilon(\xi) = \xi \cdot A\xi + i\varepsilon|\xi|^2 = \xi \cdot (A + i\varepsilon I)\xi$. Hence

$$d_A = \lim_{\varepsilon \searrow 0} \det(\varepsilon I - iA)^{-1/2},$$

the right side determined by analytic continuation in A from $\det(\varepsilon I)^{1/2} = \varepsilon^{N/2} > 0$, for $\varepsilon > 0$. Suppose $A \sim \operatorname{diag}(a_1, \dots, a_L, -b_1, \dots, -b_M)$, with $a_\mu, b_\nu > 0,\ L + M = N$. Hence $a_1 \cdots a_L b_1 \cdots b_M = 1,\ \det A = (-1)^M$, and

$$\det(\varepsilon I - iA)^{-1/2} = (\varepsilon - ia_1)^{-1/2} \cdots (\varepsilon - ia_L)^{-1/2} (\varepsilon + ib_1)^{-1/2} \cdots (\varepsilon + ib_M)^{-1/2},$$

so

$$d_A = e^{(L-M)\pi i/4}. \tag{15}$$

For example, when $N = 1$ and $A = I$, we have $d_A = e^{\pi i/4} = (1+i)/\sqrt{2}$.

Let us denote the left side of (11) by $G_A(m, n, j)$, i.e.,

$$G_A(m, n, j) = \sum_{\ell \in [0,n-1]^N} e^{2\pi i (m\ell \cdot A\ell + \ell \cdot j)/n}, \tag{16}$$

where $m, n \in \mathbb{Z}^+,\ j \in \mathbb{Z}^N$. Note that the right side of (16) depends only on the class of j in $(\mathbb{Z}/(n))^N$. The formula (6) takes the form

$$S(2\pi i m/n, x) = \Big(\frac{1}{n}\Big)^N \sum_{j \in [0,n-1]^N} G_A(m, n, j)\, \delta_{2\pi j/n}(x). \tag{17}$$

Clearly the expression (11) is even in j. Hence, on the right side one can replace $e^{\pi i \ell \cdot Bj/m}$ by $e^{-\pi i \ell \cdot Bj/m}$.

One still has a somewhat different looking sum on the right side of (11), and we are motivated to define

$$\Gamma_A(m, k, j) = \Big(\frac{1}{2k}\Big)^N \sum_{\ell \in [0,2m-1]^N} e^{\pi i (m\ell \cdot A\ell + \ell \cdot j)/k}. \tag{18}$$

Comparing this with (16), we have

$$\Gamma_A(m,k,j) = \left(\frac{1}{2k}\right)^N G_A(m,2k,j). \tag{19}$$

Note that for nonzero $a \in \mathbb{Z}$, $\Gamma_A(am,ak,aj) = \Gamma_A(m,k,j)$. If we set $n = 2k$ in (17), we have $G_A(m,2k,j) = (2k)^N \Gamma_A(m,k,j)$, and hence

$$S_A(\pi m/k, x) = \sum_{j\in[0,2k-1]^N} \Gamma_A(m,k,j)\,\delta_{\pi j/k}(x). \tag{20}$$

Meanwhile, the reciprocity formula (11) takes the form

$$\Gamma_A(m,k,j) = d_A\left(\frac{m}{k}\right)^{N/2} e^{-\pi i(j\cdot Bj)/2mk}\,\overline{\Gamma_B(k,m,j)}, \tag{21}$$

when $n = 2k$. As before, $B = A^{-1}$ here.

The reciprocity formula (11) was first established by A. Krazer [K]. More general reciprocity results have been given by several people; see [T].

References

[K] A. Krazer, Zur Theorie der mehrfachen Gausschen Summen, *H. Weber Fetschrift*, Leipzig, 1912, pp. 181–.

[T] V. Turaev, Reciprocity for Gauss sums on finite abelian groups, Math. Proc. Cambridge Phil. Soc. 124(1998), 205–214.