COMMUTATOR ESTIMATES FOR HÖLDER CONTINUOUS AND BMO-SOBOLEV MULTIPLIERS

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Abstract. We discuss conditions on a function \( f \) under which the commutator \([P, f]\) of a pseudodifferential operator \( P \) of order \( m \) with the operation of multiplication by \( f \) is an operator of order \( m - r \) on various function spaces, namely Hölder-Zygmund spaces and \( L^p \)-Sobolev spaces, given \( 0 < r < 1 \). We also establish an endpoint case involving \( r = 1 \), and we extend the scope to all \( r > 0 \) for a particularly significant case in 1 dimension.

1. Introduction

The fact that the commutator of a pseudodifferential operator \( P \) of order \( m \) with the operation of multiplication by a smooth function \( f \) is an operator of order \( m - 1 \) is a central result, which has had important refinements. In particular, such a result holds for \( f \in \text{Lip}^1 \), for a certain range of \( m \); cf. [4], [6], [10], [1], [17]. Here we examine when a Hölder hypothesis on \( f \), or some variant, implies \([P, f]\) has order \( m - r \), for some \( r \in (0, 1) \). Here is one sample result, when \( m = 0 \), \( 0 < r < 1 \):

\[
\| [P, f]u \|_{C^r} \leq C \| f \|_{C^r} (\| u \|_{L^\infty} + \| Pu \|_{L^\infty}).
\]

Such a result is useful in the regularity theory of vortex patches; cf. [2], [5]. A proof when \( P \) is a classical singular integral operator of convolution type is given in [11], pp. 355–356. The estimate (1.1) is valid more generally for \( P \in \text{OPS}^0_{1, \delta} \), \( \delta < 1 \), and even more generally for \( P \in \text{OPBS}^0_{1, 1} \), as we will see below. We recall that \( P = p(x, D) \) belongs to \( \text{OPBS}^m_{1, \delta} \) if and only if its symbol \( p(x, \xi) \) satisfies

\[
|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|+\delta|\beta|}.
\]

We say \( p(x, \xi) \in \text{BS}^m_{1, 1} \) provided \( p(x, \xi) \in \text{S}^m_{1, 1} \) and the partial Fourier transform \( \hat{p}(\eta, \xi) \) has the property

\[
\text{supp} \hat{p} \subset \{ (\eta, \xi) : |\eta| \leq \rho |\xi| \},
\]

for some \( \rho < 1 \). This class was introduced in [13]. We remark that \( \text{OPBS}^m_{1, 1} \) contains \( \text{OPBS}^m_{1, \delta} \) (modulo smoothing operators) for each \( \delta < 1 \).

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The analogue of (1.1) for \([P,f]\) acting on \(L^p\)-Sobolev spaces \(H^{r,p}\) requires an hypothesis on \(f\) slightly stronger than \(f \in C^r\), namely \(f \in h^{r,\infty}\), the bmo-Sobolev space,

\[(1.4)\quad h^{r,\infty} = \Lambda^{-r} \text{bmo}, \quad \Lambda = (1 - \Delta)^{1/2},\]

where bmo is the inhomogeneous variant of the John-Nirenberg space BMO (cf. [9]). See Appendix A for a precise definition of the spaces \(h^{r,\infty}\), and some of their basic properties. Parallel to (1.1), we will show that

\[(1.5)\quad \|[P,f]u\|_{H^{r,p}} \leq C\|f\|_{h^{r,\infty}}\|u\|_{L^p},\]

given \(P \in OPBS^0_{1,1}, 0 < r < 1, 1 < p < \infty\). We will establish the following further estimates, complementing (1.1) and (1.5). See also Appendix A for a definition of the spaces \(C^*_s\) arising in (1.7), and a description of some of their basic properties.

**Proposition 1.1.** Let \(P \in OPBS^m_{1,1}\). Assume \(0 < r < 1\) and

\[(1.6)\quad -r < s < 0, \quad s < m < r + s.\]

Then

\[(1.7)\quad f \in C^r \implies [P,f] : C^*_s \to C^*_{r+s-m}.\]

**Proposition 1.2.** Let \(P \in OPBS^m_{1,1}\). Assume \(0 < r < 1, 1 < p < \infty\), and

\[(1.8)\quad -r < s \leq 0, \quad s \leq m \leq r + s.\]

Then

\[(1.9)\quad f \in h^{r,\infty} \implies [P,f] : H^{s,p} \to H^{r+s-m,p}.\]

Note that Proposition 1.2 provides a strict extension of (1.5) (where \(m = 0\)), while (with \(m = 0\)) the \(s = 0\) limit of Proposition 1.1, given in (1.1), takes \(u, Pu \in L^\infty\) rather than in \(C^0\).

In §2 we start the proofs of the results stated above, using paraproducts. Crucial paraproduct estimates are established in §3, to complete the proofs of these results. In §4 we supplement (1.9) with estimates on \(\|[P,f]u\|_{h^{r,\infty}}\). We also get such an estimate for \(f \in \text{Lip}^1\), supplementing estimates of Calderon-Coifman-Meyer type. In §5 we extend the scope of such estimates to all \(r > 0\) when \(P\) is a particularly important singular integral integral operator on the circle (essentially the Hilbert transform). We end with Appendix A, advertised above, which provides some material on the spaces \(h^{r,\infty}\) and \(C^*_s\), appearing in Propositions 1.1 and 1.2.

**Remark.** The case \(r = 0\) of (1.5) is also valid. This was established in [7] for a classical pseudodifferential operator of order zero, and in [1] for \(P \in OPBS^0_{1,1}\). The case \(m = r + s, s = 0\) of Proposition 1.2 was established in [14], for classical pseudodifferential operators of convolution type, in dimension 1.

### 2. Paraproduct decompositions and preliminary estimates

An essential ingredient in our analysis is the paraproduct operation of J.-M. Bony:

\[(2.1)\quad T_f u = \sum_{k \geq 0} f_k \psi_k(D)u,\]
where \( \{ \psi_k \} \) is a Littlewood-Paley partition of unity and \( f_k = \sum_{j \leq k-3} \psi_j(D)f \).

Along the lines of arguments used in [15] and [1], we write

\[
P(f,u) = PT_f u + PT_u f + PR(f,u),
\]

(2.2)

Here

\[
P(f,u) = \sum_{|j-k| \leq 2} \psi_j(D)f \cdot \psi_k(D)u.
\]

(2.3)

To begin, we have, for \( 0 < r < 1 \), \( P \in OPBS_{1,1}^m \), with \( \rho \) in (1.3) sufficiently small,

(2.4)

provisionally \( \implies \quad [P, T_f] \in OPBS_{1,1}^{m-\rho} \).

Such a result follows from the paradifferential operator calculus initiated in [3] and [13]; cf. also Proposition 7.3 in Chapter I of [16]. From (2.4) we have

\[
[P, T_f]: C^r_\ast \rightarrow C^{r+s-m}_\ast, \quad [P, T_f]: H^p \rightarrow H^{r+s-m,p},
\]

(2.5)

for all \( s, m \in \mathbb{R}, p \in (1, \infty) \), given \( f \in C^r_\ast, 0 < r < 1 \).

To proceed, we have the following information on the operator \( R_f u = R(f, u) \):

\[
R_f u \in OPBS_{1,1}^m \quad \text{if} \quad R \in OPBS_{1,1}^m \;
\]

this holds for all \( r \in \mathbb{R} \); cf. [15], (3.5.11). Hence, given \( P \in OPBS_{1,1}^m \),

\[
\| PR(f,u) \|_{C^{r+s-m}_\ast} \leq C \| f \|_{C^r_\ast} \| u \|_{C^s_\ast},
\]

(2.7)

provided \( r + s > 0 \), and

\[
\| R(f,u) \|_{C^{r+s-m}_\ast} \leq C \| f \|_{C^r_\ast} \| u \|_{C^s_\ast},
\]

(2.8)

provided \( r + s > m \). Regarding Sobolev estimates, if \( 1 < p < \infty \), we have

\[
\| PR(f,u) \|_{H^{r+s-m,p}} \leq C \| f \|_{C^r_\ast} \| u \|_{H^{s,p}},
\]

(2.9)

provided \( r + s > 0 \), and

\[
\| R(f,u) \|_{H^{r+s-m,p}} \leq C \| f \|_{C^r_\ast} \| u \|_{H^{s,p}},
\]

(2.10)

provided \( r + s > m \).

To complete the proofs of the results stated in §1, it remains to estimate \( PT_u f \) and \( TP_u f \), and to supplement (2.10) by

\[
\| R(f,u) \|_{L^p} \leq C \| f \|_{H^{r,\infty}} \| u \|_{H^{s,p}}, \quad m = r + s, \quad r \geq 0.
\]

(2.11)

We undertake these estimates in the next section.

### 3. Complementary Paraproduct Estimates

Here we complete the proof of the results stated in §1, via estimates on \( T_v f \).

One basic estimate comes from

\[
v \in L^{\infty} \implies T_v \in OPBS_{1,1}^0.
\]

(3.1)

In particular, given \( P \in OPBS_{1,1}^0, r \in \mathbb{R} \),

\[
\| PT_u f \|_{C^r} \leq C \| u \|_{L^\infty} \| f \|_{C^r}, \quad \| TP_u f \|_{C^r} \leq C \| Pu \|_{L^\infty} \| f \|_{C^r};
\]

cf. [15], (3.5.5). This, together with estimates of §2, is enough to establish (1.1).

Another basic estimate comes from

\[
v \in C^{r-s}_\ast, \quad s > 0 \implies T_v \in OPBS_{1,1}^s;
\]

(3.3)
cf. [15], (3.5.7). Thus, given \( P \in OPBS_{1,1}^m \), \( 0 < r < 1 \),
\[
(3.4) \quad s < 0 , \quad u \in C_s^\star , \quad f \in C^r \Rightarrow \quad PT_u f \in C_s^{\ast m + s + r} ,
\]
and
\[
(3.5) \quad s - m < 0 , \quad u \in C_s^\star , \quad f \in C^r \Rightarrow \quad TP_u f \in C_s^{\ast m + s + r} .
\]
These results complete the proof of Proposition 1.1.

To obtain paraproduct estimates to establish (1.5) and Proposition 1.2, we start with the Coifman-Meyer estimate (cf. [6])
\[
(3.6) \quad \| \tau(f, v) \|_{L^p} \leq C \| f \|_{BMO} \| v \|_{L^p} ,
\]
valid for \( p \in (1, \infty) \), for a number of paraproduct operators, including
\[
(3.7) \quad \tau(f, v) = T_v f , \quad \tau(f, v) = R(f, v).
\]
The following consequence of (3.6) was demonstrated in Proposition 3.5.F of [15].

Lemma 3.1. For \( \tau(f, v) \) as in (3.7), we have, for each \( p \in (1, \infty) \), \( r \in \mathbb{Z}^+ \),
\[
(3.8) \quad \| \tau(f, v) \|_{H^{r,p}} \leq C \| f \|_{H^{r,\infty}} \| v \|_{H^{r,p}} , \quad 0 \leq s \leq r .
\]
We produce further extensions of this result. To rephrase (3.8) in case \( s = r = k \),
if we set \( \tau_v f = \tau(f, v) \), then, for \( p \in (1, \infty) \),
\[
(3.9) \quad v \in L^p \Rightarrow \quad \tau_v : H^{k,\infty} \rightarrow H^{k,p} , \quad k \in \mathbb{Z}^+ .
\]
Interpolation (cf. [20], Proposition 5.3) gives
\[
(3.10) \quad v \in L^p \Rightarrow \quad \tau_v : \mathfrak{h}^{r,\infty} \rightarrow H^{r,p} , \quad r \in [0, \infty) .
\]
In case \( \tau_v f = T_v f \), we deduce that for \( P \in OPBS_{1,1}^0 \), \( r > 0 \), \( p \in (1, \infty) \),
\[
(3.11) \quad \| PT_u f \|_{H^{r,p}} \leq C \| f \|_{H^{r,\infty}} \| u \|_{L^p} , \quad \| TP_u f \|_{H^{r,p}} \leq C \| f \|_{H^{r,\infty}} \| u \|_{L^p} .
\]
This, together with estimates of \( \S 2 \), establishes (1.5).

To complete the proof of Proposition 1.2, we will establish the following extension of Lemma 3.1.

Proposition 3.2. Given \( \tau \) as in Lemma 3.1 and \( p \in (1, \infty) \), the conclusion of Lemma 3.1 holds for all \( r \in \{0, \infty\} \).

Proof. We find it convenient to change notation slightly, and show that
\[
(3.12) \quad s, \sigma \geq 0 , \quad f \in \mathfrak{h}^{s+\sigma,\infty} , \quad v \in H^{-\sigma,p} \Rightarrow \quad \tau(f, v) \in H^{s,p} .
\]
For \( \sigma = 0 \), this follows from (3.10). We next claim it holds for each \( s \in \{0, \infty\} \)
and \( \sigma = k \in \mathbb{Z}^+ \). The proof is inductive. If (3.12) is valid for \( \sigma = k \leq \ell \) and if \( v \in H^{-\frac{\ell-1}{p},p} \), with \( v = \partial_j v_j \), \( v_j \in H^{-\ell,p} \),
use
\[
(3.13) \quad \tau(f, v) = \partial_j \tau(f, v_j) - \tau(\partial_j f, v_j)
\]
to get (3.12) for \( \sigma = \ell + 1 \).

To finish the proof of Proposition 3.2, let us rephrase the result (3.12) as
\[
(3.14) \quad \| \tau(\Lambda^{-\sigma} g, \Lambda^s u) \|_{H^{r,p}} \leq C \| g \|_{H^{r,\infty}} \| u \|_{L^p} , \quad s, \sigma \geq 0 .
\]
So far we have this for \( \sigma = k \in \mathbb{Z}^+ \). Let us set
\[
(3.15) \quad \Phi(z) = \tau(\Lambda^{-z} g, \Lambda^z u) , \quad \Re z \geq 0 .
\]
Then, for \( k \in \mathbb{Z}^+, y \in \mathbb{R} \),
\[
(3.16) \quad \Phi(k + iy) = \tau(\Lambda^{-k-iy} g, \Lambda^{k+iy} u) \in H^{s,p} ,
\]
with mild bounds as $|y| \to \infty$. Hence a maximum principle argument for vector-valued holomorphic functions yields (3.14) and completes the proof of Proposition 3.2.

Applying (3.8) with $\tau(f, v) = T_v f$ to $PT_u f$ and $TP_u f$, we have the following. Assume $P \in OPBS_{1,1}^0$. Change notation in (3.8), replacing $s$ by $-s$, then we have

\begin{align}
(3.17) & \quad r > 0, s \leq 0, s + r \geq 0 \implies \|PT_u f\|_{H^{r+s-m,p}} \leq C\|f\|_{b^{r,\infty}}\|u\|_{H^{r,p}}, \\
(3.18) & \quad r > 0, s \leq m, s + r \geq m \implies \|TP_u f\|_{H^{r+s-m,p}} \leq C\|f\|_{b^{r,\infty}}\|u\|_{H^{r,p}}.
\end{align}

On the other hand, using $\tau(f, v) = R(f, v)$ and taking $s = 0$, we have

\begin{align}
(4.1) & \quad r > 0, s \geq 0 \implies \|R(f, v)\|_{L^p} \leq C\|f\|_{b^{r,\infty}}\|v\|_{H^{r,p}},
\end{align}

whenever $r \geq 0$, $p \in (1, \infty)$, which implies (2.11). This completes the proof of Proposition 1.2.

**Remark.** The $s = r$ case of (3.8) also gives an endpoint case of (3.11) in [17].

### 4. bmo-Sobolev space estimates

Here we estimate $[P, f] u$ in the $b^{r,\infty}$-norm, providing a $p = \infty$ endpoint case of (1.9). For simplicity we take the order $m$ of $P$ to be zero. We establish the following.

**Proposition 4.1.** Let $P \in OPBS_{1,1}^0$. Then

\begin{align}
(4.1) & \quad \|[P, f] u\|_{b^{r,\infty}} \leq C_r \|f\|_{b^{r,\infty}} \left(\|u\|_{L^\infty} + \|Pu\|_{L^\infty}\right), \quad \text{for } 0 < r < 1, \\
(4.2) & \quad \|[P, f] u\|_{b^{1,\infty}} \leq C\|f\|_{\text{Lip}^1} \left(\|u\|_{L^\infty} + \|Pu\|_{L^\infty}\right).
\end{align}

The estimate (4.2) is an endpoint case of an estimate of Calderon-Coifman-Meyer type:

\begin{align}
(4.3) & \quad \|[P, f] u\|_{b^{1,p}} \leq C\|f\|_{\text{Lip}^1}\|u\|_{L^p}, \quad 1 < p < \infty.
\end{align}

To perform these estimates, we again use (2.2). We also make use of the fact that

\begin{align}
(4.4) & \quad P \in OPS_{m}^{1,1}, \quad s - m > 0 \implies P : b^{s,\infty} \to b^{s-m,\infty},
\end{align}

which is the endpoint case of the well known behavior on $L^p$-Sobolev spaces. This result is the case $p = \infty$, $q = 2$ of Theorem 1 of [21]. In light of this, (2.4) yields

\begin{align}
(4.5) & \quad \|[P, T_f] u\|_{b^{r,\infty}} \leq C\|f\|_{C_r}\|u\|_{\text{bmo}}, \quad 0 < r < 1.
\end{align}

Furthermore, complementary to (2.4), we have

\begin{align}
(4.6) & \quad f \in \text{Lip}^1 \implies [T_f, P] \in OPBS_{1,1}^{-1};
\end{align}

cf. [1], Proposition 4.2, or [16], Proposition 7.4 of Chapter I. Hence

\begin{align}
(4.7) & \quad \|[P, T_f] u\|_{b^{s,\infty}} \leq C\|f\|_{\text{Lip}^1}\|u\|_{\text{bmo}}.
\end{align}

Also we can use (2.6) to obtain

\begin{align}
(4.8) & \quad f \in C^r_\ast \implies R_f : \text{bmo} \to b^{r,\infty}, \quad r > 0.
\end{align}

Hence

\begin{align}
(4.9) & \quad \|PR(f, u)\|_{b^{r,\infty}}, \quad \|R(f, Pu)\|_{b^{r,\infty}} \leq C\|f\|_{C^r_\ast}\|u\|_{\text{bmo}}, \quad r > 0.
\end{align}
Finally an application of (3.1) gives
\[ (4.10) \quad \| PTu f \|_{b^{r, \infty}} \leq C \| f \|_{b^{r, \infty}} \| u \|_{L^\infty}, \quad r \in \mathbb{R}, \]
and
\[ (4.11) \quad \| TPu f \|_{b^{r, \infty}} \leq C \| f \|_{b^{r, \infty}} \| Pu \|_{L^\infty}, \quad r \in \mathbb{R}. \]
These estimates together establish (4.1)–(4.2).

5. Special estimates in one dimension

Estimates given in Propositions 1.1–1.2 and in Proposition 4.1 hold for a wider range of \( r \) when \( P \) is a particularly important singular integral operator on the circle \( S^1 \), namely
\[ (5.1) \quad P_+ \sum_{\nu \in \mathbb{Z}} a_\nu e^{i\nu \theta} = \sum_{\nu \geq 0} a_\nu e^{i\nu \theta}. \]
What is special about this operator is that its symbol is constant on each connected component of \( T^* S^1 \setminus 0 \). Thus, standard symbol asymptotics give, for any \( \delta < 1 \),
\[ (5.2) \quad A \in OPS_{1, \delta}^n(S^1) \implies [A, P_+] \in OPS^{-\infty}. \]
The following related result was established in (15.14), Chapter I, of [16]:
\[ (5.3) \quad f \in L^\infty(S^1) \implies [T_f, P_+] \in OPS^{-\infty}. \]
For use in Proposition 5.2, we mention that this argument in [16] extends, and we can take \( f \in \mathcal{D}'(S^1) \) in (5.3). Using (5.3), we will prove the following.

**Proposition 5.1.** With \( P_+ \) given by (5.1), the following commutator estimates hold:
\[ (5.4) \quad \|[P_+, f]u\|_{C^r_1} \leq C \|f\|_{C^r_1} (\|u\|_{L^\infty} + \|P_+ u\|_{L^\infty}), \quad r > 0, \]
\[ (5.5) \quad \|[P_+, f]u\|_{H^{r,p}} \leq C \|f\|_{b^{r, \infty}} \|u\|_{L^p}, \quad r > 0, \quad 1 < p < \infty, \]
\[ (5.6) \quad \|[P_+, f]u\|_{b^{r, \infty}} \leq C \|f\|_{b^{r, \infty}} (\|u\|_{L^\infty} + \|P_+ u\|_{L^\infty}), \quad r > 0. \]

**Proof.** Going back to (2.2), we see that adequate estimates on \( P_+ R(f, u) \) and \( R(f, P_+ u) \) already follow from (2.7)–(2.10) and (4.8), while adequate estimates on \( P_+ Tuf \) and \( T_{P_+} u f \) follow from (3.2), (3.11), and (4.10)–(4.11). This just leaves estimates on \([T_f, P_+]\), which follow immediately from (5.3). \( \square \)

Proposition 5.1 has an application to loop group factorization, given in [18]. The following commutator estimate also has an application there.

**Proposition 5.2.** For \( 1 < p < \infty \),
\[ (5.7) \quad \|[P_+, f]u\|_{H^{r,p}} \leq C \|f\|_{H^{r,p}} (\|u\|_{L^\infty} + \|P_+ u\|_{L^\infty}), \quad r > 0. \]

**Proof.** This follows from estimates of \( P_+ Tuf \) and \( T_{P_+} u f \) in \( H^{r,p} \), estimates of \( P_+ R(f, u) \) and \( R(f, P_+ u) \) in \( H^{r,p} \), and of \([T_f, P_+]u\) (using (5.3), strengthened to allow \( f \in \mathcal{D}'(S^1) \)), in a similar fashion to the arguments given above. We merely replace information on \( Rf \) in (2.6) by the implication \( u \in L^\infty \Rightarrow Ru \in OPS_{0,1}^1 \). \( \square \)

We mention a version of Proposition 5.1 that holds when \( S^1 \) is replaced by \( \mathbb{R} \). Namely, take
\[ (5.8) \quad q \in C^\infty(\mathbb{R}), \quad q(\xi) = 0 \text{ for } \xi \leq -1, \quad q(\xi) = 0 \text{ for } \xi \geq 1, \]
and
and set
\begin{equation}
Q_+ = q(D), \quad Q_+ \in OPS^0(\mathbb{R}).
\end{equation}
Parallel to (5.2), we have, for $\delta < 1$,
\begin{equation}
A \in OPS^m_{1,\delta}(\mathbb{R}) \implies [A, Q_+] \in OPS^{-\infty}(\mathbb{R}).
\end{equation}
Also, parallel to (5.3),
\begin{equation}
f \in L^\infty(\mathbb{R}) \implies [T_f, Q_+] \in OPS^{-\infty}(\mathbb{R}).
\end{equation}
In fact, the proof of (15.14) in Chapter 1 of [16] just relies on Proposition 6.1 in Chapter 1 of [16], which works in the setting of $\mathbb{R}^n$. Consequently, analogues of the estimates (5.4)–(5.6) hold for $[Q_+, f]u$, with $f$ and $u$ defined on $\mathbb{R}$.

**Appendix A. The spaces $C^r_*$ and $h^{r,\infty}$**

The spaces $C^r_*(\mathbb{R}^n)$, sometimes called Zygmund spaces, extend to all $r \in \mathbb{R}$ the family of spaces $C^r(\mathbb{R}^n)$, defined for $r \in (0, \infty) \setminus \mathbb{Z}^+$ as follows. If $0 < r < 1$, $C^r(\mathbb{R}^n)$ consists of Hölder continuous functions with exponent $r$. If $r = k + s$, $k \in \mathbb{Z}^+$, $0 < s < 1$, then $u \in C^r(\mathbb{R}^n)$ if and only if $\partial^\alpha u \in C^s(\mathbb{R}^n)$ whenever $|\alpha| \leq k$.

The spaces $C^r(\mathbb{R}^n)$ are conveniently defined using a Littlewood-Paley partition of unity, $\{\psi_k : k \geq 0\}$. Take $\psi_0 \in C^\infty(\mathbb{R}^n)$, $\psi_0(\xi) = 1$ for $|\xi| \leq 1$, $0$ for $|\xi| \geq 2$, set $\phi_k(\xi) = \psi_0(2^{-k}\xi)$, and set $\psi_k(\xi) = \phi_k(\xi) - \phi_{k-1}(\xi)$ for $k \geq 1$. Then, given a tempered distribution $u$ on $\mathbb{R}^n$,
\begin{equation}
u \in C^r_*(\mathbb{R}^n) \iff \sup_{k \geq 0} \|\psi_k(D)u\|_{L^\infty} < \infty.
\end{equation}
One has (cf. [15], pp. 183–184) that $C^r_*(\mathbb{R}^n) = C^r(\mathbb{R}^n)$ whenever $r \in (0, \infty) \setminus \mathbb{Z}^+$.

One also has
\begin{equation}
P \in OPS^m_{1,1}(\mathbb{R}^n) \implies P : C^r_*(\mathbb{R}^n) \to C^{r-m}_*(\mathbb{R}^n), \quad \text{if } r-m > 0,
\end{equation}
and
\begin{equation}
O \in OPS^m_{1,1}(\mathbb{R}^n) \implies O : C^r_*(\mathbb{R}^n) \to C^{r-m}_*(\mathbb{R}^n), \quad \forall m, r \in \mathbb{R}.
\end{equation}
In particular, if $0 \leq \delta < 1$,
\begin{equation}
P \in OPS^m_{1,\delta}(\mathbb{R}^n) \implies P : C^r_*(\mathbb{R}^n) \to C^{r-m}_*(\mathbb{R}^n), \quad \forall m, r \in \mathbb{R}.
\end{equation}
It follows that, for $\Lambda = (1 - \Delta)^{1/2}$, i.e., $\Lambda u(\xi) = (1 + |\xi|^2)^{1/2} \hat{u}(\xi)$,
\begin{equation}
\Lambda^m : C^r_*(\mathbb{R}^n) \xrightarrow{\sim} C^{r-m}_*(\mathbb{R}^n), \quad \forall r, m \in \mathbb{R}.
\end{equation}
See [15], Chapter 2, for more operator results. The characterization (A.1) also presents $C^r_*(\mathbb{R}^n)$ as a Besov space:
\begin{equation}
C^r_*(\mathbb{R}^n) = B^r_{\infty,\infty}(\mathbb{R}^n), \quad \forall r \in \mathbb{R}.
\end{equation}
For more on this perspective, see [22], pp. 89–91.

We turn to the spaces $h^{r,\infty}(\mathbb{R}^n)$, defined in terms of $\text{bmo}(\mathbb{R}^n)$. To start, we recall the John-Nirenberg space
\begin{equation}
\text{BMO}(\mathbb{R}^n) = \{u \in L^1_{\text{loc}}(\mathbb{R}^n) : u^\# \in L^\infty(\mathbb{R}^n)\}
\end{equation}
where
\begin{equation}
|u^\#(x)| = \sup_{B \subset B(x)} \frac{1}{|B|} \int_B |u(y) - u_B| dy,
\end{equation}
with \( B(x) = \{ B_r(x) : 0 < r < \infty \} \), \( B_r(x) \) being the ball centered at \( x \) of radius \( r \), and \( u_B \) the mean value of \( u \) on \( B \). There are variants, giving the same space. For example, one could use cubes containing \( x \) instead of balls centered at \( x \), and one could replace \( u_B \) in (A.8) by \( c_B \), chosen to minimize the integral. We set \( \|u\|_{\text{BMO}} = \|u^#\|_{L^\infty} \). This is not a norm, since \( \|c\|_{\text{BMO}} = 0 \) if \( c \) is a constant; it is a seminorm. The space \( \text{bmo}(\mathbb{R}^n) \), introduced in [9], is defined by
\[
\text{bmo}(\mathbb{R}^n) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \# u \in L^\infty(\mathbb{R}^n) \},
\]
where
\[
\# u(x) = \sup_{B \in B_r(x)} \left( \frac{1}{V(B)} \int_B |u(y) - u_B| dy + \frac{1}{V(B_1(x))} \int_{B_1(x)} |u(y)| dy \right),
\]
with \( B_r(x) = \{ B_r(x) : 0 < r \leq 1 \} \). We set \( \|u\|_{\text{bmo}} = \|u^#\|_{L^\infty} \). This is a norm, and it has good localization properties. For example,
\[
f \in C^r(\mathbb{R}^n), \ u \in \text{bmo}(\mathbb{R}^n), \ r > 0 \implies fu \in \text{bmo}(\mathbb{R}^n).
\]
Also (cf. [16], p. 30),
\[
P \in OPBS^0_{1,1}(\mathbb{R}^n) \implies P : \text{bmo}(\mathbb{R}^n) \to \text{bmo}(\mathbb{R}^n),
\]
so in particular, if \( 0 \leq \delta < 1 \),
\[
P \in OPBS^0_{1,\delta}(\mathbb{R}^n) \implies P : \text{bmo}(\mathbb{R}^n) \to \text{bmo}(\mathbb{R}^n).
\]
Now, given \( r \in \mathbb{R} \), we define
\[
\text{h}^{\epsilon,\infty}(\mathbb{R}^n) = \{ \Lambda^{-\epsilon} u : u \in \text{bmo}(\mathbb{R}^n) \},
\]
with \( \Lambda \) as in (A.5). Thus \( \text{h}^{0,\infty}(\mathbb{R}^n) = \text{bmo}(\mathbb{R}^n) \). It follows from (A.12)–(A.13) and pseudodifferential operator calculus that, given \( r, m \in \mathbb{R} \), \( 0 \leq \delta < 1 \),
\[
P \in OPBS^m_{1,1}(\mathbb{R}^n) \implies P : \text{h}^{\epsilon,\infty}(\mathbb{R}^n) \to \text{h}^{\epsilon-m,\infty}(\mathbb{R}^n),
\]
\[
P \in OPBS^m_{1,\delta}(\mathbb{R}^n) \implies P : \text{h}^{\epsilon,\infty}(\mathbb{R}^n) \to \text{h}^{\epsilon-m,\infty}(\mathbb{R}^n).
\]
We briefly indicate how to define these spaces on a compact Riemannian manifold \( M \). The spaces \( C^\epsilon_r(M) \) can be defined via a partition of unity and local coordinate charts, leading to elements of \( C^\epsilon_r(\mathbb{R}^n) \). In case \( r \in (0, \infty) \setminus \mathbb{Z}^+ \), one clearly has \( C^\epsilon_r(M) = C^r(M) \), classically defined. Also, one can deduce from (A.4) that
\[
P \in OPBS^m_{1,0}(M) \implies P : C^\epsilon_r(M) \to C^{\epsilon-m}_r(M), \ \forall r, m \in \mathbb{R}.
\]
The spaces \( \text{h}^{\epsilon,\infty}(M) \) can also be defined via a partition of unity and local coordinate charts. We refer to [19] for details, worked out there for the more general class of complete Riemannian manifolds with bounded geometry. Parallel to (A.16), we have
\[
P \in OPBS^m_{1,0}(M) \implies P : \text{h}^{\epsilon,\infty}(M) \to \text{h}^{\epsilon-m,\infty}(M), \ \forall r, m \in \mathbb{R}.
\]

References


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