Hypoelliptic (and Non-Hypoelliptic) Hodge Theory

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1. First results

Let M be a compact manifold without boundary, and let $E \to M$ be a smooth vector bundle. We assume M has a Riemannian metric and E an Hermitian metric. We will denote $C^\infty(M,E)$ simply by $C^\infty(M)$. Let $d : C^\infty(M) \to C^\infty(M)$ be a first order differential operator satisfying $d^2 = 0$, and set

$$L = dd^* + d^*d. \tag{1.1}$$

We want to establish the existence of a Hodge decomposition under an hypothesis on (1.1) weaker than ellipticity.

One possibility is to assume that there exists $\varepsilon > 0$ such that

$$\begin{aligned} &u \in \mathcal{D}'(M),\ Lu \in L^2(M) \Rightarrow u \in H^\varepsilon(M),\ \text{and} \\ &\|u\|_{H^\varepsilon} \le C\|Lu\|_{L^2} + C\|u\|_{L^2}. \end{aligned} \tag{1.2}$$

Such conditions hold for scalar operators satisfying Hörmander's sum-of-squares condition for hypoellipticity, though for systems a further analysis is necessary. When (1.2) does hold, it follows that L has compact resolvent on $L^2(M)$, hence discrete spectrum, and the Hodge decomposition is easily established.

Here, we work in greater generality. We assume L is (globally) C^∞-hypoelliptic and

$$\begin{aligned} &u \in \mathcal{D}'(M),\ Lu \in H^k(M) \Rightarrow u \in H^\varepsilon(M),\ \text{and} \\ &\|u\|_{H^\varepsilon} \le C\|Lu\|_{H^k} + C\|u\|_{L^2}. \end{aligned} \tag{1.3}$$

perhaps with $k > 0$. In this setting,

$$\operatorname{Ker} L = \{u \in \mathcal{D}'(M) : Lu = 0\} \subset C^\infty(M), \tag{1.4}$$

and

$$\|u\|_{H^\varepsilon} \le C\|u\|_{L^2}, \quad \forall\, u \in \operatorname{Ker} L, \tag{1.5}$$

so by Rellich's lemma $\operatorname{Ker} L$ is finite dimensional. Note that

$$(Lu, u) = (du, du) + (d^*u, d^*u), \tag{1.6}$$

so

$$u \in \operatorname{Ker} L \Longleftrightarrow u \in C^\infty(M),\ du = 0,\ \text{and } d^*u = 0. \tag{1.7}$$

Here and below, we use (,) to denote both the L^2-inner product and the sesquilinear $\mathcal{D}'(M) \times C^\infty(M)$ pairing (or the $C^\infty(M) \times \mathcal{D}'(M)$ pairing), as appropriate. Given $u \in \mathcal{D}'(M)$, we say

$$u \perp \operatorname{Ker} L \Longleftrightarrow (u, v) = 0,\ \forall\, v \in \operatorname{Ker} L. \tag{1.8}$$

In the setting of (1.3), let

$$\mathcal{V} = \{u \in \mathcal{D}'(M) : Lu \in H^k(M) \text{ and } u \perp \operatorname{Ker} L\}, \tag{1.9}$$

$$\mathcal{W} = \{f \in H^k(M) : f \perp \operatorname{Ker} L\}. \tag{1.10}$$

Note that $\mathcal{V} \subset H^\varepsilon(M)$. We claim that

$$L : \mathcal{V} \longrightarrow \mathcal{W} \ \text{ is bijective.} \tag{1.11}$$

Clearly L is injective on $\mathcal{V}$, and $L : \mathcal{V} \to H^k(M)$. Also

$$v \in \mathcal{V},\ g \in \operatorname{Ker} L \Longrightarrow (Lv, g) = (v, Lg) = 0, \tag{1.12}$$

so $L : \mathcal{V} \to \mathcal{W}$. As a step towards proving (1.11), we next claim that, for some $A \in (0, \infty)$,

$$\|u\|_{H^\varepsilon} \le A \|Lu\|_{H^k}, \quad \forall\, u \in \mathcal{V}. \tag{1.13}$$

If (1.13) fails for all $A > 0$, then there exist $u_j \in \mathcal{V}$ such that $\|u_j\|_{H^\varepsilon} \equiv 1$ but $\|Lu_j\|_{H^k} \to 0$. By (1.3) this implies $\|u_j\|_{L^2}$ is bounded away from 0. Now Rellich's lemma implies there is a subsequence (which we continue to denote (u_j)) converging in L^2-norm, to a limit u_0, which must have the following properties: $\|u_0\|_{L^2} > 0$, $u_0 \perp \operatorname{Ker} L$, and $Lu_0 = 0$. These are contradictory, so there must be an A such that (1.13) holds.

The estimate (1.13) implies that $L : \mathcal{V} \to \mathcal{W}$ has closed range in $\mathcal{W}$, which is itself a closed linear subspace of $H^k(M)$. Next note that

$$L(\mathcal{V}) \supset L(C^\infty(M)), \tag{1.13A}$$

since we can take any $u \in C^\infty(M)$ and subtract $h = Pu$, given by (1.16)–(1.17) below, obtaining an element of $\mathcal{V}$. Hence the annihilator of $L(\mathcal{V})$ in $H^k(M)$ consists of

$$\{v \in H^{-k}(M) : Lv = 0\} = \operatorname{Ker} L, \tag{1.14}$$

(again invoking C^∞-hypoellipticity), so in fact the range is $\mathcal{W}$, and (1.11) is established.

Now for the C^∞-Hodge decomposition. Take

$$u \in C^\infty(M) \subset H^k(M) \subset L^2(M). \tag{1.15}$$

Set

$$h = Pu \in \operatorname{Ker} L, \tag{1.16}$$

where

$$P : L^2(M) \longrightarrow \operatorname{Ker} L \tag{1.17}$$

is the L^2-orthogonal projection. We have $h \in C^\infty(M)$ and hence $u-h \in C^\infty(M) \subset H^k(M)$, orthogonal to $\operatorname{Ker} L$, i.e.,

$$u - h \in \mathcal{W}. \tag{1.18}$$

By (1.11), we have

$$u - h = Lv, \tag{1.19}$$

with $v \in \mathcal{V} \subset H^\varepsilon(M)$. In fact, C^∞-hypoellipticity gives $v \in C^\infty(M)$, and we have

$$\begin{aligned} u &= dd^*v + d^*dv + h \\ &= dv_1 + d^*w_1 + h, \quad h \in \operatorname{Ker} L, \end{aligned} \tag{1.20}$$

with $v_1, w_1 \in C^\infty(M)$.

Finally, we note that this decomposition is unique. In fact, if also

$$u = dv_2 + d^*w_2 + h_2, \quad h_2 \in \operatorname{Ker} L, \tag{1.21}$$

then

$$\begin{aligned} dv_1 \perp d^*w_2 + h_2 &\Longrightarrow dv_1 = dv_2, \\ d^*w_1 \perp dv_2 + h_2 &\Longrightarrow d^*w_1 = d^*w_2, \end{aligned} \tag{1.22}$$

and hence $h = h_2$.

2. Further results

Frequently E is graded, $E = \oplus_{j=0}^{m} E_j$, and we have $d : C^\infty(M, E_j) \to C^\infty(M, E_{j+1})$, and $d^* : C^\infty(M, E_{j+1}) \to C^\infty(M, E_j)$, and hence $L_j = L|_{C^\infty(M,E_j)}$ self adjoint on $L^2(M, E_j)$. In some cases, (1.2) holds for some j but not all j.

Example 2.1. $M =$ boundary of a strongly pseudoconvex domain in $\mathbb{C}^{n+1}$, d the $\overline{\partial}_b$ complex. Then $L = \Box_b$ is the Kohn Laplacian, acting on $C^\infty(M, E_j)$, $0 \leq j \leq n$, the space of forms of type $(0, j)$, and (1.2) holds for L_j, *except* for $j = 0$ and $j = n$.

When (1.2) holds for L_j, then L_j has compact resolvent, and we have

$$I = P_j + L_j G_j \quad \text{on} \quad L^2(M, E_j), \tag{2.1}$$

where P_j is the orthogonal projection of $L^2(M, E_j)$ onto $\operatorname{Ker} L_j$ and G_j the inverse of L_j on $\operatorname{Ker} P_j$. Hypoellipticity implies P_j projects $L^2(M, E_j)$ onto a finite dimensional space of C^∞ sections and G_j takes C^∞ sections to C^∞ sections, so $u \in C^\infty(M, E_j)$ has the decomposition with smooth pieces,

$$u = dd^* G_j u + d^* d G_j u + P_j u. \tag{2.2}$$

Suppose, as in Example 2.1, (1.2) fails for L_0 but holds for L_1. (In Example 2.1, this requires $n \geq 2$.) Then we have

$$I = P_1 + L_1 G_1 \quad \text{on} \quad L^2(M, E_1). \tag{2.3}$$

We claim that

$$I = P_0 + d^* G_1 d \quad \text{on} \quad C^\infty(M, E_0). \tag{2.4}$$

Indeed, taking $u \in C^\infty(M, E_0)$, since $d^* G_1 du \perp \operatorname{Ker} L_0$, it suffices to show that $u - d^* G_1 du \in \operatorname{Ker} L_0$. Applying (2.3) to du gives $du = L_1 G_1 du$ (since $du \perp \operatorname{Ker} L_1$), hence

$$d(u - d^* G_1 du) = L_1 G_1 du - dd^* G_1 du = 0, \tag{2.5}$$

since $L_1 G_1 du = dd^* G_1 du + d^* d G_1 du$ and $d G_1 du = 0$. Here, the range $\mathcal{R}(P_0)$ might be infinite dimensional.

More generally, if (1.2) holds for L_{j-1} and L_{j+1} but not for L_j, we have

$$I = P_j + d G_{j-1} d^* + d^* G_{j+1} d \quad \text{on} \quad C^\infty(M, E_j). \tag{2.6}$$

Again, $\mathcal{R}(P_j)$ might be infinite dimensional.

3. Another example

We consider an example related to complexes studied by M. Green and P. Griffiths. Let M be a contact manifold, with contact form θ. Let $\Omega^*(M)$ denote the ring of smooth forms on M, $\mathcal{I} \subset \Omega^*(M)$ the ideal generated by θ and $d\theta$. Both $\mathcal{I}$ and the quotient $\mathcal{Q}^*(M) = \Omega^*(M)/\mathcal{I}$ inherit a grading from that of $\Omega^*(M)$, and we have a complex $d : \mathcal{Q}^k(M) \to \mathcal{Q}^{k+1}(M)$, with cohomology $H^*_{\mathcal{I}}(M)$. It is of interest to study

$$L = d^*d + dd^* \quad \text{on} \quad \mathcal{Q}^*(M). \tag{3.1}$$

To take a specific example, let $\mathcal{H}^3$ be the 3D Heisenberg group, with coordinates (t, q, p) and right-invariant contact form

$$\theta = dt - \frac{1}{2} q\, dp + \frac{1}{2} p\, dq. \tag{3.2}$$

Note that

$$d\theta = dp \wedge dq. \tag{3.3}$$

Let M be a compact quotient of $\mathcal{H}^3$ by a discrete subgroup. Then M inherits a contact form, which we also denote θ. We have

$$\begin{aligned} \mathcal{I}^0 &= 0, \\ \mathcal{I}^1 &= \{f\theta : f \in \Omega^0(M)\}, \\ \mathcal{I}^2 &= \Omega^2(M), \\ \mathcal{I}^3 &= \Omega^3(M), \end{aligned} \tag{3.4}$$

and consequently

$$\begin{aligned} \mathcal{Q}^0(M) &\approx \Omega^0(M) = C^\infty(M), \\ \mathcal{Q}^1(M) &\approx C^\infty(M, E_1), \\ \mathcal{Q}^2(M) &\approx 0, \\ \mathcal{Q}^3(M) &\approx 0, \end{aligned} \tag{3.5}$$

where $E_1 \to M$ is a real vector bundle of rank 2. Thus the complex is

$$C^\infty(M) \xrightarrow{d} C^\infty(M, E_1), \tag{3.6}$$

and we have $L = L_0 \oplus L_1$, with

$$\begin{aligned} L_0 &= d^*d : C^\infty(M) \longrightarrow C^\infty(M), \\ L_1 &= dd^* : C^\infty(M, E_1) \longrightarrow C^\infty(M, E_1). \end{aligned} \tag{3.7}$$

The scalar operator L_0 satisfies Hörmander's condition for hypoellipticity. However, it seems that L_1 is not hypoelliptic. In fact, it seems that the implication

$$u \in L^2(U, E_1),\ d^* u \in C^\infty(U) \Longrightarrow u \in C^\infty(U, E_1) \tag{3.8}$$

(given $U \subset M$ open) fails.

Indeed, pick $U \subset M$ such that E_1 trivializes over U, $E_1|_U \approx \mathbb{R} \oplus \mathbb{R}$, so we can write $u = (u_1, u_2)^t$ and

$$d^* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = X_1 u_1 + X_2 u_2, \tag{3.9}$$

where X_1 and X_2 are first order scalar differential operators on U, with real coefficients. Thus

$$d^* \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = X_1 u_1, \tag{3.10}$$

and the implication

$$u_1 \in L^2(U),\ X_1 u_1 \in C^\infty(U) \Longrightarrow u_1 \in C^\infty(U) \tag{3.11}$$

certainly fails.

Methods of §2 give the following Hodge decomposition.

$$I = P_0 + L_0 G_0 \quad \text{on} \quad L^2(M), \tag{3.12}$$

and

$$I = P_1 + d G_0 d^* \quad \text{on} \quad C^\infty(M, E_1). \tag{3.13}$$

The rage $\mathcal{R}(P_0)$ is finite dimensional, but it may be that $\mathcal{R}(P_1)$ is infinite dimensional.