Manifolds, Vector Bundles, and Lie Groups

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Introduction

This appendix provides background material on manifolds, vector bundles, and Lie groups, which are used throughout the book. We begin with a section on metric spaces and topological spaces, defining some terms that are necessary for the concept of a manifold, defined in §2, and for that of a vector bundle, defined in §3. These sections contain mostly definitions; however, a few results about compactness are proved.

In §4 we establish the easy case of a theorem of Sard, a useful result in manifold theory. It is used only once in the text, in the development of degree theory in Chapter 1, §19.

In §5 we introduce the concept of a Lie group G and its Lie algebra $\mathfrak{g}$ and establish the correspondence between Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$. We also define a Haar measure on a Lie group. In §6 we establish an important relation between Lie groups and Lie algebras, known as the Campbell-Hausdorff formula.

In §7 we discuss representations of a Lie group and associated representations of its Lie algebra. Some basic results on representations of compact Lie groups are given in §8, and in §9 we specialize to the groups SU(2) and SO(3) and to some related groups, such as SO(4). Material in §9 is useful in Chapter 8, Spectral Theory, particularly in its study of the simplest quantum mechanical model of the hydrogen atom.

1. Metric spaces and topological spaces

A metric space is a set X together with a distance function $d : X \times X \to [0, \infty)$, having the properties that

$$\begin{aligned} &d(x, y) = 0 \Longleftrightarrow x = y, \\ &d(x, y) = d(y, x), \\ &d(x, y) \le d(x, z) + d(y, z). \end{aligned} \tag{1.1}$$

The third of these properties is called the *triangle inequality*. An example of a metric space is the set of rational numbers $\mathbb{Q}$, with $d(x, y) = |x - y|$. Another example is $X = \mathbb{R}^n$, with $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

If (x_ν) is a sequence in X, indexed by $\nu = 1, 2, 3, \dots$ (i.e., by $\nu \in \mathbb{Z}^+$), one says $x_\nu \to y$ if $d(x_\nu, y) \to 0$, as $\nu \to \infty$. One says (x_ν) is a Cauchy sequence if $d(x_\nu, x_\mu) \to 0$ as $\mu, \nu \to \infty$. One says X is a complete metric space if every Cauchy sequence converges to a limit in X. Some metric spaces are not complete; for example, $\mathbb{Q}$ is not complete. One can take a sequence (x_ν) of rational numbers such that $x_\nu \to \sqrt{2}$, which is not rational. Then (x_ν) is Cauchy in $\mathbb{Q}$, but it has no limit in $\mathbb{Q}$.

If a metric space X is not complete, one can construct its completion $\widehat{X}$ as follows. Let an element ξ of $\widehat{X}$ consist of an equivalence class of Cauchy sequences in X, where we say $(x_\nu) \sim (y_\nu)$, provided $d(x_\nu, y_\nu) \to 0$. We write the equivalence class containing (x_ν) as $[x_\nu]$. If $\xi = [x_\nu]$ and $\eta = [y_\nu]$, we can set $d(\xi, \eta) = \lim_{\nu\to\infty} d(x_\nu, y_\nu)$ and verify that this is well defined and makes $\widehat{X}$ a complete metric space.

If the completion of $\mathbb{Q}$ is constructed by this process, you get $\mathbb{R}$, the set of real numbers.

A metric space X is said to be compact provided any sequence (x_ν) in X has a convergent subsequence. Clearly, every compact metric space is complete. There are two useful conditions, each equivalent to the characterization of compactness just stated, on a metric space. The reader can establish the equivalence, as an exercise.

(i) If $S \subset X$ is a set with infinitely many elements, then there is an *accumulation point*, that is, a point $p \in X$ such that every neighborhood U of p contains infinitely many points in S.

Here, a neighborhood of $p \in X$ is a set containing the ball

$$B_\varepsilon(p) = \{x \in X : d(x, p) < \varepsilon\}, \tag{1.2}$$

for some $\varepsilon > 0$.

(ii) Every open cover $\{U_\alpha\}$ of X has a finite subcover.

Here, a set $U \subset X$ is called open if it contains a neighborhood of each of its points. The complement of an open set is said to be closed. Equivalently, $K \subset X$ is closed provided that

$$x_\nu \in K, \ x_\nu \to p \in X \Longrightarrow p \in K. \tag{1.3}$$

It is clear that any closed subset of a compact metric space is also compact.

If X_j, $1 \le j \le m$, is a finite collection of metric spaces, with metrics d_j, we can define a product metric space

$$X = \prod_{j=1}^{m} X_j, \quad d(x,y) = d_1(x_1,y_1) + \cdots + d_m(x_m,y_m). \tag{1.4}$$

Another choice of metric is $\delta(x,y) = \sqrt{d_1(x_1,y_1)^2 + \cdots + d_m(x_m,y_m)^2}$. The metrics d and δ are equivalent; that is, there exist constants $C_0, C_1 \in (0,\infty)$ such that

$$C_0\delta(x,y) \le d(x,y) \le C_1\delta(x,y), \quad \forall\ x,y \in X. \tag{1.5}$$

We describe some useful classes of compact spaces.

Proposition 1.1. *If X_j are compact metric spaces, $1 \le j \le m$, so is the product space $X = \prod_{j=1}^m X_j$.*

Proof. Suppose (x_ν) is an infinite sequence of points in X; let us write $x_\nu = (x_{1\nu}, \dots, x_{m\nu})$. Pick a convergent subsequence $(x_{1\nu})$ in X_1, and consider the corresponding subsequence of (x_ν), which we relabel (x_ν). Using this, pick a convergent subsequence $(x_{2\nu})$ in X_2. Continue. Having a subsequence such that $x_{j\nu} \to y_j$ in X_j for each $j = 1, \dots, m$, we then have a convergent subsequence in X.

The following result is called the *Heine-Borel theorem*:

Proposition 1.2. *If K is a closed bounded subset of $\mathbb{R}^n$, then K is compact.*

Proof. The discussion above reduces the problem to showing that any closed interval $I = [a,b]$ in $\mathbb{R}$ is compact. Suppose S is a subset of I with infinitely many elements. Divide I into two equal subintervals, $I_1 = [a,b_1]$, $I_2 = [b_1,b]$, $b_1 = (a+b)/2$. Then either I_1 or I_2 must contain infinitely many elements of S. Say I_j does. Let x_1 be any element of S lying in I_j. Now divide I_j in two equal pieces, $I_j = I_{j1} \cup I_{j2}$. One of these intervals (say I_{jk}) contains infinitely many points of S. Pick $x_2 \in I_{jk}$ to be one such point (different from x_1). Then subdivide I_{jk} into two equal subintervals, and continue. We get an infinite sequence of distinct points $x_\nu \in S$, and $|x_\nu - x_{\nu+k}| \le 2^{-\nu}(b-a)$, for $k \ge 1$. Since $\mathbb{R}$ is complete, (x_ν) converges, say to $y \in I$. Any neighborhood of y contains infinitely many points in S, so we are done.

If X and Y are metric spaces, a function $f : X \to Y$ is said to be continuous provided $x_\nu \to x$ in X implies $f(x_\nu) \to f(x)$ in Y.

Proposition 1.3. *If X and Y are metric spaces, $f : X \to Y$ continuous, and $K \subset X$ compact, then $f(K)$ is a compact subset of Y.*

Proof. If (y_ν) is an infinite sequence of points in $f(K)$, pick $x_\nu \in K$ such that $f(x_\nu) = y_\nu$. If K is compact, we have a subsequence $x_{\nu_j} \to p$ in X, and then $y_{\nu_j} \to f(p)$ in Y.

If $F : X \to \mathbb{R}$ is continuous, we say $f \in C(X)$. A corollary of Proposition 1.3 is the following:

Proposition 1.4. *If X is a compact metric space and $f \in C(X)$, then f assumes a maximum and a minimum value on X.*

A function $f \in C(X)$ is said to be uniformly continuous provided that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in X,\ d(x,y) \le \delta \Longrightarrow |f(x) - f(y)| \le \varepsilon. \tag{1.6}$$

An equivalent condition is that f have a modulus of continuity, in other words, a monotonic function $\omega : [0,1) \to [0,\infty)$ such that $\delta \searrow 0 \Rightarrow \omega(\delta) \searrow 0$ and such that

$$x, y \in X,\ d(x,y) \le \delta \le 1 \Longrightarrow |f(x) - f(y)| \le \omega(\delta). \tag{1.7}$$

Not all continuous functions are uniformly continuous. For example, if $X = (0,1) \subset \mathbb{R}$, then $f(x) = \sin(1/x)$ is continuous, but not uniformly continuous, on X. There *is* a case where continuity implies uniform continuity:

Proposition 1.5. *If X is a compact metric space and $f \in C(X)$, then f is uniformly continuous.*

Proof. If not, there exist $x_\nu, y_\nu \in X$ and $\varepsilon > 0$ such that $d(x_\nu, y_\nu) \le 2^{-\nu}$ but

$$|f(x_\nu) - f(y_\nu)| \ge \varepsilon. \tag{1.8}$$

Taking a convergent subsequence $x_{\nu_j} \to p$, we also have $y_{\nu_j} \to p$. Now continuity of f at p implies $f(x_{\nu_j}) \to f(p)$ and $f(y_{\nu_j}) \to f(p)$, contradicting (1.8).

If X and Y are metric spaces, the space $C(X, Y)$ of continuous maps $f : X \to Y$ has a natural metric structure, under some additional hypotheses. We use

$$D(f, g) = \sup_{x \in X} d\big(f(x), g(x)\big). \tag{1.9}$$

This sup exists provided $f(X)$ and $g(X)$ are bounded subsets of Y, where to say $B \subset Y$ is bounded is to say $d : B \times B \to [0, \infty)$ has bounded image. In particular, this supremum exists if X is compact. The following result is useful in the proof of the fundamental local existence theorem for ODE, in Chapter 1.

Proposition 1.6. *If X is a compact metric space and Y is a complete metric space, then $C(X, Y)$, with the metric (1.9), is complete.*

We leave the proof as an exercise.

The following extension of Proposition 1.1 is a special case of Tychonov's theorem.

Proposition 1.7. *If $\{X_j : j \in \mathbb{Z}^+\}$ are compact metric spaces, so is the product $X = \prod_{j=1}^{\infty} X_j$.*

Here, we can make X a metric space by setting

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(p_j(x), p_j(y))}{1 + d_j(p_j(x), p_j(y))}, \tag{1.10}$$

where $p_j : X \to X_j$ is the projection onto the jth factor. It is easy to verify that if $x_\nu \in X$, then $x_\nu \to y$ in X, as $\nu \to \infty$, if and only if, for each j, $p_j(x_\nu) \to p_j(y)$ in X_j.

Proof. Following the argument in Proposition 1.1, if (x_ν) is an infinite sequence of points in X, we obtain a nested family of subsequences

$$(x_\nu) \supset (x^1{}_\nu) \supset (x^2{}_\nu) \supset \cdots \supset (x^j{}_\nu) \supset \cdots \tag{1.11}$$

such that $p_\ell(x^j{}_\nu)$ converges in X_ℓ, for $1 \le \ell \le j$. The next step is a "diagonal construction." We set

$$\xi_\nu = x^\nu{}_\nu \in X. \tag{1.12}$$

Then, for each j, after throwing away a finite number $N(j)$ of elements, one obtains from (ξ_ν) a subsequence of the sequence $(x^j{}_\nu)$ in (1.11), so $p_\ell(\xi_\nu)$ converges in X_ℓ for all ℓ. Hence (ξ_ν) is a convergent subsequence of (x_ν).

We turn now to the notion of a topological space. This is a set X, together with a family $\mathcal{O}$ of subsets, called "open," satisfying the following conditions:

$$
\begin{aligned}
&X, \emptyset \;\; \text{open}, \\
&U_j \text{ open}, 1 \le j \le N \Rightarrow \bigcap_{j=1}^{N} U_j \text{ open}, \\
&U_\alpha \text{ open}, \alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha \text{ open},
\end{aligned}
\tag{1.13}
$$

where A is *any* index set. It is obvious that the collection of open subsets of a metric space, defined above, satisfies these conditions. As before, a set $S \subset X$ is closed provided $X \setminus S$ is open. Also, we say a subset $N \subset X$ containing p is a neighborhood of p provided N contains an open set U that in turn contains p.

If X is a topological space and S is a subset, S gets a topology as follows. For each U open in X, $U \cap S$ is declared to be open in S. This is called the *induced topology.*

A topological space X is said to be Hausdorff provided that any distinct $p, q \in X$ have disjoint neighborhoods. Clearly, any metric space is Hausdorff. Most important topological spaces are Hausdorff.

A Hausdorff topological space is said to be compact provided the following condition holds. If $\{U_\alpha : \alpha \in A\}$ is any family of open subsets of X, covering X (i.e., $X = \bigcup_{\alpha \in A} U_\alpha$), then there is a finite subcover, that is, a finite subset $\{U_{\alpha_1}, \dots, U_{\alpha_N} : \alpha_j \in A\}$ such that $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}$. An equivalent formulation is the following, known as the *finite intersection property.* Let $\{S_\alpha : \alpha \in A\}$ be any collection of closed subsets of X. If each finite collection of these closed sets has nonempty intersection, then the complete intersection $\bigcap_{\alpha \in A} S_\alpha$ is nonempty. It is not hard to show that any compact metric space satisfies this condition.

Any closed subset of a compact space is compact. Furthermore, any compact subset of a Hausdorff space is necessarily closed.

Most of the propositions stated above for compact metric spaces have extensions to compact Hausdorff spaces. We mention one nontrivial result, which is the general form of Tychonov's theorem; for a proof, see [Dug].

Theorem 1.8. *If S is any nonempty set (possibly uncountable) and if, for any $\alpha \in S$, X_α is a compact Hausdorff space, then so is $X = \prod_{\alpha \in S} X_\alpha$.*

A Hausdorff space X is said to be locally compact provided every $p \in X$ has a neighborhood N that is compact (with the induced topology).

A Hausdorff space is said to be paracompact provided every open cover $\{U_\alpha : \alpha \in A\}$ has a locally finite refinement, that is, an open cover $\{V_\beta : \beta \in B\}$ such that each V_β is contained in some U_α and each $p \in X$ has a neighborhood N_p such that $N_p \cap V_\beta$ is nonempty for only finitely many $\beta \in B$. A typical example of a paracompact space is a locally compact Hausdorff space X that is also σ-compact (i.e., $X = \bigcup_{n=1}^\infty X_n$ with X_n compact). Paracompactness is a natural condition under which to construct partitions of unity, as will be illustrated in the next two sections.

A map $F : X \to Y$ between two topological spaces is said to be continuous provided $F^{-1}(U)$ is open in X whenever U is open in Y. If $F : X \to Y$ is one-to-one and onto, and both F and F^{-1} are continuous, F is said to be a homeomorphism. For a bijective map $F : X \to Y$, the continuity of F^{-1} is equivalent to the statement that $F(V)$ is open in Y whenever V is open in X; another equivalent statement is that $F(S)$ is closed in Y whenever S is closed in X.

If X and Y are Hausdorff, and $F : X \to Y$ is continuous, then $F(K)$ is compact in Y whenever K is compact in X. In view of the discussion above, there arises the following useful sufficient condition for a continuous map $F : X \to Y$ to be a homeomorphism. Namely, if X is compact, Y is Hausdorff, and F is one-to-one and onto, then F is a homeomorphism.

2. Manifolds

A *manifold* is a Hausdorff topological space with an "atlas," that is, a covering by open sets U_j together with homeomorphisms $\varphi_j : U_j \to V_j$, V_j open in $\mathbb{R}^n$. The number n is called the dimension of M. We say that M is a smooth manifold provided the atlas has the following property. If $U_{jk} = U_j \cap U_k \neq \emptyset$, then the map

$$\psi_{jk} : \varphi_j(U_{jk}) \to \varphi_k(U_{jk}), \tag{2.1}$$

given by $\varphi_k \circ \varphi_j^{-1}$, is a smooth diffeomorphism from the open set $\varphi_j(U_{jk})$ to the open set $\varphi_k(U_{jk})$ in $\mathbb{R}^n$. By this, we mean that ψ_{jk} is C^∞, with a C^∞-inverse. If the ψ_{jk} are all C^ℓ-smooth, M is said to be C^ℓ-smooth. The pairs (U_j, φ_j) are called local coordinate charts.

A continuous map from M to another smooth manifold N is said to be smooth if it is smooth in local coordinates. Two different atlases on M, giving a priori two structures of M as a smooth manifold, are said to be equivalent if the identity map on M is smooth from each one of these two manifolds to the other. Actually, a smooth manifold is considered to be defined by equivalence classes of such atlases, under this equivalence relation.

One way manifolds arise is the following. Let $f_1, \dots, f_k$ be smooth functions on an open set $U \subset \mathbb{R}^n$. Let $M = \{x \in U : f_j(x) = c_j\}$, for a given $(c_1, \dots, c_k) \in \mathbb{R}^k$. Suppose that $M \neq \emptyset$ and, for each $x \in M$, the gradients ∇f_j are linearly independent at x. It follows easily from the implicit function theorem that M has a natural structure of a smooth manifold of dimension $n - k$. We say M is a submanifold of U. More generally, let $F : X \to Y$ be a smooth map between smooth manifolds, $c \in Y$, $M = F^{-1}(c)$, and assume that $M \neq \emptyset$ and that, at each point $x \in M$, there is a coordinate neighborhood U of x and V of c such that the derivative DF at x has rank k. More pedantically, (U, φ) and (V, ψ) are the coordinate charts, and we assume the derivative of $\psi \circ F \circ \varphi^{-1}$ has rank k at $\varphi(x)$; there is a natural notion of $DF(x) : T_xX \to T_cY$, which will be defined in the next section. In such a case, again the implicit function theorem gives M the structure of a smooth manifold.

We mention a couple of other methods for producing manifolds. For one, given any connected smooth manifold M, its universal covering space $\tilde{M}$ has the natural structure of a smooth manifold. $\tilde{M}$ can be described as follows. Pick a base point $p \in M$. For $x \in M$, consider smooth paths from p to x, $\gamma : [0,1] \to M$. We say two such paths γ_0 and γ_1 are equivalent if they are homotopic, that is, if there is a smooth map $\sigma : I \times I \to M (I = [0,1])$ such that $\sigma(0,t) = \gamma_0(t)$, $\sigma(1,t) = \gamma_1(t)$, $\sigma(s,0) = p$, and $\sigma(s,1) = x$. Points in $\tilde{M}$ lying over any given $x \in M$ consist of such equivalence classes.

Another construction produces quotient manifolds. In this situation, we have a smooth manifold M and a discrete group Γ of diffeomorphisms on M. The quotient space $\Gamma \setminus M$ consists of equivalence classes of points of M, where we set $x \sim \gamma(x)$ for each $x \in M$, $\gamma \in \Gamma$. If we assume that each $x \in M$ has a neighborhood U containing no $\gamma(x)$, for $\gamma \neq e$, the identity element of Γ, then $\Gamma \setminus M$ has a natural smooth manifold structure.

We next discuss partitions of unity. Supose M is paracompact. In this case, using a locally finite covering of M by coordinate neighborhoods, we can construct $\psi_j \in C_0^\infty(M)$ such that, for any compact $K \subset M$, only finitely many ψ_j are nonzero on K (we say the sequence ψ_j is locally finite) and such that, for any $p \in M$, some $\psi_j(p) \neq 0$. Then

$$\varphi_j(x) = \Big(\sum_k \psi_k(x)^2\Big)^{-1} \psi_j(x)^2 \tag{2.2}$$

is a locally finite sequence of functions in $C_0^\infty(M)$, satisfying $\sum_j \varphi_j(x) = 1$. Such a sequence is called a *partition of unity.* It has many uses.

Using local coordinates plus such cut-offs as appear in (2.2), one can easily prove that any smooth, compact manifold M can be smoothly imbedded in some Euclidean space $\mathbb{R}^N$, though one does not obtain so easily Whitney's optimal value of N ($N = 2\dim M + 1$, valid for paracompact M, not just compact M), proved in [Wh].

A more general notion than manifold is that of a smooth *manifold with boundary.* In this case, $\overline{M}$ is again a Hausdorff topological space, and there are two types of coordinate charts (U_j, φ_j). Either φ_j takes U_j to an open subset V_j of $\mathbb{R}^n$ as before, or φ_j maps U_j homeomorphically onto an open subset of $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Again appropriate transition maps are required to be smooth. In case $\overline{M}$ is paracompact, there is again the construction of partitions of unity. For one simple but effective application of this construction, see the proof of the Stokes formula in §13 of Chapter I.

3. Vector bundles

We begin with an intrinsic definition of a tangent vector to a smooth manifold M, at a point $p \in M$. It is an equivalence class of smooth curves through p, that is, of smooth maps $\gamma : I \to M$, I an interval containing 0, such that $\gamma(0) = p$. The equivalence relation is $\gamma \sim \gamma_1$ provided that, for some coordinate chart (U, φ) about p, $\varphi : U \to V \subset \mathbb{R}^n$, we have

$$\frac{d}{dt}\,(\varphi \circ \gamma)(0) = \frac{d}{dt}\,(\varphi \circ \gamma_1)(0). \tag{3.1}$$

This equivalence is independent of the choice of coordinate chart about p.

If $V \subset \mathbb{R}^n$ is open, we have a natural identification of the set of tangent vectors to V at $p \in V$ with $\mathbb{R}^n$. In general, the set of tangent vectors to M at p is denoted T_pM. A coordinate cover of M induces a coordinate cover of TM, the disjoint union of T_pM as p runs over M, making TM a smooth manifold. TM is called the *tangent bundle* of M. Note that each T_pM has the natural structure of a vector space of dimension n, if n is the dimension of M. If $F : X \to M$ is a smooth map between manifolds, $x \in X$, there is a natural linear map $DF(x) : T_xX \to T_pM$, $p = F(x)$, which agrees with the derivative as defined in §1 of Chapter 1, in local coordinates. $DF(x)$ takes the equivalence class of a smooth curve γ through x to that of the curve $F \circ \gamma$ through p.

The tangent bundle TM of a smooth manifold M is a special case of a vector bundle. Generally, a smooth vector bundle $E \to M$ is a smooth manifold E, together with a smooth map $\pi : E \to M$ with the following properties. For each $p \in M$, the "fiber" $E_p = \pi^{-1}(p)$ has the structure of a vector space, of dimension k, independent of p. Furthermore, there exists a cover of M by open sets U_j, and diffeomorphisms $\Phi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$ with the property that, for each $p \in U_j$, $\Phi_j : E_p \to \{p\} \times \mathbb{R}^k \to \mathbb{R}^k$ is a linear isomorphism, and if $U_{j\ell} = U_j \cap U_\ell \neq \emptyset$, we have smooth "transition functions"

$$\Phi_\ell \circ \Phi_j^{-1} = \Psi_{j\ell} : U_{j\ell} \times \mathbb{R}^k \to U_{j\ell} \times \mathbb{R}^k, \tag{3.2}$$

which are the identity on the first factor and such that for each $p \in U_{j\ell}$, $\Psi_{j\ell}(p)$ is a linear isomorphism on $\mathbb{R}^k$. In the case of complex vector bundles, we systematically replace $\mathbb{R}^k$ by $\mathbb{C}^k$ in the discussion above.

The structure above arises for the tangent bundle as follows. Let (U_j, φ_j) be a coordinate cover of M, $\varphi_j : U_j \to V_j \subset \mathbb{R}^n$. Then $\Phi_j : TU_j \to U_j \times \mathbb{R}^n$ takes the equivalence class of smooth curves through $p \in U_j$ containing an element γ to the pair $\big(p, (\varphi_j \circ \gamma)'(0)\big) \in U_j \times \mathbb{R}^n$.

A *section* of a vector bundle $E \to M$ is a smooth map $\beta : M \to E$ such that $\pi(\beta(p)) = p$ for all $p \in M$. For example, a section of the tangent bundle $TM \to M$ is a vector field on M. If X is a vector field on M, generating a flow $\mathcal{F}^t$, then $X(p) \in T_pM$ coincides with the equivalence class of $\gamma(t) = \mathcal{F}^t p$.

Any smooth vector bundle $E \to M$ has associated a vector bundle $E^* \to M$, the "dual bundle" with the property that there is a natural duality of E_p and E_p^* for each $p \in M$. In case E is the tangent bundle TM, this dual bundle is called the *cotangent bundle* and is denoted T^*M.

More generally, given a vector bundle $E \to M$, other natural constructions involving vector spaces yield other vector bundles over M, such as tensor bundles $\otimes^j E \to M$ with fiber $\otimes^j E_p$, mixed tensor bundles with fiber $\bigl(\otimes^j E_p\bigr) \otimes \bigl(\otimes^k E_p^*\bigr)$, exterior algebra bundles with fiber ΛE_p, and so forth. Note that a k-form, as defined in Chapter 1, is a section of $\Lambda^k T^*M$. A section of $\bigl(\otimes^j T\bigr) \otimes \bigl(\otimes^k T^*\bigr)M$ is called a tensor field of type (j,k).

A Riemannian metric tensor on a smooth manifold M is a smooth, symmetric section g of $\otimes^2 T^*M$ that is positive-definite at each point $p \in M$; that is, $g_p(X,X) > 0$ for each nonzero $X \in T_pM$. For any fixed $p \in M$, using a local coordinate patch (U,φ) containing p, one can construct a positive, symmetric section of $\otimes^2 T^*U$. Using a partition of unity, we can hence construct a Riemannian metric tensor on any smooth, paracompact manifold M. If we define the length of a path $\gamma : [0,1] \to M$ to be

$$L(\gamma) = \int_0^1 g\bigl(\gamma'(t), \gamma'(t)\bigr)^{1/2} dt,$$

then

$$d(p,q) = \inf\{L(\gamma) : \gamma(0) = p, \gamma(1) = q\} \tag{3.3}$$

is a distance function making M a metric space, provided M is connected.

The notion of vector bundle often aids in making intrinsic definitions of important mathematical concepts. As an illustration, we note the following intrinsic characterization of the contact form κ on T^*M, which was specified in local coordinates in (14.17) of Chapter 1. Let $z \in T^*M$; if $\pi : T^*M \to M$ is the natural projection, let $p = \pi(z)$, so $z \in T_p^*M$. To define κ at z, as $\kappa(z) \in T_z^*(T^*M)$, we specify how it acts on a tangent vector $v \in T_z(T^*M)$. The specification is

$$\langle v, \kappa(z)\rangle = \langle (D\pi)v, z\rangle, \tag{3.4}$$

where $D\pi : T_z(T^*M) \to T_pM$ is the derivative of π, and the right side of (3.4) is defined by the usual dual pairing of T_pM and T_p^*M. It is routine to check that this agrees with (14.17) of Chapter 1 in any coordinate system on M. This establishes again the result of §14 of Chapter 1, that the symplectic form $\sigma = d\kappa$ is well defined on a cotangent bundle T^*M.

4. Sard's theorem

Let $F : \Omega \to \mathbb{R}^n$ be a C^1-map, with Ω open in $\mathbb{R}^n$. If $p \in \Omega$ and $DF(p) : \mathbb{R}^n \to \mathbb{R}^n$ is not surjective, then p is said to be a critical point and $F(p)$ a critical value. The set C of critical points can be a large subset of Ω, even all of it, but the set of critical values $F(C)$ must be small in $\mathbb{R}^n$. This is part of Sard's theorem.

Theorem 4.1. *If $F : \Omega \to \mathbb{R}^n$ is a C^1-map, then the set of critical values of F has measure 0 in $\mathbb{R}^n$.*

Proof. If $K \subset \Omega$ is compact, cover $K \cap C$ with m-dimensional cubes Q_j, with disjoint interiors, of side δ_j. Pick $p_j \in C \cap Q_j$, so $L_j = DF(p_j)$ has rank $\le n-1$. Then, for $x \in Q_j$,

$$F(p_j + x) = F(p_j) + L_j x + R_j(x), \quad \|R_j(x)\| \le \rho_j = \eta_j \delta_j,$$

where $\eta_j \to 0$ as $\delta_j \to 0$. Now $L_j(Q_j)$ is certainly contained in an $(n-1)$-dimensional cube of side $C_0 \delta_j$, where C_0 is an upper bound for $\sqrt{m}\|DF\|$ on K. Since all points of $F(Q_j)$ are a distance $\le \rho_j$ from (a translate of) $L_j(Q_j)$, this implies

$$\text{meas } F(Q_j) \le 2\rho_j (C_0 \delta_j + 2\rho_j)^{n-1} \le C_1 \eta_j \delta_j^n,$$

provided δ_j is sufficiently small that $\rho_j \le \delta_j$. Now $\sum_j \delta_j^n$ is the volume of the cover of $K \cap C$. For fixed K, this can be assumed to be bounded. Hence

$$\text{meas } F(C \cap K) \le C_K \, \eta,$$

where $\eta = \max \{\eta_j\}$. Picking a cover by small cubes, we make η arbitrarily small, so meas $F(C \cap K) = 0$. Letting $K_j \nearrow \Omega$, we complete the proof.

Sard's theorem also treats the more difficult case when Ω is open in $\mathbb{R}^m, m > n$. Then a more elaborate argument is needed, and one requires more differentiability, namely that F is class C^k, with $k = m - n + 1$. A proof can be found in [Stb]. The theorem also clearly extends to smooth mappings between separable manifolds.

Theorem 4.1 is applied in Chapter 1, in the study of degree theory. We give another application of Theorem 4.1, to the existence of lots of Morse functions. This application gives the typical flavor of how one uses Sard's theorem, and it is used in a Morse theory argument in Appendix C. The proof here is adapted from one in [GP]. We begin with a special case:

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in C^\infty(\Omega)$. For $a \in \mathbb{R}^n$, set $f_a(x) = f(x) - a \cdot x$. Then, for almost every $a \in \mathbb{R}^n$, f_a is a Morse function, that is, it has only nondegenerate critical points.*

Proof. Consider $F(x) = \nabla f(x);\ F : \Omega \to \mathbb{R}^n$. A point $x \in \Omega$ is a critical point of f_a if and only if $F(x) = a$, and this critical point is degenerate only if, in addition, a is a critical value of F. Hence the desired conclusion holds for all $a \in \mathbb{R}^n$ that are not critical values of F.

Now for the result on manifolds:

Proposition 4.3. *Let M be an n-dimensional manifold, imbedded in $\mathbb{R}^K$. Let $f \in C^\infty(M)$, and, for $a \in \mathbb{R}^K$, let $f_a(x) = f(x) - a\cdot x$, for $x \in M \subset \mathbb{R}^K$. Then, for almost all $a \in \mathbb{R}^K$, f_a is a Morse function.*

Proof. Each $p \in M$ has a neighborhood Ω_p such that some n of the coordinates x_ν on $\mathbb{R}^K$ produce coordinates on Ω_p. Let's say $x_1, \dots, x_n$ do it. Let $(a_{n+1}, \dots, a_K)$ be fixed, but arbitrary. Then, by Proposition 4.2, for almost every $(a_1, \dots, a_n) \in \mathbb{R}^n$, f_a has only nondegenerate critical points on Ω_p. By Fubini's theorem, we deduce that, for almost every $a \in \mathbb{R}^K$, f_a has only nondegenerate critical points on Ω_p. (The set of bad $a \in \mathbb{R}^K$ is readily seen to be a countable union of closed sets, hence measurable.) Covering M by a countable family of such sets Ω_p, we finish the proof.

5. Lie groups

A *Lie group* G is a group that is also a smooth manifold, such that the group operations $G \times G \to G$ and $G \to G$ given by $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth maps. Let e denote the identity element of G. For each $g \in G$, we have left and right translations, L_g and R_g, diffeomorphisms on G, defined by

$$L_g(h) = gh, \quad R_g(h) = hg. \tag{5.1}$$

The set of left-invariant vector fields X on G, that is, vector fields satisfying

$$(DL_g)X(h) = X(gh), \tag{5.2}$$

is called the *Lie algebra* of G, and is denoted $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then the Lie bracket $[X, Y]$ belongs to $\mathfrak{g}$. Evaluation of $X \in \mathfrak{g}$ at e provides a linear isomorphism of $\mathfrak{g}$ with T_eG.

A vector field X on G belongs to $\mathfrak{g}$ if and only if the flow $\mathcal{F}_X^t$ it generates commutes with L_g for all $g \in G$, that is, $g(\mathcal{F}_X^t h) = \mathcal{F}_X^t(gh)$ for all $g, h \in G$. If we set

$$\gamma_X(t) = \mathcal{F}_X^t e, \tag{5.3}$$

we obtain $\gamma_X(t+s) = \mathcal{F}_X^s(\mathcal{F}_X^t e)\cdot e = (\mathcal{F}_X^t e)(\mathcal{F}_X^s e)$, and hence

$$\gamma_X(s+t) = \gamma_X(s)\gamma_X(t), \tag{5.4}$$

for $s, t \in \mathbb{R}$; we say γ_X is a smooth, one-parameter subgroup of G. Clearly,

$$\gamma_X'(0) = X(e). \tag{5.5}$$

Conversely, if γ is any smooth, one-parameter group satisfying $\gamma'(0) = X(e)$, then $\mathcal{F}^t g = g \cdot \gamma(t)$ defines a flow generated by the vector field $X \in \mathfrak{g}$ coinciding with $X(e)$ at e.

The exponential map

$$\mathrm{Exp} : \mathfrak{g} \longrightarrow G \tag{5.6}$$

is defined by

$$\mathrm{Exp}(X) = \gamma_X(1). \tag{5.7}$$

Note that $\gamma_{sX}(t) = \gamma_X(st)$, so $\mathrm{Exp}(tX) = \gamma_X(t)$. In particular, under the identification $\mathfrak{g} \to T_eG$,

$$D\ \mathrm{Exp}(0) : T_eG \longrightarrow T_eG \text{ is the identity map.} \tag{5.8}$$

The fact that each element $X \in \mathfrak{g}$ generates a one-parameter group has the following generalization, to a fundamental result of S. Lie. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra, that is, $\mathfrak{h}$ is a linear subspace and $X_j \in \mathfrak{h} \Rightarrow [X_1, X_2] \in \mathfrak{h}$. By Frobenius's theorem (established in §9 of Chapter 1), through each point p of G there is a smooth manifold M_p of dimension $k = \dim \mathfrak{h}$, which is an integral manifold for $\mathfrak{h}$ (i.e., $\mathfrak{h}$ spans the tangent space of M_p at each $q \in M_p$). We can take M_p to be the maximal such (connected) manifold, and then it is unique. Let H be the maximal integral manifold of $\mathfrak{h}$ containing the identity element e.

Proposition 5.1. *H is a subgroup of G.*

Proof. Take $h_0 \in H$ and consider $H_0 = h_0^{-1}H$; clearly, $e \in H_0$. By left invariance, H_0 is also an integral manifold of $\mathfrak{h}$, so $H_0 = H$. This shows that $h_0, h_1 \in H \Rightarrow h_0^{-1}h_1 \in H$, so H is a group.

In addition to left-invariant vector fields on G, one can consider all left-invariant differential operators on G. This is an algebra, isomorphic to the "universal enveloping algebra" $\mathfrak{U}(\mathfrak{g})$, which can be defined as

$$\mathfrak{U}(\mathfrak{g}) = \bigotimes \mathfrak{g}_{\mathbb{C}}/J, \tag{5.9}$$

where $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ and J is the two-sided ideal in the tensor algebra $\bigotimes \mathfrak{g}_{\mathbb{C}}$ generated by $\{XY - YX - [X, Y] : X, Y \in \mathfrak{g}\}$.

There are other classes of objects whose left-invariant elements are of particular interest, such as tensor fields (particularly metric tensors) and differential forms.

Given any $\alpha_0 \in \Lambda^k T_e^*G$, there is a unique k-form α on G, invariant under L_g, that is, satisfying $L_g^*\alpha = \alpha$ for all $g \in G$, equal to α_0 at e. In case

$k = n = \dim G$, if ω_0 is a nonzero element of $\Lambda^n T_e^* G$, the corresponding left-invariant n-form ω on G defines also an orientation on G, and hence a left-invariant volume form on G, called a (left) Haar measure. It is uniquely defined up to a constant multiple. Similarly one has a right Haar measure. It is very important to be able to integrate over a Lie group using Haar measure.

In many but not all cases left Haar measure is also right Haar measure; then G is said to be *unimodular*. Note that if $\omega \in \Lambda^n(G)$ gives a left Haar measure, then, for each $g \in G$, $R_g^* \omega$ is also a left Haar measure, so we must have

$$R_g^* \omega = \mu(g)\omega, \quad \mu : G \to (0, \infty). \tag{5.10}$$

Furthermore, $\mu(gg') = \mu(g)\mu(g')$. If G is compact, this implies $\mu(g) = 1$ for all g, so all compact Lie groups are unimodular.

There are some particular Lie groups that we want to mention. Let $n \in \mathbb{Z}^+$ and $F = \mathbb{R}$ or $\mathbb{C}$. Then $\mathrm{Gl}(n, F)$ is the group of all invertible $n \times n$ matrices with entries in F. We set

$$\mathrm{Sl}(n, F) = \{A \in \mathrm{Gl}(n, F) : \det A = 1\}. \tag{5.11}$$

We also set

$$\begin{aligned} \mathrm{O}(n) &= \{A \in \mathrm{Gl}(n, \mathbb{R}) : A^t = A^{-1}\}, \\ \mathrm{SO}(n) &= \{A \in \mathrm{O}(n) : \det A = 1\}, \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \mathrm{U}(n) &= \{A \in \mathrm{Gl}(n, \mathbb{C}) : A^* = A^{-1}\}, \\ \mathrm{SU}(n) &= \{A \in \mathrm{U}(n) : \det A = 1\}. \end{aligned} \tag{5.13}$$

The Lie algebras of the groups listed above also have special names. We have $\mathrm{gl}(n, F) = M(n, F)$, the set of $n \times n$ matrices with entries in F. Also,

$$\begin{aligned} \mathrm{sl}(n, F) &= \{A \in M(n, F) : \mathrm{Tr}\, A = 0\}, \\ \mathrm{o}(n) = \mathrm{so}(n) &= \{A \in M(n, \mathbb{R}) : A^t = -A\}, \\ \mathrm{u}(n) &= \{A \in M(n, \mathbb{C}) : A^* = -A\}, \\ \mathrm{su}(n) &= \{A \in \mathrm{u}(n) : \mathrm{Tr}\, A = 0\}. \end{aligned} \tag{5.14}$$

There are many other important matrix Lie groups and Lie algebras with special names, but we will not list any more here. See [Helg], [T], or [Var1] for such lists.

6. The Campbell-Hausdorff formula

The Campbell-Hausdorff formula has the form

$$\mathrm{Exp}(X)\ \mathrm{Exp}(Y) = \mathrm{Exp}\big(\mathcal{C}(X, Y)\big), \tag{6.1}$$

where G is any Lie group, with Lie algebra $\mathfrak{g}$, and Exp: $\mathfrak{g} \to G$ is the exponential map defined by (5.7); X and Y are elements of $\mathfrak{g}$ in a sufficiently small neighborhood U of zero. The map $\mathcal{C} : U \times U \to \mathfrak{g}$ has a universal form, independent of $\mathfrak{g}$. We give a demonstration similar to one in [HS], which was also independently discovered by [Str].

We begin with the case $G = \mathrm{Gl}(n, \mathbb{C})$ and produce an explicit formula for the matrix-valued analytic function $X(s)$ of s in the identity

$$e^{X(s)} = e^X e^{sY}, \tag{6.2}$$

near $s = 0$. Note that this function satisfies the ODE

$$\frac{d}{ds} e^{X(s)} = e^{X(s)} Y. \tag{6.3}$$

We can produce an ODE for $X(s)$ by using the following formula, derived in Exercises 7–10 of §4, Chapter 1:

$$\frac{d}{ds} e^{X(s)} = e^{X(s)} \int_0^1 e^{-\tau X(s)} X'(s) e^{\tau X(s)} \, d\tau. \tag{6.4}$$

As shown there, we can rewrite this as

$$\frac{d}{ds} e^{X(s)} = e^{X(s)} \Xi\big(\mathrm{ad}\ X(s)\big) X'(s). \tag{6.5}$$

Here, ad is defined as a linear operator on the space of $n \times n$ matrices by

$$\mathrm{ad}\ X(Y) = XY - YX; \tag{6.6}$$

the function Ξ is

$$\Xi(z) = \int_0^1 e^{-\tau z} \, d\tau = \frac{1 - e^{-z}}{z}, \tag{6.7}$$

an entire holomorphic function of z; and a holomorphic function of an operator is defined either as in Exercise 10 of that set, or as in §5 of Appendix A. Comparing (6.3) and (6.5), we obtain

$$\Xi\big(\mathrm{ad}\ X(s)\big) X'(s) = Y, \quad X(0) = X. \tag{6.8}$$

We can obtain a more convenient ODE for $X(s)$ as follows. Note that

$$e^{\mathrm{ad}\ X(s)} = \mathrm{Ad}\ e^{X(s)} = \mathrm{Ad}\ e^X \cdot \mathrm{Ad}\ e^{sY} = e^{\mathrm{ad}\ X}\, e^{s\ \mathrm{ad}\ Y}. \tag{6.9}$$

Now let $\Psi(\zeta)$ be holomorphic near $\zeta = 1$ and satisfy

$$\Psi(e^a) = \frac{1}{\Xi(a)} = \frac{a}{1 - e^{-a}}, \tag{6.10}$$

explicitly,

$$\Psi(\zeta) = \frac{\zeta\ \log \zeta}{\zeta - 1}. \tag{6.11}$$

It follows that

$$\Psi\big(e^{\mathrm{ad}\ X} e^{s\ \mathrm{ad}\ Y}\big) \Xi\big(\mathrm{ad}\ X(s)\big) = I, \tag{6.12}$$

so we can transform (6.8) to

$$X'(s) = \Psi\big(e^{\mathrm{ad}\,X} e^{s\,\mathrm{ad}\,Y}\big)Y, \quad X(0) = X. \tag{6.13}$$

Integrating gives the Campbell-Hausdorff formula for $X(s)$ in (6.2):

$$X(s) = X + \int_0^s \Psi\big(e^{\mathrm{ad}\,X} e^{t\,\mathrm{ad}\,Y}\big)Y\,dt. \tag{6.14}$$

This is valid for $\|sY\|$ small enough, if also X is close enough to 0.

Taking the $s = 1$ case, we can rewrite this formula as

$$e^X e^Y = e^{\mathcal{C}(X,Y)}, \quad \mathcal{C}(X,Y) = X + \int_0^1 \Psi\big(e^{\mathrm{ad}\,X} e^{t\,\mathrm{ad}\,Y}\big)Y\,dt. \tag{6.15}$$

The formula (6.15) gives a power series in ad X and ad Y which is norm-summable provided

$$\|\mathrm{ad}\,X\| \le x, \quad \|\mathrm{ad}\,Y\| \le y, \tag{6.16}$$

with $e^{x+y} - 1 < 1$, that is,

$$x + y < \log 2. \tag{6.17}$$

We can extend the analysis above to the case where X and Y are vector fields on a manifold M, asking for a vector field $X(s)$ such that

$$\mathcal{F}^1_{X(s)} = \mathcal{F}^1_X \mathcal{F}^s_Y, \tag{6.18}$$

where $\mathcal{F}^t_X$ is the flow generated by X, evaluated at time t. If there is such a family $X(s)$, depending smoothly on s, material in §6 of Chapter 1, in place of material in §4 cited above, leads to a formula parallel to (6.4), and hence to (6.8), in this context. However, we cannot always solve (6.8), because ad $X(s)$ tends not to act as a bounded operator on a Banach space of vector fields, and in fact one cannot always solve (6.18) for $X(s)$ is this case. However, if there is a *finite-dimensional* Lie algebra $\mathfrak{g}$ of vector fields containing X and Y, then the analysis (6.9)–(6.17) extends. We have

$$\mathcal{F}^t_X \mathcal{F}^t_Y = \mathcal{F}^t_{\mathcal{C}(t,X,Y)}, \tag{6.19}$$

with

$$\mathcal{C}(t,X,Y) = X + \int_0^1 \Psi\big(e^{\mathrm{ad}\,tX} e^{\mathrm{ad}\,stY}\big)Y\,ds, \tag{6.20}$$

provided $\|\mathrm{ad}\,tX\| + \|\mathrm{ad}\,tY\| < \log 2$, the operator norm $\|\mathrm{ad}\,X\|$ being computed using any convenient norm on $\mathfrak{g}$. In particular, if $M = G$ is a Lie group with Lie algebra $\mathfrak{g}$, and $X, Y \in \mathfrak{g}$, this analysis applies to yield the Campbell-Hausdorff formula for general Lie groups.

7. Representations of Lie groups and Lie algebras

We define a representation of a Lie group G on a finite-dimensional vector space V to be a smooth map

$$\pi : G \longrightarrow \operatorname{End}(V) \tag{7.1}$$

such that

$$\pi(e) = I, \quad \pi(gg') = \pi(g)\pi(g'), \quad g, g' \in G. \tag{7.2}$$

If $F \in C_0(G)$, that is, if F is continuous with compact support, we can define $\pi(F) \in \operatorname{End}(V)$ by

$$\pi(F)v = \int_G F(g)\pi(g)v \, dg. \tag{7.3}$$

We get different results depending on whether left or right Haar measure is used. Right now, let us use *right* Haar measure. Then, for $g \in G$, we have

$$\pi(F)\pi(g)v = \int_G F(x)\pi(xg)v \, dx = \int_G F(xg^{-1})\pi(x)v \, dx. \tag{7.4}$$

We also define the derived representation

$$d\pi : \mathfrak{g} \longrightarrow \operatorname{End}(V) \tag{7.5}$$

by

$$d\pi = D\pi(e) : T_eG \longrightarrow \operatorname{End}(V), \tag{7.6}$$

using the identification $\mathfrak{g} \approx T_eG$. Thus, for $X \in \mathfrak{g}$,

$$d\pi(X)v = \lim_{t\to 0} \frac{1}{t}\big[\pi(\operatorname{Exp} tX)v - v\big]. \tag{7.7}$$

The following result states that $d\pi$ is a Lie algebra homomorphism.

Proposition 7.1. *For $X, Y \in \mathfrak{g}$, we have*

$$\big[d\pi(X), d\pi(Y)\big] = d\pi\big([X, Y]\big). \tag{7.8}$$

Proof. We will first produce a formula for $\pi(F)d\pi(X)$, given $F \in C_0^\infty(G)$. In fact, making use of (7.4), we have

$$\begin{aligned}
\pi(F)d\pi(X)v &= \lim_{t\to 0} \frac{1}{t} \int_G \big[F(g)\pi(g)\pi(\operatorname{Exp} tX) - F(g)\pi(g)\big]v \, dg \\
&= \lim_{t\to 0} \frac{1}{t} \int_G \big[F\big(g \cdot \operatorname{Exp}(-tX)\big) - F(g)\big]\pi(g)v \, dg \\
&= -\pi(XF)v,
\end{aligned} \tag{7.9}$$

where XF denotes the left-invariant vector field X applied to F. It follows that

$$\begin{aligned}\pi(F)\big[d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)\big]v \\ = \pi(YXF - XYF)v = -\pi\big([X,Y]F\big)v,\end{aligned} \tag{7.10}$$

which by (7.9) is equal to $\pi(F)d\pi\big([X,Y]\big)v$. Now, if F is supported near $e \in G$ and integrates to 1, is is easily seen that $\pi(F)$ is close to the identity I, so this implies (7.8).

There is a representation of G on $\mathfrak{g}$, called the *adjoint representation*, defined as follows. Consider

$$K_g : G \longrightarrow G, \quad K_g(h) = ghg^{-1}. \tag{7.11}$$

Then $K_g(e) = e$, and we set

$$\mathrm{Ad}(g) = DK_g(e) : T_eG \longrightarrow T_eG, \tag{7.12}$$

identifying $T_eG \approx \mathfrak{g}$. Note that $K_g \circ K_{g'} = K_{gg'}$, so the chain rule implies $\mathrm{Ad}(g)\mathrm{Ad}(g') = \mathrm{Ad}(gg')$.

Note that $\gamma(t) = g\ \mathrm{Exp}(tX)g^{-1}$ is a one-parameter subgroup of G satisfying $\gamma'(0) = \mathrm{Ad}(g)X$. Hence

$$\mathrm{Exp}(t\ \mathrm{Ad}(g)X) = g\ \mathrm{Exp}(tX)\ g^{-1}. \tag{7.13}$$

In particular,

$$\mathrm{Exp}\big((\mathrm{Ad}\ \mathrm{Exp}\ sY)tX\big) = \mathrm{Exp}(sY)\ \mathrm{Exp}(tX)\ \mathrm{Exp}(-sY). \tag{7.14}$$

Now, the right side of (7.15) is equal to $\mathcal{F}_Y^{-s} \circ \mathcal{F}_X^t \circ \mathcal{F}_Y^s(e)$, so by results on the Lie derivative of a vector field given in (8.1)–(8.3) of Chapter 1, we have

$$\mathrm{Ad}(\mathrm{Exp}\ sY)X = \mathcal{F}^s_{Y\#}X. \tag{7.15}$$

If we take the s-derivative at $s = 0$, we get a formula for the derived representation of Ad, which is denoted ad, rather than d Ad. Using (8.3)–(8.5) of Chapter 1, we have

$$\mathrm{ad}(Y)X = [Y,X]. \tag{7.16}$$

In other words, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is given by the Lie bracket. We mention that Jacobi's identity for Lie algebras is equivalent to the statement that

$$\mathrm{ad}\big([X,Y]\big) = \big[\mathrm{ad}(X), \mathrm{ad}(Y)\big], \quad \forall\ X, Y \in \mathfrak{g}. \tag{7.17}$$

If V has a positive-definite inner product, we say that the representation (7.1) is unitary provided $\pi(g)$ is a unitary operator on V, for each $g \in G$ (i.e., $\pi(g)^{-1} = \pi(g)^*$).

We say the representation (7.1) is irreducible if V has no proper linear subspace invariant under $\pi(g)$ for all $g \in G$. Irreducible unitary representations are particularly important. The following version of Schur's lemma is useful.

Proposition 7.2. *A unitary representation π of G on V is irreducible if and only if, for any $A \in End(V)$,*

$$\pi(g)A = A\pi(g), \quad \forall\, g \in G \Longrightarrow A = \lambda I. \tag{7.18}$$

Proof. First, suppose π is irreducible and A commutes with $\pi(g)$ for all g. Then so does A^*, hence $A + A^*$ and $(1/i)(A - A^*)$, so we may as well suppose $A = A^*$. Now, any polynomial $p(A)$ commutes with $\pi(g)$ for all g, so it follows that each projection P_λ onto an eigenspace of A commutes with all $\pi(g)$. Hence the range of P_λ is invariant under π, so if $P_\lambda \neq 0$, it must be I, and $A = \lambda I$.

Conversely, suppose the implication (7.18) holds. Then if $W \subset V$ is invariant under π, the orthogonal projection P of V onto W must commute with all $\pi(g)$, so P is a scalar multiple of I, hence either 0 or I. This completes the proof.

Corollary 7.3. *Assume G is connected. Then a unitary representation of G on V is irreducible if and only if, for any $A \in End(V)$,*

$$d\pi(X)A = A\; d\pi(X), \quad \forall\, X \in \mathfrak{g} \Longrightarrow A = \lambda I. \tag{7.19}$$

Proof. We mention that

$$\pi(\mathrm{Exp}\; tX) = e^{t\; d\pi(X)} \tag{7.20}$$

and leave the details to the reader.

Given a representation π of G on V, there is also a representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, defined as follows. If

$$P = \sum_{\mu \le m} c_{i_1 \cdots i_\mu} X_{i_1} \cdots X_{i_\mu}, \quad X_j \in \mathfrak{g}, \tag{7.21}$$

with $c_{i_1 \cdots i_\mu} \in \mathbb{C}$, we have

$$d\pi(P) = \sum_{\mu \le m} c_{i_1 \cdots i_\mu} d\pi(X_{i_1}) \cdots d\pi(X_{i_\mu}). \tag{7.22}$$

Proposition 7.4. *Suppose G is connected. Let $P \in \mathfrak{U}(\mathfrak{g})$, and assume*

$$PX = XP, \quad \forall\, X \in \mathfrak{g}. \tag{7.23}$$

If π is an irreducible unitary representation of G on V, then $d\pi(P)$ is a scalar multiple of the identity, that is,

$$d\pi(P) = \lambda I.$$

Proof. Immediate from Corollary 7.3.

So far in this section we have concentrated on finite-dimensional representations. It is also of interest to consider infinite-dimensional representations. One example is the right-regular representation of G on $L^2(G)$:

$$R(g)f(x) = f(xg). \tag{7.24}$$

If G has right-invariant Haar measure, then $R(g)$ is a unitary operator on $L^2(G)$ for each $g \in G$, and one readily verifies that $R(g)R(g') = R(gg')$. However, the smoothness hypothesis made on π in (7.1) does not hold here. When working with an infinite-dimensional representation π of G on a Banach space V, one makes instead the hypothesis of strong continuity: For each $v \in V$, the map $g \mapsto \pi(g)v$ is continuous from G to V, with its norm topology. If the map is C^∞, one says v is a smooth vector for the representation v. For example, each $f \in C_0^\infty(G)$ is a smooth vector for the representation (7.24). Of course, $C_0^\infty(G)$ is dense in $L^2(G)$. More generally, the set of smooth vectors for any strongly continuous representation π of G on a Banach space V is dense in V. In fact, for $F \in C_0^\infty(G)$, $\pi(F)$ is still well defined by (7.3), and the space

$$\mathcal{G}_\pi = \{\pi(F)v : F \in C_0^\infty(G), v \in V\} \tag{7.25}$$

is readily verified to be a dense subspace of V consisting of smooth vectors. If V is finite dimensional, this implies that $\mathcal{G}_\pi = V$, so any strongly continuous, finite-dimensional representation of a Lie group automatically possesses the smoothness property used above.

The occasional use made of Lie group representations in this book will not require much development of the theory of infinite-dimensional representations, so we will not go further into it here. One can find treatments in many places, including [HT], [Kn], [T], [Var2], and [Wal1].

8. Representations of compact Lie groups

Throughout this section, G will be a compact Lie group. If π is a representation of G on a finite-dimensional complex vector space V, we can always put an inner product on V so that π is unitary. Indeed, let $((u, v))$ be any Hermitian inner product on V, and set

$$(u, v) = \int_G ((\pi(g)u, \pi(g)v))\, dg. \tag{8.1}$$

Note that if V_1 is a subspace of V invariant under $\pi(g)$ for all $g \in G$, and if π is unitary, then the orthogonal complement of V_1 is also invariant. Thus, if π is not irreducible on V, we can decompose it, and we can obviously continue this process only a finite number of times if dim V is finite. Thus π breaks up into a direct sum of irreducible unitary representations of G.

Let π and λ be two representations of G, on V and W, respectively. We say they are equivalent if there is $A \in \mathcal{L}(V, W)$, invertible, such that

$$\pi(g) = A^{-1}\lambda(g)A, \quad \forall\, g \in G. \tag{8.2}$$

If these representations are unitary, we say they are unitarily equivalent if A can be taken to be unitary.

Suppose that π and λ are irreducible and unitary, and (8.2) holds. Then $\pi(g)^* = A^*\lambda(g)^*(A^{-1})^*$, for all $g \in G$, so $\pi(g) = (A^*A)\pi(g)(A^*A)^{-1}$. By Schur's lemma, A^*A must be a (positive) scalar, say b^2. Replacing A by $b^{-1}A$ makes it unitary. Breaking up a general π into irreducible representations, we deduce that whenever π and λ are finite-dimensional, unitary representations, if they are equivalent, then they are unitarily equivalent.

We now derive some results known as *Weyl orthogonality relations*, which play an important role in the study of representations of compact Lie groups. To begin, let π and λ be two irreducible representations of a compact group G, on finite-dimensional spaces V and W, respectively. Consider the representation $\nu = \pi \otimes \overline{\lambda}$ on $V \otimes W' \approx \mathcal{L}(W, V)$, defined by

$$\nu(g)(A) = \pi(g)A\lambda(g)^{-1}, \quad g \in G,\ A \in \mathcal{L}(W, V). \tag{8.3}$$

Let Z be the linear subspace of $\mathcal{L}(V, W)$ on which ν acts trivially. We want to specify Z. Note that $A_0 \in Z$ if and only if

$$\pi(g)A_0 = A_0\pi(g), \quad \forall\, g \in G. \tag{8.4}$$

Since this implies that the range of A_0 is invariant under π and Ker A_0 is invariant under λ, we see that either $A_0 = 0$ or A_0 is an isomorphism from W to V. In the latter case, we have $\pi(g) = A_0\lambda(g)A_0^{-1}$, so the representations π and λ would have to be equivalent. In this case, for arbitrary $A \in Z$, we would have

$$\pi(g)A = A\lambda(g) = AA_0^{-1}\pi(g)A_0,$$

or $\pi(g)AA_0^{-1} = AA_0^{-1}\pi(g)$, so Schur's lemma implies that AA_0^{-1} is a scalar. We have proved the following result:

Proposition 8.1. *If π and λ are finite-dimensional, irreducible representations of G and if $\nu = \pi \otimes \overline{\lambda}$, then the trivial representation occurs not at all in ν if π and λ are not equivalent, and it occurs acting on a one-dimensional subspace of $V \otimes W'$ if π and λ are equivalent.*

The next ingredient for the orthogonality relation is the study of the operator

$$P = \int_G \pi(g)\ dg. \tag{8.5}$$

Here π is a finite-dimensional representation of the compact group G, not necessarily irreducible, and dg denotes Haar measure, with total mass 1. Note that

$$\pi(y)P = \int_G \pi(yg)\ dg = P = P\pi(y), \tag{8.6}$$

for all $y \in G$. Hence

$$P^2 = P\int_G \pi(g)\ dg = \int_G P\pi(g)\ dg = P, \tag{8.7}$$

so P is a projection. Also, if π is unitary, we see that $P = P^*$.

Now, if π is unitary, it gives a representation both on the range $\mathcal{R}(P)$ and on the kernel Ker P. It is clear from (8.5) that, given $v \in V$, $\|Pv\| < \|v\|$ unless $\pi(g)v = v$, for all $g \in G$. Consequently, π operates like the identity on $\mathcal{R}(P)$, but we do not have $\pi(g)v = v$ for all $g \in G$, for any nonzero $v \in \text{Ker } P$. We have proved:

Proposition 8.2. *If π is a unitary representation of G on V, then P, given by (8.5), is the orthogonal projection onto the subspace of V on which π acts trivially.*

The following is a special case:

Corollary 8.3. *If π is a nontrivial, irreducible, unitary representation, and P is given by (8.5), then $P = 0$.*

We apply Proposition 8.2 to

$$Q = \int_G \pi(g) \otimes \overline{\lambda}(g)\ dg, \tag{8.8}$$

with π and λ irreducible. By Proposition 8.1, we see that

$$Q = 0 \quad \text{if } \pi \text{ and } \lambda \text{ are not equivalent.} \tag{8.9}$$

On the other hand, if $\lambda = \pi$, then Q has as its range the set of scalar multiples of the identity operator on V (if π acts on V). Note that $\pi \otimes \overline{\pi}$ leaves invariant the space of elements $A \in \mathcal{L}(V, V)$ of trace zero, which is the orthogonal complement (with respect to the Hilbert-Schmidt inner product) of the space of scalars, so Q must annihilate this space. Thus Q is given by

$$Q(A) = (d^{-1} \text{ Tr } A)I, \quad \pi = \lambda, \; d = \dim V. \tag{8.10}$$

The identities (8.9) and (8.10) are equivalent to the Weyl orthogonality relations. If we express π and λ as matrices, with respect to some orthonormal bases, we get the following theorem:

Theorem 8.4. *Let π and λ be inequivalent irreducible, unitary representations of G, on V and W, with matrix entries π_{ij} and $\lambda_{k\ell}$, respectively. Then*

$$\int_G \pi_{ij}(g)\lambda_{k\ell}(g)\; dg = 0. \tag{8.11}$$

Also,

$$\int_G \pi_{ij}(g)\overline{\pi_{k\ell}(g)}\; dg = 0, \qquad \text{unless } i = k \text{ and } j = \ell. \tag{8.12}$$

Furthermore,

$$\int_G |\pi_{ij}(g)|^2\; dg = d^{-1}, \tag{8.13}$$

where $d = \dim V = \text{Tr } \pi(e)$.

Hence, if $\{\pi^k\}$ is a complete set of inequivalent, irreducible, unitary representations of G on spaces V_k, of dimension d_k, then

$$d_k^{1/2}\, \pi_{ij}^k(g) \tag{8.14}$$

forms an orthonormal set in $L^2(G)$. The following is the Peter-Weyl theorem:

Theorem 8.5. *The orthonormal set (8.14) is complete.*

In other words, the linear span of (8.14) is dense. If G is given as a group of unitary $N \times N$ matrices, this result is elementary. In fact, the linear span of (8.14) is an algebra (take tensor products of π^k and π^ℓ and decompose into irreducibles), and is closed under complex conjugates (pass from π to $\overline{\pi}$), so if we know it separates points (which is clear if $G \subset U(N)$), the Stone-Weierstrass theorem applies.

If we do not know a priori that $G \subset U(N)$, we can prove the theorem by considering the right-regular representation of G on $L^2(G)$:

$$R(g)f(x) = f(xg). \tag{8.15}$$

If we endow G with a bi-invariant Riemannian metric and consider the associated Laplace operator Δ, which is then a bi-invariant differential operator, we see that the representation R leaves invariant each eigenspace E_ℓ of Δ. Now, E_ℓ is finite-dimensional, and the restriction R_ℓ of R to E_ℓ splits into irreducibles:

$$E_\ell = E_{\ell 1} \oplus \cdots \oplus E_{\ell N}, \quad N = N(\ell), \tag{8.16}$$

say $R_\ell\big|_{E_{\ell m}} = R_{\ell m}$. Thus there is a unitary map $A : E_{\ell m} \to V_k$, for some $k = k(\ell, m)$, such that $R_{\ell m} = A\pi^k A^{-1}$. If $\{e_i\}$ is an orthonormal basis of V_k with respect to which the matrix of $\pi^k(g)$ is $\bigl(\pi^k_{ij}(g)\bigr)$, then $u_i = A^{-1}e_i$ gives an orthonormal basis of $E_{\ell m}$, and we have

$$u_i(xg) = \sum_j \pi^k_{ij}(g) u_j(x). \tag{8.17}$$

In particular, taking $x = e$,

$$u_i(g) = \sum_j c_j \pi^k_{ij}(g), \quad c_j = u_j(e). \tag{8.18}$$

This shows that each space $E_{\ell m}$ consists of finite linear combinations of the functions in (8.14). Since

$$L^2(G) = \bigoplus_\ell E_\ell = \bigoplus_\ell \bigoplus_m E_{\ell m},$$

this proves Theorem 8.5.

The following corollary will be useful in the next section.

Corollary 8.6. *If G_1 and G_2 are two compact Lie groups, then the irreducible, unitary representations of $G = G_1 \times G_2$ are, up to unitary equivalence, precisely those of the form*

$$\pi(g) = \pi_1(g_1) \otimes \pi_2(g_2), \tag{8.19}$$

where $g = (g_1, g_2) \in G$, and $\pi_j \in \widehat{G}_j$ is a general, irreducible, unitary representation of G_j.

Proof. Given irreducible, unitary representations π_j of G_j, the irreducibility and unitarity of (8.19) are clear. It remains to prove the completeness of the set of such representations. For this, it suffices to show that the matrix entries of such representations have dense linear span in $L^2(G_1 \times G_2)$. This follows from the general elementary fact that tensor products of orthonormal bases of $L^2(G_1)$ and $L^2(G_2)$ form an orthonormal basis of $L^2(G_1 \times G_2)$.

9. Representations of SU(2) and related groups

The group SU(2) is the group of 2×2, complex, unitary matrices of determinant 1, that is,

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix} : |z_1|^2 + |z_2|^2 = 1,\ z_j \in \mathbb{C} \right\}. \tag{9.1}$$

As a set, SU(2) is naturally identified with the unit sphere S^3 in $\mathbb{C}^2$. Its Lie algebra su(2) consists of 2×2, complex, skew-adjoint matrices of trace zero. A basis of su(2) is formed by

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{9.2}$$

Note the commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \tag{9.3}$$

The group SO(3) is the group of linear isometries of $\mathbb{R}^3$ with determinant 1. Its Lie algebra so(3) is spanned by elements J_ℓ, $\ell = 1, 2, 3$, which generate rotations about the x_ℓ-axis. One readily verifies that these satisfy the same commutation relations as in (9.3). Thus SU(2) and SO(3) have isomorphic Lie algebras. There is an explicit homomorphism

$$p : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3), \tag{9.4}$$

which exhibits SU(2) as a double cover of SO(3). One way to construct p is the following. The linear span $\mathfrak{g}$ of (9.2) over $\mathbb{R}$ is a three-dimensional, real vector space, with an inner product given by $(X, Y) = -\operatorname{Tr} XY$. It is clear that the representation p of SU(2) by a group of linear transformations on $\mathfrak{g}$ given by $p(g) = gXg^{-1}$ preserves this inner product and gives (9.4). Note that $\operatorname{Ker} p = \{I, -I\}$.

If we regard X_j as left-invariant vector fields on SU(2), set

$$\Delta = X_1^2 + X_2^2 + X_3^2, \tag{9.5}$$

a second-order, left-invariant differential operator. It follows easily from (9.3) that X_j and Δ commute:

$$\Delta X_j = X_j \Delta, \quad 1 \le j \le 3. \tag{9.6}$$

Suppose π is an irreducible unitary representation of SU(2) on V. Then π induces a skew-adjoint representation $d\pi$ of the Lie algebra su(2) and an algebraic representation of the universal enveloping algebra. By (9.6), $d\pi(\Delta)$ commutes with $d\pi(X_j)$, $j = 1, \dots, 3$. Thus, if π is irreducible, Proposition 7.4 implies

$$d\pi(\Delta) = -\lambda^2 I, \tag{9.7}$$

for some $\lambda \in \mathbb{R}$. (Since $d\pi(\Delta)$ is a sum of squares of skew-adjoint operators, it must be negative.) Let

$$L_j = d\pi(X_j). \tag{9.8}$$

Now we will diagonalize L_1 on V. Set

$$V_\mu = \{v \in V : L_1 v = i\mu v\}, \quad V = \bigoplus_{i\mu \in \text{ spec } L_1} V_\mu. \tag{9.9}$$

The structure of π is defined by how L_2 and L_3 behave on V_μ. It is convenient to set

$$L_\pm = L_2 \mp iL_3. \tag{9.10}$$

We have the following key identity, as a direct consequence of (9.3):

$$[L_1, L_\pm] = \pm i L_\pm. \tag{9.11}$$

Using this, we can establish the following:

Lemma 9.1. *We have*

$$L_\pm : V_\mu \longrightarrow V_{\mu\pm 1}. \tag{9.12}$$

In particular, if $i\mu \in$ spec L_1, then either $L_+ = 0$ on V_μ or $\mu + 1 \in$ spec L_1, and also either $L_- = 0$ on V_μ or $\mu - 1 \in$ spec L_1.

Proof. Let $v \in V_\mu$. By (9.11) we have

$$L_1 L_\pm v = L_\pm L_1 v \pm i L_\pm v = i(\mu \pm 1) L_\pm v,$$

which establishes the lemma. The operators $L_\pm$ are called *ladder operators.*

To continue, if π is irreducible on V, we claim that spec $i^{-1}L_1$ must consist of a sequence

$$\text{spec } i^{-1} L_1 = \{\mu_0, \mu_0 + 1, \dots, \mu_0 + k = \mu_1\}, \tag{9.13}$$

with

$$L_+ : V_{\mu_0 + j} \to V_{\mu_0 + j + 1} \quad \text{isomorphism, for } \ 0 \le j \le k - 1, \tag{9.14}$$

and

$$L_- : V_{\mu_1 - j} \to V_{\mu_1 - j - 1} \quad \text{isomorphism, for } \ 0 \le j \le k - 1. \tag{9.15}$$

In fact, we can compute

$$L_- L_+ = L_2^2 + L_3^2 + i[L_3, L_2] = -\lambda^2 - L_1^2 - iL_1 \tag{9.16}$$

on V, and

$$L_+ L_- = -\lambda^2 - L_1^2 + iL_1 \tag{9.17}$$

on V, so

$$L_-L_+ = \mu(\mu+1) - \lambda^2 \ \text{ on } \ V_\mu, \qquad (9.18)$$
$$L_+L_- = \mu(\mu-1) - \lambda^2 \ \text{ on } \ V_\mu.$$

Note that since L_2 and L_3 are skew-adjoint, $L_+ = -L_-^*$, so

$$L_+L_- = -L_-^*L_-, \quad L_-L_+ = -L_+^*L_+.$$

Thus

$$\text{Ker } L_+ = \text{ Ker } L_-L_+, \quad \text{Ker } L_- = \text{ Ker } L_+L_-.$$

These observations establish (9.13)–(9.15).

Considering that $d\pi$ acts on the linear span of $\{v, L_+v, \dots L_+^{\mu_1-\mu_0}v\}$ for any nonzero $v \in V_{\mu_0}$, and that irreducibility implies this must be all of V, we have

$$\dim V_\mu = 1, \quad \mu_0 \le \mu \le \mu_1. \qquad (9.19)$$

From (9.18) we see that $\mu_1(\mu_1+1) = \lambda^2 = \mu_0(\mu_0-1)$. Hence,

$$\mu_1 - \mu_0 = k \Longrightarrow \mu_0 = -\frac{k}{2},\ \mu_1 = \frac{k}{2}, \qquad (9.20)$$

and we have

$$\dim V = k+1, \quad \lambda^2 = \frac{1}{4}k(k+2) = \frac{1}{4}(\dim V^2 - 1). \qquad (9.21)$$

A nonzero element $v \in V$ such that $L_+v = 0$ is called a “highest-weight vector” for the representation π of SU(2) on V. It follows from the analysis above that all highest-weight vectors for an irreducible representation on V belong to the one-dimensional space V_{μ_1}.

The calculations above establish that an irreducible, unitary representation π of SU(2) on V is determined uniquely up to equivalence by dim V. We are ready to prove the following:

Proposition 9.2. *There is precisely one equivalence class of irreducible, unitary representations of SU(2) on* $\mathbb{C}^{k+1}$, *for each* $k = 0, 1, 2, \dots$.

We will realize each such representation, which is denoted $D_{k/2}$, on the space

$$\mathcal{P}_k = \{p(z) : p \text{ homogeneous polynomial of degree } k \text{ on } \mathbb{C}^2\}, \qquad (9.22)$$

with SU(2) acting on $\mathcal{P}_k$ by

$$D_{k/2}(g)f(z) = f(g^{-1}z), \quad g \in SU(2),\ z \in \mathbb{C}^2. \qquad (9.23)$$

Note that, for $X \in \text{su}(2)$,

$$dD_{k/2}(X)f(z) = \frac{d}{dt}f\left(e^{-tX}z\right)\big|_{t=0} = -(\partial_1 f, \partial_2 f)\cdot X\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad (9.24)$$

where $\partial_j f = \partial f/\partial z_j$. A calculation gives

$$
\begin{aligned}
L_1 f(z) &= -\frac{i}{2}(z_1\partial_1 f - z_2\partial_2 f),\\
L_2 f(z) &= -\frac{1}{2}(z_2\partial_1 f - z_1\partial_2 f),\\
L_3 f(z) &= -\frac{i}{2}(z_2\partial_1 f + z_1\partial_2 f).
\end{aligned} \tag{9.25}
$$

In particular, for

$$\varphi_{kj}(z) = z_1^{k-j} z_2^j \in \mathcal{P}_k, \quad 0 \le j \le k, \tag{9.26}$$

we have

$$L_1\varphi_{kj} = i\Bigl(-\frac{k}{2} + j\Bigr)\varphi_{kj}, \tag{9.27}$$

so

$$V = \mathcal{P}_k \Longrightarrow \text{ span } \varphi_{kj} = V_{-k/2+j}, \quad 0 \le j \le k. \tag{9.28}$$

Note that

$$L_+ f(z) = -z_2\partial_1 f(z), \quad L_- f(z) = z_1\partial_2 f(z), \tag{9.29}$$

so

$$L_+\varphi_{kj} = -(k-j)\varphi_{k,j+1}, \quad L_-\varphi_{kj} = j\varphi_{k,j-1}. \tag{9.30}$$

We see that the structure of the representation $D_{k/2}$ of SU(2) on $\mathcal{P}_k$ is as described in (9.12)–(9.21). The last detail is to show that $D_{k/2}$ is irreducible. If not, then $\mathcal{P}_k$ splits into a direct sum of several irreducible subspaces, each of which has a one-dimensional space of highest-weight vectors, annihilated by L_+. But as seen above, within $\mathcal{P}_k$, only multiples of z_2^k are annihilated by L_+, so the representaiton $D_{k/2}$ of SU(2) on $\mathcal{P}_k$ is irreducible.

We can deduce the classification of irreducible, unitary representations of SO(3) from the result above as follows. We have the covering homomorphism (9.4), and Ker $p = \{\pm I\}$. Now each irreducible representation d_j of SO(3) defines an irreducible representation $d_j \circ p$ of SU(2), which must be equivalent to one of the representations $D_{k/2}$ described above. On the other hand, $D_{k/2}$ factors through to yield a representation of SO(3) if and only if $D_{k/2}$ is the identity on Ker p, that is, if and only if $D_{k/2}(-I) = I$. Clearly, this holds if and only if k is even. Thus all the irreducible, unitary representations of SO(3) are given by representations $\widetilde{D}_j$ on $\mathcal{P}_{2j}$, uniquely defined by

$$\widetilde{D}_j\bigl(p(g)\bigr) = D_j(g), \quad g \in \mathrm{SU}(2). \tag{9.31}$$

It is conventional to use D_j instead of $\widetilde{D}_j$ to denote such a representation of SO(3). Note that D_j represents SO(3) on a space of dimension $2j + 1$, and

$$dD_j(\Delta) = -j(j+1). \tag{9.32}$$

Also, we can classify the irreducible representations of U(2), using the results on SU(2). To do this, use the exact sequence

$$1 \to K \to S^1 \times \mathrm{SU}(2) \to \mathrm{U}(2) \to 1, \tag{9.33}$$

where "1" denotes the trivial multiplicative group, and

$$K = \{(\omega, g) \in S^1 \times \mathrm{SU}(2) : g = \omega^{-1} I, \omega^2 = 1\}. \tag{9.34}$$

The irreducible representations of $S^1 \times \mathrm{SU}(2)$ are given by

$$\pi_{mk}(\omega, g) = \omega^m D_{k/2}(g) \ \text{ on } \ \mathcal{P}_k, \tag{9.35}$$

with $m, k \in \mathbb{Z}$, $k \geq 0$. Those giving a complete set of irreducible representations of U(2) are those for which $\pi_{mk}(K) = I$, that is, those for which $(-1)^m D_{k/2}(-I) = I$. Since $D_{k/2}(-I) = (-1)^k I$, we see the condition is that $m + k$ be an even integer.

We now consider the representations of SO(4). First note that SO(4) is covered by SU(2)×SU(2). To see this, equate the unit sphere $S^3 \subset \mathbb{R}^4$, with its standard metric, to SU(2), with a bi-invariant metric. Then SO(4) is the connected component of the identity in the isometry group of S^3. Meanwhile, SU(2)×SU(2) acts as a group of isometries, by

$$(g_1, g_2) \cdot x = g_1 x g_2^{-1}, \quad g_j \in \mathrm{SU}(2). \tag{9.36}$$

Thus we have a map

$$\tau : \mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SO}(4). \tag{9.37}$$

This is a group homomorphism. Note that $(g_1, g_2) \in \operatorname{Ker} \tau$ implies $g_1 = g_2 = \pm I$. Furthermore, a dimension count shows τ must be surjective, so

$$\mathrm{SO}(4) \approx \mathrm{SU}(2) \times \mathrm{SU}(2) / \{\pm(I, I)\}. \tag{9.38}$$

As shown in §8, if G_1 and G_2 are compact Lie groups, and $G = G_1 \times G_2$, then the set of all irreducible, unitary representations of G, up to unitary equivalence, is given by

$$\{\pi(g) = \pi_1(g_1) \otimes \pi_2(g_2) : \pi_j \in \widehat{G}_j\}, \tag{9.39}$$

where $g = (g_1, g_2) \in G$ and $\widehat{G}_j$ parameterizes the irreducible, unitary representations of G_j. In particular, the irreducible unitary representations of SU(2)×SU(2), up to equivalence, are precisely the representations of the form

$$\gamma_{k\ell}(g) = D_{k/2}(g_1) \otimes D_{\ell/2}(g_2), \quad k, \ell \in \{0, 1, 2, \dots\}, \tag{9.40}$$

acting on $\mathcal{P}_k \otimes \mathcal{P}_\ell \approx \mathbb{C}^{k+1} \otimes \mathbb{C}^{\ell+1}$. By (9.38), the irreducible, unitary representations of SO(4) are given by all $\gamma_{k\ell}$ such that $k + \ell$ is even, since, for $p_0 = (-I, -I) \in \mathrm{SU}(2){\times}\mathrm{SU}(2)$, $\gamma_{k\ell}(p_0) = (-1)^{k+\ell} I$.

We next consider the problem of decomposing the tensor-product representations $D_{k/2} \otimes D_{\ell/2}$ of SU(2) (i.e., the composition of (9.40) with the diagonal map SU(2)$\hookrightarrow$SU(2)$\times$SU(2)) into irreducible representations. We may as well assume that $\ell \le k$. Note that $\pi_{k\ell} = D_{k/2} \otimes D_{\ell/2}$ acts on

$$\mathcal{P}_{k\ell} = \{f(z,w) : \text{polynomial on } \mathbb{C}^2 \times \mathbb{C}^2, \\ \text{homogeneous of degree } k \text{ in } z,\ \ell \text{ in } w\}, \tag{9.41}$$

as

$$\pi_{k\ell}(g)f(z,w) = f(g^{-1}z, g^{-1}w). \tag{9.42}$$

Parallel to (9.25) and (9.29), we have, on $\mathcal{P}_{k\ell}$,

$$\begin{aligned} L_1 f &= -\frac{i}{2}(z_1\partial_{z_1}f - z_2\partial_{z_2}f + w_1\partial_{w_1}f - w_2\partial_{w_2}f), \\ L_+ f &= -z_2\partial_{z_1}f - w_2\partial_{w_1}f, \quad L_- f = z_1\partial_{z_2}f + w_1\partial_{w_2}f. \end{aligned} \tag{9.43}$$

To decompose $\mathcal{P}_{k\ell}$ into irreducible subspaces, we specify Ker L_+. In fact, a holomorphic function $f(z,w)$ annihilated by L_+ is of the form

$$f(z,w) = g(z_2, w_2, w_2z_1 - z_2w_1), \tag{9.44}$$

and the kernel of L_+ in $\mathcal{P}_{k\ell}$ is the linear span of

$$\psi_{k\ell\mu}(z,w) = z_2^{k-\mu} w_2^{\ell-\mu}(w_2z_1 - z_2w_1)^\mu, \quad 0 \le \mu \le \ell. \tag{9.45}$$

A calculation gives

$$L_1\psi_{k\ell\mu} = \frac{i}{2}(k+\ell-2\mu)\psi_{k\ell\mu}. \tag{9.46}$$

It follows that, for fixed k, ℓ, $0 \le \ell \le k$, and for each $\mu = 0, \dots, \ell$, $\psi_{k\ell\mu}$ is the highest-weight vector of a representation equivalent to $D_{(k+\ell-2\mu)/2}$, so we have
(9.47)

$$D_{k/2} \otimes D_{\ell/2} \approx \bigoplus_{\mu=0}^{\ell} D_{(k+\ell-2\mu)/2} = D_{(k-\ell)/2} \oplus D_{(k-\ell)/2+1} \oplus \cdots \oplus D_{(k+\ell)/2}.$$

This is called the *Clebsch-Gordon series.*

Extensions of the results presented here to more general compact Lie groups, due mainly to E. Cartan and H. Weyl, can be found in a number of places, including [T], [Var1], and [Wal1].

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