# Basic Theory of ODE and Vector Fields

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## Introduction

This text examines basic topics in the field of ordinary differential equations (ODE), as it has developed from the era of Newton into modern times. This is closely tied to the development of a number of concepts in advanced calculus. We begin with a brief discussion of the derivative of a vector-valued function of several variables as a linear map. We then establish in §2 the fundamental local existence and uniqueness of solutions to ODE, of the form

$$\frac{dy}{dt} = F(t, y), \quad y(t_0) = y_0, \tag{0.1}$$

where $F(t, y)$ is continuous in both arguments and Lipschitz in $y$, and $y$ takes values in $\mathbb{R}^k$. The proof uses a nice tool known as the contraction mapping principle; next we use this principle to establish the inverse and implicit function theorems in §3. After a discussion of constant-coefficient linear equations, in which we recall the basic results of linear algebra, in §4, we treat variable-coefficient linear ODE in §5, emphasizing a result known as Duhamel's principle, and then use this to examine smooth dependence on parameters for solutions to nonlinear ODE in §6.

The first six sections have a fairly purely analytic character and present ODE from a perspective similar to that seen in introductory courses. It is expected that the reader has seen much of this material before. Beginning in §7, the material begins to acquire a geometrical flavor as well. This section interprets solutions to (0.1) in terms of a flow generated by a vector field. The next two sections examine the Lie derivative of vector fields and some of its implications for ODE. While we initially work on domains in $\mathbb{R}^n$, here we begin a transition to global constructions, involving working on manifolds and hence making use of concepts that are invariant under changes of coordinates. By the end of §13, this transition is complete. Appendix B collects some of the basic facts about manifolds which are useful for such an approach to analysis.

Physics is a major source of differential equations, and in §10 we discuss some of the basic ODE arising from Newton's force law, converting the resulting second-order ODE to first-order systems known as Hamiltonian systems. The study of Hamiltonian vector fields is a major focus for the subsequent sections in this chapter. In §11 we deal with an apparently disjoint topic, the equations of geodesics on a Riemannian manifold. We introduce the covariant derivative as a tool for expressing the geodesic equations, and later show that these equations can also be cast in Hamiltonian form. In §12 we study a general class of variational problems, giving rise to both the equations of mechanics and the equations of geodesics, all expressible in Hamiltonian form.

In §13 we develop the theory of differential forms, one of E. Cartan's great contributions to analysis. There is a differential operator, called the exterior derivative, acting on differential forms. In beginning courses in multivariable calculus, one learns of div, grad, and curl as the major first-order differential operators; from a more advanced perspective, it is reasonable to think of the Lie derivative, the covariant derivative, and the exterior derivative as filling this role. The relevance of differential forms to ODE has many roots, but its most direct relevance for Hamiltonian systems is through the symplectic form, discussed in §14.

Results on Hamiltonian systems are applied in §15 to the study of first-order nonlinear PDE for a single unknown. The next section studies "completely integrable" systems, reversing the perspective, to apply solutions to certain nonlinear PDE to the study of Hamiltonian systems. These two sections comprise what is known as Hamilton-Jacobi theory. In §17 we make a further study of integrable systems arising from central force problems, particularly the one involving the gravitational attraction of two bodies, the solution to which was Newton's triumph. Section 18 gives a brief relativistic treatment of the equations of motion arising from the electromagnetic force, which ushered in Einstein's theory of relativity.

In §19 we apply material from §13 on differential forms to some topological results, such as the Brouwer fixed-point theorem, the study of the degree of a map between compact oriented manifolds, and the Jordan-Brouwer separation theorem. We apply the degree theory in §20 to a study of the index of a vector field, which reflects the behavior of its critical points.

The appendix at the end of this text discusses the existence and uniqueness of solutions to (0.1) when $F$ satisfies a condition weaker than Lipschitz in $y$. Results established here are applicable to the study of ideal fluid flow, as will be seen in Chapter 17.

## 1. The derivative

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and let $F : \mathcal{O} \to \mathbb{R}^m$ be a continuous function. We say that $F$ is differentiable at a point $x \in \mathcal{O}$, with derivative $L$, if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that, for small $y \in \mathbb{R}^n$,

$$F(x+y) = F(x) + Ly + R(x,y), \tag{1.1}$$

with

$$\frac{\|R(x,y)\|}{\|y\|} \to 0 \quad \text{as } y \to 0. \tag{1.2}$$

We denote the derivative at $x$ by $DF(x) = L$. With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, $DF(x)$ is simply the matrix of partial derivatives,

$$DF(x) = \Bigl(\frac{\partial F_j}{\partial x_k}\Bigr), \tag{1.3}$$

so that, if $v = (v_1, \dots, v_n)$ (regarded as a column vector), then

$$DF(x)v = \Bigl(\sum_k \frac{\partial F_1}{\partial x_k} v_k, \dots, \sum_k \frac{\partial F_m}{\partial x_k} v_k\Bigr). \tag{1.4}$$

It will be shown that $F$ is differentiable whenever all the partial derivatives exist and are *continuous* on $\mathcal{O}$. In such a case we say that $F$ is a $C^1$-function on $\mathcal{O}$. In general, $F$ is said to be $C^k$ if all its partial derivatives of order $\le k$ exist and are continuous.

In (1.2) we can use the *Euclidean* norm on $\mathbb{R}^n$ and $\mathbb{R}^m$. This norm is defined by

$$\|x\| = \bigl(x_1^2 + \cdots + x_n^2\bigr)^{1/2} \tag{1.5}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Any other norm would do equally well. Some basic results on the Euclidean norm are derived in §4.

More generally, the definition of the derivative given by (1.1) and (1.2) extends to a function $F : \mathcal{O} \to Y$, where $\mathcal{O}$ is an open subset of $X$, and $X$ and $Y$ are Banach spaces. Basic material on Banach spaces appears in Appendix A, Functional Analysis. In this case, we require $L$ to be a bounded linear map from $X$ to $Y$. The notion of differentiable function in this context is useful in the study of nonlinear PDE.

We now derive the *chain rule* for the derivative. Let $F : \mathcal{O} \to \mathbb{R}^m$ be differentiable at $x \in \mathcal{O}$, as above; let $U$ be a neighborhood of $z = F(x)$ in $\mathbb{R}^m$; and let $G : U \to \mathbb{R}^k$ be differentiable at $z$. Consider $H = G \circ F$. We have

$$\begin{aligned} H(x+y) &= G(F(x+y)) \\ &= G\big(F(x) + DF(x)y + R(x,y)\big) \\ &= G(z) + DG(z)\big(DF(x)y + R(x,y)\big) + R_1(x,y) \\ &= G(z) + DG(z)DF(x)y + R_2(x,y), \end{aligned} \tag{1.6}$$

with

$$\frac{\|R_2(x,y)\|}{\|y\|} \to 0 \quad \text{as } y \to 0.$$

Thus $G \circ F$ is differentiable at $x$, and

$$D(G \circ F)(x) = DG(F(x)) \cdot DF(x). \tag{1.7}$$

This result works equally well if $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^k$ are replaced by general Banach spaces.

Another useful remark is that, by the fundamental theorem of calculus, applied to $\varphi(t) = F(x+ty)$,

$$F(x+y) = F(x) + \int_0^1 DF(x+ty)y\,dt, \tag{1.8}$$

provided $F$ is $C^1$. For a typical application, see (6.6).

A closely related application of the fundamental theorem of calculus is that if we assume that $F : \mathcal{O} \to \mathbb{R}^m$ is differentiable in each variable separately, and that each $\partial F/\partial x_j$ is continuous on $\mathcal{O}$, then

$$\begin{aligned} F(x+y) &= F(x) + \sum_{j=1}^n \big[F(x+z_j) - F(x+z_{j-1})\big] \\ &= F(x) + \sum_{j=1}^n A_j(x,y)y_j, \\ A_j(x,y) &= \int_0^1 \frac{\partial F}{\partial x_j}\big(x + z_{j-1} + ty_je_j\big)\,dt, \end{aligned} \tag{1.9}$$

where $z_0 = 0,\ z_j = (y_1, \dots, y_j, 0, \dots, 0)$, and $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. Now (1.9) implies that $F$ is differentiable on $\mathcal{O}$, as we stated beneath (1.4). As is shown in many calculus texts, by using the mean value theorem instead of the fundamental theorem of calculus, one can obtain a slightly sharper result. We leave the reconstruction of this argument to the reader.

We now describe two convenient notations to express higher-order derivatives of a $C^k$-function $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open. In the first, let $J$ be a $k$-tuple of integers between 1 and $n$; $J = (j_1, \dots, j_k)$. We set

$$f^{(J)}(x) = \partial_{j_k} \cdots \partial_{j_1} f(x), \quad \partial_j = \frac{\partial}{\partial x_j}. \tag{1.10}$$

Also, we set $|J| = k$, the total order of differentiation. As will be seen in the exercises, $\partial_i \partial_j f = \partial_j \partial_i f$, provided $f \in C^2(\Omega)$. Hence, if $f \in C^k(\Omega)$, then $\partial_{j_k} \cdots \partial_{j_1} f = \partial_{\ell_k} \cdots \partial_{\ell_1} f$ whenever $\{\ell_1, \dots, \ell_k\}$ is a permutation of $\{j_1, \dots, j_k\}$. Thus, another convenient notation to use is the following. Let $\alpha$ be an $n$-tuple of nonnegative integers, $\alpha = (\alpha_1, \dots, \alpha_n)$. Then we set

$$f^{(\alpha)}(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f(x), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{1.11}$$

Note that if $|J| = |\alpha| = k$ and $f \in C^k(\Omega)$, then

$$f^{(J)}(x) = f^{(\alpha)}(x), \quad \text{with } \alpha_i = \#\{\ell : j_\ell = i\}. \tag{1.12}$$

Correspondingly, there are two expressions for monomials in $x$:

$$x^J = x_{j_1} \cdots x_{j_k}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1.13}$$

and $x^J = x^\alpha$, provided $J$ and $\alpha$ are related as in (1.12). Both of these notations are called “multi-index” notations.

We now derive Taylor’s formula with remainder for a smooth function $F : \Omega \to \mathbb{R}$, making use of these multi-index notations. We will apply the one-variable formula,

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{1}{2}\varphi''(0)t^2 + \cdots + \frac{1}{k!}\varphi^{(k)}(0)t^k + r_k(t), \tag{1.14}$$

with

$$r_k(t) = \frac{1}{k!}\int_0^t (t-s)^k \varphi^{(k+1)}(s)\, ds, \tag{1.15}$$

given $\varphi \in C^{k+1}(I),\ I = (-a, a)$. Let us assume that $0 \in \Omega$ and that the line segment from 0 to $x$ is contained in $\Omega$. We set $\varphi(t) = F(tx)$ and apply (1.14) and (1.15) with $t = 1$. Applying the chain rule, we have

$$\varphi'(t) = \sum_{j=1}^n \partial_j F(tx) x_j = \sum_{|J|=1} F^{(J)}(tx) x^J. \tag{1.16}$$

Differentiating again, we have

$$\varphi''(t) = \sum_{|J|=1, |K|=1} F^{(J+K)}(tx) x^{J+K} = \sum_{|J|=2} F^{(J)}(tx) x^J, \tag{1.17}$$

where, if $|J| = k$ and $|K| = \ell$, we take $J + K = (j_1, \dots, j_k, k_1, \dots, k_\ell)$. Inductively, we have

$$\varphi^{(k)}(t) = \sum_{|J|=k} F^{(J)}(tx)x^J. \tag{1.18}$$

Hence, from (1.14), with $t = 1$,

$$F(x) = F(0) + \sum_{|J|=1} F^{(J)}(0)x^J + \cdots + \frac{1}{k!} \sum_{|J|=k} F^{(J)}(0)x^J + R_k(x),$$

or, more briefly,

$$F(x) = \sum_{|J| \le k} \frac{1}{|J|!} F^{(J)}(0)x^J + R_k(x), \tag{1.19}$$

where

$$R_k(x) = \frac{1}{k!} \sum_{|J|=k+1} \Big( \int_0^1 (1-s)^k F^{(J)}(sx)\, ds \Big) x^J. \tag{1.20}$$

This gives Taylor's formula with remainder for $F \in C^{k+1}(\Omega)$, in the $J$-multi-index notation.

We also want to write the formula in the $\alpha$-multi-index notation. We have

$$\sum_{|J|=k} F^{(J)}(tx)x^J = \sum_{|\alpha|=k} \nu(\alpha) F^{(\alpha)}(tx)x^\alpha, \tag{1.21}$$

where

$$\nu(\alpha) = \#\{J : \alpha = \alpha(J)\}, \tag{1.22}$$

and we define the relation $\alpha = \alpha(J)$ to hold provided (1.12) holds or, equivalently, provided $x^J = x^\alpha$. Thus, $\nu(\alpha)$ is uniquely defined by

$$\sum_{|\alpha|=k} \nu(\alpha)x^\alpha = \sum_{|J|=k} x^J = (x_1 + \cdots + x_n)^k. \tag{1.23}$$

One sees that, if $|\alpha| = k$, then $\nu(\alpha)$ is equal to the product of the number of combinations of $k$ objects, taken $\alpha_1$ at a time, times the number of combinations of $k - \alpha_1$ objects, taken $\alpha_2$ at a time, and so on, times the number of combinations of $k - (\alpha_1 + \cdots + \alpha_{n-1})$ objects, taken $\alpha_n$ at a time. Thus

$$\nu(\alpha) = \binom{k}{\alpha_1} \binom{k-\alpha_1}{\alpha_2} \cdots \binom{k - \alpha_1 - \cdots - \alpha_{n-1}}{\alpha_n} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!}. \tag{1.24}$$

In other words, for $|\alpha| = k$,

$$\nu(\alpha) = \frac{k!}{\alpha!}, \quad \text{where } \alpha! = \alpha_1! \cdots \alpha_n!. \tag{1.25}$$

Thus, the Taylor formula (1.19) can be rewritten as

$$F(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_k(x), \tag{1.26}$$

where

$$R_k(x) = \sum_{|\alpha|=k+1} \frac{k+1}{\alpha!} \Bigl( \int_0^1 (1-s)^k F^{(\alpha)}(sx)\, ds \Bigr) x^\alpha. \tag{1.27}$$

## Exercises

1. Let $M_{n\times n}$ be the space of complex $n \times n$ matrices, and let $\det : M_{n\times n} \to \mathbb{C}$ denote the determinant. Show that if $I$ is the identity matrix, then

$$D\ \det(I)B = \operatorname{Tr} B,$$

   i.e.,

$$\frac{d}{dt} \det(I + tB)|_{t=0} = \operatorname{Tr} B.$$

2. If $A(t) = \bigl(a_{jk}(t)\bigr)$ is a curve in $M_{n\times n}$, use the expansion of $(d/dt)\det A(t)$ as a sum of $n$ determinants, in which the rows of $A(t)$ are successively differentiated, to show that, for $A \in M_{n\times n}$,

$$D\ \det(A)B = \operatorname{Tr}\ \bigl(\operatorname{Cof}(A)^t \cdot B\bigr),$$

   where $\operatorname{Cof}(A)$ is the cofactor matrix of $A$.
3. Suppose $A \in M_{n\times n}$ is invertible. Using

$$\det(A + tB) = (\det A)\ \det(I + tA^{-1}B),$$

   show that

$$D\ \det(A)B = (\det A)\ \operatorname{Tr}\ (A^{-1}B).$$

   Comparing the result of Exercise 2, deduce Cramer's formula:

$$(\det A)A^{-1} = \operatorname{Cof}(A)^t. \tag{1.28}$$

4. Identify $\mathbb{R}^2$ and $\mathbb{C}$ via $z = x + iy$. Then multiplication by $i$ on $\mathbb{C}$ corresponds to applying

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Let $\mathcal{O} \subset \mathbb{R}^2$ be open, and let $f : \mathcal{O} \to \mathbb{R}^2$ be $C^1$. Say $f = (u, v)$. Regard $Df(x, y)$ as a $2\times 2$ real matrix. One says $f$ is *holomorphic*, or complex analytic, provided the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.29}$$

Show that this is equivalent to the condition

$$Df(x,y)J = JDf(x,y).$$

Generalize to $\mathcal{O}$ open in $\mathbb{C}^m$, $f : \mathcal{O} \to \mathbb{C}^n$.

5. If $R(x)$ is a $C^\infty$-function near the origin in $\mathbb{R}^n$, satisfying $R(0) = 0$ and $DR(0) = 0$, show that there exist smooth functions $r_{jk}(x)$ such that

$$R(x) = \sum r_{jk}(x) x_j x_k.$$

(*Hint*: Using (1.8), write $R(x) = \Phi(x)x$, $\Phi(x) = \int_0^1 DR(tx)dt$, since $R(0) = 0$. Then $\Phi(0) = DR(0) = 0$, so (1.8) can be applied again, to give $\Phi(x) = \Psi(x)x$.)

6. If $f$ is $C^1$ on a region in $\mathbb{R}^2$ containing $[a,b] \times \{y\}$, show that

$$\frac{d}{dy} \int_a^b f(x,y)\, dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\, dx.$$

(*Hint*: Show that the left side is equal to

$$\lim_{h \to 0} \int_a^b \frac{1}{h} \int_0^h \frac{\partial f}{\partial y}(x, y+s)\, ds\, dx.)$$

7. Suppose $F : \mathcal{O} \to \mathbb{R}^m$ is a $C^2$-function. Applying the fundamental theorem of calculus, first to

$$G_j(x) = F(x + he_j) - F(x)$$

(as a function of $h$) and then to

$$H_{jk}(x) = G_j(x + he_k) - G_j(x),$$

where $\{e_j\}$ is the standard basis of $\mathbb{R}^n$, show that if $x \in \mathcal{O}$ and $h$ is small, then

$$\begin{aligned} &F(x + he_j + he_k) - F(x + he_k) - F(x + he_j) + F(x) \\ &\qquad = \int_0^h \int_0^h \frac{\partial}{\partial x_k} \frac{\partial F}{\partial x_j} \Big( x + se_j + te_k \Big)\, ds\, dt. \end{aligned}$$

Similarly, show that this quantity is equal to

$$\int_0^h \int_0^h \frac{\partial}{\partial x_j} \frac{\partial F}{\partial x_k} \Big( x + se_j + te_k \Big)\, dt\, ds.$$

Deduce that

$$\frac{\partial}{\partial x_k} \frac{\partial F}{\partial x_j}(x) = \frac{\partial}{\partial x_j} \frac{\partial F}{\partial x_k}(x).$$

(*Hint*: Use Exercise 6.)

Arguments that use the mean value theorem instead of the fundamental theorem of calculus can be found in many calculus texts.

## 2. Fundamental local existence theorem for ODE

The goal of this section is to establish the existence of solutions to an ODE:

$$\frac{dy}{dt} = F(t,y), \quad y(t_0) = y_0. \tag{2.1}$$

We will prove the following fundamental result.

**Theorem 2.1.** *Let* $y_0 \in \mathcal{O}$, *an open subset of* $\mathbb{R}^n$, $I \subset \mathbb{R}$ *an interval containing* $t_0$. *Suppose* $F$ *is continuous on* $I \times \mathcal{O}$ *and satisfies the following Lipschitz estimate in* $y$:

$$\|F(t,y_1) - F(t,y_2)\| \le L\|y_1 - y_2\|, \tag{2.2}$$

*for* $t \in I,\ y_j \in \mathcal{O}$. *Then the equation (2.1) has a unique solution on some* $t$*-interval containing* $t_0$.

To begin the proof, we note that the equation (2.1) is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(s,y(s))\, ds. \tag{2.3}$$

Existence will be established via the Picard iteration method, which is the following. Guess $y_0(t)$, e.g., $y_0(t) = y_0$. Then set

$$y_k(t) = y_0 + \int_{t_0}^{t} F\big(s, y_{k-1}(s)\big)\, ds. \tag{2.4}$$

We aim to show that, as $k \to \infty$, $y_k(t)$ converges to a (unique) solution of (2.3), at least for $t$ close enough to $t_0$. To do this, we will use the following tool, known as the *contraction mapping principle.*

**Theorem 2.2.** *Let* $X$ *be a complete metric space, and let* $T : X \to X$ *satisfy*

$$\mathit{dist}(Tx,Ty) \le r\ \mathit{dist}(x,y), \tag{2.5}$$

*for some* $r < 1$. *(We say that* $T$ *is a contraction.) Then* $T$ *has a unique fixed point* $x$. *For any* $y_0 \in X$, $T^k y_0 \to x$ *as* $k \to \infty$.

**Proof.** Pick $y_0 \in X$, and let $y_k = T^k y_0$. Then

$$\mathrm{dist}\,(y_{k+1}, y_k) \le r^k\, \mathrm{dist}\,(y_1, y_0),$$

so

$$\begin{aligned} \mathrm{dist}(y_{k+m}, y_k) \le\ & \mathrm{dist}(y_{k+m}, y_{k+m-1}) + \cdots + \mathrm{dist}(y_{k+1}, y_k) \\ \le\ & \big(r^k + \cdots + r^{k+m-1}\big)\ \mathrm{dist}(y_1, y_0) \\ \le\ & r^k \big(1 - r\big)^{-1}\ \mathrm{dist}(y_1, y_0). \end{aligned} \tag{2.6}$$

It follows that $(y_k)$ is a Cauchy sequence, so it converges; $y_k \to x$. Since $Ty_k = y_{k+1}$ and $T$ is continuous, it follows that $Tx = x$, that is, $x$ is a fixed point. The uniqueness of the fixed point is clear from the estimate $\mathrm{dist}(Tx,Tx') \le r\ \mathrm{dist}(x,x')$, which implies $\mathrm{dist}(x,x') = 0$ if $x$ and $x'$ are fixed points. This completes the proof.

Tackling the solvability of (2.3), we look for a fixed point of $T$, defined by

$$(Ty)(t) = y_0 + \int_{t_0}^{t} F(s, y(s))\, ds. \tag{2.7}$$

Let

$$X = \big\{ u \in C(J, \mathbb{R}^n) : u(t_0) = y_0,\ \sup_{t \in J} \|u(t) - y_0\| \le K \big\}. \tag{2.8}$$

Here $J = [t_0 - \varepsilon, t_0 + \varepsilon]$, where $\varepsilon$ will be chosen, sufficiently small, below. $K$ is picked so $\{y : \|y - y_0\| \le K\}$ is contained in $\mathcal{O}$, and we also suppose $J \subset I$. Then there exists an $M$ such that

$$\sup_{s \in J, \|y - y_0\| \le K} \|F(s, y)\| \le M. \tag{2.9}$$

Then, provided

$$\varepsilon \le \frac{K}{M}, \tag{2.10}$$

we have

$$T : X \to X. \tag{2.11}$$

Now, using the Lipschitz hypothesis (2.2), we have, for $t \in J$,

$$\|(Ty)(t) - (Tz)(t)\| \le \int_{t_0}^{t} L \|y(s) - z(s)\|\, ds \le \varepsilon\, L \sup_{s \in J} \|y(s) - z(s)\|, \tag{2.12}$$

assuming $y$ and $z$ belong to $X$. It follows that $T$ is a contraction on $X$ provided one has

$$\varepsilon < \frac{1}{L}, \tag{2.13}$$

in addition to the hypotheses above. This proves Theorem 2.1.

In view of the lower bound on the length of the interval $J$ on which the existence theorem works, it is easy to show that the only way a solution can fail to be globally defined, that is, to exist for all $t \in I$, is for $y(t)$ to "explode to infinity" by leaving every compact set $K \subset \mathcal{O}$, as $t \to t_1$, for some $t_1 \in I$.

We remark that the local existence proof given above works if $\mathbb{R}^n$ is replaced by any Banach space.

Often one wants to deal with a higher-order ODE. There is a standard method of reducing an $n$th-order ODE

$$y^{(n)}(t) = f(t, y, y', \dots, y^{(n-1)}) \tag{2.14}$$

to a first-order system. One sets $u = (u_0, \dots, u_{n-1})$, with

$$u_0 = y, \quad u_j = y^{(j)}, \tag{2.15}$$

and then

$$\frac{du}{dt} = \big(u_1, \dots, u_{n-1}, f(t, u_0, \dots, u_{n-1})\big) = g(t, u). \tag{2.16}$$

If $y$ takes values in $\mathbb{R}^k$, then $u$ takes values in $\mathbb{R}^{kn}$.

If the system (2.1) is nonautonomous, that is, if $F$ explicitly depends on $t$, it can be converted to an autonomous system (one with no explicit $t$-dependence) as follows. Set $z = (t, y)$. We then have

$$z' = (1, y') = (1, F(z)) = G(z). \tag{2.17}$$

Sometimes this process destroys important features of the original system (2.1). For example, if (2.1) is linear, (2.17) might be nonlinear. Nevertheless, the trick of converting (2.1) to (2.17) has some uses.

Many systems of ODE are difficult to solve explicitly. One very basic class of ODE can be solved explicitly, in terms of integrals, namely the single first-order linear ODE:

$$\frac{dy}{dt} = a(t)y + b(t), \quad y(0) = y_0, \tag{2.18}$$

where $a(t)$ and $b(t)$ are continuous real- or complex-valued functions. Set

$$A(t) = \int_0^t a(s)\, ds. \tag{2.19}$$

Then (2.18) can be written as

$$e^{A(t)} \frac{d}{dt}\big(e^{-A(t)} y\big) = b(t), \tag{2.20}$$

which yields

$$y(t) = e^{A(t)} y_0 + e^{A(t)} \int_0^t e^{-A(s)} b(s)\, ds. \tag{2.21}$$

Compare this result with formulas (4.42) and (5.8), in subsequent sections of this chapter.

## Exercises

1. Solve the initial-value problem
$$y' = y^2, \quad y(0) = a,$$
given $a \in \mathbb{R}$. On what $t$-interval is the solution defined?
2. Under the hypotheses of Theorem 2.1, if $y$ solves (2.1) for $t \in [T_0, T_1]$, and $y(t) \in K$, compact in $\mathcal{O}$, for all such $t$, prove that $y(t)$ extends to a solution for $t \in [S_0, S_1]$, with $S_0 < T_0$, $T_1 > T_0$, as stated beneath (2.13).
3. Let $M$ be a compact, smooth surface in $\mathbb{R}^n$. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map (vector field) such that, for each $x \in M$, $F(x)$ is tangent to $M$,

that is, the line $\gamma_x(t) = x + tF(x)$ is tangent to $M$ at $x$, at $t = 0$. Show that if $x \in M$, then the initial-value problem

$$y' = F(y), \quad y(0) = x$$

has a solution for all $t \in \mathbb{R}$, and $y(t) \in M$ for all $t$.
(*Hint*: Locally, straighten out $M$ to be a linear subspace of $\mathbb{R}^n$, to which $F$ is tangent. Use uniqueness. Material in §3 will help do this local straightening.) Reconsider this problem after reading §7.

4. Show that the initial-value problem

$$\frac{dx}{dt} = -x(x^2 + y^2), \ \frac{dy}{dt} = -y(x^2 + y^2), \quad x(0) = x_0, \ \ y(0) = y_0$$

has a solution for all $t \geq 0$, but not for all $t < 0$, unless $(x_0, y_0) = (0, 0)$.

## 3. Inverse function and implicit function theorems

We will use the contraction mapping principle to establish the inverse function theorem, which together with its corollary, the implicit function theorem, is a fundamental result in multivariable calculus. First we state the inverse function theorem.

**Theorem 3.1.** *Let $F$ be a $C^k$-map from an open neighborhood $\Omega$ of $p_0 \in \mathbb{R}^n$ to $\mathbb{R}^n$, with $q_0 = F(p_0)$. Suppose the derivative $DF(p_0)$ is invertible. Then there is a neighborhood $U$ of $p_0$ and a neighborhood $V$ of $q_0$ such that $F : U \to V$ is one-to-one and onto, and $F^{-1} : V \to U$ is a $C^k$-map. (One says that $F : U \to V$ is a diffeomorphism.)*

**Proof.** Using the chain rule, it is easy to reduce to the case $p_0 = q_0 = 0$ and $DF(p_0) = I$, the identity matrix, so we suppose this has been done. Thus,

$$F(u) = u + R(u), \quad R(0) = 0, \quad DR(0) = 0. \tag{3.1}$$

For $v$ small, we want to solve

$$F(u) = v. \tag{3.2}$$

This is equivalent to $u + R(u) = v$, so let

$$T_v(u) = v - R(u). \tag{3.3}$$

Thus, solving (3.2) is equivalent to solving

$$T_v(u) = u. \tag{3.4}$$

We look for a fixed point $u = K(v) = F^{-1}(v)$. Also, we want to prove that $DK(0) = I$, that is, that $K(v) = v + r(v)$, with $r(v) = o(\|v\|)$. If we succeed in doing this, it follows easily that, for general $x$ close to 0,

$$DK(x) = \Big(DF\big(K(x)\big)\Big)^{-1},$$

and a simple inductive argument shows that $K$ is $C^k$ if $F$ is $C^k$. Now consider

$$T_v : X_v \longrightarrow X_v, \tag{3.5}$$

with

$$X_v = \{u \in \Omega : \|u - v\| \le A_v\}, \tag{3.6}$$

where we set

$$A_v = \sup_{\|w\| \le 2\|v\|} \|R(w)\|. \tag{3.7}$$

We claim that (3.5) holds if $\|v\|$ is sufficiently small. To prove this, note that $T_v(u) - v = -R(u)$, so we need to show that, provided $\|v\|$ is small, $u \in X_v$ implies $\|R(u)\| \le A_v$. But indeed, if $u \in X_v$, then $\|u\| \le \|v\| + A_v$, which is $\le 2\|v\|$ if $\|v\|$ is small, so then

$$\|R(u)\| \le \sup_{\|w\| \le 2\|v\|} \|R(w)\| = A_v;$$

this establishes (3.5).

Note that if $\|v\|$ is small enough, the map (3.5) is a contraction map, so there exists a unique fixed point $u = K(v) \in X_v$. Also note that since $u \in X_v$,

$$\|K(v) - v\| \le A_v = o(\|v\|). \tag{3.8}$$

Hence, the inverse function theorem is proved.

Thus, if $DF$ is invertible on the domain of $F$, then $F$ is a local diffeomorphism, although stronger hypotheses are needed to guarantee that $F$ is a global diffeomorphism onto its range. Here is one result along these lines.

**Proposition 3.2.** *If $\Omega \subset \mathbb{R}^n$ is open and convex, $F : \Omega \to \mathbb{R}^n$ is $C^1$, and the symmetric part of $DF(u)$ is positive-definite for each $u \in \Omega$, then $F$ is one-to-one on $\Omega$.*

**Proof.** Suppose that $F(u_1) = F(u_2)$, where $u_2 = u_1 + w$. Consider $\varphi : [0,1] \to \mathbb{R}$, given by

$$\varphi(t) = w \cdot F(u_1 + tw).$$

Thus $\varphi(0) = \varphi(1)$, so $\varphi'(t_0)$ must vanish for some $t_0 \in (0,1)$, by the mean value theorem. But $\varphi'(t) = w \cdot DF(u_1 + tw)w > 0$, if $w \neq 0$, by the hypothesis on $DF$. This shows that $F$ is one-to-one.

We can obtain the following implicit function theorem as a consequence of the inverse function theorem.

**Theorem 3.3.** *Suppose $U$ is a neighborhood of $x_0 \in \mathbb{R}^k$, $V$ is a neighborhood of $z_0 \in \mathbb{R}^\ell$, and*

$$F : U \times V \longrightarrow \mathbb{R}^\ell \tag{3.9}$$

*is a $C^k$-map. Assume $D_z F(x_0, z_0)$ is invertible; say $F(x_0, z_0) = u_0$. Then the equation $F(x, z) = u_0$ defines $z = f(x, u_0)$ for $x$ near $x_0$, with $f$ a $C^k$-map.*

**Proof.** Consider $H : U \times V \to \mathbb{R}^k \times \mathbb{R}^\ell$ defined by

$$H(x, z) = \bigl(x, F(x, z)\bigr). \tag{3.10}$$

We have

$$DH = \begin{pmatrix} I & D_x F \\ 0 & D_z F \end{pmatrix}. \tag{3.11}$$

Thus $DH(x_0, z_0)$ is invertible, so $J = H^{-1}$ exists and is $C^k$, by the inverse function theorem. It is clear that $J(x, u_0)$ has the form

$$J(x, u_0) = \bigl(x, f(x, u_0)\bigr), \tag{3.12}$$

and $f$ is the desired map.

As in §2, we remark that the inverse function theorem generalizes. One can replace $\mathbb{R}^n$ by any Banach space and the proof of Theorem 3.1 given above extends with no change. Such generalizations are useful in nonlinear PDE, as we will see in Chapter 14.

## Exercises

1. Suppose that $F : U \to \mathbb{R}^n$ is a $C^2$-map, $U$ is open in $\mathbb{R}^n$, $p \in U$, and $DF(p)$ is invertible. With $q = F(p)$, define a map $N$ on a neighborhood of $p$ by

$$N(x) = x + DF(x)^{-1}\Bigl(q - F(x)\Bigr). \tag{3.13}$$

   Show that there exists $\varepsilon > 0$ and $C < \infty$ such that, for $0 \le r < \varepsilon$,

$$\|x - p\| \le r \Longrightarrow \|N(x) - p\| \le C\, r^2.$$

   Conclude that if $\|x_1 - p\| \le r$, with $r < \min(\varepsilon, 1/2C)$, then $x_{j+1} = N(x_j)$ defines a sequence converging very rapidly to $p$. This is the basis of *Newton's method*, for solving $F(p) = q$ for $p$.
   (*Hint*: Write $x = p + y$, $F(x) = F(p) + DF(x)y + R$, with $R$ given as in (1.27), with $k = 2$. Then $N(x) = p + \tilde{y}$, $\tilde{y} = -DF(x)^{-1}R$.)
2. Applying Newton's method to $f(x) = 1/x$, show that you get a fast approximation to division using only addition and multiplication.
   (*Hint*: Carry out the calculation of $N(x)$ in this case and notice a "miracle.")

3. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $z = x + iy$, as in Exercise 4 of §1. Let $U \subset \mathbb{R}^{2n}$ be open, and let $F : U \to \mathbb{R}^{2n}$ be $C^1$. Assume that $p \in U$ and $DF(p)$ is invertible. If $F^{-1} : V \to U$ is given as in Theorem 3.1, show that $F^{-1}$ is holomorphic provided $F$ is.
4. Let $\mathcal{O} \subset \mathbb{R}^n$ be open. We say that a function $f \in C^\infty(\mathcal{O})$ is *real analytic* provided that, for each $x_0 \in \mathcal{O}$, we have a convergent power-series expansion

$$f(x) = \sum_{\alpha \ge 0} \frac{1}{\alpha!} f^{(\alpha)}(x_0)(x - x_0)^\alpha, \tag{3.14}$$

valid in a neighborhood of $x_0$. Show that we can let $x$ be complex in (3.14), and obtain an extension $f$ to a neighborhood of $\mathcal{O}$ in $\mathbb{C}^n$. Show that the extended function is holomorphic, that is, satisfies the Cauchy-Riemann equations.
*Remark.* It can be shown that, conversely, any holomorphic function has a power-series expansion. See (2.30) of Chapter 3 for one such proof. For the next exercise, assume this to be known.
5. Let $\mathcal{O} \subset \mathbb{R}^n$ be open, $p \in \mathcal{O}$, and $f : \mathcal{O} \to \mathbb{R}^n$ be real analytic, with $Df(p)$ invertible. Take $f^{-1} : V \to U$ as in Theorem 3.1. Show $f^{-1}$ is real analytic. (*Hint*: Consider a holomorphic extension $F : \Omega \to \mathbb{C}^n$ of $f$, and apply Exercise 3.)

## 4. Constant-coefficient linear systems; exponentiation of matrices

Let $A$ be an $n \times n$ matrix, real or complex. We consider the linear ODE

$$y' = Ay, \quad y(0) = y_0. \tag{4.1}$$

In analogy to the scalar case, we can produce the solution in the form

$$y(t) = e^{tA} y_0, \tag{4.2}$$

where we define the matrix exponential

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \tag{4.3}$$

We will establish estimates implying the convergence of this infinite series for all real $t$, indeed for all complex $t$. Then term-by-term differentiation is valid and gives (4.1). To discuss convergence of (4.3), we need the notion of the *norm* of a matrix. This is a special case of results discussed in Appendix A, Functional Analysis.

If $u = (u_1, \dots, u_n)$ belongs to $\mathbb{R}^n$ or to $\mathbb{C}^n$, set, as in (1.5),

$$\|u\| = \bigl(|u_1|^2 + \cdots + |u_n|^2\bigr)^{1/2}. \tag{4.4}$$

Then, if $A$ is an $n \times n$ matrix, set

$$\|A\| = \sup\{\|Au\| : \|u\| \le 1\}. \tag{4.5}$$

The norm (4.4) possesses the following properties:

$$\|u\| \geq 0, \quad \|u\| = 0 \text{ if and only if } u = 0, \tag{4.6}$$

$$\|cu\| = |c| \ \|u\|, \quad \text{for real or complex } c, \tag{4.7}$$

$$\|u + v\| \leq \|u\| + \|v\|. \tag{4.8}$$

The last property, known as the *triangle inequality*, follows from Cauchy's inequality:

$$|(u, v)| \leq \|u\| \cdot \|v\|, \tag{4.9}$$

where the inner product is $(u, v) = u_1\overline{v}_1 + \cdots + u_n\overline{v}_n$. To deduce (4.8) from (4.9), just square both sides of (4.8). To prove (4.9), use $(u - v, u - v) \geq 0$ to get

$$2 \ \mathrm{Re}\,(u, v) \leq \|u\|^2 + \|v\|^2.$$

Then replace $u$ by $e^{i\theta}u$ to deduce

$$2|(u, v)| \leq \|u\|^2 + \|v\|^2.$$

Next, replace $u$ by $tu$ and $v$ by $t^{-1}v$, to get

$$2|(u, v)| \leq t^2\|u\|^2 + t^{-2}\|v\|^2,$$

for any $t > 0$. Picking $t$ so that $t^2 = \|v\|/\|u\|$, we have Cauchy's inequality (4.9).

Given (4.6)–(4.8), we easily get

$$\begin{gathered} \|A\| \geq 0, \\ \|cA\| = |c| \ \|A\|, \\ \|A + B\| \leq \|A\| + \|B\|. \end{gathered} \tag{4.10}$$

Also, $\|A\| = 0$ if and only if $A = 0$. The fact that $\|A\|$ is the smallest constant $K$ such that $\|Au\| \leq K\|u\|$ gives

$$\|AB\| \leq \|A\| \cdot \|B\|. \tag{4.11}$$

In particular,

$$\|A^k\| \leq \|A\|^k. \tag{4.12}$$

This makes it easy to check the convergence of the power-series (4.3).

Power-series manipulations can be used to establish the identity

$$e^{sA}e^{tA} = e^{(s+t)A}. \tag{4.13}$$

Another way to prove this is as follows. Regard $t$ as fixed; denote the left side of (4.13) as $X(s)$ and the right side as $Y(s)$. Then differentiation with respect to $s$ gives, respectively,

$$\begin{aligned} X'(s) &= AX(s), \quad X(0) = e^{tA}, \\ Y'(s) &= AY(s), \quad Y(0) = e^{tA}, \end{aligned} \tag{4.14}$$

so the uniqueness of solutions to the ODE implies $X(s) = Y(s)$ for all $s$. We note that (4.13) is a special case of the following.

**Proposition 4.1.** *$e^{t(A+B)} = e^{tA}e^{tB}$ for all $t$, if and only if $A$ and $B$ commute.*

**Proof.** Let

$$Y(t) = e^{t(A+B)}, \quad Z(t) = e^{tA}e^{tB}. \tag{4.15}$$

Note that $Y(0) = Z(0) = I$, so it suffices to show that $Y(t)$ and $Z(t)$ satisfy the same ODE, to deduce that they coincide. Clearly,

$$Y'(t) = (A + B)Y(t). \tag{4.16}$$

Meanwhile,

$$Z'(t) = Ae^{tA}e^{tB} + e^{tA}Be^{tB}. \tag{4.17}$$

Thus we get the equation (4.16) for $Z(t)$ provided we know that

$$e^{tA}B = Be^{tA} \ \text{ if } AB = BA. \tag{4.18}$$

This follows from the power-series expansion for $e^{tA}$, together with the fact that

$$A^kB = BA^k, \quad \forall\, k \ge 0, \ \text{ if } AB = BA. \tag{4.19}$$

For the converse, if $Y(t) = Z(t)$ for all $t$, then $e^{tA}B = Be^{tA}$, by (4.17), and hence, taking the $t$-derivative, $e^{tA}AB = BAe^{tA}$; setting $t = 0$ gives $AB = BA$.

If $A$ is in diagonal form,

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \tag{4.20}$$

then clearly

$$e^{tA} = \begin{pmatrix} e^{ta_1} & & \\ & \ddots & \\ & & e^{ta_n} \end{pmatrix}. \tag{4.21}$$

The following result makes it useful to diagonalize $A$ in order to compute $e^{tA}$.

**Proposition 4.2.** *If $K$ is an invertible matrix and $B = KAK^{-1}$, then*

$$e^{tB} = K\ e^{tA}\ K^{-1}. \tag{4.22}$$

**Proof.** This follows from the power-series expansion (4.3), given the observation that

$$B^k = K\ A^k\ K^{-1}. \tag{4.23}$$

In view of (4.20)–(4.22), it is convenient to record a few standard results about eigenvalues and eigenvectors here. Let $A$ be an $n \times n$ matrix over $F$, $F = \mathbb{R}$ or $\mathbb{C}$. An eigenvector of $A$ is a nonzero $u \in F^n$ such that

$$Au = \lambda u, \tag{4.24}$$

for some $\lambda \in F$. Such an eigenvector exists if and only if $A - \lambda I : F^n \to F^n$ is not invertible, that is, if and only if

$$\det(A - \lambda I) = 0. \tag{4.25}$$

Now (4.25) is a polynomial equation, so it always has a complex root. This proves the following.

**Proposition 4.3.** *Given an $n \times n$ matrix $A$, there exists at least one (complex) eigenvector $u$.*

Of course, if $A$ is real, and we know there is a real root of (4.25) (e.g., if $n$ is odd), then a real eigenvector exists. One important class of matrices guaranteed to have real eigenvalues is the class of self-adjoint matrices. The adjoint of an $n \times n$ complex matrix is specified by the identity $(Au, v) = (u, A^*v)$.

**Proposition 4.4.** *If $A = A^*$, then all eigenvalues of $A$ are real.*

**Proof.** $Au = \lambda u$ implies

$$\lambda \|u\|^2 = (\lambda u, u) = (Au, u) = (u, Au) = (u, \lambda u) = \overline{\lambda} \|u\|^2. \tag{4.26}$$

Hence $\lambda = \overline{\lambda}$, if $u \neq 0$.

We now establish the following important result.

**Theorem 4.5.** *If $A = A^*$, then there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of $A$.*

**Proof.** Let $u_1$ be one unit eigenvector; $Au_1 = \lambda u_1$. Existence is guaranteed by Proposition 4.3. Let $V = (u_1)^\perp$ be the orthogonal complement of the linear span of $u_1$. Then dim $V$ is $n - 1$ and

$$A : V \to V, \quad \text{if } A = A^*. \tag{4.27}$$

The result follows by induction on $n$.

**Corollary 4.6.** *If $A = A^t$ is a real symmetric matrix, then there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$.*

**Proof.** By Proposition 4.4 and the remarks following Proposition 4.3, there is one unit eigenvector $u_1 \in \mathbb{R}^n$. The rest of the proof is as above.

The proofs of the last four results rest on the fact that every nonconstant polynomial has a complex root. This is the fundamental theorem of algebra. A proof is given in §19 (Exercise 5), and another after Corollary 4.7 of Chapter 3. An alternative approach to Proposition 4.3 when $A = A^*$, yielding Proposition 4.4–Corollary 4.6, is given in one of the exercises at the end of this section.

Given an ODE in upper triangular form,

$$\frac{dy}{dt} = \begin{pmatrix} a_{11} & * & * \\ & \ddots & * \\ & & a_{nn} \end{pmatrix} y, \tag{4.28}$$

you can solve the last ODE for $y_n$, as it is just $dy_n/dt = a_{nn}y_n$. Then you get a single nonhomogeneous ODE for $y_{n-1}$, which can be solved as demonstrated in (2.18)–(2.21), and you can continue inductively to solve. Thus, it is often useful to be able to put an $n \times n$ matrix $A$ in upper triangular form, with respect to a convenient choice of basis. We will establish two results along these lines. The first is due to Schur.

**Theorem 4.7.** *For any $n \times n$ matrix $A$, there is an orthonormal basis $u_1, \dots, u_n$ of $\mathbb{C}^n$ with respect to which $A$ is in upper triangular form.*

This result is equivalent to the following proposition.

**Proposition 4.8.** *For any $A$, there is a sequence of vector spaces $V_j$ of dimension $j$, contained in $\mathbb{C}^n$, with*

$$V_n \supset V_{n-1} \supset \cdots \supset V_1 \tag{4.29}$$

*and*

$$A : V_j \longrightarrow V_j. \tag{4.30}$$

To see the equivalence, if we are granted (4.29)–(4.30), pick $u_n \perp V_{n-1}$, a unit vector, then pick $u_{n-1} \in V_{n-1}$ such that $u_{n-1} \perp V_{n-2}$, and so forth. Meanwhile, Proposition 4.8 is a simple inductive consequence of the following result.

**Lemma 4.9.** *For any matrix $A$ acting on $V_n$, there is a linear subspace $V_{n-1}$, of codimension 1, such that $A : V_{n-1} \to V_{n-1}$.*

**Proof.** Use Proposition 4.3, applied to $A^*$. There is a vector $v_1$ such that $A^* v_1 = \lambda v_1$. Let $V_{n-1} = (v_1)^\perp$. This completes the proof of the lemma, hence of Theorem 4.7.

Let us look more closely at what you can say about solutions to an ODE that has been put in the form (4.28). As mentioned, we can obtain $y_j$ inductively by solving nonhomogeneous scalar ODE

$$\frac{dy_j}{dt} = a_{jj} y_j + b_j(t), \tag{4.31}$$

where $b_j(t)$ is a linear combination of $y_{j+1}(t), \dots, y_n(t)$, and the formula (2.21) applies, with $A(t) = a_{jj} t$. We have $y_n(t) = Ce^{a_{nn} t}$, so $b_{n-1}(t)$ is a multiple of $e^{a_{nn} t}$. If $a_{n-1,n-1} \neq a_{nn}$, $y_{n-1}(t)$ will be a linear combination of $e^{a_{nn} t}$ and $e^{a_{n-1,n-1} t}$, but if $a_{n-n,n-1} = a_{nn}$, $y_{n-1}(t)$ may be a linear combination of $e^{a_{nn} t}$ and $te^{a_{nn} t}$. Further integration will involve $\int p(t) e^{\alpha t} dt$, where $p(t)$ is a polynomial. That no other sort of function will arise is guaranteed by the following result.

**Lemma 4.10.** *If* $p(t) \in \mathcal{P}_n$*, the space of polynomials of degree* $\le n$*, and* $\alpha \neq 0$*, then*

$$\int p(t) e^{\alpha t} \, dt = q(t) e^{\alpha t} + C, \tag{4.32}$$

*for some* $q(t) \in \mathcal{P}_n$.

**Proof.** The map $p = Tq$ defined by $(d/dt)(q(t)e^{\alpha t}) = p(t) e^{\alpha t}$ is a map on $\mathcal{P}_n$; in fact, we have

$$Tq(t) = \alpha q(t) + q'(t). \tag{4.33}$$

It suffices to show that $T : \mathcal{P}_n \to \mathcal{P}_n$ is invertible. But $D = d/dt$ is *nilpotent* on $\mathcal{P}_n$; $D^{n+1} = 0$. Hence

$$T^{-1} = \alpha^{-1}(I + \alpha^{-1} D)^{-1} = \alpha^{-1}\big(I - \alpha^{-1} D + \cdots + \alpha^{-n}(-D)^n\big).$$

Note that this gives a neat formula for the integral (4.32). For example,

$$\begin{aligned} \int t^n e^{-t} \, dt &= -(t^n + nt^{n-1} + \cdots + n!) e^{-t} + C \\ &= -n!\Big(1 + t + \frac{1}{2} t^2 + \cdots + \frac{1}{n!} t^n\Big) e^{-t} + C. \end{aligned} \tag{4.34}$$

This could also be established by integration by parts and induction. Of course, when $\alpha = 0$ in (4.32), the result is different; $q(t)$ is a polynomial of degree $n + 1$.

Now the implication for the solution to (4.28) is that all the components of $y(t)$ are products of polynomials and exponentials. By Theorem 4.7, we can draw the same conclusion about the solution to $dy/dt = Ay$ for any $n \times n$ matrix $A$. We can formally state the result as follows.

**Proposition 4.11.** *For any* $n \times n$ *matrix* $A$,

$$e^{tA}v = \sum e^{\lambda_j t} v_j(t), \tag{4.35}$$

*where* $\{\lambda_j\}$ *is the set of eigenvalues of* $A$ *and* $v_j(t)$ *are* $\mathbb{C}^n$*-valued polynomials. All the* $v_j(t)$ *are constant when* $A$ *is diagonalizable.*

To see that the $\lambda_j$ are the eigenvalues of $A$, note that in the upper triangular case only the exponentials $e^{a_{jj}t}$ arise, and in that case the eigenvalues are precisely the diagonal elements.

If we let $\mathcal{E}_\lambda$ denote the space of $\mathbb{C}^n$-valued functions of the form $V(t) = e^{\lambda t}v(t)$, where $v(t)$ is a $\mathbb{C}^n$-valued polynomial, then $\mathcal{E}_\lambda$ is invariant under the action of both $d/dt$ and $A$, hence of $d/dt - A$. Hence, if a sum $V_1(t) + \cdots + V_k(t),\ V_j(t) \in \mathcal{E}_{\lambda_j}$ (with $\lambda_j$s distinct), is annihilated by $d/dt - A$, so is each term in this sum.

Therefore, if (4.35) is a sum over the distinct eigenvalues $\lambda_j$ of $A$, it follows that each term $e^{\lambda_j t}v_j(t)$ is annihilated by $d/dt - A$ or, equivalently, is of the form $e^{tA}w_j$, where $w_j = v_j(0)$. This leads to the following conclusion. Set

$$G_\lambda = \{v \in \mathbb{C}^n : e^{tA}v = e^{t\lambda}v(t),\ v(t) \text{ polynomial}\}. \tag{4.36}$$

Then $\mathbb{C}^n$ has a direct-sum decomposition

$$\mathbb{C}^n = G_{\lambda_1} + \cdots + G_{\lambda_k}, \tag{4.37}$$

where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of $A$. Furthermore, each $G_{\lambda_j}$ is invariant under $A$, and

$$A_j = A|_{G_{\lambda_j}} \text{ has exactly one eigenvalue, } \lambda_j. \tag{4.38}$$

This last statement holds because $e^{tA}v$ involves only the exponential $e^{\lambda_j t}$, when $v \in G_{\lambda_j}$. We say that $G_{\lambda_j}$ is the *generalized eigenspace* of $A$, with eigenvalue $\lambda_j$. Of course, $G_{\lambda_j}$ contains $\ker(A - \lambda_j I)$. Now $B_j = A_j - \lambda_j I$ has only 0 as an eigenvalue. It is subject to the following result.

**Lemma 4.12.** *If* $B : \mathbb{C}^k \to \mathbb{C}^k$ *has only* 0 *as an eigenvalue, then* $B$ *is nilpotent; in fact,*

$$B^m = 0 \quad \text{for some } m \le k. \tag{4.39}$$

**Proof.** Let $W_j = B^j(\mathbb{C}^k)$; then $\mathbb{C}^k \supset W_1 \supset W_2 \supset \cdots$ is a sequence of finite-dimensional vector spaces, each invariant under $B$. This sequence must stabilize, so for some $m$, $B : W_m \to W_m$ bijectively. If $W_m \neq 0$, $B$ has a nonzero eigenvalue.

We next discuss the famous Jordan normal form of a complex $n \times n$ matrix. The result is the following.

**Theorem 4.13.** *If $A$ is an $n \times n$ matrix, then there is a basis of $\mathbb{C}^n$ with respect to which $A$ becomes a direct sum of blocks of the form*

$$\begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}. \tag{4.40}$$

In light of the decomposition (4.37) and Lemma 4.12, it suffices to establish the Jordan normal form for a nilpotent matrix $B$. Given $v_0 \in \mathbb{C}^k$, let $m$ be the smallest integer such that $B^m v_0 = 0$; $m \le k$. If $m = k$, then $\{v_0, Bv_0, \dots, B^{m-1}v_0\}$ gives a basis of $\mathbb{C}^k$, putting $B$ in Jordan normal form. We then say $v_0$ is a *cyclic* vector for $B$, and $\mathbb{C}^k$ is generated by $v_0$. We call $\{v_0, \dots, B^{m-1}v_0\}$ a *string.*

We will have a Jordan normal form precisely if we can write $\mathbb{C}^k$ as a direct sum of cyclic subspaces. We establish that this can be done by induction on the dimension.

Thus, inductively, we can suppose that $W_1 = B(\mathbb{C}^k)$ is a direct sum of cyclic subspaces, so $W_1$ has a basis that is a union of strings, let's say a union of $d$ strings $\{v_j, Bv_j, \dots, B^{\ell_j}v_j\}$, $1 \le j \le d$. In this case, ker $B \cap W_1 = N_1$ has dimension $d$, and the vectors $B^{\ell_j}v_j$, $1 \le j \le d$, span $N_1$. Furthermore, each $v_j$ has the form $v_j = Bw_j$ for some $w_j \in \mathbb{C}^k$.

Now dim ker $B = k - r \ge d$, where $r =$ dim $W_1$. Let $\{z_1, \dots, z_{k-r-d}\}$ span a subspace of ker $B$ complementary to $N_1$. Then the strings $\{w_j, v_j = Bw_j, \dots, B^{\ell_j}v_j\}$, $1 \le j \le d$, and $\{z_1\}, \dots, \{z_{k-r-d}\}$ generate cyclic subspaces whose direct sum is $\mathbb{C}^k$, giving the Jordan normal form.

The argument above is part of an argument of Filippov. In fact, Filippov's proof contains a further clever twist, enabling one to prove Theorem 4.13 without using the decomposition (4.37). However, since we got this decomposition almost for free as a byproduct of the ODE analysis in Proposition 4.11, this author decided to make use of it. See Strang [Str] for Filippov's proof.

We have seen how constructing $e^{tA}$ solves the equation (4.1). We can also use it to solve a nonhomogeneous equation, of the form

$$y' = Ay + b(t), \quad y(0) = y_0. \tag{4.41}$$

Direct calculation shows that the solution is given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}\, b(s)\, ds. \tag{4.42}$$

Note how this partially generalizes the formula (2.21). This formula is a special case of Duhamel's principle, which will be discussed further in §5.

We remark that the definition of $e^{tA}$ by power series (4.3) extends to the case where $A$ is a bounded linear operator on a Banach space. In that case, $e^{tA}$ furnishes the simplest sort of example of a one-parameter group of operators. Compare §9 in Appendix A, Functional Analysis, for a further discussion of semigroups of operators. A number of problems in PDE amount to exponentiating various *unbounded* operators. The discussion of eigenvalues, eigenvectors, and normal forms above relies heavily on finite dimensionality, although a good deal of it carries over to compact operators on infinite-dimensional Banach and Hilbert spaces; see §6 of Appendix A. Also, there is a somewhat more subtle extension of Theorem 4.5 for general self-adjoint operators on a Hilbert space, which is discussed in §1 of Chapter 8.

## Exercises

1. In addition to the operator norm $\|A\|$ of an $n \times n$ matrix, defined by (4.5), we consider the Hilbert-Schmidt norm $\|A\|_{\mathrm{HS}}$, defined by
$$\|A\|_{\mathrm{HS}}^2 = \sum_{j,k} |a_{jk}|^2,$$
if $A = \Big(a_{jk}\Big)$. Show that
$$\|A\| \le \|A\|_{\mathrm{HS}}.$$
(*Hint*: If $r_1, \dots, r_n$ are the rows of $A$, then for $u \in \mathbb{C}^n$, $Au$ has entries $r_j \cdot u$, $1 \le j \le n$. Use Cauchy's inequality (4.9) to estimate $|r_j \cdot u|^2$.)

   Show also that
$$\sum_j |a_{jk}|^2 \le \|A\|^2 \quad \text{for each } k,$$
and hence
$$\|A\|_{\mathrm{HS}}^2 \le n\|A\|^2.$$
(*Hint*: $\|A\| \ge \|Ae_k\|$ for each standard basis vector $e_k$.)
2. Show that, in analogy with (4.11), we have
$$\|AB\|_{\mathrm{HS}} \le \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}.$$
Indeed, show that
$$\|AB\|_{\mathrm{HS}} \le \|A\| \cdot \|B\|_{\mathrm{HS}},$$
where the first factor on the right is the operator norm $\|A\|$.
3. Let $X$ be an $n \times n$ matrix. Show that
$$\det e^X = e^{Tr\ X}.$$

(*Hint*: Use a normal form.)

Let $M_n$ denote the space of complex $n \times n$ matrices. If $A \in M_n$ and $\det A = 1$, we say that $A \in \mathrm{SL}(n, \mathbb{C})$. If $X \in M_n$ and $\mathrm{Tr}\ X = 0$, we say that $X \in \mathrm{sl}(n, \mathbb{C})$.

4. Let $X \in \mathrm{sl}(2, \mathbb{C})$. Suppose $X$ has eigenvalues $\{\lambda, -\lambda\}$, $\lambda \neq 0$. Such an $X$ can be diagonalized, so we know that there exist matrices $Z_j \in M_2$ such that
$$e^{tX} = Z_1\, e^{t\lambda} + Z_2\, e^{-t\lambda}.$$
Evaluating both sides at $t = 0$, and the $t$-derivative at $t = 0$, show that $Z_1 + Z_2 = I$, $\lambda Z_1 - \lambda Z_2 = X$, and solve for $Z_1$, $Z_2$. Deduce that
$$e^{tX} = (\cosh t\lambda)I + \lambda^{-1}(\sinh t\lambda)X.$$
5. Define holomorphic functions $C(z)$ and $S(z)$ by
$$C(z) = \cosh \sqrt{z}, \quad S(z) = \frac{\sinh \sqrt{z}}{\sqrt{z}}.$$
Deduce from Exercise 4 that, for $X \in \mathrm{sl}(2, \mathbb{C})$,
$$e^X = C(-\det X)I + S(-\det X)X.$$
Show that this identity is also valid when 0 is an eigenvalue of $X$.
6. Rederive the formula above for $e^X$, $X \in \mathrm{sl}(2, \mathbb{C})$, by using the power series for $e^X$ together with the identity
$$X^2 = -(\det X)I, \quad X \in \mathrm{sl}(2, \mathbb{C}).$$

The next set of exercises examines the derivative of the map
$$\mathrm{Exp} : M_n \to M_n, \quad \mathrm{Exp}(X) = e^X.$$

7. Set $U(t, s) = e^{t(X+sY)}$, where $X$ and $Y$ are $n \times n$ matrices, and set $U_s = \partial U/\partial s$. Show that $U_s$ satisfies
$$\frac{\partial U_s}{\partial t} = (X + sY)U_s + YU, \quad U_s(0, s) = 0.$$
8. Use Duhamel's principle, formula (4.42), to show that
$$U_s(t, s) = \int_0^t e^{(t-\tau)(X+sY)}\ Y\ e^{\tau(X+sY)}\ d\tau.$$
Deduce that
$$\frac{d}{ds}\ e^{X+sY}\bigg|_{s=0} = e^X \int_0^1 e^{-\tau X} Y e^{\tau X}\ d\tau. \tag{4.43}$$
9. Given $X \in M_n$, define $\mathrm{ad}\ X \in \mathrm{End}(M_n)$, that is,
$$\mathrm{ad}\ X : M_n \to M_n,$$
by
$$\mathrm{ad}\ X(Y) = XY - YX.$$
Show that
$$e^{-tX} Y e^{tX} = e^{-t\ \mathrm{ad}\ X} Y.$$

(*Hint*: If $V(t)$ denotes either side, show that $dV/dt = -(\mathrm{ad}\ X)V$, $V(0) = Y$.)

10. Deduce from Exercise 8 that

$$\frac{d}{ds}\, e^{X+sY}\Big|_{s=0} = e^X\ \Xi(\mathrm{ad}\ X)Y, \tag{4.44}$$

where $\Xi(z)$ is the entire holomorphic function

$$\Xi(z) = \int_0^1 e^{-\tau z}\, d\tau = \frac{1-e^{-z}}{z}. \tag{4.45}$$

The operator $\Xi(\mathrm{ad}\ X)$ is defined in the following manner. For any $L \in \mathrm{End}(\mathbb{C}^m) = M_m$, any function $F(z)$ holomorphic on $|z| < a$, with $a > \|L\|$, define $F(L)$ by power series:

$$F(L) = \sum_{n=0}^{\infty} f_n L^n, \quad \text{where } F(z) = \sum_{n=0}^{\infty} f_n z^n. \tag{4.46}$$

For further material on holomorphic functions of operators, see §5 in Appendix A.

11. With $\mathrm{Exp} : M_n \to M_n$ as defined above, describe the set of matrices $X$ such that the transformation $D\ \mathrm{Exp}(X)$ is not invertible.
12. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric, and let $Q(x) = (Ax, x)$. Let $v_1 \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be a point where $Q\big|_{S^{n-1}}$ assumes a maximum. Show that $v_1$ is an eigenvector of $A$.
(*Hint*: Show that $\nabla Q(v_1)$ is parallel to $\nabla E(v_1)$, where $E(x) = (x, x)$.)
Use this result to give an alternative proof of Corollary 4.6. Extend this argument to establish Theorem 4.5.

## 5. Variable-coefficient linear systems of ODE: Duhamel's principle

Let $A(t)$ be a continuous, $n \times n$ matrix-valued function of $t \in I$. We consider the general linear, homogeneous ODE

$$\frac{dy}{dt} = A(t)y, \quad y(0) = y_0. \tag{5.1}$$

The general theory of §2 gives local solutions. We claim that the solutions here exist for all $t \in I$. This follows from the discussion after the proof of Theorem 2.1, together with the following *estimate* on the solution to (5.1).

**Proposition 5.1.** *If $\|A(t)\| \le M$ for $t \in I$, then the solution to (5.1) satisfies*

$$\|y(t)\| \le e^{M|t|}\|y_0\|. \tag{5.2}$$

It suffices to prove this for $t \ge 0$. Then $z(t) = e^{-Mt}y(t)$ satisfies

$$z' = C(t)z, \quad z(0) = y_0, \tag{5.3}$$

with $C(t) = A(t) - M$. Hence $C(t)$ satisfies

$$\text{Re}\,(C(t)u, u) \le 0, \quad \text{for all } u \in \mathbb{C}^n. \tag{5.4}$$

Thus (5.2) is a consequence of the following energy estimate, which is of independent interest.

**Proposition 5.2.** *If $z$ solves (5.3) and if (5.4) holds for $C(t)$, then*

$$\|z(t)\| \le \|z(0)\|, \quad \text{for } \ t \ge 0.$$

**Proof.** We have

$$\begin{aligned} \frac{d}{dt}\|z(t)\|^2 &= (z'(t), z(t)) + (z(t), z'(t)) \\ &= 2\ \text{Re}\,(C(t)z(t), z(t)) \\ &\le 0. \end{aligned} \tag{5.5}$$

Thus we have global existence for (5.1). There is a matrix-valued function $S(t, s)$ such that the unique solution to (5.1) satisfies

$$y(t) = S(t, s)y(s). \tag{5.6}$$

Using this solution operator, we can treat the nonhomogeneous equation

$$y' = A(t)y + b(t), \quad y(0) = y_0. \tag{5.7}$$

Indeed, direct calculation yields

$$y(t) = S(t, 0)y_0 + \int_0^t S(t, s)b(s)\ ds. \tag{5.8}$$

This identity is known as *Duhamel's principle*.

Next we prove an identity that might be called the "noncommutative fundamental theorem of calculus."

**Proposition 5.3.** *If $A(t)$ is a continuous matrix function and $S(t, 0)$ is defined as above, then*

$$S(t, 0) = \lim_{n\to\infty} e^{(t/n)A((n-1)t/n)} \cdots e^{(t/n)A(0)}, \tag{5.9}$$

*where there are $n$ factors on the right.*

**Proof.** To prove this at $t = T$, divide the interval $[0, T]$ into $n$ equal parts. Set $y = S(t, 0)y_0$, and define $z_n(t)$ by $z_n(0) = y_0$ and

$$z_n' = A(jT/n)z_n, \quad \text{for } \ t \in \big(jT/n, (j + 1)T/n\big), \tag{5.10}$$

requiring continuity across each endpoint of these intervals. We see that

$$z_n' = A(t)z_n + R_n(t), \tag{5.11}$$

with

$$\|R_n(t)\| \le \delta_n \|z_n(t)\|, \quad \delta_n \to 0 \text{ as } n \to \infty. \tag{5.12}$$

Meanwhile we see that $\|z_n(t)\| \le C_T \|y_0\|$ on $[0,T]$. We want to compare $z_n(t)$ and $y(t)$. We have

$$\frac{d}{dt}(z_n - y) = A(t)(z_n - y) + R_n(t); \quad z_n(0) - y(0) = 0. \tag{5.13}$$

Hence Duhamel's principle gives

$$z_n(t) - y(t) = \int_0^t S(t,s) R_n(s)\, ds, \tag{5.14}$$

and since we have an a priori bound $\|S(t,s)\| \le K$ for $|s|, |t| \le T$, we get

$$\|z_n(t) - y(t)\| \le KTC_T \delta_n \|y_0\| \to 0 \text{ as } n \to \infty, \ |t| \le T. \tag{5.15}$$

In particular, $z_n(T) \to y(T)$ as $n \to \infty$. Since $z_n(T)$ is given by the right side of (5.9) with $t = T$, this proves (5.9).

## Exercises

1. Let $A(t)$ and $X(t)$ be $n \times n$ matrices satisfying

$$\frac{dX}{dt} = A(t)X.$$

We form the *Wronskian* $W(t) = \det X(t)$. Show that $W$ satisfies the ODE

$$\frac{dW}{dt} = a(t)W, \quad a(t) = \operatorname{Tr} A(t).$$

(*Hint*: Use Exercise 2 of §1 to write $dW/dt = \operatorname{Tr}(\operatorname{Cof}(X)^t dX/dt)$, and use Cramer's formula, $(\det X)X^{-1} = \operatorname{Cof}(X)^t$. *Alternative*: Write $X(t+h) = e^{hA(t)}X(t) + O(h^2)$ and use Exercise 3 of §4 to write $\det e^{hA(t)} = e^{ha(t)}$, hence $W(t+h) = e^{ha(t)}W(t) + O(h^2)$.)

2. Let $u(t) = \|y(t)\|^2$, for a solution $y$ to (5.1). Show that

$$u' \le M(t)u(t), \tag{5.16}$$

provided $\|A(t)\| \le M(t)/2$. Such a differential inequality implies the integral inequality

$$u(t) \le A + \int_0^t M(s)u(s)\, ds, \quad t \ge 0, \tag{5.17}$$

with $A = u(0)$. The following is a *Gronwall inequality*; namely, if (5.17) holds for a real-valued function $u$, then provided $M(s) \ge 0$, we have, for $t \ge 0$,

$$u(t) \le Ae^{N(t)}, \quad N(t) = \int_0^t M(s)\, ds. \tag{5.18}$$

Prove this. Note that the quantity dominating $u(t)$ in (5.18) is equal to $U$, solving $U(0) = A$, $dU/dt = M(t)U(t)$.

3. Generalize the Gronwall inequality of Exercise 2 as follows. Assume $F(t,u)$ and $\partial_u F(t,u)$ are continuous, let $U$ be a real-valued solution to

$$U' = F(t,U), \quad U(0) = A, \tag{5.19}$$

and let $u$ satisfy the integral inequality

$$u(t) \le A + \int_0^t F(s,u(s))\,ds. \tag{5.20}$$

Then prove that

$$u(t) \le U(t), \quad \text{for } t \ge 0, \tag{5.21}$$

provided $\partial F/\partial u \ge 0$. Show that this continues to hold if we replace (5.19) by

$$U(t) \ge A + \int_0^t F(s,U(s))\,ds. \tag{5.19a}$$

(*Hint*: Set $v = u - U$. Then (5.19a) and (5.20) imply

$$v(t) \le \int_0^t \Big[F(s,u(s)) - F(s,U(s))\Big]\,ds = \int_0^t M(s)v(s)\,ds,$$

where

$$M(s) = \int_0^1 F_u\Big(s, \tau u(s) + (1-\tau)U(s)\Big)\,d\tau.$$

Thus (5.17) applies, with $A = 0$.)

4. Let $x(t)$ be a smooth curve in $\mathbb{R}^3$; assume it is parameterized by arc length, so $T(t) = x'(t)$ has unit length; $T(t)\cdot T(t) = 1$. Differentiating, we have $T'(t) \perp T(t)$. The *curvature* is defined to be $\kappa(t) = \|T'(t)\|$. If $\kappa(t) \ne 0$, we set $N(t) = T'/\|T'\|$, so

$$T' = \kappa N,$$

and $N$ is a unit vector orthogonal to $T$. We define $B(t)$ by

$$B = T \times N. \tag{5.22}$$

Note that $(T,N,B)$ form an orthonormal basis of $\mathbb{R}^3$ for each $t$, and

$$T = N \times B, \quad N = B \times T. \tag{5.23}$$

By (5.22) we have $B' = T \times N'$. Deduce that $B'$ is orthogonal to both $T$ and $B$, hence parallel to $N$. We set

$$B' = -\tau N,$$

for smooth $\tau(t)$, called the *torsion*.

5. From $N' = B' \times T + B \times T'$ and the formulas for $T'$ and $B'$ given in Exercise 4, deduce the following system, called the *Frenet-Serret formula*:

$$\begin{aligned} T' &= \phantom{-\kappa T} \quad \kappa N \\ N' &= -\kappa T \qquad\qquad + \tau B \\ B' &= \phantom{-\kappa T} \; - \tau N \end{aligned} \tag{5.24}$$

Form the $3 \times 3$ matrix

$$A(t) = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \tag{5.25}$$

and deduce that the $3 \times 3$ matrix $F(t)$ whose columns are $T, N, B$,

$$F = (T, N, B),$$

satisfies the ODE

$$F' = F\,A(t).$$

6. Derive the following *converse* to the Frenet-Serret formula. Let $T(0), N(0)$, and $B(0)$ be an orthonormal set in $\mathbb{R}^3$, such that $B(0) = T(0) \times N(0)$; let $\kappa(t)$ and $\tau(t)$ be given smooth functions; and solve the system (5.24). Show that there is a unique curve $x(t)$ such that $x(0) = 0$ and $T(t), N(t)$, and $B(t)$ are associated to $x(t)$ by the construction in Exercise 4, so in particular the curve has curvature $\kappa(t)$ and torsion $\tau(t)$.
(*Hint*: To prove that (5.22) and (5.23) hold for all $t$, consider the next exercise.)
7. Let $A(t)$ be a smooth, $n \times n$ real matrix function that is *skew-adjoint* for all $t$ (of which (5.25) is an example). Suppose $F(t)$ is a real $n \times n$ matrix function satisfying

$$F' = F\,A(t).$$

If $F(0)$ is an orthogonal matrix, show that $F(t)$ is orthogonal for all $t$.
(*Hint*: Set $J(t) = F(t)^* F(t)$. Show that $J(t)$ and $J_0(t) = I$ both solve the initial-value problem

$$J' = [J, A(t)], \quad J(0) = I.)$$

8. Let $U_1 = T,\ U_2 = N$ and $U_3 = B$, and set $\omega(t) = \tau T + \kappa B$. Show that (5.24) is equivalent to $U_j' = \omega \times U_j,\ 1 \le j \le 3$.
9. Suppose $\tau$ and $\kappa$ are constant. Show that $\omega$ is constant, so $T(t)$ satisfies the constant-coefficient ODE

$$T'(t) = \omega \times T(t).$$

Note that $\omega \cdot T(0) = \tau$. Show that after a translation and rotation, $x(t)$ takes the form

$$\gamma(t) = \Big(\lambda^{-2}\kappa \cos \lambda t, \lambda^{-2}\kappa \sin \lambda t, \lambda^{-1}\tau\, t\Big), \quad \lambda^2 = \kappa^2 + \tau^2.$$

## 6. Dependence of solutions on initial data and on other parameters

We consider how a solution to an ODE depends on the initial conditions. Consider a nonlinear system

$$y' = F(y), \quad y(0) = x. \tag{6.1}$$

As noted in §2, we can consider an autonomous system, such as (6.1), without loss of generality. Suppose $F : U \to \mathbb{R}^n$ is smooth, $U \subset \mathbb{R}^n$ open; for simplicity we assume $U$ is convex. Say $y = y(t, x)$. We want to examine smoothness in $x$.

Note that *formally* differentiating (6.1) with respect to $x$ suggests that $W = D_x y(t, x)$ satisfies an ODE called the *linearization* of (6.1):

$$W' = DF(y)W, \quad W(0) = I. \tag{6.2}$$

In other words, $w(t, x) = D_x y(t, x)w_0$ satisfies

$$w' = DF(y)w, \quad w(0) = w_0. \tag{6.3}$$

To justify this, we want to compare $w(t)$ and

$$z(t) = y_1(t) - y(t) = y(t, x + w_0) - y(t, x). \tag{6.4}$$

It would be convenient to show that $z$ satisfies an ODE similar to (6.3). Indeed, $z(t)$ satisfies

$$z' = F(y_1) - F(y) = \Phi(y_1, y)z, \quad z(0) = w_0, \tag{6.5}$$

where

$$\Phi(y_1, y) = \int_0^1 DF\big(\tau y_1 + (1 - \tau)y\big)\, d\tau. \tag{6.6}$$

If we assume that

$$\|DF(u)\| \le M, \quad \text{for } u \in U, \tag{6.7}$$

then the solution operator $S(t, 0)$ of the linear ODE $d/dt - B(t)$, with $B(y) = \Phi(y_1(t), y(t))$, satisfies a bound $\|S(t, 0)\| \le e^{|t|M}$ as long as $y(t)$ and $y_1(t)$ belong to $U$. Hence

$$\|y_1(t) - y(t)\| \le e^{|t|M}\|w_0\|. \tag{6.8}$$

This establishes that $y(t, x)$ is *Lipschitz* in $x$.

To continue, since $\Phi(y, y) = DF(y)$, we rewrite (6.5) as

$$z' = \Phi(y + z, y)z = DF(y)z + R(y, z), \quad w(0) = w_0, \tag{6.9}$$

where

$$F \in C^1(U) \Longrightarrow \|R(y, z)\| = o(\|z\|) = o(\|w_0\|). \tag{6.10}$$

Now comparing (6.9) with (6.3), we have

$$\frac{d}{dt}(z - w) = DF(y)(z - w) + R(y, z), \quad (z - w)(0) = 0. \tag{6.11}$$

Then Duhamel's principle yields

$$z(t) - w(t) = \int_0^t S(t, s)R\big(y(s), z(s)\big)\, ds, \tag{6.12}$$

so by the bound $\|S(t,s)\| \le e^{|t-s|M}$ and (6.10), we have

$$z(t) - w(t) = o(\|w_0\|). \tag{6.13}$$

This is precisely what is required to show that $y(t,x)$ is differentiable with respect to $x$, with derivative $W = D_x y(t,x)$ satisfying (6.2). We state our first result.

**Proposition 6.1.** *If $F \in C^1(U)$, and if solutions to (6.1) exist for $t \in (-T_0, T_1)$, then for each such $t$, $y(t,x)$ is $C^1$ in $x$, with derivative $D_x y(t,x) = W(t,x)$ satisfying (6.2).*

So far we have shown that $y(t,x)$ is both Lipschitz and differentiable in $x$, but the continuity of $W(t,x)$ in $x$ follows easily by comparing the ODEs of the form (6.2) for $W(t,x)$ and $W(t,x+w_0)$, in the spirit of the analysis of (6.11).

If $F$ possesses further smoothness, we can obtain higher differentiability of $y(t,x)$ in $x$ by the following trick. *Couple* (6.1) and (6.2), to get an ODE for $(y,W)$:

$$\begin{aligned} y' &= F(y), \\ W' &= DF(y)W, \end{aligned} \tag{6.14}$$

with initial conditions

$$y(0) = x, \quad W(0) = I. \tag{6.15}$$

We can reiterate the preceeding argument, getting results on $D_x(y,W)$, that is, on $D_x^2 y(t,x)$, and continue, proving:

**Proposition 6.2.** *If $F \in C^k(U)$, then $y(t,x)$ is $C^k$ in $x$.*

Similarly, we can consider dependence of the solution to a system of the form

$$\frac{dy}{dt} = F(\tau, y), \quad y(0) = x \tag{6.16}$$

on a parameter $\tau$, assuming $F$ is smooth jointly in $\tau, y$. This result can be deduced from the previous one by the following trick: Consider the ODE

$$y' = F(z,y), \ z' = 0; \quad y(0) = x, \ z(0) = \tau. \tag{6.17}$$

Thus we get smoothness of $y(t,\tau,x)$ in $(\tau,x)$. As one special case, let $F(\tau,y) = \tau F(y)$. In this case $y(t_0,\tau,x) = y(\tau t_0, 1, x)$, so we can improve the conclusion of Proposition 6.2 to the following:

$$F \in C^k(U) \Longrightarrow y \in C^k \text{ jointly in } (t,x). \tag{6.18}$$

It is also true that if $F$ is analytic, then one has the analytic dependence of solutions on parameters, especially on $t$, so that power-series techniques work in that case. One approach to the proof of this is given in the exercises below, and another at the end of §9.

## Exercises

1. Let $\Omega$ be open in $\mathbb{R}^{2n}$, identified with $\mathbb{C}^n$, via $z = x + iy$. Let $X : \Omega \to \mathbb{R}^{2n}$ have components $X = (a_1, \dots, a_n, b_1, \dots, b_n)$, where $a_j(x, y)$ and $b_j(x, y)$ are real-valued. Denote the solution to $du/dt = X(u), u(0) = z$ by $u(t, z)$. Assume $f_j(z) = a_j(z) + ib_j(z)$ is holomorphic in $z$, that is, its derivative commutes with $J$, acting on $\mathbb{R}^{2k} = \mathbb{C}^k$ as multiplication by $i$. Show that, for each $t$, $u(t, z)$ is holomorphic in $z$, that is, $D_z u(t, z)$ commutes with $J$. (*Hint*: Use the linearized equation (6.2) to show that $K(t) = [W(t), J]$ satisfies the ODE
$$K' = DX(z)K, \quad K(0) = 0.)$$
2. If $\mathcal{O} \subset \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}^n$ is real analytic, show that the solution $y(t, x)$ to (6.1) is real analytic in $x$.
(*Hint*: With $F = (a_1, \dots, a_n)$, take holomorphic extensions $f_j(z)$ of $a_j(x)$ and use Exercise 1.)
Using the trick leading to (6.18), show that $y(t, x)$ is real analytic jointly in $(t, x)$.

In the next set of problems, consider a linear ODE of the form

$$A(x)\frac{du}{dx} = B(x)u, \quad 0 < x < 1, \tag{6.19}$$

where we assume that the $n \times n$ matrix functions $A$ and $B$ have holomorphic extensions to $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, such that det $A(z) = 0$ at $z = 0$, but at no other point of $\Delta$. We say $z = 0$ is a singular point. Let $u_1(x), \dots, u_n(x)$ be $n$ linearly independent solutions to (6.19), obtained, for example, by specifying $u$ at $x = 1/2$.

3. Show that each $u_j$ has a unique holomorphic extension to the universal covering surface $\mathcal{M}$ of $\Delta \setminus 0$, and show that there are $c_{jk} \in \mathbb{C}$ such that
$$u_j(e^{2\pi i}x) = \sum_k c_{jk}\, u_k(x), \quad 0 < x < 1.$$
4. Suppose the matrix $C = (c_{jk})$ is diagonalizable, with eigenvalues $\lambda_\ell \in \mathbb{C}$, $1 \le \ell \le n$. Show that there is a basis of solutions $v_\ell$ to (6.19) such that
$$v_\ell(e^{2\pi i}x) = \lambda_\ell\, v_\ell(x),$$
and hence, picking $\alpha_\ell \in \mathbb{C}$ such that $e^{2\pi i \alpha_\ell} = \lambda_\ell$,
$$v_\ell(x) = x^{\alpha_\ell} w_\ell(x); \quad w_\ell \text{ holomorphic on } \Delta \setminus 0.$$
5. Suppose $\|A(z)^{-1}B(z)\| \le K|z|^{-1}$. Show that $\|v_\ell(z)\| \le C|z|^{-K}$. Deduce that each $w_\ell(z)$ has at most a pole at $z = 0$; hence, shifting $\alpha_\ell$ by an integer, we can assume that $w_\ell$ is holomorphic on $\Delta$. (*Hint*: Recall the statement of Gronwall's inequality, in Exercises 2 and 3 of §5.)

6. Suppose that instead of $C$ being diagonalizable, it has the Jordan normal form
$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$
(in case $n = 2$). What can you say? Generalize.
7. If $a(z)$ and $b(z)$ are holomorphic on $\Delta$, convert
$$x^2 u''(x) + xa(x)u'(x) + b(x)u(x) = 0$$
to a first-order system to which Exercises 3–6 apply. (*Hint.* Take $v = xu'$ rather than $v = u'$.)

The next set of exercises deals with certain small perturbations of the system $\dot{x} = -y,\ \dot{y} = x$, whose solution curves are circles centered at the origin.

8. Let $x = x_\varepsilon(t),\ y = y_\varepsilon(t)$ solve
$$\dot{x} = -y + \varepsilon(x^2 + y^2), \quad \dot{y} = x,$$
with initial data $x(0) = 1,\ y(0) = 0$. Knowing smooth dependence on $\varepsilon$, find ODEs for the coefficients $x_j(t), y_j(t)$ in power-series expansions
$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots, \quad y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots.$$
9. Making use of the substitution $\xi(t) = -x(-t),\ \eta(t) = y(-t)$, show that, for fixed initial data and $\varepsilon$ sufficiently small, the orbits of the ODE in Exercise 8 are periodic.
10. Show that, for $\varepsilon$ small, the period of the orbit in Exercise 8 is a smooth function of $\varepsilon$. Compute the first three terms in its power-series expansion.

## 7. Flows and vector fields

Let $U \subset \mathbb{R}^n$ be open. A vector field on $U$ is a smooth map
$$X : U \longrightarrow \mathbb{R}^n. \tag{7.1}$$
Consider the corresponding ODE:
$$y' = X(y), \quad y(0) = x, \tag{7.2}$$
with $x \in U$. A curve $y(t)$ solving (7.2) is called an integral curve of the vector field $X$. It is also called an *orbit*. For fixed $t$, write
$$y = y(t, x) = \mathcal{F}_X^t(x). \tag{7.3}$$
The locally defined $\mathcal{F}_X^t$, mapping (a subdomain of) $U$ to $U$, is called the *flow* generated by the vector field $X$.

The vector field $X$ defines a differential operator on scalar functions, as follows:
$$\mathcal{L}_X f(x) = \lim_{h \to 0} h^{-1} \big[ f(\mathcal{F}_X^h x) - f(x) \big] = \frac{d}{dt} f(\mathcal{F}_X^t x)\big|_{t=0}. \tag{7.4}$$

We also use the common notation

$$\mathcal{L}_X f(x) = Xf, \tag{7.5}$$

that is, we apply $X$ to $f$ as a first-order differential operator.

Note that if we apply the chain rule to (7.4) and use (7.2), we have

$$\mathcal{L}_X f(x) = X(x) \cdot \nabla f(x) = \sum a_j(x) \frac{\partial f}{\partial x_j}, \tag{7.6}$$

if $X = \sum a_j(x) e_j$, with $\{e_j\}$ the standard basis of $\mathbb{R}^n$. In particular, using the notation (7.5), we have

$$a_j(x) = X x_j. \tag{7.7}$$

In the notation (7.5),

$$X = \sum a_j(x) \frac{\partial}{\partial x_j}. \tag{7.8}$$

We note that $X$ is a *derivation*, that is, a map on $C^\infty(U)$, linear over $\mathbb{R}$, satisfying

$$X(fg) = (Xf)g + f(Xg). \tag{7.9}$$

Conversely, any derivation on $C^\infty(U)$ defines a vector field, namely, has the form (7.8), as we now show.

**Proposition 7.1.** *If $X$ is a derivation on $C^\infty(U)$, then $X$ has the form (7.8).*

**Proof.** Set $a_j(x) = X x_j$, $X^\# = \sum a_j(x) \partial/\partial x_j$, and $Y = X - X^\#$. Then $Y$ is a derivation satisfying $Y x_j = 0$ for each $j$; we aim to show that $Yf = 0$ for all $f$. Note that whenever $Y$ is a derivation,

$$1 \cdot 1 = 1 \Rightarrow Y \cdot 1 = 2Y \cdot 1 \Rightarrow Y \cdot 1 = 0,$$

that is, $Y$ annihilates constants. Thus, in this case $Y$ annihilates all polynomials of degree $\le 1$.

Now we show that $Yf(p) = 0$ for all $p \in U$. Without loss of generality, we can suppose $p = 0$, the origin. Then, by (1.8), we can take $b_j(x) = \int_0^1 (\partial_j f)(tx)\, dt$, and write

$$f(x) = f(0) + \sum b_j(x) x_j.$$

It immediately follows that $Yf$ vanishes at 0, so the proposition is proved.

If $U$ is a manifold, it is natural to regard a vector field $X$ as a section of the tangent bundle of $U$, as explained in Appendix B. Of course, the characterization given in Proposition 7.1 makes good invariant sense on a manifold.

A fundamental fact about vector fields is that they can be “straightened out” near points where they do not vanish. To see this, suppose a smooth vector field $X$ is given on $U$ such that, for a certain $p \in U$, $X(p) \neq 0$. Then near $p$ there is a hypersurface $M$ that is nowhere tangent to $X$. We can choose coordinates near $p$ so that $p$ is the origin and $M$ is given by $\{x_n = 0\}$. Thus, we can identify a point $x' \in \mathbb{R}^{n-1}$ near the origin with $x' \in M$. We can define a map

$$\mathcal{F} : M \times (-t_0, t_0) \longrightarrow U \tag{7.10}$$

by

$$\mathcal{F}(x', t) = \mathcal{F}^t_X(x'). \tag{7.11}$$

This is $C^\infty$ and has surjective derivative and so by the inverse function theorem is a local diffeomorphism. This defines a new coordinate system near $p$, in which the flow generated by $X$ has the form

$$\mathcal{F}^s_X(x', t) = (x', t + s). \tag{7.12}$$

If we denote the new coordinates by $(u_1, \dots, u_n)$, we see that the following result is established.

**Theorem 7.2.** *If $X$ is a smooth vector field on $U$ with $X(p) \neq 0$, then there exists a coordinate system $(u_1, \dots, u_n)$ centered at $p$ (so $u_j(p) = 0$) with respect to which*

$$X = \frac{\partial}{\partial u_n}. \tag{7.13}$$

We now make some elementary comments on vector fields in the plane. Here the object is to find the integral curves of

$$f(x, y) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y}, \tag{7.14}$$

that is, to solve

$$x' = f(x, y), \quad y' = g(x, y). \tag{7.15}$$

This implies

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}, \tag{7.16}$$

or, written in differential-form notation (which will be discussed more thoroughly in §13),

$$g(x, y)\, dx - f(x, y)\, dy = 0. \tag{7.17}$$

Suppose we manage to find an explicit solution to (7.16):

$$y = \varphi(x), \quad x = \psi(y). \tag{7.18}$$

Often it is not feasible to do so, but ODE texts frequently give methods for doing so in some cases. Then the original system becomes

$$x' = f(x, \varphi(x)), \quad y' = g(\psi(y), y). \tag{7.19}$$

In other words, we have reduced ourselves to integrating vector fields on the line. We have

$$\begin{aligned} \int \big[f(x, \varphi(x))\big]^{-1} dx &= t + C_1, \\ \int \big[g(\psi(y), y)\big]^{-1} dy &= t + C_2. \end{aligned} \tag{7.20}$$

If (7.18) can be explicitly achieved, it may be that one integral or the other in (7.20) is easier to evaluate. With either $x$ or $y$ solved as a function of $t$, the other is determined by (7.18).

One case when the planar vector field can be integrated explicitly (locally) is when there is a smooth $u$, with nonvanishing gradient, explicitly given, such that

$$Xu = 0, \tag{7.21}$$

where $X$ is the vector field (7.14). One says $u$ is a *conserved quantity.* In such a case, let $w$ be any smooth function such that $(u, w)$ form a local coordinate system. In this coordinate system,

$$X = b(u, w) \frac{\partial}{\partial w} \tag{7.22}$$

by (7.7), so

$$Xv = 1, \tag{7.23}$$

with

$$v(u, w) = \int_{w_0}^{w} b(u, s)^{-1} \, ds, \tag{7.24}$$

and the local coordinate system $(u, v)$ linearizes $X$.

## Exercises

1. Suppose $h(x, y)$ is homogeneous of degree 0, that is, $h(rx, ry) = h(x, y)$, so $h(x, y) = k(x/y)$. Show that the ODE
$$\frac{dy}{dx} = h(x, y)$$
is changed to a separable ODE for $u = u(x)$, if $u = y/x$.
2. Using Exercise 1, discuss constructing the integral curves of a vector field
$$X = f(x, y) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y}$$

when $f(x, y)$ and $g(x, y)$ are homogeneous of degree $a$, that is,

$$f(rx, ry) = r^a\, f(x, y) \quad \text{for } r > 0,$$

and similarly for $g$.

3. Describe the integral curves of

$$(x^2 + y^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

4. Describe the integral curves of

$$A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$$

when $A(x, y) = a_1x + a_2y + a_3,\ B(x, y) = b_1x + b_2y + b_3$.

5. Let $X = f(x, y)(\partial/\partial x) + g(x, y)(\partial/\partial y)$ be a vector field on a disc $\Omega \subset \mathbb{R}^2$. Suppose that $\operatorname{div} X = 0$, that is, $\partial f/\partial x + \partial g/\partial y = 0$. Show that a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = g, \qquad \frac{\partial u}{\partial y} = -f$$

is given by a line integral. Show that $Xu = 0$, and hence integrate $X$. Reconsider this problem after reading §13.

6. Find the integral curves of the vector field

$$X = (2xy + y^2 + 1)\frac{\partial}{\partial x} + (x^2 + 1 - y^2)\frac{\partial}{\partial y}.$$

7. Show that

$$\operatorname{div}(e^v X) = e^v(\operatorname{div} X + Xv).$$

Hence, if $X$ is a vector field on $\Omega \subset \mathbb{R}^2$, as in Exercise 5, show that you can integrate $X$ if you can construct a function $v(x, y)$ such that $Xv = -\operatorname{div} X$. Construct such $v$ if either

$$\frac{\operatorname{div} X}{f(x, y)} = \varphi(x) \quad \text{or} \quad \frac{\operatorname{div} X}{g(x, y)} = \psi(y).$$

For now, we define $\operatorname{div} X = \partial X_1/\partial x_1 + \cdots + \partial X_n/\partial x_n$. See Chapter 2, §2, for another definition.

8. Find the integral curves of the vector field

$$X = 2xy\frac{\partial}{\partial x} + (x^2 + y^2 - 1)\frac{\partial}{\partial y}.$$

Let $X$ be a vector field on $\mathbb{R}^n$, with a critical point at 0, that is, $X(0) = 0$. Suppose that for $x \in \mathbb{R}^n$ near 0,

$$X(x) = Ax + R(x), \qquad \|R(x)\| = O(\|x\|^2), \tag{7.25}$$

where $A$ is an $n \times n$ matrix. We call $Ax$ the linearization of $X$ at 0.

9. Suppose all the eigenvalues of $A$ have negative real part. Construct a quadratic polynomial $Q : \mathbb{R}^n \to [0, \infty)$, such that $Q(0) = 0$, $\left(\partial^2 Q/\partial x_j \partial x_k\right)$ is positive-definite, and for any integral curve $x(t)$ of $X$ as in (7.25),

$$\frac{d}{dt}Q(x(t)) < 0 \ \text{ if } t \geq 0,$$

provided $x(0) = x_0 (\neq 0)$ is close enough to 0. Deduce that for small enough $C$, if $\|x_0\| \leq C$, then $x(t)$ exists for all $t \geq 0$ and $x(y) \to 0$ as $t \to \infty$. (*Hint*: Take $Q(x) = \langle x, x\rangle$, using Exercise 10 below.)

10. Let $A$ be an $n \times n$ matrix, all of whose eigenvalues $\lambda_j$ have *negative* real part. Show that there exists a Hermitian inner product $\langle\, ,\rangle$ on $\mathbb{C}^n$ such that Re $\langle Au, u\rangle < 0$ for nonzero $u \in \mathbb{C}^n$. (*Hint*: Put $A$ in Jordan normal form, but with $\varepsilon$s instead of 1s above the diagonal, where $\varepsilon$ is small compared with $|\text{Re } \lambda_j|$.)

## 8. Lie brackets

If $F : V \to W$ is a diffeomorphism between two open domains in $\mathbb{R}^n$, or between two smooth manifolds, and $Y$ is a vector field on $W$, we define a vector field $F_{\#}Y$ on $V$ so that

$$\mathcal{F}^t_{F_{\#}Y} = F^{-1} \circ \mathcal{F}^t_Y \circ F, \tag{8.1}$$

or equivalently, by the chain rule,

$$F_{\#}Y(x) = \big(DF^{-1}\big)\big(F(x)\big)Y\big(F(x)\big). \tag{8.2}$$

In particular, if $U \subset \mathbb{R}^n$ is open and $X$ is a vector field on $U$ defining a flow $\mathcal{F}^t$, then for a vector field $Y$, $\mathcal{F}^t_{\#}Y$ is defined on most of $U$, for $|t|$ small, and we can define the Lie derivative,

$$\mathcal{L}_X Y = \lim_{h\to 0} h^{-1}\big(\mathcal{F}^h_{\#}Y - Y\big) = \frac{d}{dt}\mathcal{F}^t_{\#}Y\big|_{t=0}, \tag{8.3}$$

as a vector field on $U$.

Another natural construction is the operator-theoretic bracket:

$$[X, Y] = XY - YX, \tag{8.4}$$

where the vector fields $X$ and $Y$ are regarded as first-order differential operators on $C^\infty(U)$. One verifies that (8.4) defines a derivation on $C^\infty(U)$, hence a vector field on $U$. The basic elementary fact about the Lie bracket is the following.

**Theorem 8.1.** *If $X$ and $Y$ are smooth vector fields, then*

$$\mathcal{L}_X Y = [X, Y]. \tag{8.5}$$

**Proof.** Let us first verify the identity in the special case

$$X = \frac{\partial}{\partial x_1}, \quad Y = \sum b_j(x)\frac{\partial}{\partial x_j}.$$

Then $\mathcal{F}^t_{\#}Y = \sum b_j(x + te_1)\,\partial/\partial x_j$, so $\mathcal{L}_X Y = \sum(\partial b_j/\partial x_1)\,\partial/\partial x_j$, and a straightforward calculation shows that this is also the formula for $[X, Y]$, in this case.

Now we verify (8.5) in general, at any point $x_0 \in U$. First, if $X$ is nonvanishing at $x_0$, we can choose a local coordinate system so the example above gives the identity. By continuity, we get the identity (8.5) on the closure of the set of points $x_0$, where $X(x_0) \neq 0$. Finally, if $x_0$ has a neighborhood where $X = 0$, clearly $\mathcal{L}_X Y = 0$ and $[X, Y] = 0$ at $x_0$. This completes the proof.

**Corollary 8.2.** *If $X$ and $Y$ are smooth vector fields on $U$, then*

$$\frac{d}{dt} \mathcal{F}^t_{X\#} Y = \mathcal{F}^t_{X\#} [X, Y], \tag{8.6}$$

*for all $t$.*

**Proof.** Since locally $\mathcal{F}^{t+s}_X = \mathcal{F}^s_X \mathcal{F}^t_X$, we have the same identity for $\mathcal{F}^{t+s}_{X\#}$, which yields (8.6) upon taking the $s$-derivative.

We make some further comments about cases when one can explicitly integrate a vector field $X$ in the plane, exploiting “symmetries” that may be apparent. In fact, suppose one has in hand a vector field $Y$ such that

$$[X, Y] = 0. \tag{8.7}$$

By (8.6), this implies $\mathcal{F}^t_{Y\#} X = X$ for all $t$; this connection will be pursued further in the next section. Suppose that one has an explicit hold on the flow generated by $Y$, so one can produce explicit local coordinates $(u, v)$ with respect to which

$$Y = \frac{\partial}{\partial u}. \tag{8.8}$$

In this coordinate system, write $X = a(u, v)\partial/\partial u + b(u, v)\partial/\partial v$. The condition (8.7) implies $\partial a/\partial u = 0 = \partial b/\partial u$, so in fact we have

$$X = a(v)\frac{\partial}{\partial u} + b(v)\frac{\partial}{\partial v}. \tag{8.9}$$

Integral curves of (8.9) satisfy

$$u' = a(v), \quad v' = b(v) \tag{8.10}$$

and can be found explicitly in terms of integrals; one has

$$\int b(v)^{-1}\, dv = t + C_1 \tag{8.11}$$

and then

$$u = \int a(v(t))\, dt + C_2. \tag{8.12}$$

More generally than (8.7), we can suppose that, for some constant $c$,

$$[X, Y] = cX, \tag{8.13}$$

which by (8.6) is the same as

$$\mathcal{F}^t_{Y\#}X = e^{-ct}X. \tag{8.14}$$

An example would be

$$X = f(x,y)\,\frac{\partial}{\partial x} + g(x,y)\,\frac{\partial}{\partial y}, \tag{8.15}$$

where $f$ and $g$ satisfy "homogeneity" conditions of the form

$$f(r^a x, r^b y) = r^{a-c} f(x,y), \quad g(r^a x, r^b y) = r^{b-c} g(x,y), \tag{8.16}$$

for $r > 0$; in such a case one can take explicitly

$$\mathcal{F}^t_Y(x,y) = (e^{at}x, e^{bt}y). \tag{8.17}$$

Now, if one again has (8.8) in a local coordinate system $(u,v)$, then $X$ must have the form

$$X = e^{cu}\Big[a(v)\frac{\partial}{\partial u} + b(v)\frac{\partial}{\partial v}\Big], \tag{8.18}$$

which can be explicitly integrated, since

$$u' = e^{cu}a(v),\ v' = e^{cu}b(v) \Longrightarrow \frac{du}{dv} = \frac{a(v)}{b(v)}. \tag{8.19}$$

The hypothesis (8.13) implies that the linear span (over $\mathbb{R}$) of $X$ and $Y$ is a two-dimensional, solvable Lie algebra. Sophus Lie devoted a good deal of effort to examining when one could use constructions of solvable Lie algebras of vector fields to integrate vector fields explicitly; his investigations led to his foundation of what is now called the theory of Lie groups.

## Exercises

1. Verify that the bracket (8.4) satisfies the "Jacobi identity"

$$[X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z],$$

i.e.,

$$[\mathcal{L}_X, \mathcal{L}_Y]Z = \mathcal{L}_{[X,Y]}Z.$$

2. Find the integral curves of

$$X = (x + y^2)\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$$

using (8.16).

3. Find the integral curves of

$$X = (x^2 y + y^5)\,\frac{\partial}{\partial x} + (x^2 + xy^2 + y^4)\,\frac{\partial}{\partial y}.$$

## 9. Commuting flows; Frobenius's theorem

Let $G : U \to V$ be a diffeomorphism. Recall from §8 the action on vector fields:

$$G_{\#}Y(x) = DG(y)^{-1}Y(y), \quad y = G(x). \tag{9.1}$$

As noted there, an alternative characterization of $G_{\#}Y$ is given in terms of the flow it generates. One has

$$\mathcal{F}^t_Y \circ G = G \circ \mathcal{F}^t_{G_{\#}Y}. \tag{9.2}$$

The proof of this is a direct consequence of the chain rule. As a special case, we have the following

**Proposition 9.1.** *If* $G_{\#}Y = Y$, *then* $\mathcal{F}^t_Y \circ G = G \circ \mathcal{F}^t_Y$.

From this, we derive the following condition for a pair of flows to commute. Let $X$ and $Y$ be vector fields on $U$.

**Proposition 9.2.** *If* $X$ *and* $Y$ *commute as differential operators, that is,*

$$[X, Y] = 0, \tag{9.3}$$

*then locally* $\mathcal{F}^s_X$ *and* $\mathcal{F}^t_Y$ *commute; in other words, for any* $p_0 \in U$, *there exists a* $\delta > 0$ *such that for* $|s|, |t| < \delta$,

$$\mathcal{F}^s_X \mathcal{F}^t_Y p_0 = \mathcal{F}^t_Y \mathcal{F}^s_X p_0. \tag{9.4}$$

**Proof.** By Proposition 9.1, it suffices to show that $\mathcal{F}^s_{X\#}Y = Y$. This clearly holds at $s = 0$. But by (8.6), we have

$$\frac{d}{ds}\mathcal{F}^s_{X\#}Y = \mathcal{F}^s_{X\#}[X, Y],$$

which vanishes if (9.3) holds. This finishes the proof.

We have stated that given (9.3), the identity (9.4) holds locally. If the flows generated by $X$ and $Y$ are not complete, this can break down globally. For example, consider $X = \partial/\partial x_1$, $Y = \partial/\partial x_2$ on $\mathbb{R}^2$, which satisfy (9.3) and generate commuting flows. These vector fields lift to vector fields on the universal covering surface $\tilde{M}$ of $\mathbb{R}^2 \setminus (0, 0)$, which continue to satisfy (9.3). The flows on $\tilde{M}$ do not commute globally. This phenomenon does not arise, for example, for vector fields on a compact manifold.

We now consider when a family of vector fields has a multidimensional integral manifold. Suppose $X_1, \dots, X_k$ are smooth vector fields on $U$ which are linearly independent at each point of a $k$-dimensional surface $\Sigma \subset U$. If each $X_j$ is tangent to $\Sigma$ at each point, $\Sigma$ is said to be an integral manifold of $(X_1, \dots, X_k)$.

**Proposition 9.3.** *Suppose $X_1, \dots, X_k$ are linearly independent at each point of $U$ and $[X_j, X_\ell] = 0$ for all $j, \ell$. Then, for each $x_0 \in U$, there is a $k$-dimensional integral manifold of $(X_1, \dots, X_k)$ containing $x_0$.*

**Proof.** We define a map $F : V \to U$, $V$ a neighborhood of 0 in $\mathbb{R}^k$, by

$$F(t_1, \dots, t_k) = \mathcal{F}_{X_1}^{t_1} \cdots \mathcal{F}_{X_k}^{t_k} x_0. \tag{9.5}$$

Clearly, $(\partial/\partial t_1)F = X_1(F)$. Similarly, since $\mathcal{F}_{X_j}^{t_j}$ all commute, we can put any $\mathcal{F}_{X_j}^{t_j}$ first and get $(\partial/\partial t_j)F = X_j(F)$. This shows that the image of $V$ under $F$ is an integral manifold containing $x_0$.

We now derive a more general condition guaranteeing the existence of integral submanifolds. This important result is due to Frobenius. We say $(X_1, \dots, X_k)$ is *involutive* provided that, for each $j, \ell$, there are smooth $b_m^{j\ell}(x)$ such that

$$[X_j, X_\ell] = \sum_{m=1}^{k} b_m^{j\ell}(x) X_m. \tag{9.6}$$

The following is Frobenius's theorem.

**Theorem 9.4.** *If $(X_1, \dots, X_k)$ are $C^\infty$ vector fields on $U$, linearly independent at each point, and the involutivity condition (9.6) holds, then through each $x_0$ there is, locally, a unique integral manifold $\Sigma$, of dimension $k$.*

We will give two proofs of this result. First, let us restate the conclusion as follows. There exist local coordinates $(y_1, \dots, y_n)$ centered at $x_0$ such that

$$\operatorname{span}(X_1, \dots, X_k) = \operatorname{span}\Big(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k}\Big). \tag{9.7}$$

**First proof.** The result is clear for $k = 1$. We will use induction on $k$. So let the set of vector fields $X_1, \dots, X_{k+1}$ be linearly independent at each point and involutive. Choose a local coordinate system so that $X_{k+1} = \partial/\partial u_1$. Now let

$$Y_j = X_j - (X_j u_1)\,\frac{\partial}{\partial u_1} \quad \text{for } 1 \le j \le k, \quad Y_{k+1} = \frac{\partial}{\partial u_1}. \tag{9.8}$$

Since in $(u_1, \dots, u_n)$ coordinates, no $Y_1, \dots, Y_k$ involves $\partial/\partial u_1$, neither does any Lie bracket, so

$$[Y_j, Y_\ell] \in \operatorname{span}(Y_1, \dots, Y_k), \quad j, \ell \le k.$$

Thus $(Y_1, \dots, Y_k)$ is involutive. The induction hypothesis implies that there exist local coordinates $(y_1, \dots, y_n)$ such that

$$\operatorname{span}(Y_1, \dots, Y_k) = \operatorname{span}\Big(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_k}\Big).$$

Now let

$$Z = Y_{k+1} - \sum_{\ell=1}^{k} (Y_{k+1} y_\ell) \frac{\partial}{\partial y_\ell} = \sum_{\ell > k} (Y_{k+1} y_\ell) \frac{\partial}{\partial y_\ell}. \tag{9.9}$$

Since, in the $(u_1, \dots, u_n)$ coordinates, $Y_1, \dots, Y_k$ do not involve $\partial/\partial u_1$, we have

$$[Y_{k+1}, Y_j] \in \operatorname{span}(Y_1, \dots, Y_k).$$

Thus $[Z, Y_j] \in \operatorname{span}(Y_1, \dots, Y_k)$ for $j \le k$, while (9.9) implies that $[Z, \partial/\partial y_j]$ belongs to the span of $(\partial/\partial y_{k+1}, \dots, \partial/\partial y_n)$, for $j \le k$. Thus we have

$$\Big[Z, \frac{\partial}{\partial y_j}\Big] = 0, \quad j \le k.$$

Proposition 9.3 implies span $(\partial/\partial y_1, \dots, \partial/\partial y_k, Z)$ has an integral manifold through each point, and since this span is equal to the span of $X_1, \dots, X_{k+1}$, the first proof is complete.

**Second proof.** Let $X_1, \dots, X_k$ be $C^\infty$ vector fields, linearly independent at each point and satisfying the condition (9.6). Choose an $(n-k)$-dimensional surface $\mathcal{O} \subset U$, transverse to $X_1, \dots, X_k$. For $V$ a neighborhood of the origin in $\mathbb{R}^k$, define $\Phi : V \times \mathcal{O} \to U$ by

$$\Phi(t_1, \dots, t_k, x) = \mathcal{F}^{t_1}_{X_1} \cdots \mathcal{F}^{t_k}_{X_k} x. \tag{9.10}$$

We claim that, for $x$ fixed, the image of $V$ in $U$ is a $k$-dimensional surface $\Sigma$ tangent to each $X_j$, at each point of $\Sigma$. Note that since $\Phi(0, \dots, t_j, \dots, 0, x) = \mathcal{F}^{t_j}_{X_j} x$, we have

$$\frac{\partial}{\partial t_j} \Phi(0, \dots, 0, x) = X_j(x), \quad x \in \mathcal{O}. \tag{9.11}$$

To establish the claim, it suffices to show that $\mathcal{F}^t_{X_j \#} X_\ell$ is a linear combination with coefficients in $C^\infty(U)$ of $X_1, \dots, X_k$. This is accomplished by the following:

**Lemma 9.5.** *Suppose* $[Y, X_j] = \sum_\ell \lambda_{j\ell}(x) X_\ell$, *with smooth coefficients* $\lambda_{j\ell}(x)$. *Then* $\mathcal{F}^t_{Y\#} X_j$ *is a linear combination of* $X_1, \dots, X_k$, *with coefficients in* $C^\infty(U)$.

**Proof.** Denote by $\Lambda$ the matrix $(\lambda_{j\ell})$, and let $\Lambda(t) = \Lambda(t,x) = (\lambda_{j\ell}(\mathcal{F}_Y^t x))$. Now let $A(t) = A(t,x)$ be the unique solution to the ODE

$$A'(t) = \Lambda(t)A(t), \quad A(0) = I. \tag{9.12}$$

Write $A = (\alpha_{j\ell})$. We claim that

$$\mathcal{F}^t_{Y\#} X_j = \sum_\ell \alpha_{j\ell}(t,x) X_\ell. \tag{9.13}$$

This formula will prove the lemma. Indeed, we have

$$\begin{aligned} \frac{d}{dt}\, (\mathcal{F}_Y^t)_\# X_j &= (\mathcal{F}_Y^t)_\# [Y, X_j] \\ &= (\mathcal{F}_Y^t)_\# \sum_\ell \lambda_{j\ell} X_\ell \\ &= \sum_\ell (\lambda_{j\ell} \circ \mathcal{F}_Y^t)(\mathcal{F}^t_{Y\#} X_\ell). \end{aligned}$$

Uniqueness of the solution to (9.12) gives (9.13), and we are done.

This completes the second proof of Frobenius's theorem.

## Exercises

1. Let $\Omega$ be open in $\mathbb{R}^{2n}$, identified with $\mathbb{C}^n$ via $z = x + iy$. Let
$$X = \sum \left[ a_j(x,y) \frac{\partial}{\partial x_j} + b_j(x,y) \frac{\partial}{\partial y_j} \right]$$
be a vector field on $\Omega$, where $a_j(x,y)$ and $b_j(x,y)$ are real-valued. Form $f_j(z) = a_j(z) + i b_j(z)$. Consider the vector field
$$Y = JX = \sum_j \left[ -b_j(x,y) \frac{\partial}{\partial x_j} + a_j(x,y) \frac{\partial}{\partial y_j} \right].$$
Show that $X$ and $Y$ commute, that is, $[X,Y] = 0$, provided $f(z)$ is holomorphic, namely if the Cauchy-Riemann equations hold:
$$\frac{\partial a_j}{\partial x_k} = \frac{\partial b_j}{\partial y_k}, \quad \frac{\partial a_j}{\partial y_k} = -\frac{\partial b_j}{\partial x_k}.$$
2. Assuming $f_j(z) = a_j(z) + i b_j(z)$ are holomorphic, show that, for $z \in \Omega$,
$$z(t,s) = \mathcal{F}_X^t \mathcal{F}_Y^s z$$
satisfies $\partial z/\partial s = J \partial z/\partial t$, and hence that $z(t,s)$ is holomorphic in $t + is$.
3. Suppose $a_j(x)$ are real analytic (and real-valued) on $\mathcal{O} \subset \mathbb{R}^n$. Let $X = \sum a_j(x) \partial/\partial x_j$. Show that, for $x \in \mathcal{O}$, $x(t) = \mathcal{F}_X^t x$ is real analytic in $t$ (for $t$ near 0), by applying Exercises 1 and 2.
Compare the proof of this indicated in Exercise 2 of §6.
4. Discuss the *uniqueness* of integral manifolds arising in Theorem 9.4.

5. Let $A_j$ be smooth $m \times m$ matrix-valued functions on $\mathcal{O} \subset \mathbb{R}^n$. Suppose the operators $L_j = \partial/\partial x_j + A_j(x)$, acting on functions with values in $\mathbb{R}^m$, all commute, $1 \le j \le n$. If $p \in \mathcal{O}$, show that there is a solution in a neighborhood of $p$ to
$$L_j u = 0, \quad 1 \le j \le n,$$
with $u(p) \in \mathbb{R}^m$ prescribed.

## 10. Hamiltonian systems

Hamiltonian systems arise from classical mechanics. As a most basic example, consider the equations of motion that arise from Newton's law $F = ma$, where the force $F$ is given by

$$F = -\operatorname{grad} V(x), \tag{10.1}$$

with $V$ the potential energy. We get the ODE

$$m\,\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}. \tag{10.2}$$

We can convert this into a first-order system for $(x, \xi)$, where

$$\xi = m\,\frac{dx}{dt} \tag{10.3}$$

is the momentum. We have

$$\frac{dx}{dt} = \frac{\xi}{m}, \quad \frac{d\xi}{dt} = -\frac{\partial V}{\partial x}. \tag{10.4}$$

Now consider the total energy

$$f(x,\xi) = \frac{1}{2m}\,|\xi|^2 + V(x). \tag{10.5}$$

Note that $\partial f/\partial \xi = \xi/m$ and $\partial f/\partial x = \partial V/\partial x$. Thus (10.4) is of the form

$$\frac{dx_j}{dt} = \frac{\partial f}{\partial \xi_j}, \quad \frac{d\xi_j}{dt} = -\frac{\partial f}{\partial x_j}. \tag{10.6}$$

Hence we're looking for the integral curves of the vector field

$$H_f = \sum_{j=1}^{n} \Big[\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big]. \tag{10.7}$$

For smooth $f(x,\xi)$, we call $H_f$, defined by (10.7), a Hamiltonian vector field. Note that, directly from (10.7),

$$H_f f = 0. \tag{10.8}$$

A useful notation is the Poisson bracket, defined by

$$\{f, g\} = H_f g. \tag{10.9}$$

One verifies directly from (10.7) that

$$\{f, g\} = -\{g, f\}, \tag{10.10}$$

generalizing (10.8). Also, a routine calculation verifies that

$$[H_f, H_g] = H_{\{f,g\}}. \tag{10.11}$$

As noted at the end of §7, if $X$ is a vector field in the plane and we explicitly have a function $u$ with nonvanishing gradient such that $Xu = 0$, then $X$ can be explicitly integrated. These comments apply to $X = H_f,\ u = f$, when $H_f$ is a planar Hamiltonian vector field. We can rephrase this description as follows. If $x \in \mathbb{R}, \xi \in \mathbb{R}$, then integral curves of

$$x' = \frac{\partial f}{\partial \xi}, \quad \xi' = -\frac{\partial f}{\partial x} \tag{10.12}$$

lie on a level set

$$f(x, \xi) = E. \tag{10.13}$$

Suppose that locally this set is described by

$$x = \varphi(\xi) \quad \text{or} \quad \xi = \psi(x). \tag{10.14}$$

Then we have one of the following ODEs:

$$x' = f_\xi(x, \psi(x)) \quad \text{or} \quad \xi' = -f_x(\varphi(\xi), \xi), \tag{10.15}$$

and hence we have

$$\int f_\xi\bigl(x, \psi(x)\bigr)^{-1} dx = t + C \tag{10.16}$$

or

$$-\int f_x\bigl(\varphi(\xi), \xi\bigr)^{-1} d\xi = t + C'. \tag{10.17}$$

Thus, solving (10.12) is reduced to a quadrature, that is, a calculation of an explicit integral, (10.16) or (10.17).

If the planar Hamiltonian vector field $H_f$ arises from describing motion in a force field on a line, via Newton's laws given in (10.2), so that

$$f(x, \xi) = \frac{1}{2m}\,\xi^2 + V(x), \tag{10.18}$$

then the second curve in (10.14) is

$$\xi = \pm\bigl[(2m)\bigl(E - V(x)\bigr)\bigr]^{1/2}, \tag{10.19}$$

and the formula (10.16) becomes

$$\pm\Bigl(\frac{m}{2}\Bigr)^{1/2} \int \bigl[E - V(x)\bigr]^{-1/2}\, dx = t + C, \tag{10.20}$$

defining $x$ implicitly as a function of $t$.

In some cases, the integral in (10.20) can be evaluated by elementary means. This includes the trivial case of a constant force, where $V(x) = cx$, and also the case of the "harmonic oscillator" or linearized spring, where $V(x) = cx^2$. It also includes the case of the motion of a rocket in space, along a line through the center of a planet, where $V(x) = -K/|x|$. This gravitational attraction problem for motion in several-dimensional space will be studied further in §§16 and 17. The case $V(x) = -K \cos x$ arises in the analysis of the pendulum (see (12.38)). In that case, (10.20) is an elliptic integral, rather than one that arises in first-year calculus.

For Hamiltonian vector fields in higher dimensions, more effort is required to understand the resulting flows. The notion of complete integrability provides a method of constructing explicit solutions in some cases, as will be discussed in §§16 and 17.

Hamiltonian vector fields arise in the treatment of many problems in addition to those derived from Newton's laws in Cartesian coordinates. In §11 we study the equations of geodesics and then show how they can be transformed to Hamiltonian systems. In §12 this is seen to be a special case of a broad class of variational problems, which lead to Hamiltonian systems, and which also encompass classical mechanics. This variational approach has many convenient features, such as allowing an easy formulation of the equations of motion in arbitrary coordinate systems, a theme that will be developed in a number of subsequent sections.

## Exercises

1. Verify that $[H_f, H_g] = H_{\{f,g\}}$.
2. Demonstrate that the Poisson bracket satisfies the Jacobi identity

$$\{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\}. \tag{10.21}$$

(*Hint*: Use Exercise 1 above and Exercise 1 of §8.)

3. Identifying $y$ and $\xi$, show that a planar vector field $X = f(x, y)(\partial/\partial x) + g(x, y)(\partial/\partial y)$ is Hamiltonian if and only if $\operatorname{div} X = 0$.
Reconsider Exercise 5 in §7.
4. Show that

$$\frac{d}{dt}\, g(x, \xi) = \{f, g\}$$

on an orbit of $H_f$.

5. If $X = \sum X_j(x)\partial/\partial x_j$ is a vector field on $U \subset \mathbb{R}^n$, associate to $X$ a function on $U \times \mathbb{R}^n \approx T^*U$:

$$s_X(x, \xi) = \langle X, \xi \rangle = \sum \xi_j X_j(x). \tag{10.22}$$

Show that

$$s_{[X,Y]} = \{s_X, s_Y\}. \tag{10.23}$$

## 11. Geodesics

Here we define the concept of a geodesic on a region with a Riemannian metric (more generally, a Riemannian manifold). A Riemannian metric on $\Omega \subset \mathbb{R}^n$ is specified by $g_{jk}(x)$, where $(g_{jk})$ is a positive-definite, smooth, $n \times n$ matrix-valued function on $\Omega$. If $U = \sum u^j(x)\partial/\partial x_j$ and $V = \sum v^j(x)\partial/\partial x_j$ are two vector fields on $\Omega$, their inner product is the smooth scalar function

$$\langle U, V\rangle = g_{jk}(x)\ u^j(x)v^k(x), \tag{11.1}$$

using the summation convention (i.e., summing over repeated indices). If $\Omega$ is a manifold, a Riemannian metric is an inner product on each tangent space $T_x\Omega$, given in local coordinates by (11.1). Thus, $(g_{jk})$ gives rise to a tensor field of type $(0, 2)$, that is, a section of the bundle $\otimes^2 T^*\Omega$.

If $\gamma(t),\ a \le t \le b$, is a smooth curve on $\Omega$, its length is

$$L = \int_a^b \|\gamma'(t)\|\ dt = \int_a^b \Big[g_{jk}(\gamma(t))\gamma_j'(t)\gamma_k'(t)\Big]^{1/2}\ dt. \tag{11.2}$$

A curve $\gamma$ is said to be a geodesic if, for $|t_1 - t_2|$ sufficiently small, $t_j \in [a, b]$, the curve $\gamma(t),\ t_1 \le t \le t_2$, has the shortest length of all smooth curves in $\Omega$ from $\gamma(t_1)$ to $\gamma(t_2)$.

We derive the ODE for a geodesic. We start with the case where $\Omega$ has the metric induced from a diffeomorphism $\Omega \to S,\ S$ a hypersurface in $\mathbb{R}^{n+1}$; we will identify $\Omega$ and $S$ here. This short computation will serve as a guide for the general case.

So let $\gamma_0(t)$ be a smooth curve in $S$ $(a \le t \le b)$, joining $p$ and $q$. Suppose $\gamma_s(t)$ is a smooth family of such curves. We look for a condition guaranteeing that $\gamma_0(t)$ has minimum length. Since the length of a curve is independent of its parameterization, we may additionally suppose that

$$\|\gamma_0'(t)\| = c_0, \quad \text{constant, for} \ \ a \le t \le b. \tag{11.3}$$

Let $N$ denote a field of normal vectors to $S$. Note that

$$V = \frac{\partial}{\partial s}\gamma_s(t) \perp N. \tag{11.4}$$

Also, any vector field $V \perp N$ over the image of $\gamma_0$ can be obtained by some variation $\gamma_s$ of $\gamma_0$, provided $V = 0$ at $p$ and $q$. Recall that we are assuming $\gamma_s(a) = p,\ \gamma_s(b) = q$. If $L(s)$ denotes the length of $\gamma_s$, we have

$$L(s) = \int_a^b \|\gamma_s'(t)\|\ dt, \tag{11.5}$$

and hence

$$
\begin{aligned}
L'(s) &= \frac{1}{2}\int_a^b \|\gamma_s'(t)\|^{-1}\frac{\partial}{\partial s}\big(\gamma_s'(t),\gamma_s'(t)\big)\;dt \\
&= \frac{1}{c_0}\int_a^b \Big(\frac{\partial}{\partial s}\,\gamma_s'(t),\gamma_s'(t)\Big)\;dt, \quad \text{at } s=0.
\end{aligned}
\tag{11.6}
$$

Using the identity

$$
\frac{d}{dt}\Big(\frac{\partial}{\partial s}\,\gamma_s(t),\gamma_s'(t)\Big) = \Big(\frac{\partial}{\partial s}\,\gamma_s'(t),\gamma_s'(t)\Big) + \Big(\frac{\partial}{\partial s}\,\gamma_s(t),\gamma_s''(t)\Big),
\tag{11.7}
$$

together with the fundamental theorem of calculus, in view of the fact that

$$
\frac{\partial}{\partial s}\,\gamma_s(t) = 0, \quad \text{at } t=a \text{ and } b,
\tag{11.8}
$$

we have

$$
L'(s) = -\frac{1}{c_0}\int_a^b \big(V(t),\gamma_s''(t)\big)\;dt, \quad \text{at } s=0.
\tag{11.9}
$$

Now, if $\gamma_0$ were a geodesic, we would have

$$
L'(0) = 0,
\tag{11.10}
$$

for all such variations. In other words, we must have $\gamma_0''(t) \perp V$ for all vector fields $V$ tangent to $S$ (and vanishing at $p$ and $q$), and hence

$$
\gamma_0''(t)\|N.
\tag{11.11}
$$

This vanishing of the tangential curvature of $\gamma_0$ is the usual geodesic equation for a hypersurface in $\mathbb{R}^{n+1}$.

We proceed to derive from (11.11) an ODE in standard form. Suppose $S$ is defined locally by $u(x) = C$, $\nabla u \neq 0$. Then (11.11) is equivalent to

$$
\gamma_0''(t) = K\nabla u(\gamma_0(t)),
\tag{11.12}
$$

for a scalar $K$ that remains to be determined. But the condition that $u(\gamma_0(t)) = C$ implies

$$
\gamma_0'(t)\cdot\nabla u(\gamma_0(t)) = 0,
$$

and differentiating this gives

$$
\gamma_0''(t)\cdot\nabla u(\gamma_0(t)) = -\gamma_0'(t)\cdot D^2u(\gamma_0(t))\cdot\gamma_0'(t),
\tag{11.13}
$$

where $D^2u$ is the matrix of second-order partial derivatives of $u$. Comparing (11.12) and (11.13) gives $K$, and we obtain the ODE

$$
\gamma_0''(t) = -\Big|\nabla u\big(\gamma_0(t)\big)\Big|^{-2}\Big[\gamma_0'(t)\cdot D^2u\big(\gamma_0(t)\big)\cdot\gamma_0'(t)\Big]\nabla u\big(\gamma_0(t)\big)
\tag{11.14}
$$

for a geodesic $\gamma_0$ lying in $S$.

We now want to parallel (11.6)–(11.11), to provide the ODE for a geodesic on $\Omega$ with a general Riemannian metric. As before, let $\gamma_s(t)$ be a one-parameter family of curves satisfying $\gamma_s(a) = p, \ \gamma_s(b) = q$, and (11.3). Then

$$V = \frac{\partial}{\partial s}\,\gamma_s(t)\big|_{s=0} \tag{11.15}$$

is a vector field defined on the curve $\gamma_0(t)$, vanishing at $p$ and $q$, and a general vector field of this sort could be obtained by a variation $\gamma_s(t)$. Let

$$T = \gamma_s'(t). \tag{11.16}$$

With the notation of (11.1), we have, parallel to (11.6),

$$\begin{aligned} L'(s) &= \int_a^b V\langle T, T\rangle^{1/2}\, dt \\ &= \frac{1}{2c_0}\int_a^b V\langle T, T\rangle\, dt, \quad \text{at } s = 0. \end{aligned} \tag{11.17}$$

Now we need a generalization of $(\partial/\partial s)\gamma_s'(t)$ and of the formula (11.7). One natural approach involves the notion of a *covariant derivative.*

If $X$ and $Y$ are vector fields on $\Omega$, the covariant derivative $\nabla_X Y$ is a vector field on $\Omega$. The following properties are to hold: We assume that $\nabla_X Y$ is additive in both $X$ and $Y$, that

$$\nabla_{fX} Y = f\nabla_X Y, \tag{11.18}$$

for $f \in C^\infty(\Omega)$, and that

$$\nabla_X (fY) = f\nabla_X Y + (Xf)Y \tag{11.19}$$

(i.e., $\nabla_X$ acts as a derivation). The operator $\nabla_X$ is required to have the following relation to the Riemannian metric:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{11.20}$$

One further property, called the "zero torsion condition," will uniquely specify $\nabla$:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{11.21}$$

If these properties hold, one says that $\nabla$ is a "Levi-Civita connection." We have the following existence result.

**Proposition 11.1.** *Associated with a Riemannian metric is a unique Levi-Civita connection, given by*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &+ \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle. \end{aligned} \tag{11.22}$$

**Proof.** To obtain the formula (11.22), cyclically permute $X, \ Y$, and $Z$ in (11.20) and take the appropriate alternating sum, using (11.21) to cancel out all terms involving $\nabla$ but two copies of $\langle \nabla_X Y, Z\rangle$. This derives the formula and establishes uniqueness. On the other hand, if (11.22) is taken as the definition of $\nabla_X Y$, then verification of the properties (11.18)–(11.21) is a routine exercise.

We can resume our analysis of (11.17), which becomes

$$L'(s) = \frac{1}{c_0} \int_a^b \langle \nabla_V T, T\rangle \, dt, \quad \text{at } s = 0. \tag{11.23}$$

Since $\partial/\partial s$ and $\partial/\partial t$ commute, we have $[V, T] = 0$ on $\gamma_0$, and (11.21) implies

$$L'(s) = \frac{1}{c_0} \int_a^b \langle \nabla_T V, T\rangle \, dt, \quad \text{at } s = 0. \tag{11.24}$$

The replacement for (11.7) is

$$T\langle V, T\rangle = \langle \nabla_T V, T\rangle + \langle V, \nabla_T T\rangle, \tag{11.25}$$

so, by the fundamental theorem of calculus,

$$L'(0) = -\frac{1}{c_0} \int_a^b \langle V, \nabla_T T\rangle \, dt. \tag{11.26}$$

If this is to vanish for all smooth vector fields over $\gamma_0$, vanishing at $p$ and $q$, we must have

$$\nabla_T T = 0. \tag{11.27}$$

This is the geodesic equation for a general Riemannian metric.

If $\Omega \subset \mathbb{R}^n$ carries a Riemannian metric $g_{jk}(x)$ and a corresponding Levi-Civita connection, the *Christoffel symbols* $\Gamma^k{}_{ij}$ are defined by

$$\nabla_{D_i} D_j = \sum_k \Gamma^k{}_{ji} D_k, \tag{11.28}$$

where $D_k = \partial/\partial x_k$. The formula (11.22) implies

$$g_{k\ell}\Gamma^\ell{}_{ij} = \frac{1}{2}\Big[\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\Big]. \tag{11.29}$$

We can rewrite the geodesic equation (11.27) for $\gamma_0(t) = x(t)$ as follows. With $x = (x_1, \dots, x_n)$ and $T = (\dot{x}^1, \dots, \dot{x}^n)$, we have

$$0 = \sum_\ell \nabla_T\big(\dot{x}^\ell D_\ell\big) = \sum_\ell \big[\ddot{x}^\ell D_\ell + \dot{x}^\ell \nabla_T D_\ell\big]. \tag{11.30}$$

In view of (11.28), this becomes

$$\ddot{x}^\ell + \dot{x}^j \, \dot{x}^k \, \Gamma^\ell{}_{jk} = 0 \tag{11.31}$$

(with the summation convention). The standard existence and uniqueness theory applies to this system of second-order ODE. We will call any smooth curve satisfying the equation (11.27), or equivalently (11.31), a geodesic. Shortly we will verify that such a curve is indeed locally length-minimizing. Note that if $T = \gamma'(t)$, then $T\langle T, T\rangle = 2\langle \nabla_T T, T\rangle$; so if (11.27) holds, $\gamma(t)$ automatically has constant speed.

For a given $p \in \Omega$, the exponential map

$$\operatorname{Exp}_p : U \longrightarrow \Omega \tag{11.32}$$

is defined on a neighborhood $U$ of $0 \in \mathbb{R}^n = T_p\Omega$ by

$$\operatorname{Exp}_p(v) = \gamma_v(1), \tag{11.33}$$

where $\gamma_v(t)$ is the unique constant-speed geodesic satisfying

$$\gamma_v(0) = p, \quad \gamma_v'(0) = v. \tag{11.34}$$

Note that $\operatorname{Exp}_p(tv) = \gamma_v(t)$. It is clear that $\operatorname{Exp}_p$ is well defined and $C^\infty$ on a sufficiently small neighborhood $U$ of $0 \in \mathbb{R}^n$, and its derivative at 0 is the identity. Thus, perhaps shrinking $U$, we have that $\operatorname{Exp}_p$ is a diffeomorphism of $U$ onto a neighborhood $\mathcal{O}$ of $p$ in $\Omega$. This provides what is called an exponential coordinate system, or a normal coordinate system. Clearly, the geodesics through $p$ are the lines through the origin in this coordinate system. We claim that in this coordinate system

$$\Gamma^\ell{}_{jk}(p) = 0. \tag{11.35}$$

Indeed, since the line through the origin in any direction $aD_j + bD_k$ is a geodesic, we have

$$\nabla_{(aD_j + bD_k)}(aD_j + bD_k) = 0, \quad \text{at } p, \tag{11.36}$$

for all $a, b \in \mathbb{R}$ and all $j, k$. This implies

$$\nabla_{D_j} D_k = 0, \quad \text{at } p \text{ for all } j, k, \tag{11.37}$$

which implies (11.35). We note that (11.35) implies $\partial g_{jk}/\partial x_\ell = 0$ at $p$, in this exponential coordinate system. In fact, a simple manipulation of (11.29) gives

$$\frac{\partial g_{jk}}{\partial x_\ell} = g_{mk}\Gamma^m{}_{j\ell} + g_{mj}\Gamma^m{}_{k\ell}. \tag{11.38}$$

As a consequence, a number of calculations in differential geometry can be simplified by working in exponential coordinate systems.

We now establish a result, known as the *Gauss lemma*, which implies that a geodesic is locally length-minimizing. For $a$ small, let $\Sigma_a = \{v \in \mathbb{R}^n : \|v\| = a\}$, and let $S_a = \operatorname{Exp}_p(\Sigma_a)$.

**Proposition 11.2.** *Any unit-speed geodesic through $p$ hitting $S_a$ at $t = a$ is orthogonal to $S_a$.*

**Proof.** If $\gamma_0(t)$ is a unit-speed geodesic, $\gamma_0(0) = p$, $\gamma_0(a) = q \in S_a$, and $V \in T_q\Omega$ is tangent to $S_a$, there is a smooth family of unit-speed geodesics,

$\gamma_s(t)$, such that $\gamma_s(0) = p$ and $(\partial/\partial s)\gamma_s(a)\big|_{s=0} = V$. Using (11.24) and (11.25) for this family, with $0 \le t \le a$, since $L(s)$ is constant, we have

$$0 = \int_0^a T\langle V, T\rangle \, dt = \langle V, \gamma_0'(a)\rangle,$$

which proves the proposition.

Though a geodesic is locally length-minimizing, it need not be globally length-minimizing. There are many simple examples of this, some of which are discussed in the exercises.

We next consider a "naive" alternative to the calculations (11.17)–(11.31), not bringing in the notion of covariant derivative, in order to compute $L'(0)$ when $L(s)$ is given by

$$L(s) = \int_a^b \Big[ g_{jk}\big(x_s(t)\big)\dot{x}_s^j(t) \; \dot{x}_s^k(t)\Big]^{1/2} \, dt. \tag{11.39}$$

We use the notation $T^j = \dot{x}_0^j(t)$, $V^j = (\partial/\partial s)x_s^j(t)|_{s=0}$. Calculating in a spirit similar to that of (11.6), we have (with $x = x_0$)

$$L'(0) = \frac{1}{c_0} \int_a^b \Big[ g_{jk} \frac{\partial}{\partial s} \dot{x}_s^j(t)\big|_{s=0} T^k + \frac{1}{2} V^j \frac{\partial g_{k\ell}}{\partial x_j} T^k T^\ell \Big] \, dt. \tag{11.40}$$

Now, in analogy with (11.7), and in place of (11.25), we can write

$$\frac{d}{dt}\Big( g_{jk}\big(x(t)\big) V^j T^k\Big) = g_{jk} \frac{\partial}{\partial s} \dot{x}_s^j(t)\big|_{s=0} T^k + g_{jk} V^j \ddot{x}^k(t) + T^\ell \frac{\partial g_{jk}}{\partial x_\ell} V^j T^k. \tag{11.41}$$

Thus, by the fundamental theorem of calculus,

$$L'(0) = -\frac{1}{c_0} \int_a^b \Big[ g_{jk} V^j \ddot{x}^k + T^\ell \frac{\partial g_{jk}}{\partial x_\ell} V^j T^k - \frac{1}{2} V^j \frac{\partial g_{k\ell}}{\partial x_j} T^k T^\ell \Big] \, dt, \tag{11.42}$$

and the stationary condition $L'(0) = 0$ for all variations of the form described before implies

$$g_{jk} \ddot{x}^k(t) = -\Big( \frac{\partial g_{jk}}{\partial x_\ell} - \frac{1}{2} \frac{\partial g_{k\ell}}{\partial x_j} \Big) T^k T^\ell. \tag{11.43}$$

Symmetrizing the quantity in parentheses with respect to $k$ and $\ell$ yields the ODE (11.31), with ${\Gamma^\ell}_{jk}$ given by (11.29).

Of the two derivations for the equations of (constant-speed) geodesics given in this section, the latter is a bit shorter and more direct. On the other hand, the slight additional complication of the first derivation paid for the introduction of the notion of covariant derivative, a fundamental object in differential geometry. As we will see in the next section, the methods of the second derivation are very flexible; there we consider a class of extremal problems, containing the problem of geodesics, and also containing problems giving rise to the equations of classical physics, via the stationary action principle.

We now show that the geodesic flow equations can be transformed to a Hamiltonian system. Let $(g^{jk})$ denote the matrix inverse of $(g_{jk})$, and relate $v \in \mathbb{R}^n$ to $\xi \in \mathbb{R}^n$ by

$$\xi_j = g_{jk}(x)v_k, \quad \text{i.e., } v_j = g^{jk}(x)\xi_k. \tag{11.44}$$

Define $f(x,\xi)$ on $\Omega \times \mathbb{R}^n$ by

$$f(x,\xi) = \frac{1}{2} g^{jk}(x)\xi_j\xi_k, \tag{11.45}$$

as before using the summation convention. For a manifold $M$, (11.44) is a local coordinate expression of the Riemannian metric tensor, providing an isomorphism of $T_xM$ with $T_x^*M$, and (11.45) defines half the square norm on $T^*M$. Then the integral curves $(x(t),\xi(t))$ of $H_f$ satisfy

$$\dot{x}_\ell = g^{\ell k}(x)\xi_k, \quad \dot{\xi}_\ell = -\frac{1}{2}\frac{\partial g^{jk}}{\partial x_\ell}\xi_j\xi_k. \tag{11.46}$$

If we differentiate the first equation and plug in the second one for $\dot{\xi}_k$, we get

$$\ddot{x}_\ell = \sum \Big[ -\frac{1}{2} g^{\ell j}\frac{\partial g^{ik}}{\partial x_j} + g^{kj}\frac{\partial g^{i\ell}}{\partial x_j} \Big] \xi_i\xi_k, \tag{11.47}$$

and using $\xi_j = \sum g_{jk}(x)\dot{x}_k$, straightforward manipulations yield the geodesic equation (11.31), with ${\Gamma^\ell}_{jk}$ given by (11.29).

We now describe a relatively noncomputational approach to the result just obtained. Identifying $(x,v)$-space and $(x,\xi)$-space via (11.44), let $Y$ be the resulting vector field on $(x,\xi)$-space defined by the geodesic flow. The result we want to reestablish is that $Y$ and $H_f$ coincide at an arbitrary point $(x_0,\xi_0) \in \Omega \times \mathbb{R}^n$. We will make use of an exponential coordinate system centered at $x_0$; recall that in this coordinate system the geodesics through $x_0$ become precisely the lines through the origin. (Of course, geodesics through nearby points are not generally straight lines in this coordinate system.) In such a coordinate system, we can arrange $g^{jk}(x_0) = \delta^{jk}$ and, by (11.35), $(\partial g^{jk}/\partial x_\ell)(x_0) = 0$. Thus, if $\xi_0 = (a_1,\dots,a_n)$, using (11.46) we have

$$H_f(x_0,\xi_0) = \sum a_k \frac{\partial}{\partial x_k} = Y(x_0,\xi_0) \tag{11.48}$$

in this coordinate system. The identity of $H_f$ and $Y$ at $(x_0,\xi_0)$ is independent of the coordinate system used, so our result is again established. Actually, there is a little cheat here. We have not shown that $H_f$ is defined independently of the choice of coordinates on $\Omega$. This will be established in §14; see (14.15)–(14.19).

In the next section there will be a systematic approach to converting variational problems to Hamiltonian systems.

## Exercises

1. Suppose $\mathrm{Exp}_p : B_a \to M$ is a diffeomorphism of $B_a = \{v \in T_pM : \|v\| \le a\}$ onto its image, $\mathcal{B}$. Use the Gauss lemma to show that, for each $q \in \mathcal{B},\ q = \mathrm{Exp}(w)$, the curve $\gamma(t) = \mathrm{Exp}(tw),\ 0 \le t \le 1$, is the unique shortest path from $p$ to $q$. If $\mathrm{Exp}_p$ is defined on $B_a$ but is *not* a diffeomorphism, show that this conclusion does not hold.
2. Let $M$ be a connected Riemannian manifold. Define $d(p, q)$ to be the infimum of lengths of smooth curves from $p$ to $q$. Show that this makes $M$ a metric space.
3. Let $p, q \in M$, and suppose there exists a *Lipschitz* curve $\gamma : [a, b] \to M,\ \gamma(a) = p,\ \gamma(b) = q$, parameterized by arc length, of length equal to $d(p, q)$. Show that $\gamma$ is a $C^\infty$-curve. (*Hint*: Make use of Exercise 1.)
4. Let $M$ be a connected Riemannian manifold that, with the metric of Exercise 2, is compact. Show that any $p, q \in M$ can be joined by a geodesic of length $d(p, q)$.
   (*Hint*: Let $\gamma_k : [0, 1] \to M,\ \gamma_k(0) = p,\ \gamma_k(1) = q$ be constant-speed curves of lengths $\ell_k \to d(p, q)$. Use Ascoli's theorem to produce a Lipschitz curve of length $d(p, q)$ as a uniform limit of a subsequence of these.)
5. Try to extend the result of Exercise 4 to the case where $M$ is assumed to be *complete*, rather than compact.
6. Verify that the definition of $\nabla_X$ given by (11.22) does indeed provide a Levi-Civita connection, having properties (11.18)–(11.21).
   (*Hint*: For example, if you interchange the roles of $Y$ and $Z$ in (11.22), and add it to the resulting formula for $2\langle Y, \nabla_X Z\rangle$, you can cancel all the terms on the right side except $X\langle Y, Z\rangle + X\langle Z, Y\rangle$; this gives (11.20).)

## 12. Variational problems and the stationary action principle

The calculus of variations consists of the study of stationary points (e.g., maxima and minima) of a real-valued function that is defined on some space of functions. Here, we let $M$ be a region in $\mathbb{R}^n$, or more generally an $n$-dimensional manifold, fix two points $p, q \in M$ and an interval $[a, b] \subset \mathbb{R}$, and consider a space of functions $\mathcal{P}$ consisting of smooth curves $u : [a, b] \to M$ satisfying $u(a) = p,\ u(b) = q$. We consider functions $I : \mathcal{P} \to \mathbb{R}$ of the form

$$I(u) = \int_a^b F\big(u(t), \dot{u}(t)\big)\, dt. \tag{12.1}$$

Here $F(x, v)$ is a smooth function on the tangent bundle $TM$, or perhaps on some open subset of $TM$. By definition, the condition for $I$ to be stationary at $u$ is that

$$\frac{d}{ds} I(u_s)\big|_{s=0} = 0 \tag{12.2}$$

for any smooth family $u_s$ of elements of $\mathcal{P}$ with $u_0 = u$. Note that

$$\frac{d}{ds}\, u_s(t)\big|_{s=0} = w(t) \tag{12.3}$$

defines a tangent vector to $M$ at $u(t)$, and precisely those tangent vectors $w(t)$ vanishing at $t = a$ and at $t = b$ arise from making some variation of $u$ within $\mathcal{P}$.

As in the last section, we can compute the left side of (12.2) by differentiating under the integral, and obtaining a formula for this involves considering $t$-derivatives of $w$. Recall the two approaches to this taken in §11. Here we will emphasize the second approach, since the data at hand do not generally pick out some distinguished covariant derivative on $M$. Thus we work in local coordinates on $M$. Since any smooth curve on $M$ can be enclosed by a single coordinate patch, this involves no loss of generality. Then, given (12.3), we have

$$\frac{d}{ds}\, I(u_s)\big|_{s=0} = \int_a^b \bigl[F_x(u,\dot{u})w + F_v(u,\dot{u})\dot{w}\bigr]\; dt. \tag{12.4}$$

Integrating the last term by parts and recalling that $w(a)$ and $w(b)$ vanish, we see that this is equal to

$$\int_a^b \Bigl[F_x(u,\dot{u}) - \frac{d}{dt}\, F_v(u,\dot{u})\Bigr] w\; dt. \tag{12.5}$$

It follows that the condition for $u$ to be stationary is precisely that $u$ satisfy the equation

$$\frac{d}{dt}\, F_v(u,\dot{u}) - F_x(u,\dot{u}) = 0, \tag{12.6}$$

a second-order ODE, called *Lagrange's equation.* Written more fully, it is

$$F_{vv}(u,\dot{u})\ddot{u} + F_{vx}(u,\dot{u})\dot{u} - F_x(u,\dot{u}) = 0, \tag{12.7}$$

where $F_{vv}$ is the $n \times n$ matrix of second-order $v$-derivatives of $F(x,v)$, acting on the vector $\ddot{u}$, etc. This is a nonsingular system as long as $F(x,v)$ satisfies the condition

$$F_{vv}(x,v) \text{ is invertible,} \tag{12.8}$$

as an $n \times n$ matrix, for each $(x,v) = (u(t),\dot{u}(t)),\ t \in [a,b]$.

The ODE (12.6) suggests a particularly important role for

$$\xi = F_v(x,v). \tag{12.9}$$

Then, for $(x,v) = (u,\dot{u})$, we have

$$\dot{\xi} = F_x(x,v), \quad \dot{x} = v. \tag{12.10}$$

We claim that this system, in $(x,\xi)$-coordinates, is in *Hamiltonian* form. Note that $(x,\xi)$ gives a local coordinate system under the hypothesis (12.8), by the inverse function theorem. In other words, we will produce a function $E(x,\xi)$ such that (12.10) is the same as

$$\dot{x} = E_\xi, \quad \dot{\xi} = -E_x, \tag{12.11}$$

so the goal is to construct $E(x,\xi)$ such that

$$E_x(x,\xi) = -F_x(x,v), \quad E_\xi(x,\xi) = v, \tag{12.12}$$

when $v = v(x,\xi)$ is defined by inverting the transformation

$$(x,\xi) = (x, F_v(x,v)) = \lambda(x,v). \tag{12.13}$$

If we set

$$E^b(x,v) = E(\lambda(x,v)), \tag{12.14}$$

then (12.12) is equivalent to

$$E^b_x(x,v) = -F_x + vF_{vx}, \quad E^b_v(x,v) = v\ F_{vv}, \tag{12.15}$$

as follows from the chain rule. This calculation is most easily performed using differential forms, details on which can be found in the next section; in the differential form notation, our task is to find $E^b(x,v)$ such that

$$dE^b = (-F_x + vF_{vx})\,dx + vF_{vv}\,dv. \tag{12.16}$$

It can be seen by inspection that this identity is satisfied by

$$E^b(x,v) = F_v(x,v)v - F(x,v). \tag{12.17}$$

Thus the ODE (12.7) describing a stationary point for (12.1) has been converted to a first-order Hamiltonian system, in the $(x,\xi)$-coordinates, given the hypothesis (12.8) on $F_{vv}$. In view of (12.13), one often writes (12.17) informally as

$$E(x,\xi) = \xi \cdot v - F(x,v).$$

We make some observations about the transformation $\lambda$ of (12.13). If $v \in T_xM$, then $F_v(x,v)$ acts naturally as a linear functional on $T_xM$. In other words, $\xi = F_v(x,v)$ is naturally regarded as an element of $T^*_xM$, in the cotangent bundle of $M$; it makes invariant sense to regard

$$\lambda : TM \longrightarrow T^*M \tag{12.18}$$

(if $F$ is defined on all of $TM$). This map is called the *Legendre transformation.* As we have already noted, the hypothesis (12.8) is equivalent to the statement that $\lambda$ is a local diffeomorphism.

As an example, suppose $M$ has a Riemannian metric $g$ and

$$F(x,v) = \frac{1}{2}g(v,v).$$

Then the map (12.18) is the identification of $TM$ and $T^*M$ associated with "lowering indices," using the metric tensor $g_{jk}$. A straightforward calculation gives, in this case, $E(x, \xi)$ equal to half the natural square norm on cotangent vectors. On the other hand, the function $F(x, v) = \sqrt{g(v, v)}$ *fails* to satisfy the hypothesis (12.8). Since this is the integrand for arc length, it is important to incorporate this case into our analysis. Recall from the previous section that obtaining equations for a geodesic involves parameterizing a curve by arc length. We now look at the following more general situation.

We say $F(x, v)$ is homogeneous of degree $r$ in $v$ if $F(x, cv) = c^r F(x, v)$ for $c > 0$. Thus $\sqrt{g(v, v)}$ above is homogeneous of degree 1. When $F$ is homogeneous of degree 1, hypothesis (12.8) is never satisfied. Furthermore, $I(u)$ is independent of the parameterization of a curve in this case; if $\sigma : [a, b] \to [a, b]$ is a diffeomorphism (fixing $a$ and $b$), then $I(u) = I(\tilde{u})$ for $\tilde{u}(t) = u(\sigma(t))$. Let us look at a function $f(x, v)$ related to $F(x, v)$ by

$$f(x, v) = \psi(F(x, v)), \quad F(x, v) = \varphi(f(x, v)). \tag{12.19}$$

Given a family $u_s$ of curves as before, we can write

$$\begin{aligned} \frac{d}{ds} I(u_s)\big|_{s=0} = \int_a^b \Big[ &\varphi'\big(f(u, \dot{u})\big) f_x(u, \dot{u}) \\ &- \frac{d}{dt} \big\{ \varphi'\big(f(u, \dot{u})\big) f_v(u, \dot{u}) \big\} \Big] w \, dt. \end{aligned} \tag{12.20}$$

If $u$ satisfies the condition

$$f(u, \dot{u}) = c, \tag{12.21}$$

with $c$ constant, this is equal to

$$c' \int_a^b \big[ f_x(u, \dot{u}) - (d/dt) f_v(u, \dot{u}) \big] w \, dt, \tag{12.22}$$

with $c' = \varphi'(c)$. Of course, setting

$$J(u) = \int_a^b f(u, \dot{u}) \, dt, \tag{12.23}$$

we have

$$\frac{d}{ds} J(u_s)\Big|_{s=0} = \int_a^b \Big[ f_x(u, \dot{u}) - \frac{d}{dt} f_v(u, \dot{u}) \Big] w \, dt. \tag{12.24}$$

Consequently, if $u$ satisfies (12.21), then $u$ is stationary for $I$ if and only if $u$ is stationary for $J$ (provided $\varphi'(c) \neq 0$).

It is possible that $f(x, v)$ satisfies (12.8) even though $F(x, v)$ does not, as the case $F(x, v) = \sqrt{g(v, v)}$ illustrates. Note that

$$f_{v_j v_k} = \psi'(F) F_{v_j v_k} + \psi''(F) F_{v_j} F_{v_k}.$$

Let us specialize to the case $\psi(F) = F^2$, so $f(x, v) = F(x, v)^2$ is homogeneous of degree 2. If $F$ is convex in $v$ and $(F_{v_j v_k})$, a positive-semidefinite matrix, annihilates only radial vectors, and if $F > 0$, then $f(x, v)$ is strictly convex (i.e., $f_{vv}$ is positive-definite), and hence (12.8) holds for $f(x,v)$. This is the case when $F(x,v) = \sqrt{g(v,v)}$ is the arc length integrand.

If $f(x,v) = F(x,v)^2$ satisfies (12.8), then the stationary condition for (12.23) is that $u$ satisfy the ODE

$$f_{vv}(u,\dot{u})\ddot{u} + f_{vx}(u,\dot{u})\dot{u} - f_x(u,\dot{u}) = 0,$$

a nonsingular ODE for which we know there is a unique local solution, with $u(a) = p,\ \dot{u}(a)$ given. We will be able to say that such a solution is also stationary for (12.1) once we know that (12.21) holds, that is, $f(u,\dot{u})$ is constant. Indeed, if $f(x,v)$ is homogeneous of degree 2, then $f_v(x,v)v = 2f(x,v)$, and hence

$$e^b(x,v) = f_v(x,v)v - f(x,v) = f(x,v). \tag{12.25}$$

But since the equations for $u$ take Hamiltonian form in the coordinates $(x,\xi) = (x, f_v(x,v))$, it follows that $e^b(u(t),\dot{u}(t))$ is constant for $u$ stationary, so (12.21) does hold in this case.

There is a general principle, known as the *stationary action principle*, or Hamilton's principle, for producing equations of mathematical physics. In this set-up, the state of a physical system at a given time is described by a pair $(x,v)$, position and velocity. One has a *kinetic energy* function $T(x,v)$ and a *potential energy* function $V(x,v)$, determining the dynamics, as follows. Form the difference

$$L(x,v) = T(x,v) - V(x,v), \tag{12.26}$$

known as the *Lagrangian.* Hamilton's principle states that a path $u(t)$ describing the evolution of the state in this system is a stationary path for the *action integral*

$$I(u) = \int_a^b L(u,\dot{u})\, dt. \tag{12.27}$$

In many important cases, the potential $V = V(x)$ is velocity independent and $T(x,v)$ is a quadratic form in $v$; say $T(x,v) = (1/2)v \cdot G(x)v$ for a symmetric matrix $G(x)$. In that case, we consider

$$L(x,v) = \frac{1}{2} v \cdot G(x)v - V(x). \tag{12.28}$$

Thus we have

$$\xi = L_v(x,v) = G(x)v, \tag{12.29}$$

and the conserved quantity (12.17) becomes

$$\begin{aligned} E^b(x,v) &= v \cdot G(x)v - \Big[\frac{1}{2} v \cdot G(x)v - V(x)\Big] \\ &= \frac{1}{2} v \cdot G(x)v + V(x), \end{aligned} \tag{12.30}$$

which is the *total energy* $T(x,v)+V(x)$. Note that the nondegeneracy condition is that $G(x)$ be invertible (in physical problems, $G(x)$ is typically positive-definite, but see (18.20)); assuming this, we have

$$E(x,\xi)=\frac{1}{2}\xi\cdot G(x)^{-1}\xi+V(x), \tag{12.31}$$

whose Hamiltonian vector field defines the dynamics. Note that, in this case, Lagrange's equation (12.6) takes the form

$$\frac{d}{dt}\big[G(u)\dot{u}\big]=\frac{1}{2}\dot{u}\cdot G_x(u)\dot{u}-V_x(u), \tag{12.32}$$

which can be rewritten as

$$\ddot{u}+\Gamma\dot{u}\dot{u}+G(u)^{-1}V_x(u)=0, \tag{12.33}$$

where $\Gamma\dot{u}\dot{u}$ is a vector whose $\ell$th component is ${\Gamma^\ell}_{jk}\dot{u}^j\dot{u}^k$, with ${\Gamma^\ell}_{jk}$ the connection coefficients defined by (11.29) with $(g_{jk})=G(x)$. In other words, (12.33) generalizes the geodesic equation for the Riemannian metric $(g_{jk})=G(x)$, which is what would arise in the case $V=0$.

We refer to [Ar] and [Go] for a discussion of the relation of Hamilton's principle to other formulations of the laws of Newtonian mechanics, but we will briefly illustrate it here with a couple of examples.

Consider the basic case of motion of a particle in Euclidean space $\mathbb{R}^n$, in the presence of a force field of potential type $F(x)=-\operatorname{grad}V(x)$, as in the beginning of §10. Then

$$T(x,v)=\frac{1}{2}m|v|^2,\quad V(x,v)=V(x). \tag{12.34}$$

This is of course the special case of (12.28) with $G(x)=mI$, and the ODE satisfied by stationary paths for (12.27) hence has the form

$$m\ddot{u}+V_x(u)=0, \tag{12.35}$$

precisely the equation (10.2) expressing Newton's law $F=ma$.

Figure 12.1

Next we consider one example where Cartesian coordinates are not used, namely the motion of a pendulum (Fig. 12.1). We suppose a mass $m$ is at the end of a (massless) rod of length $\ell$, swinging under the influence of gravity. In this case, we can express the potential energy as

$$V(\theta) = -mg\ell \cos\theta, \tag{12.36}$$

where $\theta$ is the angle the rod makes with the downward vertical ray, and $g$ denotes the strength of gravity. The speed of the mass at the end of the pendulum is $\ell|\dot{\theta}|$, so the kinetic energy is

$$T(\theta, \dot{\theta}) = \frac{1}{2} m\ell^2 |\dot{\theta}|^2. \tag{12.37}$$

In this case we see that Hamilton's principle leads to the ODE

$$\ell\ddot{\theta} + g \sin\theta = 0, \tag{12.38}$$

describing the motion of a pendulum.

Next we consider a very important physical problem that involves a *velocity-dependent* force, leading to a Lagrangian of a form different from (12.28), namely the (nonrelativistic) motion of a charged particle (with charge $e$) in an electromagnetic field $(E, B)$. One has Newton's law

$$m \frac{dv}{dt} = F, \tag{12.39}$$

where $v = dx/dt$ and $F$ is the *Lorentz force*, given by

$$F = e(E + v \times B). \tag{12.40}$$

Certainly $F$ here is not of the form $-\nabla V(x)$. To construct a replacement for the potential $V$, one makes use of two of Maxwell's equations for $E$ and $B$:

$$\text{curl } E = -\frac{\partial B}{\partial t}, \quad \text{div } B = 0, \tag{12.41}$$

in units where the speed of light is 1. We will return to Maxwell's equations later on. As we will show in §18, these equations imply the existence of a real-valued $\varphi(t, x)$ and a vector-valued $A(t, x)$ such that

$$B = \text{ curl } A, \quad E = -\text{ grad } \varphi - \frac{\partial A}{\partial t}. \tag{12.42}$$

Given these quantities, we set

$$V(x, v) = e(\varphi - A \cdot v), \tag{12.43}$$

and use the Lagrangian $L = T - V$, with $T = (1/2)m|v|^2$. We have

$$L_v = mv + eA, \quad L_x = -e\varphi_x + e \text{ grad}\,(A \cdot v).$$

Consequently, $(d/dt)L_v = m\ dv/dt + e\partial A/\partial t + eA_x v$. Using (12.42), we can obtain

$$\frac{d}{dt}L_v - L_x = m\,\frac{dv}{dt} - e(E + v\times\ \text{curl}\ A), \tag{12.44}$$

showing that Lagrange's equation

$$\frac{d}{dt}L_v - L_x = 0 \tag{12.45}$$

is indeed equivalent to (12.39)–(12.40).

If the electromagnetic field varies with $t$, then the Lagrangian $L$ produced by (12.43) has explicit $t$-dependence:

$$L = L(t, x, v). \tag{12.46}$$

The equation (12.45) is still the stationary condition for the integral

$$I(u) = \int_a^b L\big(t, u(t), \dot{u}(t)\big)\ dt, \tag{12.47}$$

as in (12.6). Of course, instead of (12.7), we have

$$L_{vv}(t,u,\dot{u})\ddot{u} + L_{vx}(t,u,\dot{u})\dot{u} - L_x(t,u,\dot{u}) + L_{tv}(t,u,\dot{u}) = 0. \tag{12.48}$$

Finally, we note that for this Lorentz force the Legendre transformation (12.13) is given by

$$(x,\xi) = (x, mv + eA), \tag{12.49}$$

and hence the Hamiltonian function $E(x,\xi)$ as in (12.11) is given by

$$E(x,\xi) = \frac{1}{2m}|\xi - eA|^2 + e\varphi. \tag{12.50}$$

A treatment of the *relativistic* motion of a charged particle in an electromagnetic field (which in an important sense is cleaner than the nonrelativistic treatment) is given in §18.

Hamilton's principle can readily be extended to produce *partial differential equations*, describing the motion of continua, such as vibrating strings, moving fluids, and numerous other important phenomena. Some of these results will be discussed in the beginning of Chapter 2, and others in various subsequent chapters.

We end this section by noting that Lagrange's equation (12.6) depends on the choice of a coordinate system. We can write down an analogue of (12.6), which depends on a choice of Riemannian metric on $M$, but not on a coordinate system.

Thus, let $M$ be a Riemannian manifold, and denote by $\nabla$ the Levi-Civita connection constructed in §11. If we have a family of curves in $TM$, that is, a map

$$u : I\times I \longrightarrow M, \quad u = u(t,s), \tag{12.51}$$

with velocity $u_t : I \times I \to TM$, we can write

$$I(s) = \int_a^b F\big(u_t(t,s)\big)\, dt, \tag{12.52}$$

for a given $F : TM \to \mathbb{R}$. We have

$$I'(s) = \int_a^b DF\big(u_t(t,s)\big)\partial_s u_t \, dt. \tag{12.53}$$

Note that $DF(u_t)$ acts on $\partial_s u_t \in T_{u_t}(TM)$. Now, given $v \in TM$, we can write

$$T_v(TM) = V_v(TM) \oplus H_v(TM). \tag{12.54}$$

Here the "vertical" space $V_v(TM)$ is simply $T_v(T_{\pi(v)}M)$, where $\pi : TM \to M$ is the usual projection. The "horizontal" space $H_v(TM)$ is a complementary space, isomorphic to $T_{\pi(v)}M$, defined as follows.

For any smooth curve $\gamma$ on $M$, such that $\gamma(0) = x = \pi(v)$, let $V(t) \in T_{\gamma(t)}M$ be given by parallel translation of $v$ along $\gamma$, that is, if $T = \gamma'(t)$, $V$ solves $\nabla_T V = 0,\ V(0) = v$. Thus $V(t)$ is a curve in $TM$, and $V(0) = v$. The map $\gamma'(0) \mapsto V'(0)$ is an injective linear map of $T_{\pi(v)}M$ into $T_v(TM)$, whose range we call $H_v(TM)$. One might compare the construction in §6 of Appendix C, Connections and Curvature. Thus we have both the decomposition (12.54) and the isomorphisms

$$V_v(TM) \approx T_{\pi(v)}M, \quad H_v(TM) \approx T_{\pi(v)}M. \tag{12.55}$$

The first isomorphism is canonical. The second isomorphism is simply the restriction of $D\pi : T_v(TM) \to T_{\pi(v)}M$ to the subspace $H_v(TM)$.

The splitting (12.54) gives

$$DF(v)(\partial_s u_t) = \langle F_v(v), (\partial_s u_t)_{\text{vert}}\rangle + \langle F_x(v), (\partial_s u_t)_{\text{horiz}}\rangle, \tag{12.56}$$

where we use this to define

$$F_v(v) \in T_{\pi(v)}M \approx V_v(TM), \quad F_x(v) \in T_{\pi(v)}M \approx H_v(TM). \tag{12.57}$$

If we set $v = u_t,\ w = u_s$, we have

$$I'(s) = \int_a^b \Big[\big\langle F_v(u_t), \nabla_v w\big\rangle + \big\langle F_x(v), w\big\rangle\Big]\, dt. \tag{12.58}$$

Parallel to (11.24)–(11.26), we have

$$\int_a^b \big\langle F_v(u_t), \nabla_v w\big\rangle \, dt = -\int_a^b \big\langle \nabla_v F_v(u_t), w\big\rangle \, dt, \tag{12.59}$$

where to apply $\nabla_v$ we regard $F_v(u_t)$ as a vector field defined over the curve $t \mapsto u(t,s)$ in $M$. Hence the stationary condition that $I'(0) = 0$ for all variations of $u(t) = u(t,0)$ takes the form

$$\nabla_{\dot u} F_v(\dot u) - F_x(\dot u) = 0. \tag{12.60}$$

Note that if $v(s)$ is a smooth curve in $TM$, with $\pi(v(s)) = u(s)$ and $u'(s) = w(s)$, then, under the identification in (12.55),

$$v'(s)_{\text{vert}} = \nabla_w v, \quad v'(s)_{\text{horiz}} = w. \tag{12.61}$$

Then, for smooth $F : TM \to \mathbb{R}$,

$$\frac{d}{ds} F\big(v(s)\big) = \big\langle F_v(v), \nabla_w v\big\rangle + \big\langle F_x(v), w\big\rangle. \tag{12.62}$$

In particular,

$$F(v) = \langle v, v\rangle \Longrightarrow F_v(v) = 2v \ \text{ and } \ F_x(v) = 0. \tag{12.63}$$

Thus, for this function $F(v)$, the Lagrange equation (12.60) becomes the geodesic equation $\nabla_v v = 0$, as expected. If, parallel to (12.28), we take $L(v) = (1/2)\langle v, v\rangle - V(x),\ x = \pi(v)$, then

$$L_v(v) = v, \quad L_x(v) = -\ \text{grad}\ V(x), \tag{12.64}$$

where grad $V(x)$ is the vector field on $M$ defined by $\langle \text{grad}\ V(x), W\rangle = \mathcal{L}_W V(x)$. The Lagrange equation becomes

$$\nabla_{\dot u} \dot u + \ \text{grad}\ V(u) = 0, \tag{12.65}$$

in agreement with (12.33).

## Exercises

1. Suppose that, more generally than (12.28), we have a Lagrangian of the form
$$L(x, v) = \frac{1}{2} v \cdot G(x) v + A(x) \cdot v - V(x).$$
Show that (12.30) continues to hold, that is,
$$E^b(x, v) = \frac{1}{2} v \cdot G(x) v + V(x),$$
and that the Hamiltonian function becomes, in place of (12.31),
$$E(x, \xi) = \frac{1}{2}(\xi - A(x)) \cdot G(x)^{-1}(\xi - A(x)) + V(x).$$
Work out the modification to (12.33) when the extra term $A(x) \cdot v$ is included. Relate this to the discussion of the motion in an electromagnetic field in (12.39)–(12.50).
2. Work out the differential equations for a planar double pendulum, in the spirit of (12.36)–(12.38). See Fig. 12.2. (*Hint*: To compute kinetic and potential energy, think of the plane as the complex plane, with the real axis pointing down. The position of particle 1 is $\ell_1 e^{i\theta_1}$ and that of particle 2 is $\ell_1 e^{i\theta_1} + \ell_2 e^{i\theta_2}$.)
3. After reading §18, show that the identity $\mathcal{F} = d\mathcal{A}$ in (18.19) implies the identity (12.42), with $\mathcal{A} = \varphi\ dx_0 + \sum_{j\ge1} A_j\ dx_j$.

FIGURE 12.2

4. If $A(x)$ is a vector field on $\mathbb{R}^3$ and $v$ is a constant vector, show that
$$\operatorname{grad}(v \cdot A) = \nabla_v A + v \times \ \operatorname{curl} A.$$
Use this to verify (12.44). How is the formula above modified if $v = v(x)$ is a function of $x$? Reconsider this last question after looking at the exercises following §8 of Chapter 5.
5. The statement before (12.4)—that any smooth curve $u(s)$ on $M$ can be enclosed by a single coordinate patch—is not strictly accurate, as the curve may have self-intersections. Give a more precise statement.

## 13. Differential forms

It is very desirable to be able to make constructions that depend as little as possible on a particular choice of coordinate system. The calculus of differential forms, whose study we now take up, is one convenient set of tools for this purpose.

We start with the notion of a 1-form. It is an object that is integrated over a curve; formally, a 1-form on $\Omega \subset \mathbb{R}^n$ is written

$$\alpha = \sum_j a_j(x)\ dx_j. \tag{13.1}$$

If $\gamma : [a, b] \to \Omega$ is a smooth curve, we set

$$\int_\gamma \alpha = \int_a^b \sum a_j\bigl(\gamma(t)\bigr)\gamma_j'(t)\ dt. \tag{13.2}$$

In other words,

$$\int_\gamma \alpha = \int_I \gamma^* \alpha, \tag{13.3}$$

where $I = [a, b]$ and $\gamma^*\alpha = \sum_j a_j(\gamma(t))\gamma_j'(t)$ is the *pull-back* of $\alpha$ under the map $\gamma$. More generally, if $F : \mathcal{O} \to \Omega$ is a smooth map ($\mathcal{O} \subset \mathbb{R}^m$ open), the pull-back $F^*\alpha$ is a 1-form on $\mathcal{O}$ defined by

$$F^*\alpha = \sum_{j,k} a_j(F(y)) \frac{\partial F_j}{\partial y_k}\, dy_k. \tag{13.4}$$

The usual change of variable for integrals gives

$$\int_\gamma \alpha = \int_\sigma F^*\alpha \tag{13.5}$$

if $\gamma$ is the curve $F \circ \sigma$.

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism, and

$$X = \sum b^j(x) \frac{\partial}{\partial x_j} \tag{13.6}$$

is a vector field on $\Omega$, recall that we have the vector field on $\mathcal{O}$:

$$F_\# X(y) = \big(DF^{-1}(p)\big) X(p), \quad p = F(y). \tag{13.7}$$

If we define a pairing between 1-forms and vector fields on $\Omega$ by

$$\langle X, \alpha \rangle = \sum_j b^j(x) a_j(x) = b \cdot a, \tag{13.8}$$

a simple calculation gives

$$\langle F_\# X, F^*\alpha \rangle = \langle X, \alpha \rangle \circ F. \tag{13.9}$$

Thus, a 1-form on $\Omega$ is characterized at each point $p \in \Omega$ as a linear transformation of *vectors* at $p$ to $\mathbb{R}$.

More generally, we can regard a $k$-form $\alpha$ on $\Omega$ as a $k$-multilinear map on vector fields:

$$\alpha(X_1, \dots, X_k) \in C^\infty(\Omega); \tag{13.10}$$

we impose the further condition of antisymmetry:

$$\alpha(X_1, \dots, X_j, \dots, X_\ell, \dots, X_k) = -\alpha(X_1, \dots, X_\ell, \dots, X_j, \dots, X_k). \tag{13.11}$$

We use a special notation for $k$-forms: If $1 \le j_1 < \cdots < j_k \le n$, $j = (j_1, \dots, j_k)$, we set

$$\alpha = \sum_j a_j(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k}, \tag{13.12}$$

where

$$a_j(x) = \alpha(D_{j_1}, \dots, D_{j_k}), \quad D_j = \frac{\partial}{\partial x_j}. \tag{13.13}$$

More generally, we assign meaning to (13.12) summed over all $k$-indices $(j_1, \dots, j_k)$, where we identify

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = (\operatorname{sgn} \sigma)\, dx_{j_{\sigma(1)}} \wedge \cdots \wedge dx_{j_{\sigma(k)}}, \tag{13.14}$$

$\sigma$ being a permutation of $\{1, \dots, k\}$. If any $j_m = j_\ell$ $(m \neq \ell)$, then (13.14) vanishes. A common notation for the statement that $\alpha$ is a $k$-form on $\Omega$ is

$$\alpha \in \Lambda^k(\Omega). \tag{13.15}$$

In particular, we can write a 2-form $\beta$ as

$$\beta = \sum b_{jk}(x)\ dx_j \wedge dx_k \tag{13.16}$$

and pick coefficients satisfying $b_{jk}(x) = -b_{kj}(x)$. According to (13.12) and (13.13), if we set $U = \sum u_j(x)\, \partial/\partial x_j$ and $V = \sum v_j(x)\, \partial/\partial x_j$, then

$$\beta(U, V) = 2 \sum b_{jk}(x) u^j(x) v^k(x). \tag{13.17}$$

If $b_{jk}$ is not required to be antisymmetric, one gets $\beta(U, V) = \sum (b_{jk} - b_{kj}) u^j v^k$.

If $F : \mathcal{O} \to \Omega$ is a smooth map as above, we define the pull-back $F^*\alpha$ of a $k$-form $\alpha$, given by (13.12), to be

$$F^*\alpha = \sum_j a_j\big(F(y)\big)(F^* dx_{j_1}) \wedge \cdots \wedge (F^* dx_{j_k}), \tag{13.18}$$

where

$$F^* dx_j = \sum_\ell \frac{\partial F_j}{\partial y_\ell}\, dy_\ell, \tag{13.19}$$

the algebraic computation in (13.18) being performed using the rule (13.14). Extending (13.9), if $F$ is a diffeomorphism, we have

$$(F^*\alpha)(F_\# X_1, \dots, F_\# X_k) = \alpha(X_1, \dots, X_k) \circ F. \tag{13.20}$$

If $B = (b_{jk})$ is an $n \times n$ matrix, then, by (13.14),

$$\begin{aligned} &\Big(\sum_k b_{1k}\, dx_k\Big) \wedge \Big(\sum_k b_{2k}\, dx_k\Big) \wedge \cdots \wedge \Big(\sum_k b_{nk}\, dx_k\Big) \\ &= \Big(\sum_\sigma (\mathrm{sgn}\ \sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}\Big)\, dx_1 \wedge \cdots \wedge dx_n \\ &= \big(\det B\big)\, dx_1 \wedge \cdots \wedge dx_n, \end{aligned} \tag{13.21}$$

Hence, if $F : \mathcal{O} \to \Omega$ is a $C^1$-map between two domains of dimension $n$, and $\alpha = A(x)\, dx_1 \wedge \cdots \wedge dx_n$ is an $n$-form on $\Omega$, then

$$F^*\alpha = \det DF(y)\ A(F(y))\ dy_1 \wedge \cdots \wedge dy_n. \tag{13.22}$$

Comparison with the change-of-variable formula for multiple integrals suggests that one has an intrinsic definition of $\int_\Omega \alpha$ when $\alpha$ is an $n$-form on $\Omega$, $n = \dim \Omega$. To implement this, we need to take into account that $\det DF(y)$ rather than $|\det DF(y)|$ appears in (13.21). We say that a smooth map $F : \mathcal{O} \to \Omega$ between two open subsets of $\mathbb{R}^n$ *preserves orientation* if $\det DF(y)$ is everywhere positive. The object called an "orientation" on $\Omega$ can be identified as an equivalence class of nowhere-vanishing

$n$-forms on $\Omega$, where two such forms are equivalent if one is a multiple of another by a positive function in $C^\infty(\Omega)$; the standard orientation on $\mathbb{R}^n$ is determined by $dx_1 \wedge \cdots \wedge dx_n$. If $S$ is an $n$-dimensional surface in $\mathbb{R}^{n+k}$, an orientation on $S$ can also be specified by a nowhere-vanishing form $\omega \in \Lambda^n(S)$. If such a form exists, $S$ is said to be orientable. The equivalence class of positive multiples $a(x)\omega$ is said to consist of "positive" forms. A smooth map $\psi : S \to M$ between oriented $n$-dimensional surfaces preserves orientation provided $\psi^*\sigma$ is positive on $S$ whenever $\sigma \in \Lambda^n(M)$ is positive. If $S$ is oriented, one can choose coordinate charts that are all orientation-preserving. Surfaces that cannot be oriented also exist.

If $\mathcal{O}, \Omega$ are open in $\mathbb{R}^n$ and $F : \mathcal{O} \to \Omega$ is an orientation-preserving diffeomorphism, we have

$$\int\limits_{\mathcal{O}} F^*\alpha = \int\limits_{\Omega} \alpha. \tag{13.23}$$

More generally, if $S$ is an $n$-dimensional manifold with an orientation, say the image of an open set $\mathcal{O} \subset \mathbb{R}^n$ by $\varphi : \mathcal{O} \to S$, carrying the natural orientation of $\mathcal{O}$, we can set

$$\int\limits_{S} \alpha = \int\limits_{\mathcal{O}} \varphi^*\alpha \tag{13.24}$$

for an $n$-form $\alpha$ on $S$. If it takes several coordinate patches to cover $S$, define $\int_S \alpha$ by writing $\alpha$ as a sum of forms, each supported on one patch.

We need to show that this definition of $\int_S \alpha$ is independent of the choice of coordinate system on $S$ (as long as the orientation of $S$ is respected). Thus, suppose $\varphi : \mathcal{O} \to U \subset S$ and $\psi : \Omega \to U \subset S$ are both coordinate patches, so that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is an orientation-preserving diffeomorphism. We need to check that if $\alpha$ is an $n$-form on $S$, supported on $U$, then

$$\int\limits_{\mathcal{O}} \varphi^*\alpha = \int\limits_{\Omega} \psi^*\alpha. \tag{13.25}$$

To see this, first note that, for any form $\alpha$ of any degree,

$$\psi \circ F = \varphi \Longrightarrow \varphi^*\alpha = F^*\psi^*\alpha. \tag{13.26}$$

It suffices to check this for $\alpha = dx_j$. Then $\psi^*\ dx_j = \sum (\partial\psi_j/\partial x_\ell)\, dx_\ell$, by (13.14), so

$$F^*\psi^*\ dx_j = \sum_{\ell,m} \frac{\partial F_\ell}{\partial x_m} \frac{\partial \psi_j}{\partial x_\ell}\ dx_m, \quad \varphi^*\ dx_j = \sum_m \frac{\partial \varphi_j}{\partial x_m}\ dx_m; \tag{13.27}$$

but the identity of these forms follows from the chain rule:

$$D\varphi = (D\psi)(DF) \Longrightarrow \frac{\partial \varphi_j}{\partial x_m} = \sum_\ell \frac{\partial \psi_j}{\partial x_\ell} \frac{\partial F_\ell}{\partial x_m}. \tag{13.28}$$

Now that we have (13.26), we see that the left side of (13.25) is equal to

$$\int_{\mathcal{O}} F^*(\psi^*\alpha), \tag{13.29}$$

which is equal to the right side of (13.25), by (13.23). Thus the integral of an $n$-form over an oriented $n$-dimensional surface is well defined.

Having discussed the notion of a differential form as something to be integrated, we now consider some operations on forms. There is a *wedge product*, or exterior product, characterized as follows. If $\alpha \in \Lambda^k(\Omega)$ has the form (13.12), and if

$$\beta = \sum_i b_i(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Lambda^\ell(\Omega), \tag{13.30}$$

define

$$\alpha \wedge \beta = \sum_{j,i} a_j(x) b_i(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \tag{13.31}$$

in $\Lambda^{k+\ell}(\Omega)$. A special case of this arose in (13.18)–(13.21). We retain the equivalence (13.14). It follows easily that

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{13.32}$$

In addition, there is an *interior product* if $\alpha \in \Lambda^k(\Omega)$ with a vector field $X$ on $\Omega$, producing $\iota_X \alpha = \alpha \rfloor X \in \Lambda^{k-1}(\Omega)$, defined by

$$(\alpha \rfloor X)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}). \tag{13.33}$$

Consequently, if $\alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$, $D_i = \partial/\partial x_i$, then

$$\alpha \rfloor D_{j_\ell} = (-1)^{\ell-1}\, dx_{j_1} \wedge \cdots \wedge \widehat{dx}_{j_\ell} \wedge \cdots \wedge dx_{j_k}, \tag{13.34}$$

where $\widehat{dx}_{j_\ell}$ denotes removing the factor $dx_{j_\ell}$. Furthermore,

$$i \notin \{j_1, \dots, j_k\} \Longrightarrow \alpha \rfloor D_i = 0.$$

If $F : \mathcal{O} \to \Omega$ is a diffeomorphism and $\alpha, \beta$ are forms and $X$ a vector field on $\Omega$, it is readily verified that

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta), \quad F^*(\alpha \rfloor X) = (F^*\alpha) \rfloor (F_\# X). \tag{13.35}$$

We make use of the operators $\wedge_k$ and $\iota_k$ on forms:

$$\wedge_k \alpha = dx_k \wedge \alpha, \quad \iota_k \alpha = \alpha \rfloor D_k. \tag{13.36}$$

There is the following useful *anticommutation relation*:

$$\wedge_k \iota_\ell + \iota_\ell \wedge_k = \delta_{k\ell}, \tag{13.37}$$

where $\delta_{k\ell}$ is 1 if $k = \ell$, 0 otherwise. This is a fairly straightforward consequence of (13.34). We also have

$$\wedge_j \wedge_k + \wedge_k \wedge_j = 0, \quad \iota_j \iota_k + \iota_k \iota_j = 0. \tag{13.38}$$

From (13.37) and (13.38) one says that the operators $\{\iota_j, \wedge_j : 1 \le j \le n\}$ generate a "Clifford algebra." For more on this, see Chapter 10.

Another important operator on forms is the *exterior derivative*:

$$d : \Lambda^k(\Omega) \longrightarrow \Lambda^{k+1}(\Omega), \tag{13.39}$$

defined as follows. If $\alpha \in \Lambda^k(\Omega)$ is given by (13.12), then

$$d\alpha = \sum_{j,\ell} \frac{\partial a_j}{\partial x_\ell}\, dx_\ell \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{13.40}$$

Equivalently,

$$d\alpha = \sum_{\ell=1}^{n} \partial_\ell \wedge_\ell \alpha, \tag{13.41}$$

where $\partial_\ell = \partial/\partial x_\ell$ and $\wedge_\ell$ is given by (13.36). The antisymmetry $dx_m \wedge dx_\ell = -dx_\ell \wedge dx_m$, together with the identity $\partial^2 a_j/\partial x_\ell \partial x_m = \partial^2 a_j/\partial x_m \partial x_\ell$, implies

$$d(d\alpha) = 0, \tag{13.42}$$

for any differential form $\alpha$. We also have a product rule:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \quad \alpha \in \Lambda^k(\Omega),\ \beta \in \Lambda^j(\Omega). \tag{13.43}$$

The exterior derivative has the following important property under pullbacks:

$$F^*(d\alpha) = dF^*\alpha, \tag{13.44}$$

if $\alpha \in \Lambda^k(\Omega)$ and $F : \mathcal{O} \to \Omega$ is a smooth map. To see this, extending (13.43) to a formula for $d(\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_\ell)$ and using this to apply $d$ to $F^*\alpha$, we have

$$\begin{aligned} dF^*\alpha = {} & \sum_{j,\ell} \frac{\partial}{\partial x_\ell}\big(a_j \circ F(x)\big)\ dx_\ell \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big) \\ & + \sum_{j,\nu} (\pm) a_j\big(F(x)\big)\big(F^* dx_{j_1}\big) \wedge \cdots \wedge d\big(F^* dx_{j_\nu}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big). \end{aligned} \tag{13.45}$$

Now

$$d\big(F^* dx_i\big) = \sum_{j,\ell} \frac{\partial^2 F_i}{\partial x_j \partial x_\ell}\ dx_j \wedge dx_\ell = 0,$$

so only the first sum in (13.45) contributes to $dF^*\alpha$. Meanwhile,

$$F^* d\alpha = \sum_{j,m} \frac{\partial a_j}{\partial x_m}\big(F(x)\big)\ \big(F^* dx_m\big) \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big), \tag{13.46}$$

so (13.44) follows from the identity

$$\sum_{\ell} \frac{\partial}{\partial x_\ell}\bigl(a_j \circ F(x)\bigr)\ dx_\ell = \sum_m \frac{\partial a_j}{\partial x_m}\bigl(F(x)\bigr)\ F^* dx_m, \tag{13.47}$$

which in turn follows from the chain rule.

If $d\alpha = 0$, we say $\alpha$ is *closed*; if $\alpha = d\beta$ for some $\beta \in \Lambda^{k-1}(\Omega)$, we say $\alpha$ is *exact*. Formula (13.42) implies that every exact form is closed. The converse is not always true globally. Consider the multivalued angular coordinate $\theta$ on $\mathbb{R}^2 \setminus (0,0)$; $d\theta$ is a single-valued, closed form on $\mathbb{R}^2 \setminus (0,0)$ that is not globally exact. As we will see shortly, every closed form is locally exact.

First we introduce another important construction. If $\alpha \in \Lambda^k(\Omega)$ and $X$ is a vector field on $\Omega$, generating a flow $\mathcal{F}_X^t$, the *Lie derivative* $\mathcal{L}_X\alpha$ is defined to be

$$\mathcal{L}_X\alpha = \frac{d}{dt}\bigl(\mathcal{F}_X^t\bigr)^*\alpha|_{t=0}. \tag{13.48}$$

Note the formal similarity to the definition (8.2) of $\mathcal{L}_X Y$ for a vector field $Y$. Recall the formula (8.4) for $\mathcal{L}_X Y$. The following is not only a computationally convenient formula for $\mathcal{L}_X\alpha$, but also an identity of fundamental importance.

**Proosition 13.1.** *We have*

$$\mathcal{L}_X\alpha = d(\alpha\rfloor X) + (d\alpha)\rfloor X. \tag{13.49}$$

**Proof.** First we compare both sides in the special case $X = \partial/\partial x_\ell = D_\ell$. Note that

$$\bigl(\mathcal{F}_{D_\ell}^t\bigr)^*\alpha = \sum_j a_j(x + te_\ell)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

so

$$\mathcal{L}_{D_\ell}\alpha = \sum_j \frac{\partial a_j}{\partial x_\ell}\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} = \partial_\ell\alpha. \tag{13.50}$$

To evaluate the right side of (13.49) with $X = D_\ell$, use (13.41) to write this quantity as

$$d(\iota_\ell\alpha) + \iota_\ell d\alpha = \sum_{j=1}^n \bigl(\partial_j \wedge_j \iota_\ell + \iota_\ell\partial_j\wedge_j\bigr)\alpha. \tag{13.51}$$

Using the commutativity of $\partial_j$ with $\wedge_j$ and with $\iota_\ell$, and the anticommutation relations (13.37), we see that the right side of (13.51) is $\partial_\ell\alpha$, which coincides with (13.50). Thus the proposition holds for $X = \partial/\partial x_\ell$.

Now we can prove the proposition in general, for a smooth vector field $X$ on $\Omega$. It is to be verified at each point $x_0 \in \Omega$. If $X(x_0) \neq 0$, choose a coordinate system about $x_0$ so that $X = \partial/\partial x_1$, and use the calculation above. This shows that the desired identity holds on the set of points $\{x_0 \in \Omega : X(x_0) \neq 0\}$, and by continuity it holds on the closure of this set. However, if $x_0 \in \Omega$ has a neighborhood on which $X$ vanishes, it is clear that $\mathcal{L}_X \alpha = 0$ near $x_0$ and also $\alpha \rfloor X$ and $d\alpha \rfloor X$ vanish near $x_0$. This completes the proof.

The identity (13.49) can furnish a formula for the exterior derivative in terms of Lie brackets, as follows. By (8.4) and (13.49), we have, for a $k$-form $\omega$,

(13.52)

$$\bigl(\mathcal{L}_X \omega\bigr)(X_1, \dots, X_k) = X \cdot \omega(X_1, \dots, X_k) - \sum_j \omega(X_1, \dots, [X, X_j], \dots, X_k).$$

Now (13.49) can be rewritten as

$$\iota_X d\omega = \mathcal{L}_X \omega - d\iota_X \omega. \tag{13.53}$$

This implies

(13.54)

$$(d\omega)(X_0, X_1, \dots, X_k) = \bigl(\mathcal{L}_{X_0} \omega\bigr)(X_1, \dots, X_k) - \bigl(d\iota_{X_0} \omega\bigr)(X_1, \dots, X_k).$$

We can substitute (13.52) into the first term on the right in (13.54). In case $\omega$ is a 1-form, the last term is easily evaluated; we get

$$(d\omega)(X_0, X_1) = X_0 \cdot \omega(X_1) - X_1 \cdot \omega(X_0) - \omega([X_0, X_1]). \tag{13.55}$$

More generally, we can tackle the last term on the right side of (13.54) by the same method, using (13.53) with $\omega$ replaced by the $(k-1)$-form $\iota_{X_0}\omega$. In this way we inductively obtain the formula

(13.56)

$$\begin{aligned}(d\omega)(X_0, \dots, X_k) = &\sum_{\ell=0}^{k} (-1)^{\ell} X_\ell \cdot \omega(X_0, \dots, \widehat{X}_\ell, \dots, X_k) \\ &+ \sum_{0 \le \ell < j \le k} (-1)^{j+\ell} \omega([X_\ell, X_j], X_0, \dots, \widehat{X}_\ell, \dots, \widehat{X}_j, \dots, X_k).\end{aligned}$$

Note that from (13.48) and the property $\mathcal{F}_X^{s+t} = \mathcal{F}_X^s \mathcal{F}_X^t$ it easily follows that

$$\frac{d}{dt} \bigl(\mathcal{F}_X^t\bigr)^* \alpha = \mathcal{L}_X \bigl(\mathcal{F}_X^t\bigr)^* \alpha = \bigl(\mathcal{F}_X^t\bigr)^* \mathcal{L}_X \alpha. \tag{13.57}$$

It is useful to generalize this. Let $F_t$ be any smooth family of diffeomorphisms from $M$ to $F_t(M) \subset M$. Define vector fields $X_t$ on $F_t(M)$ by

$$\frac{d}{dt} F_t(x) = X_t(F_t(x)). \tag{13.58}$$

Then it easily follows that, for $\alpha \in \Lambda^k M$,

$$\begin{aligned} \frac{d}{dt}\, F_t^* \alpha &= F_t^* \mathcal{L}_{X_t} \alpha \\ &= F_t^* \bigl[ d(\alpha \rfloor X_t) + (d\alpha) \rfloor X_t \bigr]. \end{aligned} \tag{13.59}$$

In particular, if $\alpha$ is *closed*, then if $F_t$ are diffeomorphisms for $0 \le t \le 1$,

$$F_1^* \alpha - F_0^* \alpha = d\beta, \quad \beta = \int_0^1 F_t^* (\alpha \rfloor X_t)\, dt. \tag{13.60}$$

Using this, we can prove the celebrated *Poincaré lemma.*

**Theorem 13.2.** *If $B$ is the unit ball in $\mathbb{R}^n$, centered at 0, $\alpha \in \Lambda^k(B)$, $k > 0$, and $d\alpha = 0$, then $\alpha = d\beta$ for some $\beta \in \Lambda^{k-1}(B)$.*

**Proof.** Consider the family of maps $F_t : B \to B$ given by $F_t(x) = tx$. For $0 < t \le 1$, these are diffeomorphisms, and the formula (13.59) applies. Note that

$$F_1^* \alpha = \alpha, \quad F_0^* \alpha = 0.$$

Now a simple limiting argument shows that (13.60) remains valid, so $\alpha = d\beta$ with

$$\beta = \int_0^1 F_t^* (\alpha \rfloor V) t^{-1}\, dt, \tag{13.61}$$

where $V = r\partial/\partial r = \sum x_j\, \partial/\partial x_j$. Since $F_0^* = 0$, the apparent singularity in the integrand is removable.

Since in the proof of the theorem we dealt with $F_t$ such that $F_0$ was not a diffeomorphism, we are motivated to generalize (13.60) to the case where $F_t : M \to N$ is a smooth family of maps, not necessarily diffeomorphisms. Then (13.58) does not work to define $X_t$ as a vector field, but we do have

$$\frac{d}{dt}\, F_t(x) = Z(t,x); \quad Z(t,x) \in T_{F_t(x)} N. \tag{13.62}$$

Now in (13.60) we see that

$$F^*(\alpha \rfloor X_t)(Y_1, \dots, Y_{k-1}) = \alpha\bigl(F_t(x)\bigr)\bigl(X_t, DF_t(x) Y_1, \dots, DF_t(x) Y_{k-1}\bigr),$$

and we can replace $X_t$ by $Z(t,x)$. Hence, in this more general case, if $\alpha$ is closed, we can write

$$F_1^* \alpha - F_0^* \alpha = d\beta, \quad \beta = \int_0^1 \gamma_t\, dt, \tag{13.63}$$

where, at $x \in M$,

$$\gamma_t(Y_1, \dots, Y_{k-1}) = \alpha\bigl(F_t(x)\bigr)\bigl(Z(t,x), DF_t(x) Y_1, \dots, DF_t(x) Y_{k-1}\bigr). \tag{13.64}$$

For an alternative approach to this homotopy invariance, see Exercise 7.

A basic result in the theory of differential forms is the generalized *Stokes formula*:

**Proposition 13.3.** *Given a compactly supported* $(k-1)$*-form* $\beta$ *of class* $C^1$ *on an oriented* $k$*-dimensional manifold* $\overline{M}$ *(of class* $C^2$*) with boundary* $\partial M$*, with its natural orientation,*

$$\int\limits_M d\beta = \int\limits_{\partial M} \beta. \tag{13.65}$$

The orientation induced on $\partial M$ is uniquely determined by the following requirement. If

$$\overline{M} = \mathbb{R}^k_- = \{x \in \mathbb{R}^k : x_1 \le 0\}, \tag{13.66}$$

then $\partial M = \{(x_2, \dots, x_k)\}$ has the orientation determined by $dx_2 \wedge \cdots \wedge dx_k$.

**Proof.** Using a partition of unity and invariance of the integral and the exterior derivative under coordinate transformations, it suffices to prove this when $\overline{M}$ has the form (13.66). In that case, we will be able to deduce (13.65) from the fundamental theorem of calculus. Indeed, if

$$\beta = b_j(x)\ dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k, \tag{13.67}$$

with $b_j(x)$ of bounded support, we have

$$d\beta = (-1)^{j-1} \frac{\partial b_j}{\partial x_j}\ dx_1 \wedge \cdots \wedge dx_k. \tag{13.68}$$

If $j > 1$, we have

$$\int\limits_M d\beta = \int \Bigl\{ \int_{-\infty}^{\infty} \frac{\partial b_j}{\partial x_j}\ dx_j \Bigr\}\ dx' = 0, \tag{13.69}$$

and also $\kappa^*\beta = 0$, where $\kappa : \partial M \to \overline{M}$ is the inclusion. On the other hand, for $j = 1$, we have

$$\begin{aligned} \int\limits_M d\beta &= \int \Bigl\{ \int_{-\infty}^{0} \frac{\partial b_1}{\partial x_1}\ dx_1 \Bigr\}\ dx_2 \cdots dx_k \\ &= \int b_1(0, x')\ dx' \\ &= \int\limits_{\partial M} \beta. \end{aligned} \tag{13.70}$$

This proves Stokes' formula (13.65).

It is useful to allow singularities in $\partial M$. We say a point $p \in \overline{M}$ is a *corner* of dimension $\nu$ if there is a neighborhood $\overline{U}$ of $p$ in $\overline{M}$ and a $C^2$-diffeomorphism of $\overline{U}$ onto a neighborhood of 0 in

$$K = \{x \in \mathbb{R}^k : x_j \le 0, \ \text{for } 1 \le j \le k-\nu\}, \tag{13.71}$$

where $k$ is the dimension of $M$. If $M$ is a $C^2$-manifold and every point $p \in \partial M$ is a corner (of some dimension), we say $\overline{M}$ is a $C^2$-manifold with corners. In such a case, $\partial M$ is a locally finite union of $C^2$-manifolds with corners. The following result extends Proposition 13.3.

**Proposition 13.4.** *If $\overline{M}$ is a $C^2$-manifold of dimension $k$, with corners, and $\beta$ is a compactly supported $(k-1)$-form of class $C^1$ on $\overline{M}$, then (13.65) holds.*

**Proof.** It suffices to establish this when $\beta$ is supported on a small neighborhood of a corner $p \in \partial M$, of the form $\overline{U}$ described above. Hence it suffices to show that (13.65) holds whenever $\beta$ is a $(k-1)$-form of class $C^1$, with compact support on $K$ in (13.71); and we can take $\beta$ to have the form (13.67). Then, for $j > k-\nu$, (13.69) still holds, while for $j \le k-\nu$, we have, as in (13.70),

(13.72)

$$\begin{aligned}\int\limits_K d\beta &= (-1)^{j-1} \int \Bigl\{ \int_{-\infty}^0 \frac{\partial b_j}{\partial x_j}\, dx_j \Bigr\}\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\ &= (-1)^{j-1} \int b_j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_k)\, dx_1 \cdots \widehat{dx_j} \cdots dx_k \\ &= \int\limits_{\partial K} \beta.\end{aligned}$$

The reason we required $\overline{M}$ to be a manifold of class $C^2$ (with corners) in Propositions 13.3 and 13.4 is the following. Due to the formulas (13.18)–(13.19) for a pull-back, if $\beta$ is of class $C^j$ and $F$ is of class $C^\ell$, then $F^*\beta$ is generally of class $C^\mu$, with $\mu = \min(j, \ell-1)$. Thus, if $j = \ell = 1$, $F^*\beta$ might be only of class $C^0$, so there is not a well-defined notion of a differential form of class $C^1$ on a $C^1$-manifold, though such a notion is well defined on a $C^2$-manifold. This problem can be overcome, and one can extend Propositions 13.3 and 13.4 to the case where $\overline{M}$ is a $C^1$-manifold (with corners) and $\beta$ is a $(k-1)$-form with the property that both $\beta$ and $d\beta$ are continuous. We will not go into the details. Substantially more sophisticated generalizations are given in [Fed].

## Exercises

1. If $F : U_0 \to U_1$ and $G : U_1 \to U_2$ are smooth maps and $\alpha \in \Lambda^k(U_2)$, (13.26) implies
$$(G \circ F)^* \alpha = F^*(G^* \alpha) \quad \text{in } \Lambda^k(U_0).$$
In the special case that $U_j = \mathbb{R}^n$, $F$ and $G$ are linear maps, and $k = n$, show that this identity implies
$$\det(GF) = (\det F)(\det G).$$
2. If $\alpha$ is a closed form and $\beta$ is exact, show that $\alpha \wedge \beta$ is exact. (*Hint*: Use (13.43).)

Let $\Lambda^k(\mathbb{R}^n)$ denote the space of $k$-forms (13.12) with constant coefficients. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then $T^*$ preserves this class of spaces; we denote the map
$$\Lambda^k T^* : \Lambda^k \mathbb{R}^n \longrightarrow \Lambda^k \mathbb{R}^m.$$
Similarly, replacing $T$ by $T^*$ yields
$$\Lambda^k T : \Lambda^k \mathbb{R}^m \longrightarrow \Lambda^k \mathbb{R}^n.$$

3. Show that $\Lambda^k T$ is uniquely characterized as a linear map from $\Lambda^k \mathbb{R}^m$ to $\Lambda^k \mathbb{R}^n$ that satisfies
$$(\Lambda^k T)(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k), \quad v_j \in \mathbb{R}^m.$$
4. If $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$, define an inner product on $\Lambda^k \mathbb{R}^n$ by declaring an orthonormal basis to be
$$\{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}.$$
Show that if $\{u_1, \dots, u_n\}$ is any other orthonormal basis of $\mathbb{R}^n$, then the set
$$\{u_{j_1} \wedge \cdots \wedge u_{j_k} : 1 \le j_1 < \cdots < j_k \le n\}$$
is an orthonormal basis of $\Lambda^k \mathbb{R}^n$.
5. Let $F$ be a vector field on $U$, open in $\mathbb{R}^3$, $F = \sum_1^3 f_j(x)\, \partial/\partial x_j$. Consider the 1-form $\varphi = \sum_1^3 f_j(x)\, dx_j$. Show that $d\varphi$ and curl $F$ are related in the following way:
$$\text{curl } F = \sum_1^3 g_j(x)\, \frac{\partial}{\partial x_j},$$
$$d\varphi = g_1(x)\, dx_2 \wedge dx_3 + g_2(x)\, dx_3 \wedge dx_1 + g_3(x)\, dx_1 \wedge dx_2.$$
6. If $F$ and $\varphi$ are related as in Exercise 5, show that curl $F$ is uniquely specified by the relation
$$d\varphi \wedge \alpha = \langle \text{curl } F, \alpha \rangle \omega$$
for all 1-forms $\alpha$ on $U \subset \mathbb{R}^3$, where $\omega = dx_1 \wedge dx_2 \wedge dx_3$ is the volume form.
7. Suppose $f_0, f_1 : X \to Y$ are smoothly homotopic maps, via $\Phi : X \times \mathbb{R} \to Y$, $\Phi(x, j) = f_j(x)$. Let $\alpha \in \Lambda^k Y$ be closed. Apply (13.60) to $\tilde{\alpha} = \Phi^* \alpha \in \Lambda^k(X \times \mathbb{R})$, with $F_t(x, s) = (x, s + t)$, to obtain $\tilde{\beta} \in \Lambda^{k-1}(X \times \mathbb{R})$ such that

$F_1^*\tilde{\alpha} - \tilde{\alpha} = d\tilde{\beta}$, and from there produce $\beta \in \Lambda^{k-1}(X)$ such that $f_1^*\alpha - f_0^*\alpha = d\beta$.
(*Hint*: Use $\beta = \iota^*\tilde{\beta}$, where $\iota(x) = (x, 0)$.)

For the next set of exercises, let $\Omega$ be a planar domain, $X = f(x,y)\,\partial/\partial x + g(x,y)\,\partial/\partial y$ a nonvanishing vector field on $\Omega$. Consider the 1-form $\alpha = g(x,y)\,dx - f(x,y)\,dy$.

8. Let $\gamma : I \to \Omega$ be a smooth curve, $I = (a,b)$. Show that the image $C = \gamma(I)$ is the image of an integral curve of $X$ if and only if $\gamma^*\alpha = 0$. Consequently, with slight abuse of notation, one describes the integral curves by $g\,dx - f\,dy = 0$. If $\alpha$ is exact (i.e., $\alpha = du$,) conclude that the level curves of $u$ are the integral curves of $X$.
9. A function $\varphi$ is called an integrating factor if $\tilde{\alpha} = \varphi\alpha$ is exact (i.e., if $d(\varphi\alpha) = 0$, provided $\Omega$ is simply connected). Show that an integrating factor always exists, at least locally. Show that $\varphi = e^v$ is an integrating factor if and only if $Xv = -\operatorname{div} X$. Reconsider Exercise 7 in §7. Find an integrating factor for $\alpha = (x^2 + y^2 - 1)\,dx - 2xy\,dy$.
10. Let $Y$ be a vector field that you know how to linearize (i.e., conjugate to $\partial/\partial x$) and suppose $\mathcal{L}_Y\alpha = 0$. Show how to construct an integrating factor for $\alpha$. Treat the more general case $\mathcal{L}_X\alpha = c\alpha$ for some constant $c$. Compare the discussion in §8 of the situation where $[X,Y] = cX$.

## 14. The symplectic form and canonical transformations

Recall from §10 that a Hamiltonian vector field on a region $\Omega \subset \mathbb{R}^{2n}$, with coordinates $\zeta = (x,\xi)$, is a vector field of the form

$$H_f = \sum_{j=1}^{n} \Big[ \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j} \Big]. \tag{14.1}$$

We want to gain an understanding of Hamiltonian vector fields, free from coordinates. In particular, we ask the following question. Let $F : \mathcal{O} \to \Omega$ be a diffeomorphism, and let $H_f$ be a Hamiltonian vector field on $\Omega$. Under what condition on $F$ is $F_\# H_f$ a Hamiltonian vector field on $\mathcal{O}$?

A central object in this study is the *symplectic form*, a 2-form on $\mathbb{R}^{2n}$ defined by

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j. \tag{14.2}$$

Note that if

$$U = \sum \Big[ u^j(\zeta)\frac{\partial}{\partial x_j} + a^j(\zeta)\frac{\partial}{\partial \xi_j} \Big], \quad V = \sum \Big[ v^j(\zeta)\frac{\partial}{\partial x_j} + b^j(\zeta)\frac{\partial}{\partial \xi_j} \Big],$$

then

$$\sigma(U,V)=\sum_{j=1}^{n}\bigl[-u^j(\zeta)b^j(\zeta)+a^j(\zeta)v^j(\zeta)\bigr]. \tag{14.3}$$

In particular, $\sigma$ satisfies the following nondegeneracy condition: If $U$ has the property that, for some $(x_0,\xi_0)\in\mathbb{R}^{2n}$, $\sigma(U,V)=0$ at $(x_0,\xi_0)$ for all vector fields $V$, then $U$ must vanish at $(x_0,\xi_0)$. The relation between the symplectic form and Hamiltonian vector fields is as follows:

**Proposition 14.1.** *The vector field $H_f$ is uniquely determined by the identity*

$$\sigma\rfloor H_f=-df. \tag{14.4}$$

**Proof.** The content of the identity is

$$\sigma(H_f,V)=-Vf, \tag{14.5}$$

for any smooth vector field $V$. If $V$ has the form used in (14.3), then that identity gives

$$\sigma(H_f,V)=-\sum_{j=1}^{n}\Bigl[\frac{\partial f}{\partial\xi_j}\,b^j(\zeta)+\frac{\partial f}{\partial x_j}\,v^j(\zeta)\Bigr],$$

which coincides with the right side of (14.5). In view of the nondegeneracy of $\sigma$, the proposition is proved. Note the special case

$$\sigma(H_f,H_g)=\{f,g\}. \tag{14.6}$$

The following is an immediate corollary.

**Proposition 14.2.** *If $\mathcal{O},\Omega$ are open in $\mathbb{R}^{2n}$, and $F:\mathcal{O}\to\Omega$ is a diffeomorphism preserving $\sigma$, that is, satisfying*

$$F^*\sigma=\sigma, \tag{14.7}$$

*then for any $f\in C^\infty(\Omega)$, $F_\# H_f$ is Hamiltonian on $\Omega$ and*

$$F_\# H_f=H_{F^*f}, \tag{14.8}$$

*where $F^*f(y)=f(F(y))$.*

A diffeomorphism satisfying (14.7) is called a *canonical transformation*, or a symplectic transformation. Let us now look at the condition on a vector field $X$ on $\Omega$ that the flow $\mathcal{F}_X^t$ generated by $X$ preserve $\sigma$ for each $t$. There is a simple general condition in terms of the Lie derivative for a given form to be preserved.

**Lemma 14.3.** *Let* $\alpha \in \Lambda^k(\Omega)$. *Then* $\left(\mathcal{F}_X^t\right)^* \alpha = \alpha$ *for all* $t$ *if and only if*

$$\mathcal{L}_X \alpha = 0.$$

**Proof.** This is an immediate consequence of (13.57).

Recall the formula (13.49):

$$\mathcal{L}_X \alpha = d(\alpha \rfloor X) + (d\alpha) \rfloor X. \tag{14.9}$$

We apply it in the case where $\alpha = \sigma$ is the symplectic form. Clearly, (14.2) implies

$$d\sigma = 0, \tag{14.10}$$

so

$$\mathcal{L}_X \sigma = d(\sigma \rfloor X). \tag{14.11}$$

Consequently, $\mathcal{F}_X^t$ preserves the symplectic form $\sigma$ if and only if $d(\sigma \rfloor X) = 0$ on $\Omega$. In view of Poincaré's lemma, at least locally, one has a smooth function $f(x, \xi)$ such that

$$\sigma \rfloor X = df, \tag{14.12}$$

provided $d(\sigma \rfloor X) = 0$. Any two $f$'s satisfying (14.12) must differ by a constant, and it follows that such $f$ exists globally provided $\Omega$ is simply connected. In view of Proposition 14.1, (14.12) is equivalent to the identity

$$X = -H_f. \tag{14.13}$$

In particular, we have established the following result.

**Proposition 14.4.** *The flow generated by a Hamiltonian vector field* $H_f$ *preserves the symplectic form* $\sigma$.

It follows a fortiori that the flow $\mathcal{F}^t$ generated by a Hamiltonian vector field $H_f$ leaves invariant the $2n$-form

$$v = \sigma \wedge \cdots \wedge \sigma \quad (n \text{ factors}),$$

which provides a *volume form* on $\Omega$. That this volume form is preserved is known as a theorem of Liouville. This result has the following refinement. Let $S$ be a level surface of the function $f$; suppose $f$ is nondegenerate on $S$. Then we can define a $(2n-1)$-form $w$ on $S$ (giving rise to a volume element on $S$) which is also invariant under the flow $\mathcal{F}^t$, as follows. Let $X$ be any vector field on $\Omega$ such that $Xf = 1$ on $S$, and define

$$w = j^*(v \rfloor X), \tag{14.14}$$

where $j : S \hookrightarrow \Omega$ is the natural inclusion. We claim this is well defined.

**Lemma 14.5.** *The form (14.14) is independent of the choice of $X$, as long as $Xf = 1$ on $S$.*

**Proof.** The difference of two such forms is $j^*(v\rfloor Y_1)$, where $Y_1 f = 0$ on $S$, that is, $Y_1$ is tangent to $S$. Now this form, acting on vectors $Y_2, \dots, Y_{2n}$, all tangent to $S$, is merely $(j^*v)(Y_1, \dots, Y_{2n})$; but obviously $j^*v = 0$ since $\dim S < 2n$.

We can now establish the invariance of the form $w$ on $S$.

**Proposition 14.6.** *The form (14.14) is invariant under the flow $\mathcal{F}^t$ on $S$.*

**Proof.** Since $v$ is invariant under $\mathcal{F}^t$, we have

$$\begin{aligned} \mathcal{F}^{t*}w &= j^*(\mathcal{F}^{t*}v\rfloor \mathcal{F}^t_\# X) \\ &= j^*(v\rfloor \mathcal{F}^t_\# X) \\ &= w + j^*\big(v\rfloor(\mathcal{F}^t_\# X - X)\big). \end{aligned}$$

Since $\mathcal{F}^{t*}f = f$, we see that $(\mathcal{F}^t_\# X)f = 1 = Xf$, so the last term vanishes, by Lemma 14.5, and the proof is complete.

Let $\mathcal{O} \subset \mathbb{R}^n$ be open; we claim that the symplectic form $\sigma$ is well defined on $T^*\mathcal{O} = \mathcal{O} \times \mathbb{R}^n$, in the following sense. Suppose $g : \mathcal{O} \to \Omega$ is a diffeomorphism (i.e., a coordinate change). The map this induces from $T^*\mathcal{O}$ to $T^*\Omega$ is

$$G(x, \xi) = \big(g(x), \big((Dg)^t\big)^{-1}(x)\xi\big) = (y, \eta). \tag{14.15}$$

Our invariance result is

$$G^*\sigma = \sigma. \tag{14.16}$$

In fact, a stronger result is true. We can write

$$\sigma = d\kappa, \quad \kappa = \sum_j \xi_j \, dx_j, \tag{14.17}$$

where the 1-form $\kappa$ is called the *contact form.* We claim that

$$G^*\kappa = \kappa, \tag{14.18}$$

which implies (14.16), since $G^*d\kappa = dG^*\kappa$. To see (14.18), note that

$$dy_j = \sum_k \frac{\partial g_j}{\partial x_k}\, dx_k, \quad \eta_j = \sum_\ell H_{j\ell}\xi_\ell,$$

where $(H_{j\ell})$ is the matrix of $\big((Dg)^t\big)^{-1}$, that is, the inverse matrix of $(\partial g_\ell/\partial x_j)$. Hence

$$\begin{aligned}\sum_j \eta_j\, dy_j &= \sum_{j,k,\ell} \frac{\partial g_j}{\partial x_k}\, H_{j\ell}\xi_\ell\, dx_k \\ &= \sum_{k,\ell} \delta_{k\ell}\xi_\ell\, dx_k \\ &= \sum_k \xi_k\, dx_k,\end{aligned} \tag{14.19}$$

which establishes (14.18).

As a particular case, a vector field $Y$ on $\mathcal{O}$, generating a flow $\mathcal{F}_Y^t$ on $\mathcal{O}$, induces a flow $\mathcal{G}_Y^t$ on $T^*\mathcal{O}$. Not only does this flow preserve the symplectic form; in fact, $\mathcal{G}_Y^t$ is generated by the Hamiltonian vector field $H_\Phi$, where

$$\Phi(x,\xi) = \langle Y(x), \xi\rangle = \sum_j \xi_j v^j(x) \tag{14.20}$$

if $Y = \sum v^j(x)\, \partial/\partial x_j$.

The symplectic form given by (14.2) can be regarded as a special case of a general symplectic form, which is a closed, nondegenerate 2-form on a domain (or manifold) $\Omega$. Often such a form $\omega$ arises naturally, in a form not a priori looking like (14.2). It is a theorem of *Darboux* that locally one can pick coordinates in such a fashion that $\omega$ does take the standard form (14.2). We present a short proof, due to J. Moser, of that theorem.

To start, pick $p \in \Omega$, and consider $B = \omega(p)$, a nondegenerate, antisymmetric, bilinear form on the vector space $V = T_p\Omega$. It is a simple exercise in linear algebra that if one has such a form, then $\dim V$ must be even, say $2n$, and $V$ has a basis $\{e_j, f_j : 1 \le j \le n\}$ such that

$$B(e_j, e_\ell) = B(f_j, f_\ell) = 0, \quad B(e_j, f_\ell) = \delta_{j\ell}, \tag{14.21}$$

for $1 \le j, \ell \le n$. Using such a basis to impose linear coordinates $(x,\xi)$ on a neighborhood of $p$, taken to the origin, we have $\omega = \omega_0 = \sum d\xi_j \wedge dx_j$ at $p$. Thus Darboux' theorem follows from:

**Proposition 14.7.** *If $\omega$ and $\omega_0$ are closed, nondegenerate 2-forms on $\Omega$, and $\omega = \omega_0$ at $p \in \Omega$, then there is a diffeomorphism $G_1$ defined on a neighborhood of $p$, such that*

$$G_1(p) = p \ \text{ and } \ G_1^*\omega = \omega_0. \tag{14.22}$$

**Proof.** For $t \in [0,1]$, let

$$\omega_t = (1-t)\omega_0 + t\omega = \omega_0 + t\alpha, \quad \alpha = \omega - \omega_0. \tag{14.23}$$

Thus $\alpha = 0$ at $p$, and $\alpha$ is a closed 2-form. We can therefore write

$$\alpha = d\beta \tag{14.24}$$

on a neighborhood of $p$, and if $\beta$ is given by the formula (13.61) in the proof of the Poincaré lemma, we have $\beta = 0$ at $p$. Since for each $t$, $\omega_t = \omega$ at $p$, we see that each $\omega_t$ is nondegenerate on some common neighborhood of $p$, for $t \in [0, 1]$.

Our strategy will be to produce a smooth family of local diffeomorphisms $G_t$, $0 \le t \le 1$, such that $G_t(p) = p$, $G_0 = id.$, and such that $G_t^* \omega_t$ is independent of $t$, hence $G_t^* \omega_t = \omega_0$. $G_t$ will be specified by a time-varying family of vector fields, via the ODE

$$\frac{d}{dt} G_t(x) = X_t(G_t(x)), \quad G_0(x) = x. \tag{14.25}$$

We will have $G_t(p) = p$ provided $X_t(p) = 0$. To arrange for $G_t^* \omega_t$ to be independent of $t$, note that, by the product rule,

$$\frac{d}{dt} G_t^* \omega_t = G_t^* \mathcal{L}_{X_t} \omega_t + G_t^* \frac{d\omega_t}{dt}. \tag{14.26}$$

By (14.23), $d\omega_t / dt = \alpha = d\beta$, and by Proposition 13.1,

$$\mathcal{L}_{X_t} \omega_t = d(\omega_t \rfloor X_t) \tag{14.27}$$

since $\omega_t$ is closed. Thus we can write (14.26) as

$$\frac{d}{dt} G_t^* \omega_t = G_t^* d(\omega_t \rfloor X_t + \beta). \tag{14.28}$$

This vanishes provided $X_t$ is defined to satisfy

$$\omega_t \rfloor X_t = -\beta. \tag{14.29}$$

Since $\omega_t$ is nondegenerate near $p$, this does indeed uniquely specify a vector field $X_t$ near $p$, for each $t \in [0, 1]$, which vanishes at $p$, since $\beta = 0$ at $p$. The proof of Darboux' theorem is complete.

## Exercises

1. Do the linear algebra exercise stated before Proposition 14.7, as a preparation for the proof of Darboux' theorem.
2. On $\mathbb{R}^2$, identify $(x, \xi)$ with $(x, y)$, so the symplectic form is $\sigma = dy \wedge dx$. Show that

   $$X = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \quad \text{and} \quad \alpha = g\, dx - f\, dy$$

   are related by

   $$\alpha = \sigma \rfloor X.$$

   Reconsider Exercises 8–10 of §13 in light of this.
3. Show that the volume form $w$ on the level surface $S$ of $f$, given by (14.14), can be characterized as follows. Let $S_h$ be the level set $\{f(x, \xi) = c + h\}$, $S = S_0$. Given any vector field $X$ transversal to $S$, any open set $\mathcal{O} \subset S$ with smooth

boundary, let $\widetilde{\mathcal{O}}_h$ be the thin set sandwiched between $S$ and $S_h$, lying on orbits of $X$ through $\mathcal{O}$. Then, with $v = \sigma \wedge \cdots \wedge \sigma$ the volume form on $\Omega$,

$$\int\limits_{\mathcal{O}} w = \lim_{h \to 0} \frac{1}{h} \int\limits_{\widetilde{\mathcal{O}}_h} v.$$

4. A manifold $M \subset \mathbb{R}^{2n}$ is said to be *coisotropic* if, for each $p \in M$, the tangent space $T_pM$ *contains* its symplectic annihilator

$$T_p^\sigma = \{w \in \mathbb{R}^{2n} : \sigma(v, w) = 0 \text{ for all } v \in T_pM\}.$$

It is said to be *Lagrangian* if $T_pM = T_p^\sigma$ for all $p \in M$. If $M$ is coisotropic, show that it is naturally foliated by manifolds $\{N_q\}$ such that, for $p \in N_q$, $T_pN_q = T_p^\sigma$. (*Hint*: Apply Frobenius's theorem.)

## 15. First-order, scalar, nonlinear PDE

This section is devoted to a study of PDE of the form

$$F(x, u, \nabla u) = 0, \tag{15.1}$$

for a real-valued $u \in C^\infty(\Omega)$, dim $\Omega = n$, given $F(x, u, \xi)$ smooth on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, or some subdomain thereof. We study local solutions of (15.1) satisfying

$$u|_S = v, \tag{15.2}$$

where $S$ is a smooth hypersurface of $\Omega$, $v \in C^\infty(S)$. The study being local, we suppose $S$ is given by $x_n = 0$. Pick a point $x_0 \in S \subset \mathbb{R}^n$, and set $\zeta_0 = (\partial v/\partial x_1, \dots, \partial v/\partial x_{n-1})$ at $x_0$. Assume

$$\begin{gathered} F(x_0, v(x_0), (\zeta_0, \tau_0)) = 0, \\ \frac{\partial F}{\partial \xi_n} \neq 0 \text{ at this point.} \end{gathered} \tag{15.3}$$

We call this the *noncharacteristic* hypothesis on $S$. We look for a solution to (15.1) near $x_0$.

In the paragraph above, $\nabla u$ denotes the $n$-tuple $(\partial u/\partial x_1, \dots, \partial u/\partial x_n)$. In view of the material in §§13 and 14, one should be used to the idea that the 1-form $du = \sum(\partial u/\partial x_j)\, dx_j$ has an invariant meaning. As we will see later, a Riemannian metric on $\Omega$ then associates to $du$ a vector field, denoted grad $u$.

Thus, we will rephrase (15.1) as

$$F(x, u, du) = 0. \tag{15.4}$$

We think of $F$ as being defined on $T^*\Omega \times \mathbb{R}$, or some open subset of this space. The first case we will treat is the case

$$F(x, du) = 0. \tag{15.5}$$

This sort of equation is known as an *eikonal* equation. From the treatment of (15.5), we will be able to deduce a treatment of the general case (15.4), using a device known as Jacobi's trick.

The equation (15.5) is intimately connected with the theory of Hamiltonian systems. We will use this theory to construct a surface $\Lambda$ in $\mathbb{R}^{2n}$, of dimension $n$, the graph of a function $\xi = \Xi(x)$, which ought to be the graph of $du$ for some smooth $u$. Thus our first goal is to produce a geometrical description of when

$$\Lambda = \text{ graph of } \xi = \Xi(x) \tag{15.6}$$

is the graph of $du$ for some smooth $u$.

**Proposition 15.1.** *The surface (15.6) is locally the graph of $du$ for some smooth $u$ if and only if*

$$\frac{\partial \Xi_j}{\partial x_k} = \frac{\partial \Xi_k}{\partial x_j}, \quad \forall\ j, k. \tag{15.7}$$

**Proof.** This follows from the Poincaré lemma, since (15.7) is the same as the condition that $\sum \Xi_j(x)\, dx_j$ be closed.

The next step is to produce the following geometrical restatement.

**Proposition 15.2.** *The surface $\Lambda$ of (15.6) is the graph of $du$ (locally) if and only if $\sigma(X, Y) = 0$ for all vectors $X, Y$ tangent to $\Lambda$, where $\sigma$ is the symplectic form.*

If $\Lambda$ satisfies this condition, and dim $\Lambda = n$, we say $\Lambda$ is a *Lagrangian* surface.

**Proof.** We may as well check $\sigma(X_j, X_k)$ for some specific set $X_1, \dots, X_n$ of linearly independent vector fields, tangent to $\Lambda$. Thus, take

$$X_j = \frac{\partial}{\partial x_j} + \sum_{\ell} \frac{\partial \Xi_\ell}{\partial x_j} \frac{\partial}{\partial \xi_\ell}. \tag{15.8}$$

In view of the formula (14.3), we have

$$\sigma(X_j, X_k) = \frac{\partial \Xi_k}{\partial x_j} - \frac{\partial \Xi_j}{\partial x_k}, \tag{15.9}$$

so the result follows from Proposition 15.1.

To continue our pursuit of the solution to (15.5), we next specify a surface $\Sigma$, of dimension $n-1$, lying over $S = \{x_n = 0\}$, namely, with $\partial_j v = \partial v / \partial x_j$,

$$\Sigma = \{(x, \xi) : x_n = 0,\ \xi_j = \partial_j v, \ \ \text{for } 1 \le j \le n-1,\ F(x, \xi) = 0\}. \tag{15.10}$$

The noncharacteristic hypothesis implies, by the implicit function theorem, that (with $x' = (x_1, \dots, x_{n-1})$), the equation

$$F(x', 0; \partial_1 v, \dots, \partial_{n-1} v, \tau) = 0$$

implicitly defines $\tau = \tau(x')$, so (15.10) defines a smooth surface of dimension $n - 1$ through the point $(x_0, (\zeta_0, \tau_0))$.

We now define $\Lambda$ to be the union of the integral curves of the Hamiltonian vector field $H_F$ through $\Sigma$. Note that the noncharacteristic hypothesis implies that $H_F$ has a nonvanishing $\partial/\partial x_n$ component over $S$, so $\Lambda$ is a surface of dimension $n$, and is the graph of a function $\xi = \Xi(x)$, at least for $x$ close to $x_0$ (Fig. 15.1). Since $F$ is constant on integral curves of $H_F$, it follows that $F = 0$ on $\Lambda$.

Figure 15.1

**Theorem 15.3.** *The surface $\Lambda$ constructed above is locally the graph of $du$, for a solution $u$ to*

$$F(x, du) = 0, \quad u|_S = v. \tag{15.11}$$

**Proof.** We will show that $\Lambda$ is Lagrangian. So let $X, Y$ be vector fields tangent to $\Lambda$ at $(x, \xi)$ in $\Lambda \subset \mathbb{R}^{2n}$. We need to examine $\sigma(X, Y)$. First suppose $x \in S$ (i.e., $(x, \xi) \in \Sigma$). Then we may decompose $X$ and $Y$ into $X = X_1 + X_2$, $Y = Y_1 + Y_2$, with $X_1, Y_1$ tangent to $\Sigma$ and $X_2, Y_2$ multiples of $H_F$ at $(x, \xi)$. It suffices to show that $\sigma(X_1, Y_1) = 0$ and $\sigma(X_1, Y_2) = 0$. Since $\Sigma$, regarded simply as projecting over $\{x_n = 0\}$, is the graph of a gradient, Proposition 15.2 implies $\sigma(X_1, Y_1) = 0$. On the other hand, $\sigma(X_1, Y_2)$ is a multiple of $\sigma(X_1, H_F) = \langle X_1, dF\rangle = X_1F$. Since $X_1$ is tangent to $\Sigma$ and $F = 0$ on $\Sigma$, $X_1F = 0$.

Thus we know that $\sigma(X, Y) = 0$ if $X$ and $Y$ are tangent to $\Lambda$ at a point in $\Sigma$. Suppose now that $X$ and $Y$ are tangent to $\Lambda$ at a point $\mathcal{F}^t(x, \xi)$, where $(x, \xi) \in \Sigma$ and $\mathcal{F}^t$ is the flow generated by $H_F$. We have

$$\sigma(X, Y) = \bigl(\mathcal{F}^{t*}\sigma\bigr)(\mathcal{F}^t_\# X, \mathcal{F}^t_\# Y).$$

Now $\mathcal{F}^t_\# X$ and $\mathcal{F}^t_\# Y$ are tangent to $\Lambda$ at $(x, \xi) \in \Sigma$. We use the important fact that the flow generated by $H_F$ leaves the symplectic form invariant to conclude that

$$\sigma(X, Y) = \sigma(\mathcal{F}^t_\# X, \mathcal{F}^t_\# Y) = 0.$$

This shows that $\Lambda$ is Lagrangian.

Thus $\Lambda$ is the graph of $du$ for some smooth $u$, uniquely determined up to an additive constant. Pick $x_0 \in S$ and set $u(x_0) = v(x_0)$. We see that, on $S$, $\partial u/\partial x_j = \partial v/\partial x_j$ for $1 \le j \le n-1$, so this forces $u|_S = v$. We have seen that $F = 0$ on $\Lambda$, so we have solved (15.11).

An important example of an eikonal equation is

$$|d\varphi|^2 = 1 \tag{15.12}$$

on a Riemannian manifold, with metric tensor $g_{jk}$. In local coordinates, (15.12) is

$$\sum_{j,k} g^{jk}(x) \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_k} = 1, \tag{15.13}$$

where, as before, $(g^{jk})$ is the matrix inverse to $(g_{jk})$. We want to give a geometrical description of solutions to this equation. Let $\varphi$ be specified on a hypersurface $S \subset M; \varphi|_S = \psi$. Assume that $|d\psi| < 1$ on $S$. Then there are two possible sections of $T^*M$ over $S$, giving the graphs of $d\varphi$ over $S$. Pick one of then; call it $\Sigma$. As we have seen, the graph of $d\varphi$ is the flow-out $\Lambda$ of $\Sigma$, via the flow generated by $H_f$, with $f(x, \xi) = (1/2)|\xi|^2 = (1/2)\sum g^{jk}(x)\xi_j\xi_k$, that is, via the "geodesic flow" on $T^*M$. The projections onto $M$ of the integral curves of $H_f$ in $T^*M$ are geodesics on $M$. The geometrical description of $\varphi$ arises from the following result.

**Proposition 15.4.** *The level surfaces of $\varphi$ are orthogonal to the geodesics that are the projections on $M$ of the integral curves of $H_f$ through $\Sigma$.*

**Proof.** If we consider a point $x \in M$ over which $\Lambda$ is the graph of $d\varphi$, we have $(x, \xi) \in \Lambda$, $\xi = d\varphi(x)$. The assertion of the proposition is that the metric tensor, inducing an isomorphism $T_x^*M \approx T_xM$, identifies $\xi$ with $\gamma'(t)$, where $\gamma'(t)$, the tangent vector to such a geodesic, is the projection onto $T_xM$ of $H_f$ at $(x, \xi)$. Since

$$H_f = \sum \Big[ \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big], \tag{15.14}$$

this projection is equal to

$$\sum \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} = \sum g^{jk}(x) \xi_k \frac{\partial}{\partial x_j}, \tag{15.15}$$

which is in fact the image of $\xi \in T_x^*$ under the natural metric isomorphism $T_x^*M \approx T_xM$. This proves the proposition.

We can restate it this way. The metric isomorphism $T^*M \approx TM$ produces from the 1-form $d\varphi$, the *gradient vector field* grad $\varphi$. In local coordinates, with $d\varphi = \sum (\partial\varphi/\partial x_j)\, dx_j$, we have

$$\text{grad } \varphi = \sum g^{jk}(x) \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_k}. \tag{15.16}$$

Thus, the content of the last proposition is the following:

**Corollary 15.5.** *If $\gamma(t)$ is the geodesic of unit speed that is the projection on $M$ of an integral curve of $H_f$ through $\Sigma$, then*

$$\textit{grad } \varphi(x) = \gamma'(t), \quad \textit{at } x = \gamma(t). \tag{15.17}$$

Suppose, for example, that for an initial condition on $\varphi$ we take $\varphi = c$ (constant) on the surface $S$. Then, near $S$, the other level sets of $\varphi$ are described as follows. For $p \in S$, let $\gamma_p(t)$ be the unit-speed geodesic through $p$, so $\gamma_p(0) = p$, orthogonal to $S$, going in one of two possible directions, corresponding to a choice of one of two possible $\Sigma$s, as mentioned above. Then

$$\varphi(x) = c + t, \quad \text{at } x = \gamma_p(t). \tag{15.18}$$

This gives a very geometrical picture of solutions to (15.12).

On flat Euclidean space, where geodesics are just straight lines, these formulas become quite explicit. Suppose, for example, that we want to solve $|d\varphi|^2 = 1$ on $\mathbb{R}^n$ (i.e., $\sum (\partial\varphi/\partial x_j)^2 = 1$), and we prescribe

$$\varphi = 0 \text{ on a surface } S \text{ defined by } \psi(x) = 0, \tag{15.19}$$

where $\psi(x)$ is given. Then it is clear that, for $|t|$ not too large, $\varphi$ is defined by

$$\varphi\big(x + t|\nabla\psi(x)|^{-1}\nabla\psi(x)\big) = t, \quad \text{for } x \in S. \tag{15.20}$$

For small $a$, $I = (-a, a)$, the map

$$\Psi : S \times I \longrightarrow \mathbb{R}^n \tag{15.21}$$

given by

$$\Psi(x,t) = x + t|\nabla\psi(x)|^{-1}\nabla\psi(x) \tag{15.22}$$

is a diffeomorphism, but simple examples show that this can break down for large $|t|$.

Having solved the special sort of first-order PDE known as the eikonal equation, we now tackle the general case (15.1)–(15.2), subject to the condition (15.3). We use a method, called Jacobi's trick, of defining $u$ implicitly by

$$V(x, u(x)) = 0 \tag{15.23}$$

and producing a PDE for $V$ of the eikonal type. Indeed (15.23) gives, with $V = V(x,z)$,

$$\nabla_x V + V_z \nabla u = 0, \quad \text{or} \quad \nabla u = -V_z^{-1}\nabla_x V, \tag{15.24}$$

so set

$$g(x, z, \xi, \zeta) = F(x, z, -\zeta^{-1}\xi). \tag{15.25}$$

Our equation for $V$ is hence $F(x, z, -V_z^{-1}\nabla_x V) = 0$, or

$$g(x, z, \nabla_{x,z} V) = 0. \tag{15.26}$$

This is of eikonal type. Our initial condition is

$$V = z - v \quad \text{on } x_n = 0. \tag{15.27}$$

This gives $V_z \neq 0$ locally, so by the implicit function theorem, (15.23) defines a function $u(x)$, which solves the system (15.1)–(15.2).

## Exercises

1. Let $X$ be a vector field on a region $\Omega$, generating a flow $\mathcal{F}^t$, which we will assume is defined everywhere. Consider the *linear* PDE

$$\frac{\partial u}{\partial t} = Xu, \quad u(0,x) = f(x). \tag{15.28}$$

Show that a solution is given by

$$u(t,x) = f(\mathcal{F}^t x).$$

Show that the equation

$$\frac{\partial u}{\partial t} = Xu + g(t,x), \quad u(0,x) = f(x) \tag{15.29}$$

is solved by

$$u(t,x) = f(\mathcal{F}^t x) + \int_0^t g\Big(s, \mathcal{F}^{t-s}x\Big)\, ds,$$

and that

$$\frac{\partial u}{\partial t} = Xu + a(t,x)u, \quad u(0,x) = f(x) \tag{15.30}$$

is solved by

$$u(t,x) = e^{\left(\int_0^t a(s,\mathcal{F}^{t-s}x)ds\right)} f(\mathcal{F}^t x).$$

(*Hint*: The solution to (15.28) is constant on integral curves of $\partial/\partial t - X$ in $\mathbb{R} \times \Omega$. Apply Duhamel's principle to (15.29). Then find $A(t,x)$ such that (15.30) is equivalent to

$$e^{-A}\Big(\frac{\partial}{\partial t} - X\Big)\Big(e^A u\Big) = 0.)$$

2. A PDE of the form

$$\frac{\partial u}{\partial t} + \sum_{j=1}^n a_j(x,u)\frac{\partial u}{\partial x_j} = 0,$$

for a real-valued $u = u(t,x)$, is a special case of a *quasilinear* equation. Show that if we set $u(0,x) = v(x) \in C^\infty(\mathbb{R}^n)$, then there is a unique smooth solution in a neighborhood of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R}^{n+1}$, and $u(t,x)$ has the following property. For each $x_0 \in \mathbb{R}^n$, consider the vector field

$$V_{x_0} = \frac{\partial}{\partial t} + \sum_{j=1}^n a_j(x, v(x_0))\,\frac{\partial}{\partial x_j}.$$

Then $u(t,x)$ is equal to $v(x_0)$ on the integral curve of $V_{x_0}$ through $(0,x_0)$. Considering the example

$$u_t + uu_x = 0, \quad u(0,x) = e^{-x^2},$$

show that this smooth solution can cease to exist globally, due to two such lines crossing.

3. Work out explicitly the solution to

$$\Big(\frac{\partial \varphi}{\partial x}\Big)^2 + \Big(\frac{\partial \varphi}{\partial y}\Big)^2 = 1,$$

satisfying $\varphi(x,y) = 0$ on the parabola $y = x^2$, and $\partial\varphi/\partial y > 0$ there, using (15.19) and (15.20). Write a computer program to graph the level curves of $\varphi$. How does the solution break down?

4. The group of dilations of $T^*M$, defined (in local coordinates) by $D(r)(x,\xi) = (x, r\xi)$, is generated by a vector field $\vartheta$ on $T^*M$, which we call the natural radial vector field. Show that $\vartheta$ is uniquely specified by the identity

$$\sigma(\vartheta, X) = \langle X, \kappa \rangle,$$

when $X$ is a vector field on $T^*M$, and $\kappa = \sum \xi_j\, dx_j$ is the contact form (14.17).

5. Suppose $\Lambda$ is a submanifold of $T^*M$ of dimension $n = \dim M$, with $\iota : \Lambda \hookrightarrow T^*M$. Show that $\Lambda$ is Lagrangian if and only if $\iota^*\kappa$ is a closed 1-form on $\Lambda$ (hence locally exact). If $\Lambda$ is Lagrangian, relate $\iota^*\kappa = df$ on $\Lambda$ to $du$, in the context of Proposition 15.1.
6. Suppose $\Lambda$ is a Lagrangian submanifold of $T^*M$, transverse to $\vartheta$. Define a subbundle $\mathcal{V}$ of $T\Lambda$ by
$$\mathcal{V}_{(x,\xi)} = (\vartheta)^\sigma \cap T_{(x,\xi)}\Lambda,$$
where $(\vartheta)^\sigma$ is the set of vectors $v \in T_{(x,\xi)}T^*M$ such that $\sigma(\vartheta, v) = 0$. Show that $\mathcal{V}$ is an *integrable* subbundle of $T\Lambda$, that is, that Frobenius's theorem applies to $\mathcal{V}$, giving a foliation of $\Lambda$. If $\Lambda$ is the graph of $du$, $u \in C^\infty(M)$, show that the inverse image, under $\pi : \Lambda \to M$, of the level sets of $u$ gives the leaves of this foliation of $\Lambda$.

## 16. Completely integrable Hamiltonian systems

Here we will examine the consequences of having $n$ "conservation laws" for a Hamiltonian system with $n$ degrees of freedom. More precisely, suppose $\mathcal{O}$ is a region in $\mathbb{R}^{2n}$, with coordinates $(x, \xi)$ and symplectic form $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$, or more generally $\mathcal{O}$ could be a symplectic manifold of dimension $2n$. Suppose we have $n$ functions $u_1, \dots, u_n$, in involution, that is,

$$\{u_j, u_k\} = 0, \quad 1 \le j, k \le n. \tag{16.1}$$

The function $u_1 = F$ could be the energy function whose Hamiltonian vector field we want to analyze, and $u_2, \dots, u_n$ auxiliary functions, constructed to reflect conservation laws. We give some examples shortly. In case one has $n$ such functions, with linearly independent gradients, one is said to have a *completely integrable* system.

Our goal here will be to show that in such a case the flows generated by the $H_{u_j}$ can be constructed *by quadrature*. We define the last concept as follows. Given a collection of functions $\{u_j\}$, a map is said to be constructed by quadrature if it is produced by a composition of the following operations:

(i) elementary algebraic manipulation,
(ii) differentiation,
(iii) integration,
(iv) constructing inverses of maps.

To begin the study of a completely integrable system, given (16.1), consider, for a given $p \in \mathbb{R}^n$, the level set

$$M_p = \{(x, \xi) \in \mathcal{O} : u_j(x, \xi) = p_j\}. \tag{16.2}$$

Assuming the $u_j$ have linearly independent gradients, each nonempty $M_p$ is a manifold of dimension $n$. Note that each vector field $H_{u_j}$ is tangent to $M_p$, by (16.1), and therefore $\{H_{u_j} : 1 \le j \le n\}$ *spans* the tangent space to

$M_p$ at each point. Since $\sigma(H_{u_j}, H_{u_k}) = \{u_j, u_k\}$, we conclude from (16.1) that

$$\text{each } M_p \text{ is Lagrangian.} \tag{16.3}$$

If we make the "generic" hypothesis

$$\pi : M_p \to \mathbb{R}^n \text{ is a local diffeomorphism,} \tag{16.4}$$

where $\pi(x, \xi) = x$, then $M_p$ is the graph of a closed 1-form $\Xi_p$ (depending smoothly on $p$); note that $\Xi_p(x)$ is constructed by inverting a map, one of the operations involved in construction by quadrature. Furthermore, $\Xi_p$ being closed, we can construct a smooth function $\varphi(x, p)$ such that

$$M_p \text{ is the graph of } x \mapsto d_x\varphi(x, p). \tag{16.5}$$

The function $\varphi(x, p)$ is constructed from $\Xi_p$ by an integration, another ingredient in construction by quadrature. Note that a statement equivalent to (16.5) is that $\varphi$ simultaneously satisfies the eikonal equations

$$u_j\big(x, d_x\varphi(x, p)\big) = p_j, \quad 1 \le j \le n. \tag{16.6}$$

Consider now the following maps:

$$\begin{tikzcd}
(x, p) \arrow[r, "F_1"] & \big(d_p\varphi(x, p), p\big) \arrow[d, "\mathcal{C}"'] \\
(x, p) \arrow[r, "F_2"] & \big(x, d_x\varphi(x, p)\big).
\end{tikzcd} \tag{16.7}$$

Since $F_2(x, p) = (x, \Xi_p(x))$, it is clear that $F_2$ is a local diffeomorphism under our hypotheses. This implies that the matrix

$$\frac{\partial^2\varphi}{\partial p_j \partial x_k} \tag{16.8}$$

is invertible, which hence implies that $F_1$ is a local diffeomorphism (by the inverse function theorem). Hence $\mathcal{C}$ is locally defined, as a diffeomorphism:

$$\mathcal{C}\big(d_p\varphi(x, p), p\big) = \big(x, d_x\varphi(x, p)\big). \tag{16.9}$$

Write $\mathcal{C}(q, p) = (x, \xi)$. Note that

$$\begin{aligned} F_2^* \sum d\xi_j \wedge dx_j &= \sum_{j,k} \frac{\partial^2\varphi}{\partial p_k \partial x_j}\, dp_k \wedge dx_j \\ &= F_1^* \sum dp_j \wedge dq_j, \end{aligned} \tag{16.10}$$

so

$$\mathcal{C}^*\Big(\sum d\xi_j \wedge dx_j\Big) = \sum dp_j \wedge dq_j, \tag{16.11}$$

that is, $\mathcal{C}$ preserves the symplectic form. One says $\mathcal{C}$ is a canonical transformation with *generating function* $\varphi(x, p)$. Now conjugation by $\mathcal{C}$ takes the Hamiltonian vector fields $H_{u_j}$ on $(x, \xi)$-space to the Hamiltonian vector fields $H_{\tilde{u}_j}$ on $(q, p)$-space, with

$$\tilde{u}_j(q, p) = u_j \circ \mathcal{C}(q, p) = p_j,$$

in view of (16.6). Thus

$$H_{\tilde{u}_j} = \frac{\partial}{\partial q_j}, \tag{16.12}$$

so $\mathcal{C}$ conjugates the flows generated by $H_{u_j}$ to simple straight-line flows. This provides the construction of the $H_{u_j}$-flows by quadrature.

Note that if $\mathcal{O}$ has dimension 2, one needs only one function $u_1$. Thus the construction above generalizes the treatment of Hamiltonian systems on $\mathbb{R}^2$ given in §10. In fact, the approach given above, specialized to $n = 1$, is closer to the analysis in §10 than it might at first appear. Using notation as in §10, let $u_1 = f,\ p_1 = E$, so

$$M_E = \{(x, \xi) : f(x, \xi) = E\}$$

is the graph of $\xi = \psi(x, E) = d_x\varphi(x, E)$, with

$$\varphi(x, E) = \int \psi(x, E)\ dx.$$

Note that $f(x, \psi(x, E)) = E \Rightarrow f_\xi \psi_E = 1$, so

$$d_E\varphi(x, E) = \int f_\xi\bigl(x, \psi(x, E)\bigr)^{-1}\ dx, \tag{16.13}$$

and $\mathcal{C}$ maps $(\int f_\xi^{-1} dx, E)$ to $(x, \psi(x, \xi))$. To say $\mathcal{C}$ conjugates $H_f$ to $H_E = \partial/\partial q$ (in $(q, E)$ coordinates) is to say that under the time-$t$ Hamiltonian flow, $\int f_\xi^{-1} dx$ is augmented by $t$; but this is precisely the content of (10.16), namely,

$$\int f_\xi\bigl(x, \psi(x, E)\bigr)^{-1}\ dx = t + C(E). \tag{16.14}$$

We also note that, for the purpose of linearizing $H_{u_1}$, it suffices to have $\varphi(x, p)$, satisfying only the eikonal equation

$$u_1\bigl(x, d_x\varphi(x, p)\bigr) = p_1, \tag{16.15}$$

such that the matrix (16.8) is invertible. The existence of $u_2, \dots, u_n$, which together with $u_1$ are in involution, provides a way to construct $\varphi(x, p)$, but any other successful attack on (16.15) is just as satisfactory. Integrating $H_{u_1}$ by perceiving solutions to (16.15) is the essence of the Hamilton-Jacobi method.

We now look at some examples of completely integrable Hamiltonian systems. First we consider geodesic flow on a two-dimensional surface of revolution $M^2 \subset \mathbb{R}^3$. Note that $T^*M^2$ is four-dimensional, so we want $u_1$ and $u_2$, in involution. The function $u_1$ is, of course, the energy funtion

$u_1 = (1/2)\sum g^{jk}(x)\xi_j\xi_k$; as we have seen, $H_{u_1}$ generates the geodesic flow. Our function $u_2$ will arise from the group of rotations $R_\theta$ of $M^2$ about its axis of symmetry, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. This produces a group $\mathcal{R}_\theta$ of canonical transformations of $T^*M^2$, generated by a Hamiltonian vector field $X = H_{u_2}$, with $u_2(x,\xi) = \langle \partial/\partial\theta, \xi\rangle$. Since $R_\theta$ is a group of isometries of $M^2$, $\mathcal{R}_\theta$ preserves $u_1$ (i.e., $Xu_1 = 0$), or equivalently, $\{u_2, u_1\} = 0$. We have our pair of functions in involution. Thus geodesics on such a surface of revolution can be constructed by quadrature.

Another important class of completely integrable Hamiltonian systems is provided by motion in a central force field in the plane $\mathbb{R}^2$. In other words, let $x(t)$, a path in $\mathbb{R}^2$, satisfy

$$\ddot{x} = -\nabla V(x), \quad V(x) = v(|x|). \tag{16.16}$$

The Hamiltonian system is

$$\dot{x} = \nabla_\xi F, \quad \dot{\xi} = -\nabla_x F, \tag{16.17}$$

with

$$F(x,\xi) = \frac{1}{2}|\xi|^2 + v(|x|). \tag{16.18}$$

We take $u_1 = F$ and look for $u_2$, in involution. Again $u_2$ arises from a group of rotations, this time rotations of $\mathbb{R}^2$ about the origin. The method we have given by which a vector field on $\Omega$ produces a Hamiltonian vector field on $T^*\Omega$ yields the formula

$$\begin{aligned} u_2(x,\xi) &= \Big\langle \frac{\partial}{\partial\theta}, \xi \Big\rangle \\ &= \Big\langle -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}, \xi \Big\rangle \\ &= x_1\xi_2 - x_2\xi_1. \end{aligned} \tag{16.19}$$

This is the "angular momentum." The symmetry of $V(x)$ implies that the group of rotations on $T^*\mathbb{R}^2$ generated by $H_{u_2}$ preserves $F = u_1$, that is,

$$\{u_1, u_2\} = 0, \tag{16.20}$$

a fact that is also easily verified from (16.18) and (16.19) by a computation. This expresses the well-known law of conservation of angular momentum. It also establishes the complete integrability of the general central force problem on $\mathbb{R}^2$. We remark that, for the general central force problem in $\mathbb{R}^n$, conservation of angular momentum forces any path to lie in a plane, so there is no loss of generality in studying planar motion.

The case

$$V(x) = -\frac{K}{|x|} \quad (K > 0) \tag{16.21}$$

of the central force problem is called the Kepler problem. It gives Newton's description of a planet traveling about a massive star, or of two celestial bodies revolving about their center of mass. We will give a direct study of central force problems, with particular attention to the Kepler problem, in the next section.

These examples of completely integrable systems have been based on only the simplest of symmetry considerations. For many other examples of completely integrable systems, see [Wh].

We have dealt here only with the local behavior of completely integrable systems. There is also an interesting "global" theory, which among other things studies the distinction between the regular behavior of completely integable systems on the one hand and varieties of "chaotic behavior" exhibited by (globally) nonintegrable systems on the other. The reader can find out more about this important topic (begun in [Poi]) in [Mos], [TS], [Wig], and references given therein.

## Exercises

1. Let $u_1(x,\xi) = (1/2)|\xi|^2 - |x|^{-1}$ be the energy function for the Kepler problem (with $K = 1$), and let $u_2(x,\xi)$ be given by (16.19). Set
$$v_j(x,\xi) = x_j|x|^{-1} - x_j|\xi|^2 + (x\cdot\xi)\xi_j, \quad j = 1, 2.$$
$(v_1, v_2)$ is called the Lenz vector. Show that the following Poisson bracket relations hold:
$$\begin{aligned}\{u_1, v_j\} &= 0, \quad j = 1, 2,\\ \{u_2, v_j\} &= \pm v_j,\\ \{v_1, v_2\} &= 2u_1u_2.\end{aligned}$$
Also show that
$$v_1^2 + v_2^2 - 2u_1u_2^2 = 1.$$
2. Deduce that the Kepler problem is integrable in several different ways. Can you relate this to the fact that all bounded orbits are periodic?

In Exercises 3–5, suppose a given $M_p$, as in (16.2), is compact, and $du_j, 1 \le j \le n$ are linearly independent at each point of $M_p$.

3. Show that there is an $\mathbb{R}^n$-action on $M_p$, defined by $\Phi(t)(\zeta) = \mathcal{F}_1^{t_1}\cdots\mathcal{F}_n^{t_n}\zeta$, for $t = (t_1,\dots,t_n), \zeta \in M_p$, where $\mathcal{F}_j^s$ is the flow generated by $H_{u_j}$. Show that $\Phi(t+s)\zeta = \Phi(t)\Phi(s)\zeta$.
4. Show that $\mathbb{R}^n$ acts transitively on $M_p$, that is, given $\zeta \in M_p$, $\mathcal{O}(\zeta) = \{\Phi(t)\zeta : t \in \mathbb{R}^n\}$ is all of $M_p$. (*Hint*: Use the linear independence to show $\mathcal{O}(\zeta)$ is open. Then, if $\zeta_1$ is on the boundary of $\mathcal{O}(\zeta)$ in $M_p$, show that $\mathcal{O}(\zeta_1) \cap \mathcal{O}(\zeta) \ne \emptyset$.)
5. Fix $\zeta_0 \in M_p$ and let $\Gamma = \{t \in \mathbb{R}^n : \Phi(t)\zeta_0 = \zeta_0\}$. Show that $M_p$ is diffeomorphic to $\mathbb{R}^n/\Gamma$ and that this is a torus.
6. If $u_1 = F$ can be extended to a completely integrable system in two different ways, with the setting of Exercises 3–5 applicable in each case, then phase

space may be foliated by tori in two different ways. Hence intersections of various tori will be invariant under $H_F$. How does this relate to Exercise 2?

## 17. Examples of integrable systems; central force problems

In the last section it was noted that central force problems give rise to a class of completely integrable Hamiltonian systems with two degrees of freedom. Here we will look at this again, from a more elementary point of view. We look at a class of Hamiltonians on a region in $\mathbb{R}^4$, of the form

$$F(y,\eta) = F(y_1,\eta_1,\eta_2), \tag{17.1}$$

that is, with no $y_2$-dependence. Thus Hamilton's equations take the form

$$\dot{y}_j = \frac{\partial F}{\partial \eta_j}, \quad \dot{\eta}_1 = -\frac{\partial F}{\partial y_1}, \quad \dot{\eta}_2 = 0. \tag{17.2}$$

In particular, $\eta_2$ is constant on any orbit, say

$$\eta_2 = L. \tag{17.3}$$

This, in addition to $F$, provides the second conservation law implying integrability; note that $\{F,\eta_2\} = 0$. If $F(y_1,\eta_1,L) = E$ on an integral curve, we write this relation as

$$\eta_1 = \psi(y_1,L,E). \tag{17.4}$$

We can now pursue an analysis that is a variant of that described by (10.14)–(10.20). The first equation in (17.2) becomes

$$\dot{y}_1 = F_{\eta_1}\big(y_1,\psi(y_1,L,E),L\big), \tag{17.5}$$

with solution given implicitly by

$$\int F_{\eta_1}\big(y_1,\psi(y_1,L,E),L\big)^{-1}\,dy_1 = t + C. \tag{17.6}$$

Once one has $y_1(t)$, then one has

$$\eta_1(t) = \psi(y_1(t),L,E), \tag{17.7}$$

and then the remaining equation in (17.2) becomes

$$\dot{y}_2 = F_{\eta_2}\big(y_1(t),\eta_1(t),L\big), \tag{17.8}$$

which is solved by an integration.

We apply this method to the central force problem, with

$$F(x,\xi) = \frac{1}{2}|\xi|^2 + v(|x|), \quad x \in \mathbb{R}^2. \tag{17.9}$$

Use of polar coordinates is clearly suggested, so we set

$$y_1 = r,\ y_2 = \theta; \quad x_1 = r\cos\theta,\ x_2 = r\sin\theta. \tag{17.10}$$

In these coordinates, the Euclidean metric $dx_1^2 + dx_2^2$ becomes $dr^2 + r^2 d\theta^2$, so, as in (12.31), the function $F$ becomes

$$F(y,\eta) = \frac{1}{2}\big(\eta_1^2 + y_1^{-2}\eta_2^2\big) + v(y_1). \tag{17.11}$$

We see that the first pair of ODEs in (17.2) takes the form

$$\dot r = \eta_1, \quad \dot\theta = L r^{-2}, \tag{17.12}$$

where $L$ is the constant value of $\eta_2$ along an integral curve, as in (17.3). The last equation, rewritten as

$$r^2 \dot\theta = L, \tag{17.13}$$

expresses conservation of angular momentum. The remaining ODE in (17.2) becomes

$$\dot\eta_1 = L^2 r^{-3} - v'(r). \tag{17.14}$$

Note that differentiating the first equation of (17.12) and using (17.14) gives

$$\ddot r = L^2 r^{-3} - v'(r), \tag{17.15}$$

an equation that can be integrated by the methods described in (10.12)–(10.20). We will not solve (17.15) by this means here, though (17.15) will be used below, to produce (17.23). For now, we instead use (17.4)–(17.6). In the present case, (17.4) takes the form

$$\eta_1 = \pm\big[2E - 2v(r) - L^2 r^{-2}\big]^{1/2}, \tag{17.16}$$

and since $F_{\eta_1} = \eta_1$, (17.6) takes the form

$$\pm\int \big[2Er^2 - 2r^2 v(r) - L^2\big]^{-1/2} r\, dr = t + C. \tag{17.17}$$

In the case of the Kepler problem (16.21), where $v(r) = -K/r$, the resulting integral

$$\pm\int \big(2Er^2 + 2Kr - L^2\big)^{-1/2} r\, dr = t + C \tag{17.18}$$

can be evaluated using techniques of first-year calculus, by completing the square in $2Er^2 + 2Kr - L^2$. Once $r = r(t)$ is given, the equation (17.13) provides an integral formula for $\theta = \theta(t)$.

One of the most remarkable aspects of the analysis of the Kepler problem is the demonstration that orbits all lie on some conic section, given in polar coordinates by

$$r\big[1 + e\ \cos(\theta - \theta_0)\big] = ed, \tag{17.19}$$

where $e$ is the "eccentricity." We now describe the famous, elementary but ingenious trick used to demonstrate this. The method involves producing a differential equation for $r$ in terms of $\theta$, from (17.13) and (17.15). More precisely, we produce a differential equation for $u$, defined by

$$u = r^{-1}. \tag{17.20}$$

By the chain rule,

$$\frac{dr}{dt} = -r^2 \frac{du}{dt} = -r^2 \frac{du}{d\theta} \frac{d\theta}{dt} = -L \frac{du}{d\theta}, \tag{17.21}$$

in light of (17.13). Differentiating this with respect to $t$ gives

$$\frac{d^2r}{dt^2} = -L \frac{d}{dt} \frac{du}{d\theta} = -L \frac{d^2u}{d\theta^2} \frac{d\theta}{dt} = -L^2u^2 \frac{d^2u}{d\theta^2}, \tag{17.22}$$

again using (17.13). Comparing this with (17.15), we get $-L^2u^2(d^2u/d\theta^2) = L^2u^3 - v'(1/u)$ or, equivalently,

$$\frac{d^2u}{d\theta^2} + u = (Lu)^{-2} v'\Bigl(\frac{1}{u}\Bigr). \tag{17.23}$$

In the case of the Kepler problem, $v(r) = -K/r$, the right side becomes the constant $K/L^2$, so in this case (17.23) becomes the *linear* equation

$$\frac{d^2u}{d\theta^2} + u = \frac{K}{L^2}, \tag{17.24}$$

with general solution

$$u(\theta) = A\cos(\theta - \theta_0) + \frac{K}{L^2}, \tag{17.25}$$

which is equivalent to the formula (17.19) for a conic section.

For more general central force problems, the equation (17.23) is typically not linear, but it is of the form treatable by the method of (10.12)–(10.20).

## Exercises

1. Solve explicitly $w''(t) = -w(t)$, for $w$ taking values in $\mathbb{R}^2 = \mathbb{C}$. Show that $|w(t)|^2 + |w'(t)|^2 = 2E$ is constant for each orbit.
2. For $w(t)$ taking values in $\mathbb{C}$, define a new curve by
$$Z(\tau) = w(t)^2, \quad \frac{d\tau}{dt} = |w(t)|^2.$$
Show that if $w''(t) = -w(t)$, then
$$Z''(\tau) = -4E \frac{Z(\tau)}{|Z(\tau)|^3},$$
that is, $Z(\tau)$ solves the Kepler problem.
3. Analyze the flow of $H_F$, for $F$ of the form (17.1), in a manner more directly parallel to the approach in §16, in a spirit similar to (16.13) and (16.14). Note

that, with $u_1 = F,\ u_2 = \eta_2,\ p_1 = E,\ p_2 = L$, the canonical transformation $\mathcal{C}$ of (16.9) is defined by

$$\mathcal{C}\Bigl(\int F_{\eta_1}^{-1}\,dy_1, y_2 - \int F_{\eta_1}^{-1}F_{\eta_2}\,dy_1; E, L\Bigr) = \Bigl(y_1, y_2; \psi(y_1, L, E), L\Bigr),$$

where the first integrand is $F_{\eta_1}\Bigl(y_1, \psi(y_1, L, E), L\Bigr)^{-1}$, and so on.

4. Analyze the equation (17.23) for $u(\theta)$ in the following cases.

   (a) $v(r) = -K/r^2$,
   (b) $v(r) = Kr^2$,
   (c) $v(r) = -K/r + \varepsilon r^2$.

   Show that, in case (c), $u(\theta)$ is typically not periodic in $\theta$.
5. Consider motion on a surface of revolution, under a force arising from a rotationally invariant potential. Show that you can choose coordinates $(r, \theta)$ so that the metric tensor is $ds^2 = dr^2 + \beta(r)^{-1}\,d\theta^2$, and then you get a Hamiltonian system of the form (17.2) with

   $$F(y_1, \eta_1, \eta_2) = \frac{1}{2}\eta_1^2 + \frac{1}{2}\beta(y_1)\eta_2^2 + v(y_1),$$

   where $y_1 = r, y_2 = \theta$. Show that, parallel to (17.16) and (17.17), you get

   $$\dot{r} = \pm\Bigl[2E - 2v(r) - L^2\beta(r)\Bigr]^{1/2}.$$

   Show that $u = 1/r$ satisfies

   $$\frac{du}{d\theta} = \mp\frac{u^2}{L\beta(1/u)}\left[2E - 2v\Bigl(\frac{1}{u}\Bigr) - L^2\beta\Bigl(\frac{1}{u}\Bigr)\right]^{1/2}.$$

## 18. Relativistic motion

Mechanical systems considered in previous sections were formulated in the Newtonian framework. The description of a particle moving subject to a force was given in terms of a curve in *space* (with a positive-definite metric), parameterized by *time*. In the relativistic set-up, one has not space and time as separate entities, but rather *spacetime*, provided with a metric of Lorentz signature. In particular, Minkowski spacetime is $\mathbb{R}^4$ with inner product

$$\langle x, y\rangle = -x_0y_0 + \sum_{j=1}^{3} x_jy_j, \tag{18.1}$$

given $x = (x_0, \dots, x_3),\ y = (y_0, \dots, y_3)$. The behavior of a particle moving in a force field is described by a curve in spacetime, which is *timelike*, that is, its tangent vector $T$ satisfies $\langle T, T\rangle < 0$. We parameterize the curve not by time, but by arc length, so we consider a curve $x(\tau)$ satisfying

$$\langle u(\tau), u(\tau)\rangle = -1, \quad u(\tau) = x'(\tau). \tag{18.2}$$

The parameter $\tau$ is often called "proper time," and $u(\tau)$ the "4-velocity." Such a curve $x(\tau)$ is sometimes called a "world line."

Relativistic laws of physics are to be formulated in a manner depending only on the Lorentz metric (18.1), but contact is made with the Newtonian picture by using the product decomposition $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, writing $x = (t, x_s)$, $t = x_0$, and $x_s = (x_1, x_2, x_3)$. The "3-velocity" is $v = dx_s/dt$. Then

$$u = \gamma(1, v), \tag{18.3}$$

where, by (18.2),

$$\gamma = \frac{dt}{d\tau} = \bigl(1 - |v|^2\bigr)^{-1/2}, \tag{18.4}$$

with $|v|^2 = v_1^2 + v_2^2 + v_3^2$. In the limit of small velocities, $\gamma$ is close to 1.

The particle whose motion is to be described is assumed to have a constant "rest mass" $m_0$, and then the "4-momentum" is defined to be

$$p = m_0 u. \tag{18.5}$$

In terms of the decomposition (18.3),

$$p = (m_0\gamma, m_0\gamma v), \tag{18.6}$$

where $m_0 v$ is the momentum in Newtonian theory. The replacement for Newton's equation $m_0 dv/dt = f$ is

$$\frac{dp}{d\tau} = F, \tag{18.7}$$

the right side being the "Minkowski 4-force."

Newtonian theory and Einstein's relativity are related as follows. Define $m$ by $m = m_0\gamma$ and, using (18.6) and (18.7), write

$$F = \Bigl(\frac{dm}{d\tau}, \frac{d(mv)}{d\tau}\Bigr) = \Bigl(\frac{dm}{d\tau}, \gamma\,\frac{d(mv)}{dt}\Bigr). \tag{18.8}$$

Then we identify $f_C = d(mv)/dt$ as the "classical force" and write the last expression as $(f^0, \gamma f_C)$. If (18.2) is to hold, we require $f^0 = \gamma f_C \cdot v$ (the dot product in Euclidean $\mathbb{R}^3$), so

$$F = \gamma(f_C \cdot v, f_C). \tag{18.9}$$

With this correspondence, the equation (18.7) yields Newton's equation in the small velocity limit.

Since the 4-velocity has constant length, by (18.2), the Minkowski 4-force $F$ must satisfy

$$\langle F, u\rangle = 0. \tag{18.10}$$

It follows that in relativity one cannot have velocity-independent forces. The simplest situation compatible with (18.10) is for $F$ to be linear in $u$, say

$$F(x, u) = \widetilde{\mathcal{F}}(x)u, \tag{18.11}$$

where for each $x \in \mathbb{R}^4$, $\widetilde{\mathcal{F}}(x)$ is a linear transformation on $\mathbb{R}^4$; in other words, $\widetilde{\mathcal{F}}$ is a tensor field of type $(1,1)$. The condition (18.10) holds provided $\widetilde{\mathcal{F}}$ is skew-adjoint with respect to the Lorentz inner product:

$$\langle \widetilde{\mathcal{F}}u, w\rangle = -\langle u, \widetilde{\mathcal{F}}w\rangle. \tag{18.12}$$

Equivalently, if we consider the related tensor $\mathcal{F}$ of type $(0,2)$,

$$\mathcal{F}(u,w) = \langle u, \widetilde{\mathcal{F}}w\rangle, \tag{18.13}$$

then $\mathcal{F}$ is antisymmetric, that is, $\mathcal{F}$ is a 2-form. In index notation, $\mathcal{F}_{jk} = h_{j\ell}\mathcal{F}^{\ell}{}_{k}$, where $h_{jk}$ defines the Lorentz metric.

The electromagnetic field is of this sort. The classical force exerted by an electric field $E$ and a magnetic field $B$ on a particle with charge $e$ is the Lorentz force

$$f_L = e(E + v \times B), \tag{18.14}$$

as in (12.40). Using this in (18.9) gives, for $u = (u^0, v)$,

$$\widetilde{\mathcal{F}}u = e(E \cdot v, Eu^0 + v \times B). \tag{18.15}$$

Consequently the 2-form $\mathcal{F}$ is $\mathcal{F}(u,w) = e\sum \mathcal{F}_{\mu\nu}u_\mu w_\nu$ with

$$\big(\mathcal{F}_{\mu\nu}\big) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{18.16}$$

In relativity it is this 2-form which is called the electromagnetic field.

To change notation slightly, let us denote by $\mathcal{F}$ the 2-form described by (18.16), namely, with $t = x_0$,

$$\mathcal{F} = \sum_{j=1}^{3} E_j\, dx_j \wedge dt + B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2. \tag{18.17}$$

Thus the force in (18.11) is now denoted by $e\widetilde{\mathcal{F}}u$.

We can construct a Lagrangian giving the equation of motion (18.7), (18.11), in a fashion similar to (12.44). The part of Maxwell's equations for the electromagnetic field recorded as (12.41) is equivalent to the statement that

$$d\mathcal{F} = 0. \tag{18.18}$$

Thus we can find a 1-form $\mathcal{A}$ on Minkowski spacetime such that

$$\mathcal{F} = d\mathcal{A}. \tag{18.19}$$

Then we can set

$$L(x,u) = \frac{1}{2}m_0\langle u, u\rangle + e\langle \mathcal{A}, u\rangle, \tag{18.20}$$

and the force law $dp/d\tau = e\widetilde{\mathcal{F}}(x)u$ is seen to be equivalent to

$$\frac{d}{d\tau} L_u - L_x = 0. \tag{18.21}$$

See Exercise 3 below. In this case, the Legendre transform (12.13) becomes, with $u^b = (-u^0, u^1, u^2, u^3)$,

$$(x, \xi) = (x, m_0 u^b + e\mathcal{A}), \tag{18.22}$$

and we get the Hamiltonian system

$$\frac{dx}{d\tau} = E_\xi, \quad \frac{d\xi}{d\tau} = -E_x, \tag{18.23}$$

with

$$E(x, \xi) = \frac{1}{2m_0} \big\langle \xi - e\mathcal{A}, \xi - e\mathcal{A} \big\rangle. \tag{18.24}$$

## Exercises

1. Consider a constant electromagnetic field of the form

$$E = (1, 0, 0), \quad B = 0.$$

Work out the solution to Newton's equation

$$m \frac{dv}{dt} = e(E + v \times B), \quad v = \frac{dx}{dt},$$

for the path $x = x(t)$ in $\mathbb{R}^3$ of a particle of charge $e$, mass $m$, moved by the Lorentz force arising from this field. Then work out the solution to the relativistic equation

$$m_0 \frac{du}{d\tau} = e(E \cdot v, Eu^0 + v \times B),$$

with $u = (u^0, v)$ (having square norm $-1$), $u = dx/d\tau$, for the path in $\mathbb{R}^4$ of a particle of charge $e$, rest mass $m_0$, moved by such an electromagnetic field. Compare the results. Do the same for

$$E = 0, \quad B = (1, 0, 0).$$

2. Take another look at Exercise 3 in §12.
3. Show that taking (18.20) for the Lagrangian implies that Lagrange's equation (18.21) is equivalent to the force law $dp/d\tau = e\widetilde{\mathcal{F}}u$, on Minkowski spacetime. (*Hint*: To compute $L_x$, use

$$d\langle \mathcal{A}, u\rangle = -(d\mathcal{A})\rfloor u + \mathcal{L}_u \mathcal{A},$$

regard $u$ as independent of $x$, and note that $d\mathcal{A}/d\tau = \nabla_u \mathcal{A} = \mathcal{L}_u \mathcal{A}$, in that case.)

Compare Exercise 4 in §12.

4. Verify formula (18.16) for $\mathcal{F}_{\mu\nu}$. Show that the matrix for $\widetilde{\mathcal{F}}$ has the same form, except all $E_j$ carry plus signs.

5. An alternative sign convention for the Lorentz metric on Minkowski spacetime is to replace (18.1) by $\langle x, y\rangle = x_0y_0 - \sum_{j\ge 1} x_jy_j$. Show that this leads to a sign change in (18.16). What other sign changes arise?
6. Suppose a 1-form $\mathcal{A}$ is given, satisfying (18.19), on a general four-dimensional Lorentz manifold $M$. Let $L : TM \to \mathbb{R}$ be given by (18.20). Use the set-up described in (12.51)–(12.65) to derive equations of motion, extending the Lorentz force law from Minkowski spacetime to any Lorentz 4-manifold. (*Hint*: In analogy with (12.64), show that $L_v$ is given by

$$L_v = m_0u + e\mathcal{A}^{\#},$$

where $\mathcal{A}^{\#}$ is the vector field corresponding to $\mathcal{A}$ via the metric (by raising indices). Taking a cue from Exercise 3, show that $L_x$ satisfies

$$L_x = e\widetilde{\mathcal{F}}u + e\nabla_u\mathcal{A}^{\#}.$$

Deduce that the equation

$$m_0\nabla_u u = e\widetilde{\mathcal{F}}u$$

is the stationary condition for this Lagrangian.)

## 19. Topological applications of differential forms

Differential forms are a fundamental tool in calculus. In addition, they have important applications to topology. We give a few here, starting with simple proofs of some important topological results of Brouwer.

**Proposition 19.1.** *There is no continuous retraction $\varphi : B \to S^{n-1}$ of the closed unit ball $B$ in $\mathbb{R}^n$ onto its boundary $S^{n-1}$.*

In fact, it is just as easy to prove the following more general result. The approach we use is adapted from [Kan].

**Proposition 19.2.** *If $\overline{M}$ is a compact, oriented manifold with nonempty boundary $\partial M$, there is no continuous retraction $\varphi : \overline{M} \to \partial M$.*

**Proof.** A retraction $\varphi$ satisfies $\varphi \circ j(x) = x$, where $j : \partial M \hookrightarrow \overline{M}$ is the natural inclusion. By a simple approximation, if there were a continuous retraction there would be a smooth one, so we can suppose $\varphi$ is smooth.

Pick $\omega \in \Lambda^{n-1}(\partial M)$ to be the volume form on $\partial M$, endowed with some Riemannian metric ($n = \dim M$), so $\int_{\partial M} \omega > 0$. Now apply Stokes' theorem to $\alpha = \varphi^*\omega$. If $\varphi$ is a retraction, $j^*\varphi^*\omega = \omega$, so we have

$$\int\limits_{\partial M} \omega = \int\limits_{M} d\varphi^*\omega. \tag{19.1}$$

But $d\varphi^*\omega = \varphi^*d\omega = 0$, so the integral (19.1) is zero. This is a contradiction, so there can be no retraction.

A simple consequence of this is the famous Brouwer fixed-point theorem.

**Theorem 19.3.** *If $F : B \to B$ is a continuous map on the closed unit ball in $\mathbb{R}^n$, then $F$ has a fixed point.*

**Proof.** We are claiming that $F(x) = x$ for some $x \in B$. If not, define $\varphi(x)$ to be the endpoint of the ray from $F(x)$ to $x$, continued until it hits $\partial B = S^{n-1}$. It is clear that $\varphi$ would be a retraction, contradicting Proposition 19.1.

We next show that an even-dimensional sphere cannot have a smooth nonvanishing vector field.

**Proposition 19.4.** *There is no smooth nonvanishing vector field on $S^n$ if $n = 2k$ is even.*

**Proof.** If $X$ were such a vector field, we could arrange it to have unit length, so we would have $X : S^n \to S^n$, with $X(v) \perp v$ for $v \in S^n \subset \mathbb{R}^{n+1}$. Thus there is a unique unit-speed geodesic $\gamma_v$ from $v$ to $X(v)$, of length $\pi/2$. Define a smooth family of maps $F_t : S^n \to S^n$ by $F_t(v) = \gamma_v(t)$. Thus $F_0(v) = v$, $F_{\pi/2}(v) = X(v)$, and $F_\pi = A$ would be the *antipodal map*, $A(v) = -v$. By (13.63), we deduce that $A^*\omega - \omega = d\beta$ is exact, where $\omega$ is the volume form on $S^n$. Hence, by Stokes' theorem,

$$\int_{S^n} A^*\omega = \int_{S^n} \omega. \tag{19.2}$$

On the other hand, it is straightforward that $A^*\omega = (-1)^{n+1}\omega$, so (19.2) is possible only when $n$ is odd.

Note that an important ingredient in the proof of both Proposition 19.2 and Proposition 19.4 is the existence of $n$-forms on a compact, oriented, $n$-dimensional manifold $M$ which are not exact (though of course they are closed). We next establish the following important counterpoint to the Poincaré lemma.

**Proposition 19.5.** *If $M$ is a compact, connected, oriented manifold of dimension $n$ and $\alpha \in \Lambda^n M$, then $\alpha = d\beta$ for some $\beta \in \Lambda^{n-1}(M)$ if and only if*

$$\int_M \alpha = 0. \tag{19.3}$$

We have already discussed the necessity of (19.3). To prove the sufficiency, we first look at the case $M = S^n$.

In that case, any $n$-form $\alpha$ is of the form $a(x)\omega$, $a \in C^\infty(S^n)$, $\omega$ the volume form on $S^n$, with its standard metric. The group $G = \mathrm{SO}(n+1)$ of rotations of $\mathbb{R}^{n+1}$ acts as a transitive group of isometries on $S^n$. In Appendix B, Manifolds, Vector Bundles, and Lie Groups, we construct the integral of functions over $\mathrm{SO}(n+1)$, with respect to Haar measure.

As noted in Appendix B, we have the map $\mathrm{Exp} : \mathrm{Skew}(n+1) \to \mathrm{SO}(n+1)$, giving a diffeomorphism from a ball $\mathcal{O}$ about 0 in $\mathrm{Skew}(n+1)$ onto an open set $U \subset \mathrm{SO}(n+1) = G$, a neighborhood of the identity. Since $G$ is compact, we can pick a finite number of elements $\xi_j \in G$ such that the open sets $U_j = \{\xi_j g : g \in U\}$ cover $G$. Pick $\eta_j \in \mathrm{Skew}(n+1)$ such that $\mathrm{Exp}\ \eta_j = \xi_j$. Define $\Phi_{jt} : U_j \to G$ for $0 \le t \le 1$ by

$$\Phi_{jt}\bigl(\xi_j\ \mathrm{Exp}(A)\bigr) = (\mathrm{Exp}\ t\eta_j)(\mathrm{Exp}\ tA), \quad A \in \mathcal{O}. \tag{19.4}$$

Now partition $G$ into subsets $\Omega_j$, each of whose boundaries has content zero, such that $\Omega_j \subset U_j$. If $g \in \Omega_j$, set $g(t) = \Phi_{jt}(g)$. This family of elements of $SO(n+1)$ defines a family of maps $F_{gt} : S^n \to S^n$. Now, as in (13.60), we have

$$\alpha = g^*\alpha - d\kappa_g(\alpha), \quad \kappa_g(\alpha) = \int_0^1 F_{gt}^*(\alpha \rfloor X_{gt})\ dt, \tag{19.5}$$

for each $g \in \mathrm{SO}(n+1)$, where $X_{gt}$ is the family of vector fields on $S^n$ generated by $F_{gt}$, as in (13.58). Therefore,

$$\alpha = \int\limits_G g^*\alpha\ dg - d\int\limits_G \kappa_g(\alpha)\ dg. \tag{19.6}$$

Now the first term on the right is equal to $\overline{\alpha}\omega$, where $\overline{\alpha} = \int a(g\cdot x)dg$ is a constant; in fact, the constant is

$$\overline{\alpha} = \frac{1}{\mathrm{Vol}\ S^n}\int\limits_{S^n} \alpha. \tag{19.7}$$

Thus, in this case, (19.3) is precisely what serves to make (19.6) a representation of $\alpha$ as an exact form. This finishes the case $M = S^n$.

For a general compact, oriented, connected $M$, proceed as follows. Cover $M$ with open sets $\mathcal{O}_1, \dots, \mathcal{O}_K$ such that each $\overline{\mathcal{O}}_j$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$. Set $U_1 = \mathcal{O}_1$, and inductively enlarge each $\mathcal{O}_j$ to $U_j$, so that $\overline{U}_j$ is also diffeomorphic to the closed ball, and such that $U_{j+1} \cap U_j \ne \emptyset$, $1 \le j < K$. You can do this by drawing a simple curve from $\overline{\mathcal{O}}_{j+1}$ to a point in $U_j$ and thickening it. Pick a smooth partition of unity $\varphi_j$, subordinate to this cover.

Given $\alpha \in \Lambda^n M$, satisfying (19.3), take $\tilde{\alpha}_j = \varphi_j\alpha$. Most likely $\int \tilde{\alpha}_1 = c_1 \ne 0$, so take $\sigma_1 \in \Lambda^n M$, with compact support in $U_1 \cap U_2$, such that $\int \sigma_1 = c_1$. Set $\alpha_1 = \tilde{\alpha}_1 - \sigma_1$, and redefine $\tilde{\alpha}_2$ to be the old $\tilde{\alpha}_2$ plus $\sigma_1$. Make a similar construction using $\int \tilde{\alpha}_2 = c_2$, and continue. When you are done, you have

$$\alpha = \alpha_1 + \cdots + \alpha_K, \tag{19.8}$$

with $\alpha_j$ compactly supported in $U_j$. By construction,

$$\int \alpha_j = 0, \tag{19.9}$$

for $1 \le j < K$. But then (19.3) implies $\int \alpha_K = 0$ too.

Now pick $p \in S^n$ and define smooth maps

$$\psi_j : M \longrightarrow S^n, \tag{19.10}$$

which map $U_j$ diffeomorphically onto $S^n \setminus p$ and map $M \setminus U_j$ to $p$. There is a unique $v_j \in \Lambda^n S^n$, with compact support in $S^n \setminus p$, such that $\psi^* v_j = \alpha_j$. Clearly

$$\int\limits_{S^n} v_j = 0,$$

so by the case $M = S^n$ of Proposition 19.5 already established, we know that $v_j = dw_j$ for some $w_j \in \Lambda^{n-1} S^n$, and then

$$\alpha_j = d\beta_j, \quad \beta_j = \psi_j^* w_j. \tag{19.11}$$

This concludes the proof.

We can sharpen and extend some of the topological results given above, using the notion of the degree of a map between compact, oriented surfaces. Let $X$ and $Y$ be compact, oriented, $n$-dimensional surfaces. We want to define the degree of a smooth map $F : X \to Y$. To do this, assume $Y$ is connected. We pick $\omega \in \Lambda^n Y$ such that

$$\int\limits_Y \omega = 1. \tag{19.12}$$

We want to define

$$\operatorname{Deg}(F) = \int\limits_X F^* \omega. \tag{19.13}$$

The following result shows that $\operatorname{Deg}(F)$ is indeed well defined by this formula. The key argument is an application of Proposition 19.5.

**Lemma 19.6.** *The quantity (19.13) is independent of the choice of $\omega$, as long as (19.12) holds.*

**Proof.** Pick $\omega_1 \in \Lambda^n Y$ satisfying $\int_Y \omega_1 = 1$, so $\int_Y \omega - \omega_1 = 0$. By Proposition 19.5, this implies

$$\omega - \omega_1 = d\alpha, \quad \text{for some } \alpha \in \Lambda^{n-1} Y. \tag{19.14}$$

Thus

$$\int\limits_X F^*\omega - \int\limits_X F^*\omega_1 = \int\limits_X dF^*\alpha = 0, \tag{19.15}$$

and the lemma is proved.

The following is a most basic property.

**Proposition 19.7.** *If $F_0$ and $F_1$ are homotopic, then Deg$(F_0) =$ Deg$(F_1)$.*

**Proof.** As noted in Exercise 7 of §13, if $F_0$ and $F_1$ are homotopic, then $F_0^*\omega - F_1^*\omega$ is exact, say $d\beta$, and of course $\int_X \, d\beta = 0$.

We next give an alternative formula for the degree of a map, which is very useful in many applications. A point $y_0 \in Y$ is called a regular value of $F$ provided that, for each $x \in X$ satisfying $F(x) = y_0$, $DF(x) : T_xX \to T_{y_0}Y$ is an isomorphism. The easy case of Sard's theorem, discussed in Appendix B, implies that *most* points in $Y$ are regular. Endow $X$ with a volume element $\omega_X$, and similarly endow $Y$ with $\omega_Y$. If $DF(x)$ is invertible, define $JF(x) \in \mathbb{R} \setminus 0$ by $F^*(\omega_Y) = JF(x)\omega_X$. Clearly the *sign* of $JF(x)$ (i.e., sgn $JF(x) = \pm 1$), is independent of the choices of $\omega_X$ and $\omega_Y$, as long as they determine the given orientations of $X$ and $Y$.

**Proposition 19.8.** *If $y_0$ is a regular value of $F$, then*

$$\text{Deg}(F) = \sum \{\text{sgn } JF(x_j) : F(x_j) = y_0\}. \tag{19.16}$$

**Proof.** Pick $\omega \in \Lambda^n Y$, satisfying (19.12), with support in a small neighborhood of $y_0$. Then $F^*\omega$ will be a sum $\sum \omega_j$, with $\omega_j$ supported in a small neighborhood of $x_j$, and $\int \omega_j = \pm 1$ as sgn $JF(x_j) = \pm 1$.

The following result is a powerful tool in degree theory.

**Proposition 19.9.** *Let $\overline{M}$ be a compact, oriented manifold with boundary. Assume that dim $M = n + 1$. Given a smooth map $F : \overline{M} \to Y$, let $f = F\big|_{\partial M} : \partial M \to Y$. Then*

$$\text{Deg}(f) = 0.$$

**Proof.** Applying Stokes' theorem to $\alpha = F^*\omega$, we have

$$\int\limits_{\partial M} f^*\omega = \int\limits_M dF^*\omega.$$

But $dF^*\omega = F^*d\omega$, and $d\omega = 0$ if dim $Y = n$, so we are done.

An easy corollary of this is another proof of Brouwer's no-retraction theorem. Compare the proof of Proposition 19.2.

**Corollary 19.10.** *If* $\overline{M}$ *is a compact, oriented manifold with nonempty boundary* $\partial M$*, then there is no smooth retraction* $\varphi : \overline{M} \to \partial M$*.*

**Proof.** Without loss of generality, we can assume that $\overline{M}$ is connected. If there were a retraction, then $\partial M = \varphi(\overline{M})$ must also be connected, so Proposition 19.9 applies. But then we would have, for the map $id. = \varphi\big|_{\partial M}$, the contradiction that its degree is both 0 and 1.

For another application of degree theory, let $X$ be a compact, smooth, oriented hypersurface in $\mathbb{R}^{n+1}$, and set $\Omega = \mathbb{R}^{n+1} \setminus X$. (Assume $n \geq 1$.) Given $p \in \Omega$, define

$$F_p : X \longrightarrow S^n, \quad F_p(x) = \frac{x-p}{|x-p|}. \tag{19.17}$$

It is clear that $\operatorname{Deg}(F_p)$ is constant on each connected component of $\Omega$. It is also easy to see that, when $p$ crosses $X$, $\operatorname{Deg}(F_p)$ jumps by $\pm 1$. Thus $\Omega$ has at least two connected components. This is most of the smooth case of the Jordan-Brouwer separation theorem:

**Theorem 19.11.** *If* $X$ *is a smooth, compact, oriented hypersurface of* $\mathbb{R}^{n+1}$*, which is connected, then* $\Omega = \mathbb{R}^{n+1} \setminus X$ *has exactly two connected components.*

**Proof.** Since $X$ is oriented, it has a smooth, global, normal vector field. Use this to separate a small collar neighborhood $\mathcal{C}$ of $X$ into two pieces; $\mathcal{C} \setminus X = \mathcal{C}_0 \cup \mathcal{C}_1$. The collar $\mathcal{C}$ is diffeomorphic to $[-1, 1] \times X$, and each $\mathcal{C}_j$ is clearly connected. It suffices to show that any connected component $\mathcal{O}$ of $\Omega$ intersects either $\mathcal{C}_0$ or $\mathcal{C}_1$. Take $p \in \partial\mathcal{O}$. If $p \notin X$, then $p \in \Omega$, which is open, so $p$ cannot be a boundary point of any component of $\Omega$. Thus $\partial\mathcal{O} \subset X$, so $\mathcal{O}$ must intersect a $\mathcal{C}_j$. This completes the proof.

Let us note that, of the two components of $\Omega$, exactly one is unbounded, say $\Omega_0$, and the other is bounded; call it $\Omega_1$. Then we claim that if $X$ is given the orientation it gets as $\partial\Omega_1$,

$$p \in \Omega_j \Longrightarrow \operatorname{Deg}(F_p) = j. \tag{19.18}$$

Indeed, for $p$ very far from $X$, $F_p : X \to S^n$ is not onto, so its degree is 0. And when $p$ crosses $X$, from $\Omega_0$ to $\Omega_1$, the degree jumps by $+1$.

For a simple closed curve in $\mathbb{R}^2$, this result is the smooth case of the Jordan curve theorem. That special case of the argument given above can be found in [Sto].

We remark that, with a bit more work, one can show that any compact, smooth hypersurface in $\mathbb{R}^{n+1}$ is orientable. For one proof, see Appendix B to Chapter 5.

The next application of degree theory is useful in the study of closed orbits of planar vector fields. Let $C$ be a simple, smooth, closed curve in $\mathbb{R}^2$, parameterized by arc length, of total length $L$. Say $C$ is given by $x = \gamma(t),\ \gamma(t+L) = \gamma(t)$. Then we have a unit tangent field to $C,\ T(\gamma(t)) = \gamma'(t)$, defining

$$T : C \longrightarrow S^1. \tag{19.19}$$

**Proposition 19.12.** *For $T$ given by (19.19), we have*

$$\mathit{Deg}(T) = 1. \tag{19.20}$$

**Proof.** Pick a tangent line $\ell$ to $C$ such that $C$ lies on one side of $\ell$, as in Fig. 19.1. Without changing $\mathrm{Deg}(T)$, you can flatten out $C$ a little, so it intersects $\ell$ along a line segment, from $\gamma(L_0)$ to $\gamma(L) = \gamma(0)$, where we take $L_0 = L - 2\varepsilon,\ L_1 = L - \varepsilon$.

Now $T$ is close to the map $T_s : C \to S^1$, given by

$$T_s\big(\gamma(t)\big) = \frac{\gamma(t+s) - \gamma(t)}{|\gamma(t+s) - \gamma(t)|}, \tag{19.21}$$

for any $s > 0$ small enough; hence $T$ and $T_s$ are homotopic, for small positive $s$. It follows that $T$ and $T_s$ are homotopic for all $s \in (0, L)$. Furthermore, we can even let $s = s(t)$ be any continuous function $s : [0, L] \to (0, L)$ such that $s(0) = s(L)$. In particular, $T$ is homotopic to the map $V : C \to S^1$, obtained from (19.21) by taking

$$s(t) = L_1 - t, \quad \text{for } t \in [0, L_0],$$

and $s(t)$ going monotonically from $L_1 - L_0$ to $L_1$, for $t \in [L_0, L]$. Note that

$$V\big(\gamma(t)\big) = \frac{\gamma(L_1) - \gamma(t)}{|\gamma(L_1) - \gamma(t)|}, \quad 0 \le t \le L_0.$$

The parts of $V$ over the ranges $0 \le t \le L_0$ and $L_0 \le t \le L$, respectively, are illustrated in Figures 19.1 and 19.2. We see that $V$ maps the segment of $C$ from $\gamma(0)$ to $\gamma(L_0)$ into the lower half of the circle $S^1$, and it maps the segment of $C$ from $\gamma(L_0)$ to $\gamma(L)$ into the upper half of the circle $S^1$. Therefore, $V$ (hence $T$) is homotopic to a one-to-one map of $C$ onto $S^1$, preserving orientation, and (19.20) is proved.

The material of this section can be cast in the language of deRham cohomology, which we now define. Let $M$ be a smooth manifold. A smooth $k$-form $u$ is said to be *exact* if $u = dv$ for some smooth $(k-1)$-form $v$, and *closed* if $du = 0$. Since $d^2 = 0$, every exact form is closed:

$$\mathcal{E}^k(M) \subset \mathcal{C}^k(M), \tag{19.22}$$

Figure 19.1

Figure 19.2

where $\mathcal{E}^k(M)$ and $\mathcal{C}^k(M)$ denote respectively the spaces of exact and closed $k$-forms. The deRham cohomology groups are defined as quotient spaces:

$$\mathcal{H}^k(M) = \mathcal{C}^k(M)/\mathcal{E}^k(M). \tag{19.23}$$

There are no nonzero $(-1)$-forms, so $\mathcal{E}^0(M) = 0$. A 0-form is a real-valued function, and it is closed if and only if it is constant on each connected component of $M$, so

$$\mathcal{H}^0(M) \approx \mathbb{R}^\nu, \quad \nu = \# \text{ connected components of } M. \tag{19.24}$$

An immediate consequence of Proposition 19.5 is the following:

**Proposition 19.13.** *If $M$ is a compact, connected, oriented manifold of dimension $n$, then*

$$\mathcal{H}^n(M) \approx \mathbb{R}. \tag{19.25}$$

Via the pull-back of forms, a smooth map $F : X \to Y$ between two manifolds induces maps on cohomology:

$$F^* : \mathcal{H}^j(Y) \longrightarrow \mathcal{H}^j(X). \tag{19.26}$$

If $X$ and $Y$ are both compact, connected, oriented, and of dimension $n$, then we have $F^* : \mathcal{H}^n(Y) \to \mathcal{H}^n(X)$, and, via the isomorphism $\mathcal{H}^n(X) \approx \mathbb{R} \approx \mathcal{H}^n(Y)$ arising from integration of $n$-forms, this map is simply multiplication by Deg $F$.

The subject of deRham cohomology plays an important role in material we develop later, such as Hodge theory, in Chapter 5, and index theory, in Chapter 10.

## Exercises

1. Show that the identity map $I : X \to X$ has degree 1.
2. Show that if $F : X \to Y$ is not onto, then $\mathrm{Deg}(F) = 0$.
3. If $A : S^n \to S^n$ is the antipodal map, show that $\mathrm{Deg}(A) = (-1)^{n-1}$.
4. Show that the homotopy invariance property given in Proposition 19.7 can be deduced as a corollary of Proposition 19.9. (*Hint*: Take $\overline{M} = X \times [0,1]$.)
5. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. Show that if we identify $S^2 \approx \mathbb{C} \cup \{\infty\}$, then $p : \mathbb{C} \to \mathbb{C}$ has a unique continuous extension $\widetilde{p} : S^2 \to S^2$, with $\widetilde{p}(\infty) = \infty$. Show that
$$\mathrm{Deg}\, \widetilde{p} = n.$$
Deduce that $\widetilde{p} : S^2 \to S^2$ is onto, and hence that $p : \mathbb{C} \to \mathbb{C}$ is onto. In particular, each nonconstant polynomial in $z$ has a complex root.
This result is the fundamental theorem of algebra.

## 20. Critical points and index of a vector field

A *critical point* of a vector field $V$ is a point at which $V$ vanishes. Let $V$ be a vector field defined on a neighborhood $\mathcal{O}$ of $p \in \mathbb{R}^n$, with a single critical point, at $p$. Then, for any small ball $B_r$ about $p$, $B_r \subset \mathcal{O}$, we have a map

$$V_r : \partial B_r \to S^{n-1}, \quad V_r(x) = \frac{V(x)}{|V(x)|}. \tag{20.1}$$

The degree of this map is called the *index* of $V$ at $p$, denoted $\mathrm{ind}_p(V)$; it is clearly independent of $r$. If $V$ has a finite number of critical points, then the index of $V$ is defined to be

$$\text{Index}(V) = \sum \text{ind}_{p_j}(V). \tag{20.2}$$

If $\psi : \mathcal{O} \to \mathcal{O}'$ is an orientation-preserving diffeomorphism, taking $p$ to $p$ and $V$ to $W$, then we claim that

$$\text{ind}_p(V) = \text{ind}_p(W). \tag{20.3}$$

In fact, $D\psi(p)$ is an element of $GL(n, \mathbb{R})$ with positive determinant, so it is homotopic to the identity, and from this it readily follows that $V_r$ and $W_r$ are homotopic maps of $\partial B_r \to S^{n-1}$. Thus one has a well-defined notion of the index of a vector field with a finite number of critical points on any oriented manifold $M$.

A vector field $V$ on $\mathcal{O} \subset \mathbb{R}^n$ is said to have a nondegenerate critical point at $p$ provided $DV(p)$ is a nonsingular $n \times n$ matrix. The following formula is convenient.

**Proposition 20.1.** *If $V$ has a nondegenerate critical point at $p$, then*

$$\text{ind}_p(V) = \text{sgn det } DV(p). \tag{20.4}$$

**Proof.** If $p$ is a nondegenerate critical point, and we set $\psi(x) = DV(p)x$, $\psi_r(x) = \psi(x)/|\psi(x)|$, for $x \in \partial B_r$, it is readily verified that $\psi_r$ and $V_r$ are homotopic, for $r$ small. The fact that $\text{Deg}(\psi_r)$ is given by the right side of (20.4) is an easy consequence of Proposition 19.8.

The following is an important global relation between index and degree.

**Proposition 20.2.** *Let $\overline{\Omega}$ be a smooth bounded region in $\mathbb{R}^{n+1}$. Let $V$ be a vector field on $\overline{\Omega}$, with a finite number of critical points $p_j$, all in the interior $\Omega$. Define $F : \partial\Omega \to S^n$ by $F(x) = V(x)/|V(x)|$. Then*

$$\text{Index}(V) = \text{Deg}(F). \tag{20.5}$$

**Proof.** If we apply Proposition 19.9 to $\overline{M} = \overline{\Omega} \setminus \bigcup_j B_\varepsilon(p_j)$, we see that $\text{Deg}(F)$ is equal to the sum of degrees of the maps of $\partial B_\varepsilon(p_j)$ to $S^n$, which gives (20.5).

Next we look at a process of producing vector fields in higher-dimensional spaces from vector fields in lower-dimensional spaces.

**Proposition 20.3.** *Let $W$ be a vector field on $\mathbb{R}^n$, vanishing only at $0$. Define a vector field $V$ on $\mathbb{R}^{n+k}$ by $V(x, y) = \big(W(x), y\big)$. Then $V$ vanishes only at $(0, 0)$. Then we have*

$$\text{ind}_0 W = \text{ind}_{(0,0)} V. \tag{20.6}$$

**Proof.** If we use Proposition 19.8 to compute degrees of maps, and choose $y_0 \in S^{n-1} \subset S^{n+k-1}$, a regular value of $W_r$, and hence also for $V_r$, this identity follows.

We turn to a more sophisticated variation. Let $X$ be a compact, oriented, $n$-dimensional submanifold of $\mathbb{R}^{n+k}$, $W$ a (tangent) vector field on $X$ with a finite number of critical points $p_j$. Let $\overline{\Omega}$ be a small tubular neighborhood of $X$, $\pi : \overline{\Omega} \to X$ mapping $z \in \overline{\Omega}$ to the nearest point in $X$. Let $\varphi(z) = \operatorname{dist}(z, X)^2$. Now define a vector field $V$ on $\overline{\Omega}$ by

$$V(z) = W(\pi(z)) + \nabla\varphi(z). \tag{20.7}$$

**Proposition 20.4.** *If* $F : \partial\Omega \to S^{n+k-1}$ *is given by* $F(z) = V(z)/|V(z)|$, *then*

$$\mathit{Deg}(F) = \mathit{Index}(W). \tag{20.8}$$

**Proof.** We see that all the critical points of $V$ are points in $X$ that are critical for $W$, and, as in Proposition 20.3, $\operatorname{Index}(W) = \operatorname{Index}(V)$. But Proposition 20.2 implies that $\operatorname{Index}(V) = \operatorname{Deg}(F)$.

Since $\varphi(z)$ is increasing as one moves away from $X$, it is clear that, for $z \in \partial\Omega$, $V(z)$ points out of $\overline{\Omega}$, provided it is a sufficiently small tubular neighborhood of $X$. Thus $F : \partial\Omega \to S^{n+k-1}$ is homotopic to the *Gauss map*

$$N : \partial\Omega \longrightarrow S^{n+k-1}, \tag{20.9}$$

given by the outward-pointing normal. This immediately gives the next result.

**Corollary 20.5.** *Let* $X$ *be a compact oriented manifold in* $\mathbb{R}^{n+k}$, $\overline{\Omega}$ *a small tubular neighborhood of* $X$, *and* $N : \partial\Omega \to S^{n+k-1}$ *the Gauss map. If* $W$ *is a vector field on* $X$ *with a finite number of critical points, then*

$$\mathit{Index}(W) = \mathit{Deg}(N). \tag{20.10}$$

Clearly, the right side of (20.10) is independent of the choice of $W$. Thus any two vector fields on $X$ with a finite number of critical points have the same index, that is, $\operatorname{Index}(W)$ is an invariant of $X$. This invariant is denoted by

$$\operatorname{Index}(W) = \chi(X), \tag{20.11}$$

and is called the *Euler characteristic* of $X$. See the exercises for more results on $\chi(X)$. A different definition of $\chi(X)$ is given in Chapter 5. These two definitions are related in §8 of Appendix C, Connections and Curvature.

## Exercises

In Exercises 1–3, $V$ is a vector field on a region $\Omega \subset \mathbb{R}^2$. A nondegenerate critical point $p$ of a vector field $V$ is said to be a *source* if the real parts of the eigenvalues of $DV(p)$ are all positive, a *sink* if they are all negative, and a *saddle* if they are all either positive or negative, and there exist some of each sign. Such a critical point is called a *center* if all orbits of $V$ close to $p$ are closed orbits, which stay near $p$; this requires all the eigenvalues of $DV(p)$ to be purely imaginary.

1. Let $V$ have a nondegenerate critical point at $p$. Show that

$$\begin{aligned} p \text{ saddle} &\Longrightarrow \operatorname{ind}_p(V) = -1, \\ p \text{ source} &\Longrightarrow \operatorname{ind}_p(V) = 1, \\ p \text{ sink} &\Longrightarrow \operatorname{ind}_p(V) = 1, \\ p \text{ center} &\Longrightarrow \operatorname{ind}_p(V) = 1. \end{aligned}$$

2. If $V$ has a closed orbit $\gamma$, show that the map $T : \gamma \to S^1$, $T(x) = V(x)/|V(x)|$, has degree $+1$. (*Hint*: Use Proposition 19.8.)
3. If $V$ has a closed orbit $\gamma$ whose inside $\mathcal{O}$ is contained in $\Omega$, show that $V$ must have at least one critical point in $\mathcal{O}$, and that the sum of the indices of such critical points must be $+1$. (*Hint*: Use Proposition 20.2.)
   If $V$ has exactly one critical point in $\mathcal{O}$, show that it cannot be a saddle.
4. Let $M$ be a compact, oriented surface. Given a triangulation of $M$, within each triangle construct a vector field, vanishing at seven points as illustrated in Fig. 20.1, with the vertices as attractors, the center as a repeller, and the midpoints of each side as saddle points. Fit these together to produce a smooth vector field $X$ on $M$. Show directly that

$$\operatorname{Index}(X) = V - E + F,$$

where

$$V = \#\text{ vertices}, \quad E = \#\text{ edges}, \quad F = \#\text{ faces},$$

in the triangulation.

FIGURE 20.1

5. More generally, construct a vector field on an $n$-simplex so that when a compact, oriented, $n$-dimensional manifold $M$ is triangulated into simplices, one produces a vector field $X$ on $M$ such that

$$\operatorname{Index}(X) = \sum_{j=0}^{n} (-1)^j \nu_j, \tag{20.12}$$

   where $\nu_j$ is the number of $j$-simplices in the triangulation, namely, $\nu_0 = \#$ vertices, $\nu_1 = \#$ edges, $\dots, \nu_n = \#$ of $n$-simplices. (See Fig. 20.2 for a picture of a 3-simplex, with its faces (i.e., 2-simplices), edges, and vertices labeled.)

   The right side of (20.12) is one definition of $\chi(M)$. As we have seen, the left side of (20.12) is independent of the choice of $X$, so it follows that the right side is independent of the choice of triangulation.

Figure 20.2

6. Let $M$ be the sphere $S^n$, which is homeomorphic to the boundary of an $(n+1)$-simplex. Computing the right side of (20.12), show that

$$\chi(S^n) = 2 \text{ if } n \text{ even}, \quad 0 \text{ if } n \text{ odd}. \tag{20.13}$$

   Conclude that if $n$ is even, there is no smooth nowhere-vanishing vector field on $S^n$, thus obtaining another proof of Proposition 19.4.
7. With $X = S^n \subset \mathbb{R}^{n+1}$, note that the manifold $\partial\Omega$ in (20.9) consists of two copies of $S^n$, with opposite orientations. Compute the degree of the map $N$ in (20.9) and (20.10), and use this to give another derivation of (20.13), granted (20.11).
8. Consider the vector field $R$ on $S^2$ generating rotation about an axis. Show that $R$ has two critical points, at the "poles." Classify the critical points, compute $\operatorname{Index}(R)$, and compare the $n = 2$ case of (20.13).
9. Show that the computation of the index of a vector field $X$ on a manifold $M$ is independent of orientation and that $\operatorname{Index}(X)$ can be defined when $M$ is not orientable.

## A. Nonsmooth vector fields

Here we establish properties of solutions to the ODE

$$\frac{dy}{dt} = F(t,y), \quad y(t_0) = x_0 \tag{A.1}$$

of a sort done in §§2–6, under weaker hypotheses than those used there; in particular, we do not require $F$ to be Lipschitz in $y$. For existence, we can assume considerably less:

**Proposition A.1.** *Let $x_0 \in \mathcal{O}$, an open subset of $\mathbb{R}^n$, $I \subset \mathbb{R}$ an interval containing $t_0$. Assume $F$ is continuous on $I \times \mathcal{O}$. Then the equation (A.1) has a solution on some $t$-interval containing $t_0$.*

**Proof.** Without loss of generality, we can assume $F$ is bounded and continuous on $\mathbb{R}\times\mathbb{R}^n$. Take $F_j \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ such that $|F_j| \le K$ and $F_j \to F$ locally uniformly, and let $y_j \in C^\infty(\mathbb{R})$ be the unique solution to

$$\frac{dy_j}{dt} = F_j(t,y), \quad y_j(t_0) = x_0, \tag{A.2}$$

whose existence is guaranteed by the material of §2. Thus

$$y_j(t) = x_0 + \int_{t_0}^t F_j\bigl(s, y_j(s)\bigr)\, ds. \tag{A.3}$$

Now

$$|F_j| \le K \Longrightarrow |y_j(t') - y_j(t)| \le K|t' - t|. \tag{A.4}$$

Hence, by Ascoli's theorem (see Proposition 6.2 in Appendix A, Functional Analysis) the sequence $(y_j)$ has a subsequence $(y_{j_\nu})$ which converges locally uniformly: $y_{j_\nu} \to y$. It follows immediately that

$$y(t) = x_0 + \int_{t_0}^t F\bigl(s, y(s)\bigr)\, ds, \tag{A.5}$$

so $y$ solves (A.1).

Under the hypotheses of Proposition A.1, a solution to (A.1) may not be unique. The following family of examples illustrates the phenomenon. Take $a \in (0,1)$ and consider

$$\frac{dy}{dt} = |y|^a, \quad y(0) = 0. \tag{A.6}$$

Then one solution on $[0,\infty)$ is given by

$$y_0(t) = (1-a)^{1/(1-a)}\, t^{1/(1-a)}, \tag{A.7}$$

and another is given by

$$y_*(t) = 0.$$

Note that, for any $\varepsilon > 0$, the problem $dy/dt = |y|^a,\ y(0) = \varepsilon$ has a unique solution on $t \in [0, \infty)$, and $\lim_{\varepsilon \to 0}\ y_\varepsilon(t) = y_0(t)$. Understanding this provides the key to the following uniqueness result, due to W. Osgood.

Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a modulus of continuity, i.e., $\omega(0) = 0$, $\omega$ is continuous, and increasing. We may as well assume $\omega$ is bounded and $C^\infty$ on $(0, \infty)$.

**Proposition A.2.** *In the setting of Proposition A.1, assume $F$ is continuous on $I \times \mathcal{O}$ and that*

$$|F(t, y_1) - F(t, y_2)| \le \omega\bigl(|y_1 - y_2|\bigr), \tag{A.8}$$

*for all $t \in I,\ y_j \in \mathcal{O}$. Then solutions to (A.1) (with range in $\mathcal{O}$) are unique, provided*

$$\int_0^1 \frac{ds}{\omega(s)} = \infty. \tag{A.9}$$

**Proof.** If $y_1(t)$ and $y_2(t)$ are two solutions to (A.1), then

$$y_1(t) - y_2(t) = \int_{t_0}^t \bigl\{F\bigl(s, y_1(s)\bigr) - F\bigl(s, y_2(s)\bigr)\bigr\}\, ds. \tag{A.10}$$

Let us set $\theta(t) = |y_1(t) - y_2(t)|$. Hence, by (A.8), for $t \ge t_0$,

$$\theta(t) \le \int_{t_0}^t \omega\bigl(\theta(s)\bigr)\, ds. \tag{A.11}$$

In particular, for each $\varepsilon > 0$, $\theta(t) \le \int_{t_0}^t \omega\bigl(\theta(s) + \varepsilon\bigr)\, ds$. Since we are assuming $\omega$ is smooth on $(0, \infty)$, we can apply the Gronwall inequality, derived in (5.19)–(5.21), to deduce that

$$\theta(t) \le \varphi_\varepsilon(t), \quad \forall\, t \ge t_0,\ \varepsilon > 0, \tag{A.12}$$

where $\varphi_\varepsilon$ is uniquely defined on $[t_0, \infty)$ by

$$\varphi_\varepsilon'(t) = \omega\bigl(\varphi_\varepsilon(t) + \varepsilon\bigr), \quad \varphi_\varepsilon(t_0) = 0. \tag{A.13}$$

Thus

$$\int_0^{\varphi_\varepsilon(t)} \frac{d\zeta}{\omega(\zeta + \varepsilon)} = t - t_0. \tag{A.14}$$

Now the hypothesis (A.9) implies

$$\lim_{\varepsilon \searrow 0}\ \varphi_\varepsilon(t) = 0, \quad \forall\, t \ge t_0, \tag{A.15}$$

so we have $\theta(t) = 0$, for all $t \ge t_0$. Similarly, one shows $\theta(t) = 0$, for $t \le t_0$, and uniqueness is proved.

An important example to which Proposition A.2 applies is

$$\omega(s) = s\ \log\frac{1}{s}, \quad s \le \frac{1}{2}. \tag{A.16}$$

This arises in the study of ideal fluid flow, as will be seen in Chapter 17.

A similar argument establishes continuous dependence on initial data. If

$$\frac{dy_j}{dt} = F(t, y_j), \quad y_j(t_0) = x_j, \tag{A.17}$$

then

$$y_1(t) - y_2(t) = x_1 - x_2 + \int_{t_0}^{t} \bigl\{F\bigl(s, y_1(s)\bigr) - F\bigl(s, y_2(s)\bigr)\bigr\}\, ds, \tag{A.18}$$

so $\theta_{12}(t) = |y_1(t) - y_2(t)|$ satisfies

$$\theta_{12}(t) \le |x_1 - x_2| + \int_{t_0}^{t} \omega\bigl(\theta_{12}(s)\bigr)\, ds. \tag{A.19}$$

An argument similar to that used above gives (for $t \ge t_0$)

$$\theta_{12}(t) \le \vartheta(|x_1 - x_2|, t), \tag{A.20}$$

where, for $a > 0,\ t \ge t_0,\ \vartheta(a,t)$ is the unique solution to

$$\partial_t \vartheta = \omega(\vartheta), \quad \vartheta(a, t_0) = a, \tag{A.21}$$

that is,

$$\int_a^{\vartheta(a,t)} \frac{d\zeta}{\omega(\zeta)} = t - t_0. \tag{A.22}$$

Again, the hypothesis (A.9) implies

$$\lim_{a \searrow 0} \vartheta(a,t) = 0, \quad \forall\ t \ge t_0. \tag{A.23}$$

By (A.20), we have

$$|y_1(t) - y_2(t)| \le \vartheta\bigl(|x_1 - x_2|, t\bigr), \tag{A.24}$$

for all $t \ge t_0$, and a similar argument works for $t \le t_0$.

## References

[AM] R. Abraham and J. Marsden, *Foundations of Mechanics*, Benjamin/Cummings, Reading, Mass., 1978.

[Ar] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[Ar2] V. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983.

[Bir] G. D. Birkhoff, *Dynamical Systems*, AMS Colloq. Publ., Vol. 9, Providence, R.I., 1927.

[Car] C. Caratheodory, *Calculus of Variations and Partial Differential Equations of the First Order*, Holden-Day, San Francisco, 1965.

[Fed] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[Go] H. Goldstein, *Classical Mechanics*, Addison-Wesley, New York, 1950.

[GS] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge Univ. Press, Cambridge, 1984.

[Hal] J. Hale, *Ordinary Differential Equations*, Wiley, New York, 1969.

[Har] P. Hartman, *Ordinary Differential Equations*, Baltimore, 1973.

[HS] M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[Ja] J. Jackson, *Classical Electrodynamics*, J. Wiley, New York, 1962.

[Kan] Y. Kannai, An elementary proof of the no-retraction theorem, *Amer. Math. Monthly* 88(1981), 264–268.

[Lef] S. Lefschetz, *Differential Equations, Geometric Theory*, J. Wiley, New York, 1957.

[LS] L. Loomis and S. Sternberg, *Advanced Calculus*, Addison-Wesley, New York, 1968.

[Mos] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton Univ. Press, Princeton, N.J., 1973.

[NS] V. Nemytskii and V. Stepanov, *Qualitative Theory of Differential Equations*, Dover, New York, 1989.

[Poi] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier-Villars, 1899.

[Poo] W. Poor, *Differential Geometric Structures*, McGraw-Hill, New York, 1981.

[Rin] W. Rindler, *Essential Relativity*, Springer, New York, 1977.

[Spi] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vols. 1–5, Publish or Perish Press, Berkeley, 1979.

[Stb] S. Sternberg, *Lectures on Differential Geometry*, Prentice Hall, New Jersey, 1964.

[Sto] J. J. Stoker, *Differential Geometry*, Wiley-Interscience, New York, 1969.

[Str] G. Strang, *Linear Algebra and its Applications*, Harcourt Brace Jovanovich, San Diego, 1988.

[TS] J. M. Thompson and H. Stewart, *Nonlinear Dynamics and Chaos*, J. Wiley, New York, 1986.

[Wh] E. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Dover, New York, 1944.

[Wig] S. Wiggins, *Introduction to Applied Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.