Positive Definite Zonal Functions

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Informal Notes

Abstract. We explore what is special about a zonal function on the sphere S^{n-1} being a non-negative linear combination of zonal spherical harmonics (normalized to be positive at the "north pole").

1. Introduction

A function u on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is said to be a zonal function if u is a function of x_n alone. Such a function can be expanded in zonal harmonics. In more detail, we have

$$L^2(S^{n-1}) = \bigoplus_{\ell \geq 0} V_\ell, \tag{1.1}$$

where each V_ℓ is an eigenspace of the Laplace-Beltrami operator Δ on S^{n-1}:

$$V_\ell = \{f \in C^\infty(S^{n-1}) : \Delta f = -\lambda_\ell f\}, \tag{1.2}$$

where $\{-\lambda_\ell : \ell \geq 0\} = \text{Spec } \Delta$, i.e., $\lambda_\ell = \ell(\ell + n - 2)$. Each space V_ℓ has a one-dimensional subspace of zonal functions, spanned by

$$\mathfrak{z}_\ell(x) = A_{\ell n} C_\ell^\alpha(x_n), \quad \alpha = \frac{n-2}{2}, \tag{1.3}$$

where C_ℓ^α is a Gegenbauer polynomial, given by the generating function

$$(1 - 2tr + r^2)^{-\alpha} = \sum_{\ell=0}^{\infty} C_\ell^\alpha(t) r^\ell. \tag{1.4}$$

(We assume $n \geq 3$.) Compare [T], Chapter 8, §4. Taking $t = 1$ gives $(1 - r)^{-2\alpha}$ for the left side, and we see that

$$C_\ell^\alpha(1) > 0, \tag{1.5}$$

for $\alpha > 0,\ \ell \geq 0$. We set $A_{\ell n} = 1/C_\ell^\alpha(1)$ (which is > 0) in (1.3), so

$$\mathfrak{z}_\ell(p_0) = 1, \tag{1.6}$$

where $p_0 = (0, \dots, 0, 1)$ is the "north pole" of s^{n-1}.

We are interested in zonal functions of the form

$$u(x) = \sum_{\ell \geq 0} c_\ell \mathfrak{z}_\ell(x), \quad c_\ell \geq 0. \tag{1.7}$$

If $u \in C(S^{n-1})$ has the form (1.7), we say

$$u \in \mathcal{PZ}(S^{n-1}). \tag{1.8}$$

At first glance, one might think the normalization $\mathfrak{z}(p_0) > 0$ is pretty arbitrary, and that the class of functions given by (1.7) is not particularly distinguished. However, as we will see, this is far from the case.

In fact $\mathcal{PZ}(S^{n-1})$ is naturally equivalent to the space of K-bi-invariant elements of $C(G)$ that are *positive-definite*, as functions on the compact group G, where

$$G = SO(n), \quad K = SO(n-1). \tag{1.9}$$

This natural class of functions on G is independent of any choice of normalized eigenfunctions. Further general properties of positive-definite functions imply that

$$u, v \in \mathcal{PZ}(S^{n-1}) \Longrightarrow uv \in \mathcal{PZ}(S^{n-1}). \tag{1.10}$$

Furthermore, the class $\mathcal{Z}(S^{n-1})$ of zonal functions on S^{n-1} has a natural convolution product, and

$$u, v \in \mathcal{PZ}(S^{n-1}) \Longrightarrow u * v \in \mathcal{PZ}(S^{n-1}). \tag{1.11}$$

In the next section we recall some facts about positive-definite functions. We work more generally on a compact Lie group G, and replace S^{n-1} by $M = G/K$, where K is a closed subgroup of G. In §§3–4 we define $\mathcal{PZ}(M)$ and identify it with the class of K-bi-invariant functions on G that are positive definite. We establish (1.10), in the more general setting of $\mathcal{PZ}(M)$. For (1.11), we require M to be a rank-one symmetric space.

2. Positive definite functions on G

Let G be a compact Lie group. A function $u \in C(G)$ is said to be positive definite provided

$$L(u) : L^2(G) \longrightarrow L^2(G), \tag{2.1}$$

defined as left convolution by u:

$$L(u)f(x) = u * f(x) = \int_G u(y) f(y^{-1}x)\, dy, \quad f \in L^2(G), \tag{2.2}$$

is a positive, self-adjoint operator. An equivalent condition is that, for each strongly continuous unitary representation π of G on a Hilbert space H, $\pi(u) : H \to H$ is a positive, self-adjoint operator, where

$$\pi(u) f = \int_G u(y) \pi(y) f\, dy, \quad f \in H. \tag{2.3}$$

Note that in (2.2), $L(y)f(x) = f(y^{-1}x)$ is the left regular representation of G on $L^2(G)$. Note also that

$$\begin{aligned} L(u)f(x) &= \int_G u(xy^{-1}) f(y)\, dy \\ &= \int_G K_u(x,y) f(y)\, dy, \end{aligned} \tag{2.4}$$

where

$$K_u(x,y) = u(xy^{-1}). \tag{2.5}$$

Hence an alternative characterization is that $u \in C(G)$ is positive definite if and only if, for each finite set $\{g_1, \dots, g_N\} \subset G$, the $N \times N$ matrix

$$A = (a_{jk}) = \big(u(g_j g_k^{-1})\big) \ \text{ is positive and self-adjoint in } \ M(N, \mathbb{C}). \tag{2.6}$$

We denote the set of positive definite elements of $C(G)$ by $\mathcal{P}(G)$. The following is a consequence of the characterization (2.6).

Proposition 2.1. *We have*

$$u, v \in \mathcal{P}(G) \Longrightarrow uv \in \mathcal{P}(G). \tag{2.7}$$

Proof. Take $\{g_1, \dots, g_N\} \subset G$. Parallel to (2.6), given $v \in \mathcal{P}(G)$, we have

$$B = \bigl(v(g_j g_k^{-1})\bigr) \ \text{ positive, self-adjoint.} \tag{2.8}$$

To establish (2.7), it suffices to deduce from (2.6) and (2.8) that

$$C = (c_{jk}), \quad c_{jk} = a_{jk} b_{jk} \tag{2.9}$$

is positive. In fact, (2.6) and (2.8) imply

$$A \otimes B \geq 0 \ \text{ in } \ M(N^2, \mathbb{C}), \tag{2.10}$$

hence

$$\sum a_{jk} b_{j'k'} \eta_{jj'} \overline{\eta}_{kk'} \geq 0, \quad \forall\, \eta_{jj'} \in \mathbb{C}. \tag{2.11}$$

Take $\eta_{jj'} = \xi_j \delta_{jj'}$. We get

$$\sum a_{jk} b_{j'k'} \eta_{jj'} \overline{\eta}_{kk'} = \sum a_{jk} b_{jk} \xi_j \overline{\xi}_k, \tag{2.12}$$

so (2.11) implies

$$\sum a_{jk} b_{jk} \xi_j \overline{\xi}_k \geq 0, \quad \forall\, \xi_j \in \mathbb{C}, \tag{2.13}$$

as desired.

We also have convolution $u*v$, defined as in (2.2), and if π is a strongly continuous unitary representation of G on H,

$$\pi(u * v) = \pi(u)\pi(v). \tag{2.14}$$

Now, if $\pi(u)$ and $\pi(v)$ *commute*, we can deduce that $\pi(u*v)$ is positive, self-adjoint, provided $\pi(u)$ and $\pi(v)$ are, but we cannot draw such a conclusion if these factors do not commute. More on this later.

The following ties in with the normalization (1.6).

Proposition 2.2. *Let* $e \in G$ *denote the identity element. Then*

$$u \in \mathcal{P}(G) \Longrightarrow u(e) \geq |u(x)|, \quad \forall \, x \in G. \tag{2.15}$$

Proof. Let $\{\pi_\alpha\}$ be a complete set of irreducible unitary representations of G on V_α, of dimension d_α. We have the following Fourier inversion formula for $u \in L^2(G)$:

$$u(x) = \sum_\alpha d_\alpha \operatorname{Tr}\bigl(\pi_\alpha(u)\pi_\alpha(x)^*\bigr), \tag{2.16}$$

with convergence in L^2-norm. To check absolute and uniform convergence, note that

$$d_\alpha |\operatorname{Tr}(\pi_\alpha(u)\pi_\alpha(x)^*)| \leq d_\alpha \|\pi_\alpha(u)\|_{TR}. \tag{2.17}$$

If $u \in C(G)$ is positive definite, then each $\pi_\alpha(u) \geq 0$, and

$$\|\pi_\alpha(u)\|_{TR} = \operatorname{Tr} \pi_\alpha(u). \tag{2.18}$$

In such a case,

$$|u(x)| \leq \sum_\alpha d_\alpha \operatorname{Tr} \pi_\alpha(u) = u(e), \tag{2.19}$$

as asserted in (2.15).

3. K-bi-invariant functions

Let G be a compact Lie group and K a closed subgroup, and set $M = G/K$, endowed with a G-invariant Riemannian metric. Set

$$p_0 = [e] \in M = G/K. \tag{3.1}$$

A function in $C(M)$ corresponds to a function $u \in C(G)$ satisfying $u(xk) = u(x)$, for each $x \in G,\ k \in K$. A function in $C(M)$ invariant under the action of K corresponds to a function $u \in C(G)$ satisfying

$$u(k_1 x k_2) = u(x), \quad \forall\, k_j \in K,\ x \in G. \tag{3.2}$$

We say u is K-bi-invariant, or $u \in C(K \setminus G/K)$.

We denote by $\mathcal{Z}(M)$ the space of continuous functions on M invariant under the action of K. We have a natural isomorphism

$$\mathcal{Z}(M) \approx C(K \setminus G/K), \tag{3.3}$$

and these spaces are also naturally isomorphic to the space of K-bi-invariant functions in $C(G)$. Note that

$$u, v \in \mathcal{Z}(M) \Longrightarrow uv \in \mathcal{Z}(M). \tag{3.4}$$

We say

$$u \in \mathcal{PZ}(M) \tag{3.5}$$

provided $u \in \mathcal{Z}(M)$ and its counterpart in $C(G)$ is positive definite. By Proposition 2.1,

$$u, v \in \mathcal{PZ}(M) \Longrightarrow uv \in \mathcal{PZ}(M). \tag{3.6}$$

Recall the convolution product of u and v in $C(G)$:

$$u * v(x) = \int_G u(y) v(y^{-1}x)\, dy. \tag{3.7}$$

If u and v are right K-invariant, $u * v$ might not be right K-invariant, except in special cases. However, if u and v are K-bi-invariant, so is $u * v$. This gives rise to a convolution product on $\mathcal{Z}(M)$:

$$u, v \in \mathcal{Z}(M) \Longrightarrow u * v \in \mathcal{Z}(M). \tag{3.8}$$

If G is non-commutative, the convolution product on $C(G)$ is non-commutative. As we will see, sometimes the convolution product on $\mathcal{Z}(M)$ is commutative (in particular, this happens when $M = S^{n-1}$).

For general $u \in C(G)$, if π is a unitary representation of G,

$$v(x) = u(k_1 x k_2) \Longrightarrow \pi(v) = \pi(k_1^{-1})\pi(u)\pi(k_2^{-1}). \tag{3.9}$$

Hence, if u is K-bi-invariant, then

$$\pi(u) = \pi(k_1)\pi(u)\pi(k_2), \quad \forall\, k_j \in K. \tag{3.10}$$

Equivalently, given $u \in C(G)$,

$$\begin{aligned} u \in C(K \setminus G/K) \Longrightarrow \pi(u) &= \pi(u)\pi(k) \\ &= \pi(k)\pi(u), \quad \forall\, k \in K. \end{aligned} \tag{3.11}$$

Consequently, if $u \in C(K \setminus G/K)$, then, for all $k \in K$,

$$\begin{aligned} &\pi(k) = \text{id. on } \mathcal{R}\pi(u), \text{ and} \\ &\pi(k) = \text{id. on } \mathcal{R}\pi(u)^* = \big(\mathcal{N}\pi(u)\big)^{\perp}, \end{aligned} \tag{3.12}$$

where $\mathcal{R}\pi(u)$ is the range of $\pi(u)$ and $\mathcal{N}\pi(u)$ is the null space of $\pi(u)$. If $\pi(u)$ is self-adjoint, the two conditions in (3.12) are equivalent to each other.

4. Specialization to $M = S^{n-1}$

We now assume G and K are given by (1.9), so $M = S^{n-1}$, with its standard metric. G acts on M by rotations, hence as a unitary group on $L^2(M)$, and then it acts on each space V_ℓ in (1.1). Call this representation π_ℓ. Since S^{n-1} is a rank-one symmstric space, dim $K \setminus G/K = 1$ and each V_ℓ has a one-dimensional subspace of zonal functions,

$$\mathcal{Z}_\ell = \text{Span}\,(\mathfrak{z}_\ell), \tag{4.1}$$

with $\mathfrak{z}_\ell$ as in (1.3) (normalized by (1.6). (Hence each π_ℓ is irreducible.) The space $\mathcal{Z}_\ell$ is the subspace of V_ℓ on which $\pi_\ell(k)$ acts as the identity for all $k \in K$. Hence, by (3.12), if $u \in \mathcal{Z}(M)$,

$$\pi_\ell(u) : V_\ell \longrightarrow \mathcal{Z}_\ell, \tag{4.2}$$

and ditto for $\pi_\ell(u)^*$. Generally, if π is a unitary representation of G and $u \in C(G)$,

$$\pi(u)^* = \pi(u^*), \quad u^*(x) = \overline{u(x^{-1})}. \tag{4.3}$$

In this setting, we have

$$\mathfrak{z}_\ell^* = \mathfrak{z}_\ell, \tag{4.4}$$

so $\pi_\ell(\mathfrak{z}_\ell)$ is a scalar muitiple of an orthogonal projection:

$$\pi_\ell(\mathfrak{z}_\ell) = \gamma_\ell Z_\ell, \tag{4.5}$$

with $\gamma_\ell \in \mathbb{R}$ and

$$Z_\ell = \text{orthogonal projection of } V_\ell \text{ onto Span}\,(\mathfrak{z}_\ell). \tag{4.6}$$

The Weyl orthogonality relations imply that if π_α is an irreducible representation of G,

$$\pi_\alpha \quad \text{not} \quad \equiv \pi_\ell \Longrightarrow \pi_\alpha(\mathfrak{z}_\ell) = 0. \tag{4.7}$$

In view of the inversion formula (2.16), this implies $\pi_\ell(\mathfrak{z}_\ell) \neq 0$, so $\gamma_\ell \neq 0$ in (4.5). Whether $\gamma_\ell > 0$ or $\gamma_\ell < 0$, either $\mathfrak{z}_\ell$ or $-\mathfrak{z}_\ell$ is mapped by π_ℓ to a positive operator, so it must be positive definite. In fact, since $\mathfrak{z}(p_0) > 0$, it follows from Proposition 2.2 that it must be $\mathfrak{z}_\ell$ that is positive definite, so $\gamma_\ell > 0$ and

$$\mathfrak{z}_\ell \text{ is positive definite.} \tag{4.8}$$

In fact, via (2.19),

$$\mathfrak{z}(p_0) = 1 \Longrightarrow \gamma_\ell = \frac{1}{d_\ell}, \quad d_\ell = \dim V_\ell. \tag{4.9}$$

We also deduce from Proposition 2.2 that

$$|\mathfrak{z}_\ell(x)| \leq 1, \quad \forall\, x \in S^{n-1}. \tag{4.10}$$

Note that (4.5) extends:

$$u \in \mathcal{Z}(S^{n-1}) \Longrightarrow \pi_\ell(u) = \Gamma_\ell(u) Z_\ell, \quad \Gamma_\ell : \mathcal{Z}(S^{n-1}) \to \mathbb{C}. \tag{4.11}$$

This observation enables us to prove the following.

Proposition 4.1. *The convolution product on* $\mathcal{Z}(S^{n-1})$ *is commutative:*

$$u, v \in \mathcal{Z}(S^{n-1}) \Longrightarrow u * v = v * u \quad in \quad \mathcal{Z}(S^{n-1}). \tag{4.12}$$

Consequently, with $\mathcal{PZ}(M)$ *defined as in §3,*

$$u, v \in \mathcal{PZ}(S^{n-1}) \Longrightarrow u * v \in \mathcal{PZ}(S^{n-1}). \tag{4.13}$$

Proof. To get (4.12), it suffices to show that

$$u, v \in \mathcal{Z}(S^{n-1}) \Longrightarrow \pi_\alpha(u)\pi_\alpha(v) = \pi_\alpha(v)\pi_\alpha(u), \tag{4.14}$$

for each irreducible unitary representation π_α of $= SO(n)$. In fact, either π_α is equivalent to π_ℓ for some ℓ (which always holds if $n = 3$), in which case (4.11) holds, or else, by the Weyl orthogonality relations, $\pi_\alpha(u) = 0$. Since each Z_ℓ is a rank-one projection, (4.14) follows. As noted previously, (4.13) is a consequence of such commutativity.

We now show that $\mathcal{PZ}(M)$, defined as in §3, coincides with the space characterized by (1.7)–(1.8) when $M = S^{n-1}$.

Proposition 4.2. *With* $\mathcal{PZ}(M)$ *defined as in §3, a function* $u \in \mathcal{Z}(S^{n-1})$ *belongs to* $\mathcal{PZ}(S^{n-1})$ *if and only if (1.7) holds.*

Proof. By (4.9)–(4.10), if (1.7) holds and the sum is bounded, then $\sum c_\ell < \infty$, and the sum converges absolutely and uniformly. That such a sum belongs to $\mathcal{PZ}(S^{n-1})$ then follows readily from (4.8).

For the converse, if $u \in \mathcal{Z}(S^{n-1})$, we can write

$$u = \sum b_\ell \mathfrak{z}_\ell, \tag{4.15}$$

with $b_\ell \in \mathbb{C}$, and convergence in L^2-norm. We need to show that if $u \in \mathcal{PZ}(S^{n-1})$, then each b_ℓ is ≥ 0. In fact, for u as in (4.15) and each $\ell \geq 0$, we have

$$\pi_\ell(u) = b_\ell \gamma_\ell Z_\ell. \tag{4.16}$$

Since $\gamma_\ell > 0$, we have $b_\ell \geq 0$ when $u \in \mathcal{PZ}(S^{n-1})$.

We hence have (1.10)–(1.11).

Remark. While we have desired to deduce (1.10) from "general principles," we can also deduce it (in the setting of $\mathcal{PZ}(S^{n-1})$ defined by (1.7)–(1.8)) from special function identities. In fact, given $\alpha > 0$ and $\ell_j \geq 0$,

$$C^\alpha_{\ell_1}(t) C^\alpha_{\ell_2}(t) = \sum_{\ell \in \mathcal{S}(\ell_1, \ell_2)} \sigma^\alpha_{\ell_1,\ell_2}(\ell) C^\alpha_\ell(t), \tag{4.17}$$

where

$$\mathcal{S}(\ell_1, \ell_2) = \{\ell \in \mathbb{Z} : |\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2 \text{ and } \ell = \ell_1 + \ell_2 \text{ mod } 2\}, \tag{4.18}$$

and

$$\sigma^\alpha_{\ell_1,\ell_2}(\ell) > 0 \quad \text{whenever} \quad \ell \in \mathcal{S}(\ell_1, \ell_2). \tag{4.19}$$

A formula for these coefficients is given in [V], p. 491. For another application of this, see [RT], §5.

References

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