# Dirac-type Operators on a Compact Riemann Surface

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**Abstract**

The following provides a short mini-course on $\overline{\partial}$ operators arising in the theory of compact Riemann surfaces, and what they have to say about spaces of holomorphic sections of holomorphic line bundles, and more generally holomorphic vector bundles over such surfaces, in light of the Riemann-Roch theorem. This is excerpted from an appendix in [5].

Here we collect some concepts and results regarding a natural $\overline{\partial}$ operator defined on sections of a holomorphic line bundle, or more generally a holomorphic vector bundle, $L$ over a compact Riemann surface $M$. The material on line bundles follows [8, §9], to which we refer for further details. Other sources include [3], [6], and, for vector bundles, [4], [2] and [10].

Some special line bundles over $M$ include the canonical bundle $\kappa \to M$, and its conjugate $\overline{\kappa}$. The bundle $\kappa$ is the cotangent bundle $T^*M$, endowed by the complex structure of $M$ with the structure of a holomorphic line bundle. The bundle $\overline{\kappa}$ is an anti-holomorphic line bundle. We can characterize a holomorphic line bundle as follows. Given a complex line bundle $L \to M$, let $\{U_j\}$ be a covering of $M$. A holomorphic structure on $L$ consists of nowhere vanishing sections $s_j$ of $L$ over $U_j$ such that

$$s_j = \sigma_{jk} s_k, \quad \text{on } U_{jk} = U_j \cap U_k, \tag{1}$$

with $\sigma_{jk}$ holomorphic, nowhere vanishing, complex-valued functions, called the transition functions of this line bundle. To define a line bundle, one needs the family $\{\sigma_{jk}\}$ to satisfy the cocycle condition

$$\sigma_{jk}\sigma_{k\ell} = \sigma_{j\ell} \text{ on } U_j \cap U_k \cap U_\ell. \tag{2}$$

If the functions $\sigma_{jk}$ are anti-holomorphic, $L$ gets the structure of an anti-holomorphic line bundle. We mention that changing the transition functions to $\overline{\sigma}_{jk}$ defines the conjugate line bundle, $\overline{L}$.

If $L \to M$ is a holomorphic line bundle, we have

$$\overline{\partial}_L : C^\infty(M, L) \longrightarrow C^\infty(M, L \otimes \overline{\kappa}), \tag{3}$$

defined as follows. Pick a local coordinate patch $U$ and a local (nowhere-vanishing) holomorphic section $S$ of $L$ over $U$. Then an arbitrary section $u$ over $U$ is of the form $u = vS$, with $v$ complex valued, and we set

$$\overline{\partial}_L u = \frac{\partial v}{\partial \overline{z}} \, S \otimes d\overline{z}. \tag{4}$$

This is independent of the choice of local holomorphic section $S$ and of local holomorphic coordinate system.

The operator $\overline{\partial}_L$ is a first order, elliptic differential operator. In addition to (3), we also have

$$\overline{\partial}_L : \mathcal{D}'(M, L) \longrightarrow \mathcal{D}'(M, L \otimes \overline{\kappa}), \tag{5}$$

and, for $s \in \mathbb{R}$, $p \in (1, \infty)$,

$$\overline{\partial}_L : H^{s+1,p}(M, L) \longrightarrow H^{s,p}(M, L \otimes \overline{\kappa}). \tag{6}$$

Standard elliptic theory implies $\overline{\partial}_L$ is Fredholm in (3), (5), and (6), and has the same index in all cases. The fact that $\overline{\partial}_L$ in (5) has range of finite codimension (say $m$) implies that any linear subspace $V \subset \mathcal{D}'(M, L \otimes \overline{\kappa})$ of dimension $> m$ has nonzero intersection with the range of $\overline{\partial}_L$. It follows that there exists a nonzero $v \in \mathcal{D}'(M, L \otimes \overline{\kappa})$, supported on a finite point set $\mathcal{F}$, and in the range of $\overline{\partial}_L$, so $v = \overline{\partial}_L u$. Such $u$ is holomorphic on $M \setminus \mathcal{F}$, and in fact is a nontrivial meromorphic section of $L$:

Each holomorphic line bundle over $M$ has a nontrivial meromorphic section. (7)

We denote by $\mathcal{M}(L)$ the space of meromorphic sections of $L$, and by $\mathcal{O}(L)$ the space of holomorphic sections of $L$, i.e.,

$$\mathcal{O}(L) = \operatorname{Ker} \overline{\partial}_L. \tag{8}$$

One also has

$$\operatorname{Ker} \overline{\partial}_L^* \approx \mathcal{O}(L' \otimes \kappa), \tag{9}$$

where $L'$ is the dual bundle to $L$. Hence

$$\operatorname{Index} \overline{\partial}_L = \dim \mathcal{O}(L) - \dim \mathcal{O}(L' \otimes \kappa). \tag{10}$$

The following formula for the index of $\overline{\partial}_L$ is known as the Riemann-Roch formula. (Cf., e.g., [8, Theorem 9.1].)

**Proposition 1** *If $L$ is a holomorphic line bundle over a compact Riemann surface $M$,*

$$\operatorname{Index} \overline{\partial}_L = c_1(L) + 1 - g. \tag{11}$$

Here $g$ is the genus of $M$, related to the Euler characteristic $\chi(M)$ by

$$\chi(M) = 2 - 2g. \tag{12}$$

Thus the Riemann sphere $S^2$ has genus 0 and a torus $\mathbb{C}/\Lambda$ has genus 1.

The integer $c_1(L)$ is called the first Chern class of $L$. For our purposes, the following characterization will suffice. Take a nontrivial meromorphic section $u \in \mathcal{M}(L)$. It has a finite number of zeros and poles. If $p$ is a zero of $u$, let $\nu_u(p)$ be the order of the zero. If $p$ is a pole of $u$, let $-\nu_u(p)$ be the order of the pole. We define the *divisor* of $u \in \mathcal{M}(L)$ to be the formal sum

$$\vartheta(u) = \sum_p \nu_u(p) \cdot p. \tag{13}$$

In such a case (cf. [8, Proposition 9.3]),

$$c_1(L) = \sum_p \nu_u(p). \tag{14}$$

This is independent of the choice of nontrivial $u \in \mathcal{M}(L)$. As a corollary,

$$c_1(L) < 0 \Longrightarrow \mathcal{O}(L) = 0. \tag{15}$$

Note that if $L_1$ and $L_2$ are holomorphic line bundles over $M$, with meromorphic sections $u_1$ and $u_2$, then $u_1 \otimes u_2$ is a meromorphic section of $L_1 \otimes L_2$. The characterization (14) readily gives

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2). \tag{16}$$

Generalizing (13), we say a divisor on $M$ is a finite formal sum

$$\vartheta = \sum_p \nu(p) \cdot p, \quad \nu(p) \in \mathbb{Z}. \tag{17}$$

To any divisor $\vartheta$ we can associate a holomorphic line bundle, denoted $E_\vartheta$. To construct $E_\vartheta$, it is convenient to use the method of transition functions, as in (1). Cover $M$ with holomorphic coordinate charts $U_j$. Pick $\psi_j$, meromorphic on $U_j$, having a pole of order exactly $|\nu(p)|$ at $p$ if $\nu(p) < 0$, a zero of order exactly $\nu(p)$ if $\nu(p) > 0$, and no other poles or zeros. The transition functions

$$\varphi_{jk} = \psi_k^{-1} \psi_j \text{ on } U_j \cap U_k \tag{18}$$

define the holomorphic line bundle $E_\vartheta$. The collection $\{\psi_j, U_j\}$ defines a meromorphic section

$$\psi \in \mathcal{M}(E_\vartheta), \quad \vartheta(\psi) = -\vartheta. \tag{19}$$

Hence

$$c_1(E_\vartheta) = -\sum_p \nu(p). \tag{20}$$

There is a natural isomorphism

$$\mathcal{M}(L, \vartheta) \approx \mathcal{O}(L \otimes E_\vartheta), \tag{21}$$

where

$$\mathcal{M}(L, \vartheta) = \{u \in \mathcal{M}(L) : \vartheta(u) \geq \vartheta\}. \tag{22}$$

The isomorphism takes $u \in \mathcal{M}(L, \vartheta)$ to $u\psi$, with $\psi$ as in (19). We also mention that if $u$ is a nontrivial element of $\mathcal{M}(L)$, then we have holomorphically equivalent line bundles

$$L \approx E_{-\vartheta(u)}. \tag{23}$$

We now specify some examples where $\overline{\partial}_L$ is invertible.

**Proposition 2** *If $M$ has genus $g = 0$ and $p \in M$, then*

$$L = E_p \Longrightarrow \overline{\partial}_L \text{ is invertible.} \tag{24}$$

*Proof.* By (20), $c_1(L) = -1$. Hence, by (11), $\operatorname{Index} \overline{\partial}_L = 0$, and by (15), $\operatorname{Ker} \overline{\partial}_L = 0$. ☐

As is well known, if $g = 0$, then $M$ is holomorphically equivalent to the Riemann sphere $S^2$. A proof can be found in [8, §9]. We mention that one ingredient is the following.

**Lemma 3** *If $M$ is a compact Riemann surface, $q \in M$, and there exists a meromorphic function $u$ on $M$ with just one pole, a simple pole at $q$, then $M$ is holomorphically equivalent to $S^2$. In fact, $u : M \to \mathbb{C} \cup \{\infty\}$ provides the equivalence.*

We move on to genus $g = 1$:

**Proposition 4** *If $M$ has genus $g = 1$, and $p, q$ are distinct points in $M$, then*

$$L = E_{p-q} \Longrightarrow \overline{\partial}_L \ \text{ is invertible.} \tag{25}$$

*Proof.* This time, $c_1(L) = 0$, hence, by (11), $\operatorname{Index} \overline{\partial}_L = 0$. It remains to show that $\operatorname{Ker} \overline{\partial}_L = 0$, i.e., $\mathcal{O}(E_{p-q}) = 0$. By (21)–(22), with $L$ the trivial line bundle, any non-trivial element of $\mathcal{O}(E_{p-q})$ would yield a meromorphic function $u$ on $M$ that vanishes at $p$ and has at most a simple pole at $q$. If $u$ actually had a simple pole at $q$, Lemma 3 would imply $M \approx S^2$, a contradiction. Otherwise, $u$ must be constant, and hence $\equiv 0$. □

If $g = 1$, then $M$ is holomorphically equivalent to some torus $\mathbb{C}/\Lambda$. A proof of this can also be found in [8, §9].

There are more general classes of holomorphic line bundles $L$ for which $\overline{\partial}_L$ is invertible. We mention the following easy generalization of Propositions 2 and 4.

**Proposition 5** *Let $L \to M$ be a holomorphic line bundle. If $M$ has genus $g = 0$, then*

$$c_1(L) = -1 \Longrightarrow \overline{\partial}_L \ \text{ is invertible.} \tag{26}$$

*If $M$ has genus $g = 1$, then*

$$c_1(L) = 0 \ \text{ and } \ \mathcal{O}(L) = 0 \Longrightarrow \overline{\partial}_L \ \text{ is invertible.} \tag{27}$$

*For general genus $g$,*

$$c_1(L) = g - 1 \ \text{ and } \ \mathcal{O}(L) = 0 \Longrightarrow \overline{\partial}_L \ \text{ is invertible.} \tag{28}$$

In case $L = E_\vartheta$, (20) specifies $c_1(E_\vartheta)$, so one easily sees when the hypothesis of (26) is satisfied. One also easily sees when $c_1(L) = 0$.

Given $c_1(L) = 0$, one may or may not have $\mathcal{O}(L) = 0$. For example, if $L$ is the trivial bundle, constants belong to $\mathcal{O}(L)$, so $\mathcal{O}(L) \approx \mathbb{C}$. Conversely, if there is a nontrivial $u \in \mathcal{O}(L)$, then, by (14), $u$ is nowhere vanishing. Hence $u$ provides a holomorphic trivialization of $L$. In case $L = E_\vartheta$, with $\vartheta$ as in (17) and $\sum \nu(p) = 0$, one can deduce from (21)–(22) that $\mathcal{O}(E_\vartheta) \neq 0$ if and only if there exists a meromorphic function $u$ on $M$ such that $\vartheta(u) = \vartheta$. When this holds is settled by a result known as Abel's Theorem, which can be found in [6].

One elementary result is that, given $p$ and $q$ distinct points of $M$,

$$\mathcal{O}(E_{p-q}) \neq 0, \ \text{hence } E_{p-q} \text{ is holomorphically trivial } \Longleftrightarrow M \approx S^2. \tag{29}$$

The proof of the right-pointing implication follows that of Proposition 4. For the converse, if $p, q \in \mathbb{C}$, take $u(z) = (z - p)/(z - q)$, for which $\vartheta(u) = p - q$.

Regarding the condition (28), for $g \geq 2$, there are many line bundles satisfying this. In more detail, the space of equivalence classes of holomorphic line bundles $L \to M$ with $c_1(L) = g - 1$ is a Jacobi variety, equivalent to a $g$-dimensional complex torus:

$$J^{g-1}(M) \approx \mathbb{C}^g/\Lambda, \tag{30}$$

See [6]. Given $L \in J^{g-1}(M)$, if $\mathcal{O}(L) \neq 0$, then by (23), $L \approx E_{-(p_1+\cdots+p_{g-1})}$, for some points $p_j \in M$, $1 \le j \le g-1$. Now the image of the $(g-1)$-fold product of copies of $M$ in $J^{g-1}(M)$,

$$(p_1, \dots, p_{g-1}) \mapsto E_{-(p_1+\cdots+p_{g-1})}, \tag{31}$$

is a $(g-1)$-dimensional variety, often denoted $\Theta$, in the $g$-dimensional space $J^{g-1}(M)$. We have

$$\text{If } L \in J^{g-1}(M) \setminus \Theta, \text{ then (28) holds.} \tag{32}$$

Moving on to vector bundles, we next record the formula for Index $\overline{\partial}_L$ when $L \to M$ is a holomorphic vector bundle, of rank $r$ (so each fiber is a complex vector space of dimension $r$). In such a case, $\sigma_{jk}$ in (2) are $r \times r$ matrices. Identity (10) continues to hold, and the Riemann-Roch theorem in this setting takes the following form.

**Proposition 6** *If $L$ is a holomorphic vector bundle of rank $r$ over a compact Riemann surface $M$, of genus $g$,*

$$\text{Index}\,\overline{\partial}_L = c_1(L) + r(1-g). \tag{33}$$

A proof can be found in [4]. Another proof, using methods from [8], is given in [9]. Here,

$$c_1(L) = c_1(\Lambda^r L), \tag{34}$$

if $L$ has rank $r$. We mention that, complementing (16), if $\widetilde{L}$ has rank $\tilde{r}$,

$$c_1(L \oplus \widetilde{L}) = c_1(L) + c_1(\widetilde{L}), \quad c_1(L \otimes \widetilde{L}) = \tilde{r} c_1(L) + r c_1(\widetilde{L}), \tag{35}$$

and, if $L'$ is the dual bundle to $L$,

$$c_1(L') = -c_1(L). \tag{36}$$

Parallel to Proposition 5, we have:

**Proposition 7** *In the setting of Proposition 6, the operator $\overline{\partial}_L$ is invertible if and only if*

$$c_1(L) = r(g-1) \ \textit{and} \ \mathcal{O}(L) = 0. \tag{37}$$

In case $g = 0$ or $g = 1$, each holomorphic vector bundle of rank $r$ is a sum of line bundles, $L = L_1 \oplus \cdots \oplus L_r$. See [4] for $g = 0$ and [1] for $g = 1$. In case $g = 0$, (37) holds if and only if $c_1(L_j) = -1$ for each $j$. In case $g = 1$, (37) holds if and only if $c_1(L_j) = 0$ and $\mathcal{O}(L_j) = 0$ for each $j$.

For $g \ge 2$, it is the case that most "semistable" holomorphic vector bundles $L$ of rank $r$ with $c_1(L) = r(g-1)$ satisfy (37). Explaining this requires a definition.

**Definition.** A holomorphic vector bundle of rank $r$, $L \to M$, is semistable provided that, if $L_0$ is a subbundle of rank $r_0$, then

$$\frac{c_1(L_0)}{r_0} \le \frac{c_1(L)}{r}. \tag{38}$$

If one always has strict inequality in (38), we say $L$ is stable.

Here is one reason to restrict to semistable vector bundles.

**Lemma 8** *If $L$ is not semistable, then* (37) *fails.*

*Proof.* Say $r = \operatorname{rank} L$, $c_1(L) = r(g-1)$. If $L$ is not semistable, there is a subbundle $L_0$, of rank $r_0$, such that (38) fails, hence

$$c_1(L_0) > \frac{c_1(L)}{r} r_0 = r_0(g-1). \tag{39}$$

Then the Riemann-Roch formula implies $\operatorname{Index} \overline{\partial}_{L_0} > 0$, so $\mathcal{O}(L_0) \neq 0$, hence $\mathcal{O}(L) \neq 0$. □

The class of semistable holomorphic vector bundles $L \to M$, of rank $r$, with $c_1(L) = k$, has a well-studied moduli space $\mathcal{M}(r, k)$ of equivalence classes. The equivalence at stable bundles is the usual holomorphic equivalence. It is a bit more subtle at the vector bundles that are not stable, and we refer to [10] for details. It has been shown that $\mathcal{M}(r, k)$ is a complex projective variety, smooth at the stable bundles (which form a dense open subset). It has further been shown (cf. [2]) that there is a "theta divisor" $\Theta \subset \mathcal{M}(r, r(g-1))$, of complex codimension 1, such that

$$L \in \mathcal{M}(r, r(g-1)) \setminus \Theta \Longrightarrow \text{(37) holds.} \tag{40}$$

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