The Zeta Function and the Prime Number Theorem

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Introduction

I was motivated to put together these notes while enjoying three books on prime numbers ([D], [J], and [S]) as 2003 Summer reading.

The Prime Number Theorem, giving the asymptotic behavior as $x \to +\infty$ of $\pi(x)$, the number of primes $\le x$, has for its proof three ingredients:

(I) Formulas for $\log \zeta(s)$ and $\zeta'(s)/\zeta(s)$ as Mellin transforms involving a function $J(x)$, closely related to $\pi(x)$, and a more subtly related function, $\psi(x)$.

(II) Information that $\zeta'(s)/\zeta(s)$ has on $\{s : \operatorname{Re} s \ge 1\}$ a singularity only at $s = 1$.

(III) A Tauberian theorem that yields the asymptotic behavior of $J(x)$ from the behavior of such a Mellin transform.

Step (I) is achieved by taking the Euler product for $\zeta(s)$, applying log, and following your nose, naturally finding $J(x)$. It is easy to show that $J(x)$ and $\pi(x)$ have similar behavior as $x \to +\infty$. Taking the s-derivative yields the relation involving $\zeta'(s)/\zeta(s)$, and naturally produces the function $\psi(x)$. Step (II) is more subtle, though there is a clever and brief proof, largely due to de la Vallée Poussin, which is commonly presented (and which we will recall in Appendix A). Step (III) can be handled by a powerful result known as Ikehara's Tauberian Theorem, though it is often treated via a variant, slightly weaker and somewhat easier result, known as the Ingham-Newman theorem. One possibly novel aspect of our treatment (in §§4–5) it to produce another variant, relying more on real Fourier analysis and the fact that $\zeta'(s)/\zeta(s)$ is C^∞ on $\{s : \operatorname{Re} s = 1\}$, except at $s = 1$. We came upon this approach while exploring similarities between arguments to implement step (III) and arguments used for "precise spectral asymptotics" of elliptic differential operators.

In §1 we implement step (I), recast step (II) as a statement about the zeros of $\zeta(s)$, state Ikehara's Tauberian Theorem, and use these results to prove the PNT. We give the Laplace transform version of Ikehara's theorem, and using it involves making a change of variable. In §2 we restate Ikehara's theorem in Mellin transform language, allowing one to avoid such a change of variable. Our calculations are done in terms of the Mellin transform of measures, often atomic measures. We avoid mention of Dirichlet series. This formalism allows us to avoid summation by parts arguments, and simply use integration by parts. We consider this an advantage, though some might regard it as merely a matter of taste.

To be a little more explicit about matters hinted about above, in §§1–2 we go from

$$\psi(x) \sim x, \quad \psi(x) = \sum_{p,k;p^k \le x} \log p$$

to

$$J(x) \sim \frac{x}{\log x}, \quad J(x) = \sum_{k\ge 1} \frac{1}{k}\pi(x^{1/k})$$

to

$$\pi(x) \sim \frac{x}{\log x}.$$

It turns out that formulas relating $\psi(x)$ to $J(x)$ bring in naturally

$$\mathrm{Li}(x) = \int_2^x \frac{dy}{\log y},$$

and that $\mathrm{Li}(x)$ is a better approximation to $J(x)$ and $\pi(x)$ than is $x/\log x$. In §3 we show how estimates on the error $E(x) = \psi(x) - x$ naturally lead to estimates on $J(x) - \mathrm{Li}(x)$. While we do not produce estimates on $E(x)$ here that imply the superiority of $\mathrm{Li}(x)$, these formulas do serve to make $\mathrm{Li}(x)$ look like the natural candidate.

In §4 we use the results of steps (I) and (II), interpreted as information on the Fourier transform of a measure H derived from $J(x)$, to produce complete asymptotic expansions of a class of mollifications of H. In §5 we apply fairly elementary methods of measure theory and Fourier analysis to derive from this the asymptotic behavior of $J(x)$ and hence a proof of the PNT.

There is an alternative approach to step (III), involving a much simpler Tauberian theorem, at the cost of a more sophisticated Mellin transform and a finer knowledge of $\zeta'(s)/\zeta(s)$ on $\{s : \mathrm{Re}\, s \ge 1\}$. We discuss this in §6, giving a treatment adapted from Chapter 13 of [A].

We have two appendices. Appendix A contains a proof of the result that $\zeta(s)$ has no zeros on $\{s : \mathrm{Re}\, s = 1\}$. Appendix B, not strongly related to the rest of these notes, gives a calculation of $\zeta(k)$ when k is an even integer, as a result of examining the Fourier series of $f_k(x) = x^k$ on $[-\pi, \pi]$.

1. The zeta function and the prime number theorem

The zeta function, given for $\mathrm{Re}\, s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

is connected to the study of primes via the Euler formula

$$\zeta(s) = \prod_p \Bigl(1 - \frac{1}{p^s}\Bigr)^{-1}. \tag{1.2}$$

Here and below, p will run over the set of primes. The formula (1.2) can be rewritten

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log\Big(1-\frac{1}{p^s}\Big)\\ &= \sum_p\sum_{k=1}^{\infty}\frac{1}{k}\,\frac{1}{p^{ks}}.\end{aligned}\tag{1.3}$$

This can be rewritten as a Mellin transform, using the counting function

$$\pi(x) = \#\{p : p \le x\},\tag{1.4}$$

satisfying

$$\pi'(x) = \sum_p \delta_p(x).\tag{1.5}$$

We see that if

$$J(x) = \sum_{k\ge 1}\frac{1}{k}\pi(x^{1/k}),\tag{1.6}$$

then

$$J'(x) = \sum_{k\ge 1}\frac{1}{k}\sum_p \delta_{p^k}(x),\tag{1.7}$$

and hence (1.3) is equivalent to

$$\begin{aligned}\log\zeta(s) &= \int_0^\infty J'(x)x^{-s}\,dx\\ &= s\int_0^\infty J(x)x^{-s-1}\,dx.\end{aligned}\tag{1.8}$$

Thus there is a hope (vaguely realized) of obtaining $J(x)$ via inversion of the Mellin transform.

We make some comments on the close relationship of $\pi(x)$ and $J(x)$. Note that in (1.6) all terms vanish for k so large that $x^{1/k} < 2$, i.e., the sum is restricted to $k \le \log_2 x$. It readily follows that

$$\begin{aligned}\pi(x) < J(x) &< \pi(x) + \pi(x^{1/2})\sum_{k=2}^{\log_2 x}\frac{1}{k}\\ &< \pi(x) + \pi(x^{1/2})\log(\log_2 x).\end{aligned}\tag{1.9}$$

While we do not make use of it here, it is interesting to know that the relationship (1.6) can be inverted, to yield

$$\pi(x) = \sum_{n \geq 1} \frac{\mu(n)}{n} J(x^{1/n}), \tag{1.10}$$

where μ is the Möbius function, given by

$$\mu(n) = \begin{array}{ll} 0 & \text{if } n \text{ has a repeated prime factor,} \\ (-1)^{\#\text{prime factors}} & \text{otherwise.} \end{array} \tag{1.11}$$

At this point we recall some lore on $\pi(x)$ and on $\zeta(s)$. First, there is Chebycheff's estimate:

$$C_1 \frac{x}{\log x} < \pi(x) < C_2 \frac{x}{\log x}, \tag{1.12}$$

which is more elementary than the Prime Number Theorem. (A direct proof of (1.12) will be contained in results we derive in §4.) Next, $\zeta(s)$ is meromorphic on $\mathbb{C}$, with just one pole, at $s = 1$. Also $\zeta(s)$ has no zeros in $\{s : \operatorname{Re} s > 1\}$, as a simple consequence of (1.2). In addition, $\zeta(s)$ has no zeros on $\{s : \operatorname{Re} s = 1\}$. This result is harder to prove, and lies at the heart of the proofs of Hadamard and de le Vallée Poussin of the Prime Number Theorem. (We will present what has become the standard proof of this fact in Appendix A.) Below we will show how the PNT follows from this fact, together with a certain Tauberian theorem, which we will also state below.

Let us proceed to take $s = 1 + it$ in (1.8) and rewrite it as a Fourier transform:

$$\frac{\log \zeta(1 + it)}{1 + it} = \int_{-\infty}^{\infty} J(e^y) e^{-y} \, e^{-ity} \, dy = \widehat{F}(t), \tag{1.13}$$

the Fourier transform of

$$F(y) = J(e^y) e^{-y}. \tag{1.14}$$

Note that F is supported on $\{y \in \mathbb{R} : y \geq \log 2\}$. By (1.9) and (1.12) we have

$$0 < F(y) \leq \frac{C_3}{y},$$

which does not quite give $F \in L^1(\mathbb{R})$. This is consistent with the fact that the left side of (1.13) has a logarithmic singularity at $t = 0$:

$$\widehat{F}(t) = \frac{\log \zeta(1 + it)}{1 + it} \sim \log \frac{1}{t}, \quad t \to 0. \tag{1.15}$$

By comparison, if we set

$$G(y) = \begin{cases} \dfrac{1}{y}, & y \geq 1, \\ 0, & y < 1, \end{cases} \tag{1.16}$$

we have $\widehat{G} \in C^\infty(\mathbb{R} \setminus 0)$ and

$$\widehat{G}(t) \sim \log \frac{1}{t}, \quad t \to 0. \tag{1.17}$$

Hence the Fourier transform of $F - G$ is continuous on $\mathbb{R}$. This leads one to believe that $F(y) - G(y)$ is relatively smaller as $y \to \infty$, or otherwise put, that

$$J(e^y) \sim \frac{e^y}{y}, \quad y \to \infty, \tag{1.18}$$

i.e.,

$$J(x) \sim \frac{x}{\log x}, \quad \text{or} \quad \pi(x) \sim \frac{x}{\log x}, \quad x \to \infty. \tag{1.19}$$

Of course, (1.19) is one version of the PNT. Unfortunately it is not so obvious how to deduce (1.18)–(1.19) rigorously from (1.15)–(1.17). One way to get asymptotics for $J(x)$ from knowledge of $\zeta(s)$ will be to exploit the following.

Ikehara Tauberian Theorem. *Let $w \nearrow$ and consider*

$$f(s) = \int_0^\infty e^{-sy}\, dw(y). \tag{1.20}$$

Assume f is holomorphic on $\{s : \operatorname{Re} s > 1\}$ and that

$$f(s) - \frac{A}{s-1} \quad \textit{is continuous on} \quad \{s : \operatorname{Re} s \geq 1\}. \tag{1.21}$$

Then

$$e^{-y} w(y) \to A \quad \textit{as} \quad y \to +\infty. \tag{1.22}$$

This result is applicable to what one gets upon applying d/ds to (1.8), i.e.,

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \int_0^\infty J'(x)(\log x) x^{-s}\, dx \\ &= \int_0^\infty J'(e^y) y\, e^{-sy} e^y\, dy. \end{aligned} \tag{1.23}$$

This has the form (1.20) with

$$\begin{aligned} w(y) &= \int_0^y x e^x J'(e^x)\, dx \\ &= \int_0^{e^y} (\log x) J'(x)\, dx \\ &= \sum_{p,k;p^k \le e^y} \log p. \end{aligned} \tag{1.24}$$

In other words,

$$w(y) = \psi(e^y), \quad \psi(x) = \sum_{p,k;p^k \le x} \log p. \tag{1.25}$$

Ikehara's theorem applies, with $A = 1$, to give:

$$\frac{\psi(x)}{x} \to 1, \quad \text{as} \quad x \to \infty. \tag{1.26}$$

Note that

$$\begin{aligned} \psi(x) &= \int_0^x (\log y) J'(y)\, dy \\ &= (\log x) J(x) - \int_0^x \frac{J(y)}{y}\, dy. \end{aligned} \tag{1.27}$$

Now the estimate (1.12) implies

$$\frac{J(y)}{y} \le \frac{C_3}{\log y}, \quad \text{hence} \quad \lim_{x \to \infty} \frac{1}{x} \int_0^x \frac{J(y)}{y}\, dy = 0, \tag{1.28}$$

so (1.26) is equivalent to

$$\lim_{x \to \infty} \frac{\log x}{x} J(x) = 1, \quad \text{i.e.,} \quad \lim_{x \to \infty} \frac{\log x}{x} \pi(x) = 1, \tag{1.29}$$

the PNT as stated in (1.19).

2. Alternative presentation of the Tauberian argument

Let us use the change of variable $w(y) = v(e^y)$ to restate Ikehara's Tauberian Theorem:

Theorem. *Let $v \nearrow$ and consider*

$$f(s) = \int_1^\infty x^{-s}\, dv(x). \tag{2.1}$$

Assume f is holomorphic on $\{s : \operatorname{Re} s > 1\}$ and that

$$f(s) - \frac{A}{s-1} \quad \text{is continuous on} \quad \{s : \operatorname{Re} s \geq 1\}. \tag{2.2}$$

Then

$$\frac{v(x)}{x} \to A \quad \text{as} \quad x \to +\infty. \tag{2.3}$$

We apply this to

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty J'(x)(\log x)\, x^{-s}\, dx = \int_1^\infty x^{-s}\, d\psi(x), \tag{2.4}$$

where

$$\psi'(x) = (\log x) J'(x), \tag{2.5}$$

i.e.,

$$\psi(x) = \int_0^x (\log y) J'(y)\, dy = \sum_{p,k;p^k \leq x} \log p. \tag{2.6}$$

The Tauberian theorem yields

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1. \tag{2.7}$$

As noted in (1.27), integration by parts gives

$$\psi(x) = (\log x) J(x) - \int_0^x \frac{J(y)}{y}\, dy. \tag{2.8}$$

Since by Chebycheff's estimate

$$\frac{J(y)}{y} \leq \frac{C_3}{\log y}, \quad \text{hence} \quad \lim_{x\to\infty} \frac{1}{x} \int_0^x \frac{J(y)}{y}\, dy = 0, \tag{2.9}$$

we see that (2.7) is equivalent to

$$\lim_{x\to\infty} \frac{\log x}{x} J(x) = \lim_{x\to\infty} \frac{\log x}{x} \pi(x) = 1, \tag{2.10}$$

which is the PNT.

As indicated in the Introduction, we will establish a slightly simpler variant of Ikehara's theorem in §§4-5 and show that it yields the PNT.

3. The natural appearance of the logarithmic integral

We have seen in §1 that the asymptotic relation $\psi(x) \sim x$ is equivalent to $J(x) \sim x/\log x$, hence to $\pi(x) \sim x/\log x$. It turns out that a better approximation to $\pi(x)$ is given by the logarithmic integral:

$$\mathrm{Li}(x) = \int_2^x \frac{dy}{\log y}. \tag{3.1}$$

In fact, $\mathrm{Li}(x)$ naturally arises to connect $\psi(x)$ to $J(x)$, as we show here. Recall the relation (1.27) between $\psi(x)$ and $J(x)$, i.e.,

$$\psi'(x) = (\log x)J'(x), \tag{3.2}$$

with ψ and J supported on $[2, \infty)$. Solving for $J'(x)$ and integrating by parts gives

$$\begin{aligned} J(x) &= \int_0^x \frac{1}{\log y}\psi'(y)\,dy \\ &= \frac{\psi(x)}{\log x} + \int_0^x \frac{1}{(\log y)^2}\frac{\psi(y)}{y}\,dy. \end{aligned} \tag{3.3}$$

Suppose we have

$$\psi(x) = x + E(x). \tag{3.4}$$

Then (3.3) yields

$$J(x) = \frac{x}{\log x} + \int_2^x \frac{dy}{(\log y)^2} + \frac{E(x)}{\log x} + \int_2^x \frac{1}{(\log y)^2}\frac{E(y)}{y}\,dy. \tag{3.5}$$

We can write this in a more compact form, noting that the same integration by parts as used in (3.3), with $\psi(y)$ replaced by y, and the integration done over $[2, \infty)$, gives

$$\mathrm{Li}(x) = \frac{x}{\log x} + \int_2^x \frac{dy}{(\log y)^2} - \frac{2}{\log 2}. \tag{3.6}$$

Hence (3.5) becomes

$$J(x) = \mathrm{Li}(x) + \frac{2}{\log 2} + \frac{E(x)}{\log x} + \int_2^x \frac{1}{(\log y)^2}\frac{E(y)}{y}\,dy. \tag{3.7}$$

As for the last term on the right side of (3.7), note that, if

$$\frac{|E(y)|}{(\log y)^2} \le \frac{F(x)}{(\log x)^2}, \quad \text{for } \ 2 \le y \le x, \tag{3.8}$$

then

$$\Bigl| \int_2^x \frac{1}{(\log y)^2} \frac{E(y)}{y} \, dy \Bigr| \le \frac{F(x)}{(\log x)^2} \int_2^x \frac{dy}{y} \le \frac{F(x)}{\log x}. \tag{3.9}$$

The result (1.9) is effective in passing from here to a comparison of $\pi(x)$ and $\mathrm{Li}(x)$.

Though we don't prove it here, it is known that there exists $a > 0$ such that

$$\psi(x) = x + O(xe^{-a\sqrt{\log x}}). \tag{3.10}$$

Cf. [P]. It then follows from (3.7), (3.9), and (1.9) that

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-a\sqrt{\log x}}). \tag{3.11}$$

Remark. Comparison of (3.3)–(3.7) with the analogous argument on pp. 54–55 of [P] reinforces my preference of integration by parts over summation by parts.

4. Some elementary asymptotics

Here we will derive some asymptotic expansions that are elementary consequences of the identity

$$-\frac{\zeta'(1+it)}{\zeta(1+it)} = \widehat{H}(t) = \int_{-\infty}^{\infty} H(y) e^{-ity} \, dy, \quad H(y) = J'(e^y) y. \tag{4.1}$$

(Note that $H \in \mathcal{S}'(\mathbb{R})$ and $\operatorname{supp} H \subset [\log 2, \infty)$.) One consequence of our analysis will be a self-contained proof of the Chebycheff-type estimate

$$J(x) \le C \, \frac{x}{\log x}, \tag{4.2}$$

though not with Chebycheff's explicit identification of C.

We begin with the relation

$$(H * \beta)\hat{}\,(t) = \widehat{H}(t) \hat{\beta}(t), \tag{4.3}$$

where

$$H * \beta(x) = \int H(y) \beta(x-y) \, dy. \tag{4.4}$$

We choose β to have the following properties:

$$\beta \in \mathcal{S}(\mathbb{R}), \quad \beta > 0, \quad \hat{\beta}(0) = 1, \quad \operatorname{supp} \hat{\beta} \subset [-A, A]. \tag{4.5}$$

We could insist that A be small and then require rather little knowledge of $\zeta(s)$, or we could allow A to be large. We can say that $\widehat{H}(t)\hat{\beta}(t)$ is a compactly supported distribution on $\mathbb{R}$, with one simple singularity, at $t = 0$:

$$\widehat{H}(t)\hat{\beta}(t) - \frac{1}{it + 0} \in C^\infty(\mathbb{R}). \tag{4.6}$$

From this follows a complete asymptotic expansion:

$$H * \beta(x) \sim 1 + \sum_{\nu \ge 1} \alpha_\nu x^{-\nu}, \quad x \to +\infty. \tag{4.7}$$

A change of variable gives

$$H * \beta(\log x) = \int_0^\infty J'(y)(\log y)\, \sigma\Big(\frac{x}{y}\Big) \frac{dy}{y}, \tag{4.8}$$

with

$$\sigma(x) = \beta(\log x), \tag{4.9}$$

and hence we have the asymptotic expansion

$$\int_0^\infty J'(y)(\log y)\, \sigma\Big(\frac{x}{y}\Big) \frac{dy}{y} \sim 1 + \sum_{\nu \ge 1} \alpha_\nu (\log x)^{-\nu}, \quad x \to +\infty. \tag{4.10}$$

We can obtain some estimates from this, recalling that J' is a positive measure supported on $[2, \infty)$. Note that

$$\sigma \ge 0, \quad \frac{1}{2} \le x \le 1 \Longrightarrow \sigma(x) \ge C_1 > 0. \tag{4.11}$$

We get

$$\int_x^{2x} J'(y)(\log y) \frac{dy}{y} \le C_2, \quad \forall\, x > 0, \tag{4.12}$$

hence

$$\int_x^{2x} J'(y)(\log y)\, dy \le 2C_2 x, \tag{4.13}$$

and then, summing a geometric series, we get

$$\psi(x) = \int_0^x J'(y)(\log y)\, dy \le C_3 x, \quad \forall x \ge 2. \tag{4.14}$$

As noted before in (1.27), integration by parts gives

$$\psi(x) = J(x)(\log x) - \int_0^x \frac{J(y)}{y}\, dy, \tag{4.15}$$

and the obvious inequality $J(y) \le y$ bounds this last integral by x. Thus we have the Chebycheff-type bound (4.2) on $J(x)$, and ditto for $\pi(x)$.

5. Further asymptotics

Here we derive further consequences of (4.1), including the PNT, with proofs that do not invoke Ikehara's theorem. We look at the following set-up:

$$\lim_{x\to+\infty} \int_{-\infty}^{\infty} \beta(x-y)\, d\mu(y) = \int_{-\infty}^{\infty} \beta(y)\, dy. \tag{5.1}$$

We are working with

$$d\mu(y) = J'(e^y) y\, dy, \tag{5.2}$$

i.e., $\mu([0,x]) = J(e^x)e^{-x}x + \int_0^x J(e^y)e^{-y}(y-1)\, dy$. The main properties we need are that μ is a positive measure, supported in $(1,\infty)$, and $\mu([0,x]) \le Cx$. Furthermore, as shown in §4, for the measure μ we are working with, (5.1) holds whenever $\hat{\beta} \in C_0^\infty(\mathbb{R})$. Note that the existence of $\beta \in \mathcal{S}(\mathbb{R})$ such that $\beta > 0$ while $\hat{\beta} \in C_0^\infty(\mathbb{R})$ implies

$$\mu([x, x+1]) \le C_0 \tag{5.3}$$

for some $C_0 < \infty$, independent of x. We desire to extend the validity of (5.1) to other classes of functions β.

We proceed in stages, starting with:

Lemma 5.1. *The result (5.1) holds whenever $\beta \in \mathcal{S}(\mathbb{R})$.*

Proof. Fix $\varphi \in \mathcal{S}(\mathbb{R})$ such that

$$\int \varphi(y)\, dy = 1, \quad \hat{\varphi} \in C_0^\infty(\mathbb{R}), \tag{5.4}$$

and let $\varphi_\nu(x) = \nu\varphi(\nu x)$, for $\nu \ge 1$, so $u \mapsto \varphi_\nu * u$ is an approximate identity. Then we know that (5.1) holds when β is replaced by $\beta_\nu = \varphi_\nu * \beta$. We examine the difference

$$\int_{-\infty}^{\infty} f_\nu(x-y)\, d\mu(y), \quad f_\nu(x) = \beta_\nu(x) - \beta(x). \tag{5.5}$$

We have $\beta_\nu \to \beta$ in the $\mathcal{S}(\mathbb{R})$-topology, hence, e.g.,

$$|f_\nu(x)| \le \frac{\varepsilon(\nu)}{1+x^2}, \quad \varepsilon(\nu) \to 0 \ \text{ as } \ \nu \to \infty. \tag{5.6}$$

Now in concert with (5.3) we have

$$\Bigl| \int f_\nu(x-y)\, d\mu(y) \Bigr| \le C \sum_{k=0}^{\infty} \frac{\varepsilon(\nu)}{1+k^2}. \tag{5.7}$$

Taking $\nu \to \infty$ gives (5.1) in this context.

In particular, (5.1) holds whenever $\beta \in C_0^\infty(\mathbb{R})$. Next we have:

Lemma 5.2. *The result (5.1) holds whenever β is continuous and compactly supported.*

Proof. Again use a mollifier and replace β by $\beta_\nu = \varphi_\nu * \beta$, but this time take $\varphi \in C_0^\infty(\mathbb{R})$, rather than $\hat{\varphi} \in C_0^\infty(\mathbb{R})$. Then we have (5.1) for β replaced by β_ν, by Lemma 5.1, and we can estimate (5.5) this time using

$$|f_\nu(x)| \le \varepsilon(\nu), \quad \operatorname{supp} f_\nu \subset [A, B], \tag{5.8}$$

together with (5.3).

We pass beyond continuity, starting with:

Lemma 5.3. *The result (5.1) holds whenever β is the characteristic function of an interval $I = [a, b]$, with $-\infty < a < b < \infty$.*

Proof. We can produce continuous functions f_ν and g_ν, compactly supported, with

$$f_\nu \le \beta \le g_\nu, \quad \int (g_\nu - f_\nu)\, dy < \frac{1}{\nu}. \tag{5.9}$$

Now (5.1) holds with β replaced by f_ν and with β replaced by g_ν, so we have

$$\begin{aligned} \int f_\nu(y)\, dy &\le \liminf_{x\to\infty} \int \beta(x-y)\, d\mu(y) \\ &\le \limsup_{x\to\infty} \int \beta(x-y)\, d\mu(y) \\ &\le \int g_\nu(y)\, dy, \end{aligned} \tag{5.10}$$

yielding the result.

We are now ready for our main extension of (5.1).

Proposition 5.4. *The result (5.1) holds whenever β has compact support, is bounded, and Riemann integrable.*

Proof. By Lemma 5.3, (5.1) holds for any compactly supported step function. Now let β satisfy the current hypotheses and take $\varepsilon > 0$. We can find compactly supported step functions f_ε and g_ε such that

$$f_\varepsilon \leq \beta \leq g_\varepsilon, \qquad \int (g_\varepsilon - f_\varepsilon)\, dy < \varepsilon. \tag{5.11}$$

Then an analogue of (5.10) finishes the proof.

We change variables and rewrite (5.1) as

$$\lim_{x\to\infty} \int_0^\infty J'(y)(\log y)\, \sigma\Big(\frac{x}{y}\Big)\, \frac{dy}{y} = \int_0^\infty \sigma(y)\, \frac{dy}{y}, \tag{5.12}$$

with $\sigma(y) = \beta(\log y)$. By Proposition 5.4 we can say that (5.12) is valid whenever σ is a bounded, Riemann integrable function with compact support in $(0,\infty)$. Let us consider some examples.

Example 1. Take $\sigma(y) = \chi_{[a,b]}(y)$, with $0 < a < b < \infty$. We get

$$\lim_{x\to\infty} \int_{x/b}^{x/a} J'(y)(\log y)\, \frac{dy}{y} = \int_a^b \frac{dy}{y} = \log \frac{b}{a},$$

or equivalently

$$\lim_{x\to\infty} \int_{ax}^{bx} J'(y)(\log y)\, \frac{dy}{y} = \log \frac{b}{a}. \tag{5.13}$$

Example 2. Take $\sigma(y) = y^{-1}\chi_{[a,b]}(y)$. We get

$$\lim_{x\to\infty} \frac{1}{x} \int_{x/b}^{x/a} J'(y)(\log y)\, dy = \int_a^b \frac{dy}{y^2} = \frac{1}{a} - \frac{1}{b},$$

or equivalently

$$\lim_{x\to\infty} \frac{1}{x} \int_{ax}^{bx} J'(y)(\log y)\, dy = b - a. \tag{5.14}$$

Noting that $\log x + \log a \leq \log y \leq \log x + \log b$ on the interval of integration in (5.14), we have

$$\lim_{s\to\infty} \frac{\log x}{x} \int_{ax}^{bx} J'(y)\, dy = b - a. \tag{5.15}$$

In light of the close agreement between $J(x)$ and $\pi(x)$, this is equivalent to

$$\pi(bx) - \pi(ax) \sim (b-a)\,\frac{x}{\log x}, \tag{5.16}$$

whenever $0 < a < b < \infty$. Note that the left side of (5.16) is equal to the number of primes in the interval $(ax, bx]$.

Remark 1. We can compare (5.13) with Mertens's estimate, which is

$$\int_0^x J'(y)(\log y)\,\frac{dy}{y} = \log x + O(1). \tag{5.17}$$

This result is "elementary;" see pp. 90-91 of [J] for a proof. Now (5.17) implies

$$\int_{ax}^{bx} J'(y)(\log y)\,\frac{dy}{y} = \log\frac{b}{a} + O(1), \tag{5.18}$$

which does not quite give (5.13). This is a pity, since passing from (5.13) to (5.15)–(5.16) merely involves the proof of Proposition 5.4, so an elementary proof of (5.13) would yield an elementary proof of (5.16).

Remark 2. It is simple enough to pass from (5.16) to the standard form of the PNT, i.e.,

$$\pi(x) \sim \frac{x}{\log x}. \tag{5.19}$$

To see this, first note that (5.16) is equivalent to

$$\pi(bx) - \pi(ax) \sim L(bx) - L(ax), \quad L(x) = \frac{x}{\log x}. \tag{5.20}$$

For example,

$$\pi(2^{k+1}) - \pi(2^k) = \Big(L(2^{k+1}) - L(2^k)\Big)(1+\varepsilon_k), \tag{5.21}$$

with $\varepsilon_k \to 0$ as $k \to \infty$. Hence

$$\frac{\pi(2^N)}{L(2^N)} = 1 + \sum_{k=1}^{N-1} \varepsilon_k\,\frac{L(2^{k+1}) - L(2^k)}{L(2^N)} + \frac{1}{L(2^N)}, \tag{5.22}$$

and it is readily verified that the right side of (5.22) tends to 1 as $N \to \infty$. Now that we have

$$\pi(2^N) \sim L(2^N), \tag{5.23}$$

passing to (5.19) follows readily from this and (5.20).

6. Alternative endgame

In this section, largely following Chapter 13 of [A], we show how the PNT can be obtained via a study of

$$\psi_1(x) = \int_0^x \psi(y)\, dy, \tag{6.1}$$

with $\psi(x)$ given by (1.27). Note that $\operatorname{supp} \psi \subset [2, \infty)$, so the lower limit of integration in (6.1) could be changed from 0 to a for any $a < 2$. We note the elementary estimate

$$\psi_1(x) \le Cx^2. \tag{6.2}$$

The following Tauberian theorem will suffice for the approach taken here.

Lemma 6.1. *If we have*

$$\lim_{x\to+\infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2}, \tag{6.3}$$

then it follows that

$$\lim_{x\to+\infty} \frac{\psi(x)}{x} = 1. \tag{6.4}$$

This result is elementary. A short proof is given on pp. 280–281 of [A]. As we have seen, (6.4) is equivalent to the PNT, so it remains to prove (6.3).

We relate ψ_1 to the zeta function. As in (2.4), we have

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \int_1^\infty \psi'(x) x^{-s}\, dx \\ &= s \int_1^\infty \psi(x) x^{-s-1}\, dx, \end{aligned} \tag{6.5}$$

and hence, using (6.1),

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= s \int_1^\infty \psi_1'(x) x^{-s-1}\, dx \\ &= s(s+1) \int_1^\infty \psi_1(x) x^{-s-2}\, dx, \end{aligned} \tag{6.6}$$

as long as $\mathrm{Re}\, s > 1$. In other words,

$$-\frac{1}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{\psi_1(x)}{x^2}\, x^{-s}\, dx. \tag{6.7}$$

We want to produce the Mellin transform of an elementary function to cancel the singularity of the left side of (6.7) at $s = 1$, and arrange that the difference be nicely behaved on $\{s : \mathrm{Re}\, s = 1\}$. Using

$$\int_1^\infty x^{-s}\, dx = \frac{1}{s-1}, \tag{6.8}$$

valid for $\mathrm{Re}\, s > 1$, we can see that

$$\begin{aligned} \frac{1}{2}\int_1^\infty \Bigl(1 - \frac{1}{x}\Bigr)^2 x^{-s}\, dx &= \frac{1}{2}\int_1^\infty (x^{-s} - 2x^{-s-1} + x^{-s-2})\, dx \\ &= \frac{1}{(s-1)s(s+1)}, \end{aligned} \tag{6.9}$$

so we achieve such cancellation with

$$-\frac{1}{s(s+1)}\Bigl(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\Bigr) = \int_1^\infty \Bigl[\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Bigl(1 - \frac{1}{x}\Bigr)^2\Bigr] x^{-s}\, dx. \tag{6.10}$$

To simplify notation, let us set

$$\Phi(x) = \Bigl[\frac{\psi_1(x)}{x^2} - \frac{1}{2}\Bigl(1 - \frac{1}{x}\Bigr)^2\Bigr]\chi_{[1,\infty)}(x). \tag{6.11}$$

We make the change of variable $x = e^y$ and set $s = 1 + it$ (so $\mathrm{Re}\, s > 1 \Leftrightarrow \mathrm{Im}\, t < 0$), and write the right side of (6.10) as

$$\int_{-\infty}^\infty \Phi(e^y) e^{-ity}\, dy. \tag{6.12}$$

Clearly $\Phi(e^y)$ is bounded, and supported on $0 \le y < \infty$, so one can pass from $\mathrm{Im}\, t < 0$ to $t \in \mathbb{R}$, interpreting (6.12) as the Fourier transform of a tempered distribution, and see that (6.12) is equal to

$$\Xi(t) = -\frac{1}{(1+it)(2+it)}\Bigl[\frac{\zeta'(1+it)}{\zeta(1+it)} + \frac{1}{it}\Bigr]. \tag{6.13}$$

Fourier inversion (for tempered distributions) implies that the inverse Fourier transform of Ξ is equal to $\Phi(e^y)$. The following result provides the final key for this approach to the PNT.

Lemma 6.2. *We have*

$$\Xi \in L^1(\mathbb{R}). \tag{6.14}$$

Given (6.14), we have $\lim_{y\to+\infty} \Phi(e^y) = 0$, by the Riemann-Lebesgue lemma, which in turn gives (6.3). As for (6.14), the fact that $1/(s-1)$ cancels the pole of $\zeta'(s)/\zeta(s)$ at $s = 1$ plus the fact that $\zeta(s)$ has no zeros with $\operatorname{Re} s = 1$ implies $\Xi \in C^\infty(\mathbb{R})$. The large t behavior of $\Xi(t)$ is a consequence of the following estimate:

$$\Bigl|\frac{\zeta'(1+it)}{\zeta(1+it)}\Bigr| \le C(\log|t|)^9, \quad t \in \mathbb{R},\ |t| \ge 2. \tag{6.15}$$

This result can also be obtained from (A.1); see [A], pp. 287–288, or [J], pp. 108–109, for a proof.

A. Zeros of the zeta function

Here we show that $\zeta(s) \neq 0$ when $\operatorname{Re} s = 1$. Our treatment follows [J], pp. 106–107, though many other sources have similar treatments. We start with:

Lemma A.1. *For all $\sigma > 1,\ t \in \mathbb{R}$,*

$$\zeta(\sigma)^3\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \ge 1. \tag{A.1}$$

Proof. Since $\log|z| = \operatorname{Re}\log z$, (A.1) is equivalent to

$$3\log\zeta(\sigma) + 4\operatorname{Re}\log\zeta(\sigma+it) + \operatorname{Re}\log\zeta(\sigma+2it) \ge 0. \tag{A.2}$$

By (1.8) we have

$$\log\zeta(s) = \sum_{n\ge1}\frac{a(n)}{n^s}, \tag{A.3}$$

with coefficients $a(n) \ge 0$ for each n. Thus the left side of (A.2) is equal to

$$\sum_{n=1}^\infty \frac{a(n)}{n^\sigma}\ \operatorname{Re}(3 + 4n^{-it} + n^{-2it}). \tag{A.4}$$

But, with $\theta_n = t\log n$,

$$\begin{aligned} \operatorname{Re}(3 + 4n^{-it} + n^{-2it}) &= 3 + 4\cos\theta_n + \cos 2\theta_n \\ &= 2 + 4\cos\theta_n + 2\cos^2\theta_n \\ &= 2(1+\cos\theta_n)^2, \end{aligned} \tag{A.5}$$

which is ≥ 0 for each n, so we have (A.2), as asserted.

We will assume it known that $\zeta(s) - 1/(s-1)$ is an entire holomorphic function. Proofs of this fact can be found in [A], [P], [S], Chapter 3 of [T1], and §19 of [T2]. Now for our result:

Theorem A.2. *For all real $t \neq 0$, $\zeta(1+it) \neq 0$.*

Proof. Suppose $t \in \mathbb{R} \setminus 0$ and $\zeta(1+it) = 0$. Then

$$\lim_{\sigma \searrow 1} \frac{\zeta(\sigma+it)}{\sigma-1} = \zeta'(1+it). \tag{A.6}$$

If $\Phi(\sigma)$ denotes the left side of (A.1), then

$$\Phi(\sigma) = \bigl((\sigma-1)\zeta(\sigma)\bigr)^3 \Bigl(\frac{|\zeta(\sigma+it)|}{\sigma-1}\Bigr)^4 \bigl|(\sigma-1)\zeta(\sigma+2it)\bigr|. \tag{A.7}$$

Note that

$$\lim_{\sigma \searrow 1} (\sigma-1)\zeta(\sigma) = 1. \tag{A.8}$$

Thus if (A.6) holds we must have $\lim_{\sigma \searrow 1} \Phi(\sigma) = 0$, contradicting (A.1).

B. Evaluation of $\zeta(k)$ via Fourier Series, for k Even

We evaluate $\zeta(k)$ inductively, for k even, by studying the Fourier series of

$$f_k(x) = x^k \quad \text{on} \quad S^1 = [-\pi, \pi]. \tag{B.1}$$

We have, by Fourier inversion,

$$\sum_{n=-\infty}^{\infty} \hat{f}_k(n) = 2\pi f_k(0) = 0, \tag{B.2}$$

where

$$\hat{f}_k(n) = \int_{-\pi}^{\pi} x^k e^{-inx}\, dx. \tag{B.3}$$

In particular,

$$\hat{f}_k(0) = \int_{-\pi}^{\pi} x^k\, dx = \frac{2\pi^{k+1}}{k+1}, \tag{B.4}$$

if k is even. To evaluate other Fourier coefficients, we recall the following trick. Set

$$T_k(x) = \sum_{j=0}^{k} \frac{x^j}{j!}. \tag{B.5}$$

Then $T_k'(x) = T_{k-1}(x)$, so

$$\frac{d}{dx}\Big(T_k(x)e^{-x}\Big) = T_{k-1}(x)e^{-x} - T_k(x)e^{-x} = -\frac{x^k}{k!}e^{-x}, \tag{B.6}$$

and hence

$$\int x^k e^{-x}\,dx = -k!\,T_k(x)e^{-x} + C. \tag{B.7}$$

Setting $x = \alpha y$ gives

$$\int y^k e^{-\alpha y}\,dy = -\frac{k!}{\alpha^{k+1}}T_k(\alpha y)e^{-\alpha y} + C, \tag{B.8}$$

and hence, for $n \neq 0$,

$$\begin{aligned} \hat{f}_k(n) &= -\frac{k!}{(in)^{k+1}}T_k(inx)e^{-inx}\Big|_{-\pi}^{\pi} \\ &= (-1)^{n+1}\frac{k!}{(in)^{k+1}}\big[T_k(in\pi) - T_k(-in\pi)\big]. \end{aligned} \tag{B.9}$$

Now, by (B.5), we have

$$T_k(in\pi) - T_k(-in\pi) = 2\sum_{0<j<k;j\text{ odd}}\frac{(in\pi)^j}{j!}. \tag{B.10}$$

Note that

$$k\text{ even} \Longrightarrow \hat{f}_k(-n) = \hat{f}_k(n). \tag{B.11}$$

Since we are assuming k is even, we can put together (B.2), (B.4), and (B.9)–(B.11) to get

$$\begin{aligned} \frac{2}{k+1}\pi^{k+1} &= 4k!\sum_{n=1}^{\infty}\frac{(-1)^n}{(in)^{k+1}}\sum_{0<j<k;j\text{ odd}}\frac{(in\pi)^j}{j!} \\ &= 4k!\sum_{0<j<k;j\text{ odd}}\frac{\pi^j}{j!}\,i^{j-k-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{k-j+1}}. \end{aligned} \tag{B.12}$$

Note that in this last double sum $j-k-1$ and $k-j+1$ are even. Note also that, for $\ell \geq 2$,

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{(-1)^n}{n^\ell} &= \sum_{n\text{ even}}\frac{1}{n^\ell} - \sum_{n\text{ odd}}\frac{1}{n^\ell} \\ &= 2\sum_{m=1}^{\infty}\frac{1}{(2m)^\ell} - \sum_{n=1}^{\infty}\frac{1}{n^\ell} \\ &= -(1-2^{1-\ell})\zeta(\ell). \end{aligned} \tag{B.13}$$

Consequently (B.12) gives, for k even,

$$\text{(B.14)} \qquad \frac{2}{k+1}\pi^{k+1} = 4k! \sum_{0<j<k;j \text{ odd}} \frac{\pi^j}{j!}\, i^{j-k+1}\,(1-2^{j-k})\zeta(k-j+1).$$

Note that $j-k+1$ is even in this sum, so $i^{j-k+1} = \pm 1$.

Take the case $k=2$. Then the right side has one term, $j=1$. We get

$$\text{(B.15)} \qquad \frac{2}{3}\pi^3 = 8\pi\Big(\frac{1}{2}\Big)\zeta(2), \quad \text{i.e.,} \quad \zeta(2) = \frac{\pi^2}{6},$$

a well known formula. From here, (B.14) inductively gives $\zeta(k)$, for k even, as a rational multiple of π^k.

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