The Cauchy Integral Theorem on Finite-Perimeter Domains

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Contents

1. Introduction

The first goal of this note is to establish the following two results.

Proposition 1.1. *Let $\Omega \subset \mathbb{C}$ be a bounded open set, and assume Ω is a finite-perimeter domain. Assume*

$$f \in \mathrm{Lip}(\overline{\Omega}), \quad \text{and } f \text{ is holomorphic on } \Omega. \tag{1.1}$$

Then

$$\int_{\partial\Omega} f(z)\, d\alpha(z) = 0. \tag{1.2}$$

Here α is a complex Borel measure on $\partial\Omega$ that will be described below; see (3.4)–(3.7).

Proposition 1.2. *Let $\Omega \subset \mathbb{C}$ be a bounded open set. Assume $\partial\Omega$ has finite 1-dimensional Hausdorff measure and only a finite number of connected components. Then (1.2) holds provided*

$$f \in C(\overline{\Omega}) \text{ and } f \text{ is holomorphic on } \Omega. \tag{1.3}$$

These results lead to the following version of the Cauchy integral formula.

Proposition 1.3. *In the setting of either Proposition 1.1 or 1.2, if* $z_0 \in \Omega$*, then*

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(z)}{z - z_0}\, d\alpha(z). \tag{1.4}$$

Proof. Propositions 1.1–1.2 apply to

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0},$$

which has a removable singularity at $z = z_0$. Hence (1.4) follows from the identity

$$\int\limits_{\partial\Omega} \frac{1}{z - z_0}\, d\alpha(z) = 2\pi i. \tag{1.5}$$

In turn, we can apply Proposition 1.1 to $f(z) = 1/(z - z_0)$, with Ω replaced by $\Omega \setminus \overline{D_\varepsilon(z_0)}$, where $D_\varepsilon(z_0)$ is a small disk centered at z_0, to get

$$\int\limits_{\partial\Omega} \frac{1}{z - z_0}\, d\alpha(z) = \int\limits_{\partial D_\varepsilon(z_0)} \frac{1}{z - z_0}\, d\alpha(z). \tag{1.6}$$

The formula (3.7) for the measure α makes it clear that the right side of (1.6) is equal to

$$\int\limits_{\partial D_\varepsilon(z_0)} \frac{dz}{z - z_0}, \tag{1.7}$$

whose identity with $2\pi i$ is standard.

Of the results described above, Proposition 1.1 is well known, and is included just as a starting point. The goal in subsequent propositions is to establish (1.2) and (1.4) under weaker hypotheses on f than (1.1).

We define some terms used above, starting in a general n-dimensional setting.

Given $\Omega \subset \mathbb{R}^n$, bounded and open, let $\chi_\Omega(x) = 1$ for $x \in \Omega$, 0 for $x \notin \Omega$, and form

$$\nabla \chi_\Omega = \mu, \tag{1.8}$$

an $\mathbb{R}^n$-valued distribution, supported on the boundary $\partial\Omega$. We say Ω is a finite-perimeter domain if μ is a finite $\mathbb{R}^n$-valued measure. In such a case, the Radon-Nikodym theorem allows us to write

$$\mu = -\nu\, \sigma, \tag{1.9}$$

where σ is a positive measure supported on $\partial\Omega$ and ν is a bounded, $\mathbb{R}^n$-valued Borel function of $\partial\Omega$, satisfying

$$|\nu(x)| = 1, \quad \sigma\text{-a.e.} \tag{1.10}$$

Via distribution theory, we can restate (1.8)–(1.9) as follows. Take $X \in C^\infty(\mathbb{R}^n, \mathbb{C}^n)$, a complex vector field. Then

$$\langle \operatorname{div} X, \chi_\Omega \rangle = -\langle X, \nabla \chi_\Omega \rangle. \tag{1.11}$$

Hence (1.8)–(1.9) is equivalent to

$$\int\limits_\Omega \operatorname{div} X \, dx = \int\limits_{\partial\Omega} \langle \nu, X \rangle \, d\sigma. \tag{1.12}$$

Applying a mollifier, $X_\varepsilon = \varphi_\varepsilon * X$, and noting that $\operatorname{div} X_\varepsilon = \varphi_\varepsilon * \operatorname{div} X$, we see that, if

$$X \in C(\mathbb{R}^n), \quad \operatorname{div} X \in L^1(\mathbb{R}^n), \tag{1.13}$$

then

$$\begin{aligned} &X_\varepsilon \longrightarrow X \quad \text{uniformly on } \overline{\Omega} \text{ (hence on } \partial\Omega\text{), and} \\ &\operatorname{div} X_\varepsilon \longrightarrow \operatorname{div} X \quad \text{in} \quad L^1\text{-norm}, \end{aligned} \tag{1.14}$$

so we have the identity (1.12) whenever X satisfies (1.13), in particular whenever $X \in \operatorname{Lip}(\mathbb{R}^n)$. This leads to the following:

Proposition 1.4. *If $\Omega \subset \mathbb{R}^n$ is a finite-perimeter domain, (1.12) holds whenever $X \in \operatorname{Lip}(\overline{\Omega})$.*

Proof. Each $X \in \operatorname{Lip}(\overline{\Omega})$ has a Lipschitz extension to $\mathbb{R}^n$.

For later use, we record the following simple consequence.

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a finite-perimeter domain. Assume*

$$X \in C(\overline{\Omega}), \quad \operatorname{div} X\big|_\Omega \in L^1(\Omega), \tag{1.15}$$

and assume there exist $X_k \in \operatorname{Lip}(\overline{\Omega})$ such that

$$X_k \to X \quad \textit{uniformly on } \overline{\Omega}, \quad \operatorname{div} X_k \to \operatorname{div} X \quad \textit{in} \quad L^1(\Omega). \tag{1.16}$$

Then X satisfies (1.12).

The rest of this note is organized as follows. Section 2 recalls basic facts about finite-perimeter domains $\Omega \subset \mathbb{R}^n$, the measure σ on $\partial\Omega$, and its relation to $(n-1)$-dimensional Hausdorff measure. In §3 we prove Proposition 1.1. Along the way to proving Proposition 1.2, we bring in the following.

Proposition 1.6. *Let $\Omega \subset \mathbb{C}$ be a bounded, open, finite-perimeter domain. Assume Ω has a "tame interior approximation." That is, assume there exist finite-perimeter domains $\Omega_k \subset \overline{\Omega}_k \subset \Omega$ and $C < \infty$ such that*

$$\chi_{\Omega_k} \to \chi_\Omega \quad a.e., \text{ and } \quad \sigma(\partial\Omega_k) \le C. \tag{1.17}$$

Then (1.2) holds whenever f satisfies (1.3).

We prove Proposition 1.6 in §4, and then address the proof of Proposition 1.2 in §5. We tie Propositions 1.2 and 1.6 together with a result on domains satisfying a "lower Ahlfors regularity" condition.

In §6 we specialize Proposition 1.2 to the case where $\Omega \subset \mathbb{C}$ is a bounded open set whose boundary $\partial\Omega$ is a disjoint union of a finite number of rectifiable Jordan curves, and make contact with a version of the Cauchy integral theorem given in [Si].

In §7 we mention some variants of Propositions 1.2–1.3, valid for holomorphic functions on Ω such that

$$\mathcal{N}f \in L^p(\partial\Omega, \sigma), \tag{1.18}$$

where $\mathcal{N}f$ is the nontangential maximal function of f. Such results are established in [HMT], [MMT], and [MMM], for certain classes of finite perimeter domains.

We have two appendices. Appendix A verifies that, when $\Omega \subset \mathbb{R}^n$ is a bounded open set, the definition of when Ω is a finite-perimeter domain (and what its perimeter is) given in [CR] is equivalent to that given in this note (which follows a number of sources, such as [EG]). Appendix B treats some removable singularity theorems, describing compact sets $K \subset \mathbb{C}$ with the property that

$$f \in C(\mathbb{C}), \quad f \text{ holomorphic on } \mathbb{C} \setminus K \Longrightarrow f \text{ holomorphic on } \mathbb{C}. \tag{1.19}$$

Results of such a nature arise from the analysis in §4.

2. Structure of finite-perimeter domains

We give a further discussion of the structure of a bounded open $\Omega \subset \mathbb{R}^n$ that is a finite-perimeter domain. Recall from §1 the vector measure μ, the positive measure σ, and the function ν on $\partial\Omega$. When $\partial\Omega$ is C^1, ν is continuous on $\partial\Omega$ and $\nu(x)$ is the outward pointing unit normal to $\partial\Omega$ at x. For general finite-perimeter domains, $\partial\Omega$ can be quite rough, and it is useful to specify some distinguished subsets. We define these subsets below and state some results, established in works of DeGiorgi and Federer, referring to [EG] for proofs. These subsets will be denoted $\partial^*\Omega \subset \partial_0\Omega \subset \partial_*\Omega \subset \partial\Omega$.

We first define the reduced boundary $\partial^*\Omega$. By the Besikovitch differentiation theorem,

$$\lim_{r\to 0} \frac{1}{\sigma(B_r(x))} \int\limits_{B_r(x)} \nu \, d\sigma = \nu(x), \tag{2.1}$$

for σ-a.e. x. If $x \in \partial\Omega$ and the limit (2.1) exists, and $|\nu(x)| = 1$, we say $x \in \partial^*\Omega$. It is an important structural result that $\partial^*\Omega$ is countably $(n-1)$-rectifiable, i.e., it is a countable disjoint union

$$\partial^*\Omega = \bigcup_k M_k \cup N, \tag{2.2}$$

where each M_k is a compact subset of an $(n-1)$-dimensional C^1 surface (to which ν is normal in the usual sense) and $\mathcal{H}^{n-1}(N) = 0$, where $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. Furthermore,

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^*\Omega. \tag{2.3}$$

Next, given a unit vector ν_E and $x \in \partial\Omega$, set

$$H^{\pm}_{\nu_E}(x) = \{y \in \mathbb{R}^n : \langle \pm\nu_E, y - x\rangle \geq 0\}. \tag{2.4}$$

Then (cf. [EG], p. 203), for $x \in \partial^*\Omega$, $\Omega^+ = \Omega$, and $\Omega^- = \mathbb{R}^n \setminus \Omega$, one has

$$\lim_{r\to 0} r^{-n} \mathcal{L}^n \Big(B_r(x) \cap \Omega^{\pm} \cap H^{\pm}_{\nu_E}(x) \Big) = 0, \tag{2.5}$$

when $\nu_E = \nu(x)$, as in (2.1). Here $\mathcal{L}^n$ denotes Lebesgue measure on $\mathbb{R}^n$. More generally, a unit vector ν_E for which (2.5) holds is called the measure-theoretic unit normal to $\partial\Omega$ at x. If such ν_E exists, it is unique. Thus we define $\partial_0\Omega$ to consist of $x \in \partial\Omega$ for which (2.5) holds, with $\nu_E(x)$ denoting the measure-theoretic outer normal. We have $\partial_0\Omega \supset \partial^*\Omega$ and $\nu_E(x) = \nu(x)$ on $\partial^*\Omega$.

Third, we define $\partial_*\Omega$, the measure-theoretic boundary of Ω, to consist of $x \in \partial\Omega$ such that

$$\limsup_{r\to 0} r^{-n}\mathcal{L}^n\Big(B_r(x)\cap\Omega^{\pm}\Big) > 0. \tag{2.6}$$

It is clear that $\partial_*\Omega \supset \partial_0\Omega$. Furthermore (cf. [EG], p. 208) one has

$$\mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0. \tag{2.7}$$

Consequently the formula (1.12) can be written

$$\int_{\Omega} \operatorname{div} X \, dx = \int_{\partial_*\Omega} \langle \nu, X\rangle \, d\mathcal{H}^{n-1}, \tag{2.8}$$

for X satisfying the conditions of Proposition 1.4 or Proposition 1.5.

We remark that $\partial_*\Omega$ is well defined whether or not Ω has finite perimeter. It is known that Ω has finite perimeter if and only if $\mathcal{H}^{n-1}(\partial_*\Omega) < \infty$ (cf. [EG], p. 222).

3. Proof of Proposition 1.1

Let us now take $n = 2$ and address Proposition 1.1. We take a complex valued $f \in C(\overline{\Omega})$ and form the vector field

$$X = \begin{pmatrix} if \\ -f \end{pmatrix}. \tag{3.1}$$

Then, as a distribution on Ω,

$$\operatorname{div} X = i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}. \tag{3.2}$$

Hence

$$\operatorname{div} X = 0 \text{ on } \Omega \Longleftrightarrow f \ \text{ is holomorphic on } \ \Omega. \tag{3.3}$$

Now if $f \in \mathrm{Lip}(\overline{\Omega})$, then $X \in \mathrm{Lip}(\overline{\Omega})$, and Proposition 1.4 applies, yielding

$$\int\limits_{\partial\Omega} f\langle \nu, E\rangle \, d\sigma = 0, \quad E = \begin{pmatrix} i \\ -1 \end{pmatrix}. \tag{3.4}$$

This proves Proposition 1.1, with the complex measure α given by

$$\alpha = \langle \nu, E\rangle \, \sigma. \tag{3.5}$$

Note that if we implement the natural identification $\mathbb{R}^2 \approx \mathbb{C}$, we obtain

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \mapsto \nu_1 + i\nu_2 = \tilde{\nu}, \quad \text{while} \ \ \langle \nu, E\rangle = i\nu_1 - \nu_2, \tag{3.6}$$

hence

$$\alpha = i\tilde{\nu}\, \sigma. \tag{3.7}$$

4. Proof of Proposition 1.6

To prepare for the proof of Proposition 1.6, we introduce a concept that arose in [HMT].

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, finite-perimeter domain. We say Ω has a tame interior approximation if there exist finite perimeter domains $\Omega_k \subset \overline{\Omega}_k \subset \Omega$ and $C < \infty$ such that

$$\chi_{\Omega_k} \to \chi_\Omega \text{ a.e., as } k \to \infty, \quad \text{and} \ \ \sigma(\partial\Omega_k) \le C. \tag{4.1}$$

We remark that it need *not* be the case that $\sigma(\partial\Omega_k) \to \sigma(\partial\Omega)$. Actually, the first part of our hypothesis in (4.1) is weaker than given in [HMT], which required $\Omega_k \nearrow \Omega$. Thus the following result is a little stronger than its counterpart in [HMT], though the proof is essentially the same.

Proposition 4.1. *If $\Omega \subset \mathbb{R}^n$ is a bounded, open finite-perimeter domain with tame interior approximation, and if*

$$X \in C(\overline{\Omega}), \quad \operatorname{div} X \in L^1(\Omega), \tag{4.2}$$

then

$$\int\limits_\Omega \operatorname{div} X \, dx = \int\limits_{\partial\Omega} \langle \nu, X \rangle \, d\sigma. \tag{4.3}$$

Proof. The argument involving (1.13)–(1.14) implies

$$\int \chi_{\Omega_k}(x) \operatorname{div} X \, dx = -\langle \nabla \chi_{\Omega_k}, X \rangle, \tag{4.4}$$

for each k. The first part of (4.1) implies, via the Lebesgue dominated convergence theorem, that the left side of (4.4) tends to

$$\int\limits_\Omega \operatorname{div} X \, dx, \quad \text{as} \ \ k \to \infty, \tag{4.5}$$

given $\operatorname{div} X \in L^1(\Omega)$. We also have

$$\chi_{\Omega_k} \longrightarrow \chi_\Omega \ \text{ in } \ L^p(\mathbb{R}^n), \quad \forall\, p \in [1, \infty), \tag{4.6}$$

all supported in $\overline{\Omega}$, hence

$$\nabla \chi_{\Omega_k} \longrightarrow \nabla \chi_\Omega, \quad \text{in} \ \ \mathcal{E}'(\mathbb{R}^n). \tag{4.7}$$

But the second part of (4.1) implies $\{\nabla\chi_{\Omega_k}\}$ is a bounded family of $\mathbb{R}^n$-valued measures, supported in $\overline{\Omega}$, and this together with (4.7) gives

$$\nabla\chi_{\Omega_k} \longrightarrow \nabla\chi_{\Omega}, \quad \text{weak}^* \text{ in } \mathcal{M}(\overline{\Omega}). \tag{4.8}$$

Thus the right side of (4.4) tends to

$$\int\limits_{\partial\Omega} \langle \nu, X\rangle \, d\sigma, \tag{4.9}$$

as $k \to \infty$, given $X \in C(\overline{\Omega})$. We hence have (4.3).

Given Proposition 4.1, arguments from §3 establish the following, which is Proposition 1.6.

Proposition 4.2. *Let $\Omega \subset \mathbb{C}$ be a bounded, open, finite-perimeter domain, and assume Ω has a tame interior approximation. Assume*

$$f \in C(\overline{\Omega}) \quad \textit{and } f \textit{ is holomorphic on } \Omega. \tag{4.10}$$

Then

$$\int\limits_{\partial\Omega} f(z)\, d\alpha(z) = 0, \tag{4.11}$$

where α is the complex Borel measure on $\partial\Omega$ given by (3.5)–(3.7).

Though we have Proposition 1.6 at this point, we will linger, and derive some more results for the class of domains with tame interior approximations. First, from (4.8) and the formula (3.5)–(3.7) for α, we have the following complement to Proposition 4.2.

Proposition 4.3. *Let $\{\Omega_k\}$ be a tame interior approximation to a bounded finite-perimeter domain $\Omega \subset \mathbb{C}$. Define the complex measure α on $\partial\Omega$ as in (3.5)–(3.7), and similarly define the measures α_k on $\partial\Omega_k$. Then*

$$\alpha_k \longrightarrow \alpha \quad \textit{weak}^* \textit{ in } \mathcal{M}(\overline{\Omega}). \tag{4.12}$$

Note. One might *not* have $\sigma_k \to \sigma$ weak*. Thus cancellation effects can play a role in (4.12). For an example, consider the slit disk,

$$\Omega = D_1(0) \setminus [0,1), \tag{4.13}$$

where $D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$. Then

$$\overline{\Omega} = \overline{D_1(0)}, \quad \partial\Omega = S^1 \cup [0,1], \quad \partial_*\Omega = S^1, \tag{4.14}$$

where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. One can readily find a tame interior approximation to Ω, e.g.,

$$\Omega_k = \{z \in \Omega : \text{dist}(z, \partial\Omega) > 2^{-k}\}. \tag{4.15}$$

Note that

$$\{\Omega_k\} \text{ is also a tame interior approximation to } D_1(0), \tag{4.16}$$

by the definition (4.1) (though not by the definition in [HMT]).

Consideration of the last example points to a removable singularities theorem:

Proposition 4.4. *Let $\mathcal{O} \subset \mathbb{C}$ be a bounded open set, $K \subset \overline{\mathcal{O}}$ a closed set of 2D Lebesgue measure 0. Assume $\Omega = \mathcal{O} \setminus K$ has a tame interior approximation, $\Omega_k \subset \Omega$ (which is automatically also a tame interior approximation to $\mathcal{O}$). Then*

$$f \in C(\overline{\mathcal{O}}),\ f \text{ holomorphic on } \Omega = \mathcal{O} \setminus K \Longrightarrow f \text{ holomorphic on } \mathcal{O}. \tag{4.17}$$

Proof. Proposition 4.2 applies to Ω, and the accompanying Cauchy integral formula holds:

$$f(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial_* \Omega} \frac{f(z)}{z - \zeta} \, d\alpha_\Omega(z), \tag{4.18}$$

for all $\zeta \in \Omega$. By (4.12), the complex measure α_Ω on $\partial\Omega$ is equal to the measure $\alpha_{\mathcal{O}}$ in $\partial\mathcal{O}$. Thus, for $\zeta \in \Omega$,

$$f(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial_* \mathcal{O}} \frac{f(z)}{z - \zeta} \, d\alpha_{\mathcal{O}}(z). \tag{4.19}$$

But the right side of (4.19) is holomorphic on $\mathbb{C} \setminus \overline{\partial_* \mathcal{O}}$, and in particular on $\mathcal{O}$.

In case $\mathcal{O} = D_1(0)$ and Ω is given by (4.13), the fact that $K = [0, 1]$ is a removable set of singularities (in the category $C(\overline{\mathcal{O}})$) can be established directly, via Morera's theorem. Proposition 4.4 deals with sets K much more irregular than a slit. One might investigate for what class of compact sets $K \subset \overline{\mathcal{O}}$ a direct proof of (4.17) via Morera's theorem can be made.

See Appendix B for comments on a related removable singularity problem.

5. Proof of Proposition 1.2

Proposition 1.2 follows directly from Proposition 1.6 and the next result:

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded open set. Assume $\mathcal{H}^1(\partial\Omega) < \infty$ and that $\partial\Omega$ has a finite number of connected components. Then Ω has a tame interior approximation. More precisely, there exist finite perimeter domains $\Omega_k \subset \overline{\Omega}_k \subset \Omega$ and $C < \infty$ such that*

$$\Omega_k \nearrow \Omega, \quad \textit{and} \quad \sigma(\partial\Omega_k) \le C. \tag{5.1}$$

Note that the condition (5.1) is stronger than (1.17). In fact, it is the condition initially used in [HMT].

The first step in the proof of Proposition 5.1 brings in the following concept, which we formulate in the general n-dimensional context. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume $\mathcal{H}^{n-1}(\partial\Omega) < \infty$. We say $\partial\Omega$ is lower Ahlfors regular if there exists $C_1 \in (0,\infty)$ such that, for all $r \in (0,1]$,

$$C_1 r^{n-1} \le \mathcal{H}^{n-1}(\partial\Omega \cap B_r(p)), \quad \forall\, p \in \partial\Omega. \tag{5.2}$$

We have the following.

Proposition 5.2. *Let $\Omega \subset \mathbb{C}$ satisfy the hypotheses of Proposition 5.1. Also assume each connected component Γ_k of $\partial\Omega$ has positive diameter, d_k (i.e., consists of more than one point). Then $\partial\Omega$ is lower Ahlfors regular.*

Proof. For such Ω, if $p \in \Gamma_k$, we have

$$\mathcal{H}^1(\partial\Omega \cap B_r(p)) \ge r, \quad \forall\, r \in (0, d_k). \tag{5.3}$$

Proposition 5.2 together with the next result establish Proposition 5.1.

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and that $\partial\Omega$ is lower Ahlfors regular. Then Ω has the tame interior approximation property, satisfying (5.1).*

In order to prove Proposition 5.3, we bring in the following lemma, which arose, in another context, in [HMT].

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^n$ satisfy the hypotheses of Proposition 5.3. Set*

$$\mathcal{O}_\delta = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \le \delta\}. \tag{5.4}$$

Then

$$\operatorname{Vol}(\mathcal{O}_\delta) \le C\delta. \tag{5.5}$$

Proof. Let B denote the unit ball centered at 0 in $\mathbb{R}^n$, with volume V_n, and set

$$\chi_\delta(x) = V_n^{-1}\delta^{-n}\chi_B(\delta^{-1}x), \tag{5.6}$$

so $\int \chi_\delta\, dx = 1$. Set $\omega = \mathcal{H}^{n-1}\rfloor\partial\Omega$ and

$$G_\delta = \omega * \chi_\delta, \tag{5.7}$$

so

$$\int G_\delta(x)\, dx = \mathcal{H}^{n-1}(\partial\Omega), \quad \forall\, \delta > 0. \tag{5.8}$$

We also have the following. If $p \in \partial\Omega$ and $|x - p| = \operatorname{dist}(x, \partial\Omega)$,

$$\begin{aligned} \operatorname{dist}(x,\partial\Omega) \le \frac{\delta}{2} \Rightarrow G_\delta(x) &\ge C\delta^{-n}\mathcal{H}^{n-1}(\partial\Omega \cap B_{\delta/2}(p)) \\ &\ge C\delta^{-1}, \end{aligned} \tag{5.9}$$

the last inequality by (5.2). Hence

$$\begin{aligned} \operatorname{Vol}(\mathcal{O}_{\delta/2}) &\le C\delta \int\limits_{\mathcal{O}_{\delta/2}} G_\delta(x)\, dx \\ &\le C\delta\mathcal{H}^{n-1}(\partial\Omega). \end{aligned} \tag{5.10}$$

This proves (5.5).

Having this lemma, we are ready for the

Proof of Proposition 5.3. Consider $\varphi \in \operatorname{Lip}(\overline{\Omega})$, given by $\varphi(x) = \operatorname{dist}(x, \partial\Omega)$, and set

$$\Omega_s = \{x \in \Omega : \varphi(x) \ge s\}. \tag{5.11}$$

For $\delta > 0$, set

$$\begin{aligned} \psi_\delta(x) = \quad &\delta, \quad &&\text{for } x \in \Omega_\delta, \\ &\varphi(x), &&\text{for } x \in \Omega \setminus \Omega_\delta. \end{aligned} \tag{5.12}$$

Thus $\nabla\psi_\delta$ is supported on $\mathcal{O}_\delta$ and $|\nabla\psi_\delta| = 1$ on $\mathcal{O}_\delta$. A version of the co-area formula (Theorem 5.4.4 of [Zie]) gives

$$\int\limits_{\mathcal{O}_\delta} |\nabla\psi_\delta|\, dx = \int_0^\delta \|\nabla\chi_{\Omega_s}\|_{\mathrm{TV}}\, ds, \tag{5.13}$$

where $\|\cdot\|_{\mathrm{TV}}$ is the total variation of a (vector-valued) measure. Now the left side of (5.13) equals $\operatorname{Vol}(\mathcal{O}_\delta)$, so (5.5) implies

$$\int_0^\delta \|\nabla\chi_{\Omega_s}\|_{\mathrm{TV}}\, ds \le C\delta. \tag{5.14}$$

Thus, for each $k \ge 1$, there exists $s \in (0, 1/k)$ such that $\|\nabla\chi_{\Omega_s}\|_{\mathrm{TV}} \le C$. This proves Proposition 5.3.

6. Domains bounded by simple, closed, rectifiable curves

Let $\Omega \subset \mathbb{C}$ be a bounded open set, and assume $\partial\Omega$ consists of a finite set of disjoint, rectifiable Jordan curves,

$$\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_m. \tag{6.1}$$

Say Γ_j is the image of

$$\gamma_j : [0, L_j] \longrightarrow \mathbb{C}, \tag{6.2}$$

with $\gamma_j(0) = \gamma_j(L_j)$. We parametrize γ_j by arc-length, so it is a Lipschitz map, and, for $f \in C(\Gamma_j)$,

$$\int_0^{L_j} f(\gamma_j(t))\, dt = \int_{\Gamma_j} f(z)\, d\lambda(z), \tag{6.3}$$

where

$$\lambda = \mathcal{H}^1 \lfloor \partial\Omega. \tag{6.4}$$

We have $\mathcal{H}^1(\partial\Omega) = \sum L_j < \infty$, and a fortiori $\mathcal{H}^1(\partial_*\Omega) < \infty$. Hence, as seen in §2, Ω is a finite-perimeter domain. Proposition 1.2 applies, and we have the following.

Proposition 6.1. *Take Ω as above, bounded by m rectifiable Jordan curves, Γ_j, and assume*

$$f \in C(\overline{\Omega}), \quad f \ \text{ holomorphic on } \Omega. \tag{6.5}$$

Then

$$\sum_{j=1}^{m} \int_{\Gamma_j} f(z)\, d\alpha(z) = 0, \tag{6.6}$$

where, as in (3.7),

$$\alpha = i\tilde{\nu}\, \sigma, \tag{6.7}$$

and σ is the positive measure on $\partial\Omega$ specified by (1.8)–(1.10).

To complement this result, we want to relate $\int_{\Gamma_j} f(z)\,d\alpha(z)$ to the integral

$$(6.8)\qquad \begin{aligned}\int\limits_{\gamma_j} f(z)\,dz &= \int_0^{L_j} f(\gamma_j(t))\gamma_j'(t)\,dt\\ &= \int\limits_{\Gamma_j} f(z)\tau(z)\,d\lambda(z),\end{aligned}$$

where

$$(6.9)\qquad \tau(\gamma_j(t)) = \gamma_j'(t)$$

defines a bounded, λ-measurable function on Γ_j. Meanwhile, by (3.4)–(3.7),

$$(6.10)\qquad \begin{aligned}\int\limits_{\Gamma_j} f(z)\,d\alpha(z) &= \int\limits_{\Gamma_j} f(z)i\tilde{\nu}(z)\,d\sigma(z)\\ &= \int\limits_{\Gamma_j} f(z)\tau(z)\,d\sigma(z),\end{aligned}$$

the last identity because, by (2.2),

$$(6.11)\qquad i\tilde{\nu}(z) = \tau(z), \quad \sigma\text{-a.e.}$$

We are hence motivated to record the following result.

Lemma 6.2. *For Ω, λ, and σ as described above,*

$$(6.12)\qquad \lambda = \sigma.$$

Equivalently (via (2.7)),

$$(6.13)\qquad \mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0.$$

Proof. This follows from Theorem I of [CR], whose part (ii) yields

$$(6.14)\qquad \sigma(\partial\Omega) = \sum_j L_j = \lambda(\partial\Omega),$$

which implies (6.13).

Now we can restate Proposition 6.1 as follows.

Corollary 6.3. *Take Ω as in Proposition 6.1 and assume f satisfies (6.5). Then*

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\, dz = 0. \tag{6.15}$$

For $m = 1$, this is Theorem 4.6.1 of [Si]. To illustrate the extra generality of Proposition 1.2, we mention that one can readily produce examples of bounded, open, connected sets $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ is connected and $\mathcal{H}^1(\partial\Omega) < \infty$, but $\mathbb{C} \setminus \overline{\Omega}$ has infinitely many connected components. Indeed, one can arrange that every neighborhood U of each point $p \in \partial\Omega$ contains infinitely many connected components of $\mathbb{C} \setminus \overline{\Omega}$. On the other hand, there is the following notable rectifiability theorem of [CR].

Theorem ([CR]). *Let $\Omega \subset \mathbb{C}$ be a bounded open set with finite perimeter. Assume*

$$\partial\Omega = \partial(\mathbb{C} \setminus \overline{\Omega}), \tag{6.16}$$

i.e., Ω is the interior of $\overline{\Omega}$, and

$$\mathbb{C} \setminus \overline{\Omega} \text{ has } m \text{ connected components}, m \in \mathbb{N}. \tag{6.17}$$

Write $\Omega = \cup_j \mathcal{O}_j$, as the disjoint union of its connected components. Then

$$\sigma(\partial\Omega) = \sum_j \sigma(\partial\mathcal{O}_j), \tag{6.18}$$

and each $\partial\mathcal{O}_j$ consists of $\le m$ rectifiable Jordan curves γ_{jk}. Furthermore, for each j,

$$\sigma(\partial\mathcal{O}_j) = \sum_k \mathcal{H}^1(\gamma_{jk}). \tag{6.19}$$

7. Further results

Here we describe some variants of Propositions 1.2–1.3, valid for a broader class of holomorphic functions on Ω, for certain classes of finite-perimeter domains Ω. Propositions 7.1–7.2 below are consequences of results of [HMT], [MMT], and [MMM], to which we refer for proofs.

Given a bounded, finite-perimeter domain $\Omega \subset \mathbb{C}$, and $p \in [1, \infty]$, we set

$$\mathfrak{H}^p(\Omega) = \{f \text{ holomorphic on } \Omega : \mathcal{N}f \in L^p(\partial\Omega, \sigma)\}, \tag{7.1}$$

where $\mathcal{N}f$ denotes the nontangential maximal function,

$$\mathcal{N}f(z) = \sup_{\zeta \in \Gamma_z} |f(\zeta)|, \tag{7.2}$$

and where, for $z \in \partial\Omega$, and some choice of $a \in (0, 1)$,

$$\Gamma_z = \{\zeta \in \Omega : \text{dist}(\zeta, \partial\Omega) \ge a|\zeta - z|\}. \tag{7.3}$$

The following result was established in [HMT] and pursued further in [MMT] and [MMT2].

Proposition 7.1. *Let $\Omega \subset \mathbb{C}$ be a bounded, finite-perimeter domain. Assume $\mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0$ and that $\partial\Omega$ is Ahlfors regular. Assume that*

$$f \in \mathfrak{H}^p(\Omega), \quad \text{for some } p > 1, \tag{7.4}$$

and that

$$f \quad \text{has a nontangential a.e. limit } f_b \in L^p(\partial\Omega, \sigma). \tag{7.5}$$

Then, with α denoting the complex measure on $\partial\Omega$ given by (3.4)–(3.7),

$$\int_{\partial\Omega} f_b(z)\, d\alpha(z) = 0, \tag{7.6}$$

and, for each $z_0 \in \Omega$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f_b(z)}{z - z_0}\, d\alpha(z). \tag{7.7}$$

Here, to say $\partial\Omega$ is Ahlfors regular is to say there exist $C_j \in (0, \infty)$ such that, for all $r \in (0, 1]$,

$$C_1 r \le \mathcal{H}^1(\partial\Omega \cap B_r(p)) \le C_2 r, \quad \forall\, p \in \partial\Omega. \tag{7.8}$$

Contrast this with the one-sided condition (5.2).

Proposition 7.1 was sharpened in [MMM] in several ways. For one, the endpoint case $p = 1$ was included. For another, the class of domains Ω was broadened. These results of [MMM] include the following.

Proposition 7.2. *Let $\Omega \subset \mathbb{C}$ be a bounded, finite-perimeter domain. Assume $\partial\Omega$ is lower Ahlfors regular (cf. (5.2)), that $\mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0$, and that σ is a doubling measure on $\partial\Omega$. Assume*

$$f \in \mathfrak{H}^1(\Omega), \tag{7.9}$$

and f satisfies (7.5), with $p = 1$. Then (7.6)–(7.7) hold.

The monograph [MMM] also has Fatou theorems, to the effect that, if $f \in \mathfrak{H}^p(\Omega)$, then (7.5) holds. Such results are shown to hold if $p > 1$ and Ω is a uniformly rectifiable domain. A stronger Fatou theorem, allowing $p = 1$, is demonstrated in [MMM] when Ω is an Ahlfors regular NTA domain.

There are other interesting variants. For example, if $\Omega \subset \mathbb{C}$ is smoothly bounded, and f is holomorphic on Ω, then

$$f \in L^2(\Omega) \Longrightarrow \operatorname{Tr} f \in H^{-1/2}(\partial\Omega), \tag{7.10}$$

and one has variants of Propositions 1.2–1.3. General results implying (7.10) can be found in [Se], using pseudodifferential operator calculus. An elementary argument, using Green's formula, is given in [T]. There are further results along these lines, making contact with the "bullet product" developed in [MMM], which we will discuss in another note.

A. Complements on finite-perimeter domains

Here we fix a mollifier, of the form

$$J_\varepsilon u = \varphi_\varepsilon * u, \quad \varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x),$$
(A.1)
$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \varphi \ge 0, \quad \int \varphi(x)\, dx = 1.$$

Given a bounded open set $\Omega \subset \mathbb{R}^n$, we set

(A.2) $$F_\varepsilon(x) = \varphi_\varepsilon * \chi_\Omega(x).$$

Our first goal here is to prove the following.

Proposition A.1. *Given $\Omega \subset \mathbb{R}^n$, open and bounded, the following are equivalent:*

(A.3) $$\Omega \text{ is a finite-perimeter domain and } \sigma(\partial\Omega) = A,$$

and

(A.4) $$\lim_{\varepsilon\to 0} \int |\nabla F_\varepsilon(x)|\, dx = A.$$

We include this result because (A.4) is given as the definition of a finite-perimeter domain in [CR], and we want to be sure this is equivalent to the definition given in §1.

To start the proof, note that F_ε, defined by (A.2), belongs to $C_0^\infty(\mathbb{R}^n)$, and

(A.5) $$\nabla F_\varepsilon = \varphi_\varepsilon * \mu,$$

with

(A.6) $$\mu = \nabla \chi_\Omega \in \mathcal{E}'(\mathbb{R}^n),$$

and

(A.7) $$\varphi_\varepsilon * \mu \longrightarrow \mu \ \text{ in } \ \mathcal{E}'(\mathbb{R}^n).$$

Now (A.4) implies $\varphi_\varepsilon * \mu$ is bounded in $L^1(\mathbb{R}^n)$. Hence there is a subsequence $\varepsilon_k \to 0$ such that

(A.8) $$\varphi_{\varepsilon_k} * \mu \longrightarrow \tilde{\mu}, \text{ weak}^* \text{ in } \mathcal{M}(K),$$

the space of finite Borel measures supported on a compact set $K \subset \mathbb{R}^n$, chosen large enough that $\operatorname{supp} F_\varepsilon \subset K$ for all $\varepsilon \in (0,1]$. Here $\tilde{\mu}$ is an $\mathbb{R}^n$-valued measure, supported on $\partial\Omega$. Furthermore, by Alaoglu's theorem, if (A.4) holds,

(A.9) $$\|\tilde{\mu}\|_{\text{TV}} \le A.$$

Comparison with (A.7) implies $\mu = \tilde{\mu}$, i.e., μ is an $\mathbb{R}^n$-valued measure, so (1.9) applies, and

(A.10) $$\|\mu\|_{\text{TV}} = \sigma(\partial\Omega) = \widetilde{A} \le A.$$

Thus we are in the setting of (A.3) (but with $\widetilde{A}$ in place of A). Hence, to prove Proposition A.1, it remains only to show that (A.3) $\Rightarrow$ (A.4).

In fact, we prove the following more precise result.

Proposition A.2. *Assume Ω is a bounded, open, finite-perimeter domain in $\mathbb{R}^n$, with $\nabla\chi_\Omega = \mu = -\nu\,\sigma$, as in (1.8)–(1.9). Take F_ε as in (A.2). Then*

$$|\nabla F_\varepsilon| \longrightarrow \sigma, \quad \text{weak}^* \text{ in } \mathcal{M}(K), \tag{A.11}$$

the space of finite Borel measures on K (the compact neighborhood of $\overline{\Omega}$ described above).

Proof. Take $\delta > 0$. We apply Lusin's theorem to $\nu \in L^1(\partial\Omega, \sigma)$ in (1.9) to see that there is a compact set $K_1 \subset \partial\Omega$ such that

$$\sigma(\partial\Omega \setminus K_1) < \delta \ \text{ and } \ \nu\big|_{K_1} \text{ is continuous.} \tag{A.12}$$

Then there exists

$$\psi \in C(K) \ \text{ such that } \ \psi = \nu \ \text{ on } \ K_1, \quad |\psi| \le 1, \tag{A.13}$$

and we can write

$$-\mu = \nu\sigma_0 + \psi\sigma_1, \quad \sigma_1 = \sigma\lfloor K_1, \quad \sigma_0 = \sigma\lfloor(\partial\Omega \setminus K_1). \tag{A.14}$$

Then

$$\begin{aligned} -\nabla F_\varepsilon = -J_\varepsilon\mu &= J_\varepsilon(\nu\sigma_0) + J_\varepsilon(\psi\sigma_1) \\ &= J_\varepsilon(\nu\sigma_0) + [J_\varepsilon, \psi]\sigma_1 + \psi J_\varepsilon\sigma_1 \\ &= g_\varepsilon^0 + g_\varepsilon^1 + \psi J_\varepsilon \sigma_1. \end{aligned} \tag{A.15}$$

We have

$$\|g_\varepsilon^0\|_{L^1} \le \|\sigma_0\|_{\mathrm{TV}} < \delta. \tag{A.16}$$

As for $g_\varepsilon^1 = [J_\varepsilon, \psi]\sigma_1$, we have the following.

Lemma A.3. *Given $\sigma_1 \in \mathcal{M}(\partial\Omega)$, $\psi \in C(K)$, the commutator $[J_\varepsilon, \psi]$ satisfies*

$$\|[J_\varepsilon, \psi]\sigma_1\|_{L^1} \longrightarrow 0 \quad \text{as} \quad \varepsilon \searrow 0. \tag{A.17}$$

Proof. If $\psi \in C_0^\infty(\mathbb{R}^n)$, $[J_\varepsilon, \psi]$ is bounded in $OPS^{-1}_{1,0}(\mathbb{R}^n)$, so $\{[J_\varepsilon, \psi]\sigma_1 : \varepsilon \in (0,1]\}$ is bounded in the Sobolev space $H^{s,p}(K)$ for some $p > 1$, $s > 0$, hence relatively compact in $L^1(K)$. Clearly $[J_\varepsilon, \psi]\sigma_1 \to 0$ in $\mathcal{E}'(\mathbb{R}^n)$, and this leads to (A.17), for $\psi \in C_0^\infty(\mathbb{R}^n)$. The result for all $\psi \in C(K)$ follows readily, via a limiting argument.

Returning to the proof of Proposition A.2, we have from (A.15)–(A.17) that

$$\limsup_{\varepsilon \to 0} \| -\nabla F_\varepsilon - \psi J_\varepsilon \sigma_1\|_{L^1} \le \delta, \tag{A.18}$$

hence

$$\limsup_{\varepsilon\to 0} \| \, |\nabla F_\varepsilon| - |\psi| J_\varepsilon \sigma_1 \|_{L^1} \le \delta. \tag{A.19}$$

Now the L^1-bounds on $|\nabla F_\varepsilon|$ imply that for each sequence $\varepsilon \to 0$, there is a subsequence $\varepsilon_k \to 0$ and a positive measure α, supported by $\partial\Omega$, such that

$$|\nabla F_{\varepsilon_k}| \longrightarrow \alpha, \quad \text{weak}^* \text{ in } \mathcal{M}(K). \tag{A.20}$$

Meanwhile,

$$|\psi| J_\varepsilon \sigma_1 \longrightarrow |\psi|\sigma_1, \quad \text{weak}^*, \tag{A.21}$$

and $|\psi|\sigma_1 = \sigma_1$. In conjunction with (A.19), and via Alaoglu's theorem, we get

$$\|\alpha - \sigma_1\|_{\mathrm{TV}} \le \delta, \tag{A.22}$$

hence

$$\|\alpha - \sigma\|_{\mathrm{TV}} \le 2\delta. \tag{A.23}$$

Since the left side of (A.23) is independent of δ, this implies $\alpha = \sigma$, and we have the desired conclusion (A.11).

B. A removable singularity problem

Motivated by Proposition 4.4, we address the following basic case of a removable singularity problem. Given $K \subset \mathbb{C}$ compact, we ask when we have the implication

$$f \in C(\mathbb{C}),\ f \text{ holomorphic on } \mathbb{C} \setminus K \Longrightarrow f \text{ holomorphic on } \mathbb{C}. \tag{B.1}$$

This is to some degree close to the question of when we have the implication

$$f \in L^\infty(\mathbb{C}),\ f \text{ holomorphic on } \mathbb{C} \setminus K \Longrightarrow f \text{ holomorphic on } \mathbb{C}, \tag{B.2}$$

which has been much studied. The fact that, when $K = [0, 1]$ is a slit, (B.1) holds but (B.2) fails illustrates the subtlety of the questions. Note that (B.1) fails if and only if there is a nonzero distribution

$$\omega \in \mathcal{E}'(\mathbb{C}), \quad \operatorname{supp} \omega \subset K, \tag{B.3}$$

such that

$$\frac{1}{z} * \omega \in C(\mathbb{C}). \tag{B.4}$$

Similarly, (B.2) fails if and only if there exists ω satisfying (B.3), with (B.4) replaced by $(1/z) * \omega \in L^\infty(\mathbb{C})$. Note that if K has positive 2D Lebesgue measure,

$$\omega = \chi_K \Longrightarrow \frac{1}{z} * \omega \in C^r(\mathbb{C}), \quad \forall\, r < 1, \tag{B.5}$$

so (B.1) fails for such K. On the other hand, if we apply Proposition 4.4 with $\mathcal{O}$ taken to be an open disk containing K, we obtain the following.

Proposition B.1. *Let $K \subset \mathbb{C}$ be a compact set with 2D Lebesgue measure 0. Assume there exist open sets $U_k \supset K$ and $C < \infty$ such that*

$$\mathcal{L}^2(U_k) \to 0, \quad \mathcal{H}^1(\partial U_k) \le C. \tag{B.6}$$

Then the implication (B.1) holds.

Corollary B.2. *Let $K \subset \mathbb{C}$ be a compact set with finite 1-dimensional Hausdorff measure,*

$$\mathcal{H}^1(K) < \infty. \tag{B.7}$$

Then the implication (B.1) holds.

Proof. We recall how 1-dimensional Hausdorff measure is constructed. Given $\delta > 0$, $S \subset \mathbb{C}$, we set

$$h^*_{1,\delta}(S) = \inf\Bigl\{\sum_{j\ge 1}(\operatorname{diam} B_j) : S \subset \bigcup_{j\ge 1} B_j, \operatorname{diam} B_j \le \delta\Bigr\}, \tag{B.8}$$

which is a monotone function of δ, and then set

$$h^*_1(S) = \lim_{\delta\to 0} h^*_{1,\delta}(S), \tag{B.9}$$

which is a metric outer measure. Every compact $K \subset \mathbb{C}$ is h^*_1-measurable, and we set

$$\mathcal{H}^1(K) = h^*_1(K). \tag{B.10}$$

One can make a construction parallel to (B.8)–(B.10), in which the countable cover of S in (B.8) is required to consist of *open disks* of diameter $\le \delta$. We denote the resulting measure by $\mathcal{H}^1_D$, which is clearly $\ge \mathcal{H}^1$. Now a set of diameter d_j might not be contained in a disk of diameter d_j, but it is certainly contained in a (closed) disk of radius d_j (i.e., diameter $2d_j$), so we have

$$\mathcal{H}^1(K) \le \mathcal{H}^1_D(K) \le 2\mathcal{H}^1(K). \tag{B.11}$$

If K is compact, any cover of K by open disks has a finite subcover. Hence, for $\delta = 2^{-k}$, we can take a finite cover of K by open disks $D_{k,1}, \dots, D_{k,N(k)}$, such that $\operatorname{diam} D_{k,\ell} \le \delta$ and

$$\sum_\ell (\operatorname{diam} D_{k,\ell}) \le \mathcal{H}^1_D(K) + 2^{-k}. \tag{B.12}$$

If we set

$$U_k = \bigcup_\ell D_{k,\ell} \supset K, \tag{B.13}$$

then

$$\sigma(\partial U_k) \le \pi \sum_\ell (\operatorname{diam} D_{k,\ell}), \tag{B.14}$$

and

$$\begin{aligned} \mathcal{L}^2(U_k) &\le \frac{\pi}{4}\sum_\ell (\operatorname{diam} D_{k,\ell})^2 \\ &\le \frac{\pi\delta}{4}\sum_\ell (\operatorname{diam} D_{k,\ell}), \end{aligned} \tag{B.15}$$

so (B.6) holds, and hence Proposition B.1 applies.

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