

# Introduction to Analysis in Several Variables (Advanced Calculus)

MICHAEL TAYLOR

This text was produced for the second part of a two-part sequence on advanced calculus, whose aim is to provide a firm logical foundation for analysis, for students who have had 3 semesters of calculus and a course in linear algebra. The first part treats analysis in one variable, and the text *Introduction to Analysis in One Variable* was written to cover that material. The text at hand treats analysis in several variables. These two texts can be used as companions, but they are written so that they can be used independently, if desired.

An introductory chapter treats background for multivariable calculus. This includes sections on one-variable calculus, on  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , on vector spaces and linear transformations, and on determinants. The text proceeds to material on analysis in several variables, from differential calculus to the  $n$ -dimensional integral, to calculus on  $n$ -dimensional surfaces in  $\mathbb{R}^k$ , and generalizations known as manifolds.

Topics covered include systems of differential equations, and their relation to vector fields, differential forms, as a coordinate-independent way to set up integrals, with numerous applications to such topics as holomorphic and harmonic functions, and to topological implications, involving “degree theory.” We also study the geometry of surfaces, from geodesics to curvature, relating the latter to degree theory via Gauss-Bonnet theorems. A chapter on Fourier analysis treats both  $n$ -dimensional Fourier series, the Fourier transform on  $\mathbb{R}^n$ , and the theory of spherical harmonics on  $n$ -dimensional spheres, as well as Fourier analysis on matrix groups. A final topic in the appendix previews an extension of degree theory, known as de Rham theory. Such topics spotlight the unity of the various analytical and geometric aspects of Advanced Calculus.

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### Chapter 1. Background

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2. Euclidean spaces
3. Vector spaces and linear transformations
4. Determinants

### Chapter 2. Multivariable differential calculus

1. The derivative
2. Inverse function and implicit function theorem

### 3. Systems of differential equations and vector fields

#### Chapter 3. Multivariable integral calculus and calculus on surfaces

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2. Surfaces and surface integrals
3. Partitions of unity
4. Sard's theorem
5. Morse functions
6. The tangent space to a manifold

#### Chapter 4. Differential forms and the Gauss-Green-Stokes formula

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3. The general Stokes formula
4. The classical Gauss, Green, and Stokes formulas
5. Differential forms and the change of variable formula

#### Chapter 5. Applications of the Gauss-Green-Stokes formula

1. Holomorphic functions and harmonic functions
2. Differential forms, homotopy, and the Lie derivative
3. Differential forms and degree theory

#### Chapter 6. Differential geometry of surfaces

1. Geometry of surfaces I: geodesics
2. Geometry of surfaces II: curvature
3. Geometry of surfaces III: the Gauss-Bonnet theorem
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#### Chapter 7. Fourier analysis

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6. Isoperimetric inequality

#### Appendix A. Complementary material

1. Metric spaces, convergence, and compactness
2. Inner product spaces
3. Eigenvalues and eigenvectors
4. Complements on power series
5. The Weierstrass theorem and the Stone-Weierstrass theorem

6. Further results on harmonic functions
7. Beyond degree theory – introduction to de Rham theory