

Introduction to Complex Analysis

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This text is designed for a first course in complex analysis, for beginning graduate students, or well prepared undergraduates, whose background includes multivariable calculus, linear algebra, and advanced calculus. In this course the student will learn that all the basic functions that arise in calculus, first derived as functions of a real variable, such as powers and fractional powers, exponentials and logs, trigonometric functions and their inverses, and also many new functions that the student will meet, are naturally defined for complex arguments. Furthermore, this expanded setting reveals a much richer understanding of these functions.

The central objects in complex analysis are functions that are complex-differentiable (i.e., holomorphic). One early goal is to establish an equivalence between being holomorphic and having a convergent power series expansion. Half of this is done in Chapter 1, the other half in Chapter 2, after the introduction of the Cauchy integral theorem as a major theoretical tool. Other applications of this essential tool occupy much of the rest of the text.

We emphasize the connection of complex analysis to other facets of analysis and geometry. As some distinguishing features of the treatment here, we mention the following.

- (1) An emphasis on Fourier analysis, both as an application of basic results and as a tool of more general applicability in analysis.
- (2) Use of geometrical techniques, clarifying the study of holomorphic functions as conformal maps, and extending the usual study to more general surfaces.
- (3) Connections with differential equations. Techniques of complex analysis are brought to bear, from the classical case of Bessel equations to more general situations. Connections are made to how some of these important differential equations arise from a study of wave equations and diffusion equations on various domains in higher dimensional Euclidean space.

Contents

Chapter 1. Basic calculus in the complex domain

1. Complex numbers, power series, and exponentials
2. Holomorphic functions, derivatives, and path integrals
3. Holomorphic functions defined by power series
4. Exponential and trigonometric functions: Euler's formula
5. Square roots, logs, and other inverse functions
6. π is irrational

Chapter 2. Going deeper – the Cauchy integral theorem and consequences

1. The Cauchy integral theorem and the Cauchy integral formula
2. The maximum principle, Liouville's theorem, and the fundamental theorem of algebra
3. Harmonic functions on planar domains
4. Morera's theorem, the Schwarz reflection principle, and Goursat's theorem
5. Infinite products
6. Uniqueness and analytic continuation
7. Singularities
8. Laurent series
9. Green's theorem
10. The fundamental theorem of algebra (elementary proof)
11. Absolutely convergent series

Chapter 3. Fourier analysis and complex function theory

1. Fourier series and the Poisson integral
2. Fourier transforms
3. Laplace transforms and Mellin transforms
4. Inner product spaces
5. The matrix exponential
5. The Weierstrass and Runge approximation theorems

Chapter 4. Residue calculus, the argument principle, and two very special functions

1. Residue calculus
2. The argument principle
3. The Gamma function
4. The Riemann zeta function and the prime number theorem
5. Euler's constant
6. Hadamard's factorization theorem

Chapter 5. Conformal maps and geometrical aspects of complex function theory

1. Conformal maps
2. Normal families
3. The Riemann sphere and other Riemann surfaces
4. The Riemann mapping theorem
5. Boundary behavior of conformal maps
6. Covering maps
7. The disk covers the twice-punctured plane
8. Montel's theorem
9. Picard's theorem
10. Harmonic functions II
11. Surfaces and metric tensors
12. Poincaré metrics

13. Groups

Chapter 6. Elliptic functions and elliptic integrals

1. Periodic and doubly periodic functions
2. The Weierstrass \wp -function in elliptic function theory
3. Theta functions and elliptic functions
4. Elliptic integrals
5. The Riemann surface of the square root of a cubic
6. Rapid evaluation of the Weierstrass \wp -function

Chapter 7. Complex analysis and differential equations

1. Bessel functions
2. Differential equations on a complex domain
3. Holomorphic families of differential equations
4. From wave equations to Bessel and Legendre equations

Appendix A. Complementary material

1. Metric spaces, convergence, and compactness
2. Derivatives and diffeomorphisms
3. The Laplace asymptotic method and Stirling's formula
4. The Stieltjes integral
5. Abelian theorems and Tauberian theorems
6. Cubics, quartics, and quintics