Karamata’s Tauberian Theorem

MICHAEL TAYLOR

1. Basics

Let \( \mu \) be a positive Borel measure on \([0, \infty)\). Assume \( e^{-s \lambda} \in L^1(\mathbb{R}^+, \mu) \) for each \( s > 0 \), and assume

\[
\int_0^\infty e^{-s \lambda} \, d\mu(\lambda) \sim A \varphi(s), \quad \text{as } s \searrow 0,
\]

where \( \varphi(s) \nearrow +\infty \) as \( s \searrow 0 \), and we say

\[
\Phi(s) \sim A \varphi(s) \iff \Phi(s) = A \varphi(s) + o(\varphi(s)), \quad \text{as } s \searrow 0.
\]

Regarding \( \varphi(s) \), we will assume that

\[
\varphi(s) = L \psi(s) = \int_0^\infty e^{-s \lambda} \psi(\lambda) \, d\lambda, \quad \psi > 0.
\]

The classical examples are

\[
\psi(\lambda) = \lambda^{\alpha - 1}, \quad \varphi(s) = \Gamma(\alpha) s^{-\alpha}, \quad \alpha > 0.
\]

Slightly more exotic examples are

\[
\psi(\lambda) = \lambda^{\alpha - 1} \log \lambda, \quad \varphi(s) = (\Gamma'(\alpha) - \Gamma(\alpha) \log s) s^{-\alpha},
\]

again for \( \alpha > 0 \). Actually, in this case \( \psi(\lambda) < 0 \) for \( \lambda \in (0, 1) \), so we would want to cut this off, obtaining

\[
\psi(\lambda) = \lambda^{\alpha - 1} (\log \lambda)_+, \quad \varphi(s) = \Gamma(\alpha) \left( \log \frac{1}{s} \right) s^{-\alpha} + O(s^{-\alpha}),
\]

as \( s \searrow 0 \). See §2 for another example, in which \( \varphi(s) \sim \log 1/s \).

Our goal is to establish (under some natural hypotheses on \( \varphi \) and \( \psi \)) that

\[
\mu([0, R]) = A \int_0^R \psi(\lambda) \, d\lambda + o(\varphi(R^{-1})), \quad \text{as } R \nearrow +\infty.
\]

In case \( \varphi \) and \( \psi \) are given by (1.4), this yields the implication

\[
\int_0^\infty e^{-s \lambda} \, d\mu(\lambda) \sim A s^{-\alpha}, \quad \text{as } s \searrow 0 \implies \mu([0, R]) \sim \frac{A}{\Gamma(\alpha + 1)} R^\alpha, \quad \text{as } R \nearrow +\infty,
\]
which is the most basic version of the Karamata Tauberian theorem.

We tackle the problem of establishing (1.7) in stages, examining when we can show that

\[ \int_0^\infty f(s\lambda) \, d\mu(\lambda) = A \int_0^\infty f(s\lambda) \psi(\lambda) \, d\lambda + o(\varphi(s)), \]

for various classes of functions \( f(\lambda) \), ultimately including

\[ \chi_I(\lambda) = \begin{cases} 1 & \text{for } 0 \leq \lambda \leq 1, \\ 0 & \text{for } \lambda > 1. \end{cases} \]

We start with the function space

\[ \mathcal{E} = \left\{ \sum_{k=1}^M \gamma_k e^{-k\lambda} : \gamma_k \in \mathbb{R}, M \in \mathbb{N} \right\}. \]

Note that this is an algebra of functions that separates the points of \([0, \infty)\), hence, by the Stone-Weierstrass theorem, it is dense in

\[ C_0([0, \infty)) = \{ f \in C([0, \infty)) : f(\infty) = 0 \}. \]

Now, if \( f \in \mathcal{E} \), say

\[ f(\lambda) = \sum_{k=1}^M \gamma_k e^{-k\lambda}, \]

then the hypothesis (1.1) implies

\[ \int_0^\infty f(s\lambda) \, d\mu(\lambda) = \sum_{k=1}^M \gamma_k \int_0^\infty e^{-sk\lambda} \, d\mu(\lambda) \]

\[ = A \sum_{k=1}^M \gamma_k \varphi(ks) + o\left( \sum_{k=1}^M \varphi(ks) \right) \]

\[ = A \int_0^\infty f(s\lambda) \psi(\lambda) \, d\lambda + o(\varphi(s)), \]

since

\[ \varphi(ks) \leq \varphi(s), \quad \text{for } k \geq 1. \]

Hence (1.9) holds for all \( f \in \mathcal{E} \). The following is the next key result.
**Lemma 1.1.** Given (1.1), the result (1.9) holds for all

\[ f \in C_0([0, \infty)) \] such that \( e^\lambda f \in C_0([0, \infty)) \).

**Proof.** Given such \( f \), and given \( \varepsilon > 0 \), take \( h \in E \) such that \( \sup |h(\lambda) - e^\lambda f(\lambda)| \leq \varepsilon \), and set \( g = e^{-\lambda} h \), so

\[ g \in E, \quad |f(\lambda) - g(\lambda)| \leq \varepsilon e^{-\lambda}. \]

This implies

\[ \int_0^\infty |f(s\lambda) - g(s\lambda)| d\mu(\lambda) \leq \varepsilon \int_0^\infty e^{-s\lambda} d\mu(\lambda) \]

and

\[ \int_0^\infty |f(s\lambda) - g(s\lambda)| \psi(\lambda) d\lambda \leq \varepsilon \int_0^\infty e^{-s\lambda} \psi(\lambda) d\lambda. \]

The fact that the right sides of (1.18) and (1.19) are both \( \leq C \varepsilon \varphi(s) \), for \( s \in (0, 1] \), follows from (1.1) and (1.3), respectively. But we know that (1.9) holds with \( g \) in place of \( f \). Hence

\[ \left| \int_0^\infty f(s\lambda) d\mu(\lambda) - A \int_0^\infty f(s\lambda) \psi(\lambda) d\lambda \right| \leq 2C \varepsilon \varphi(s) + o(\varphi(s)), \]

for each \( \varepsilon > 0 \). Taking \( \varepsilon \searrow 0 \) yields the lemma.

We now tackle (1.9) for \( f = \chi_I \), given by (1.10). For each \( \delta \in (0, 1/2] \), take \( f_\delta, g_\delta \in C_0([0, \infty)) \) such that

\[ 0 \leq f_\delta \leq \chi_I \leq g_\delta \leq 1, \]

with

\[ f_\delta(\lambda) = 1 \quad \text{for} \quad 0 \leq \lambda \leq 1 - \delta, \]

\[ 0 \quad \text{for} \quad \lambda \geq 1, \]

and

\[ g_\delta(\lambda) = 1 \quad \text{for} \quad 0 \leq \lambda \leq 1, \]

\[ 0 \quad \text{for} \quad \lambda \geq 1 + \delta. \]

Note that Lemma 1.1 is applicable to each \( f_\delta \) and \( g_\delta \). Hence

\[ \int_0^\infty \chi_I(s\lambda) d\mu(\lambda) \leq \int_0^\infty g_\delta(s\lambda) d\mu(\lambda) \]

\[ = A \int_0^\infty g_\delta(s\lambda) \psi(\lambda) d\lambda + o(\varphi(s)), \]
and

\[
\int_0^\infty \chi_I(s\lambda) \, d\mu(\lambda) \geq \int_0^\infty f_\delta(s\lambda) \, d\mu(\lambda) = A \int_0^\infty f_\delta(s\lambda)\psi(\lambda) \, d\lambda + o(\varphi(s)).
\]

Next,

\[
\int_0^\infty [g_\delta(s\lambda) - f_\delta(s\lambda)]\psi(\lambda) \, d\lambda \\
\leq \int_{(1-\delta)/s}^{(1+\delta)/s} \psi(\lambda) \, d\lambda \\
\leq \frac{2\delta}{s} \max\{|\psi(\lambda)| : |\lambda - \frac{1}{s}| \leq \frac{\delta}{s}\}.
\]

We now make the hypothesis that, for some \(\varepsilon > 0, b > 0, B < \infty\),

\[
\max\{|\psi(\lambda)| : |\lambda - \frac{1}{s}| \leq \frac{\varepsilon}{s}\} \leq Bs\varphi(s), \quad \text{for } 0 < s \leq b.
\]

Note that such a condition holds in cases (1.4) and (1.6). When such an estimate holds, (1.26) yields

\[
\int_0^\infty [g_\delta(s\lambda) - f_\delta(s\lambda)]\psi(\lambda) \, d\lambda \leq 2B\delta \varphi(s), \quad \text{for } \delta \leq \varepsilon, s \leq b.
\]

It then follows from (1.24)–(1.26) that

\[
\lim_{s \to 0}^\infty \varphi(s)^{-1}\left|\int_0^\infty \chi_I(s\lambda) \, d\mu(\lambda) - A \int_0^\infty \chi_I(s\lambda)\psi(\lambda) \, d\lambda\right| \\
\leq \inf_{\delta \leq \varepsilon} 2B\delta = 0.
\]

We have the following conclusion.

**Proposition 1.2.** Let \(\mu\) be a positive measure on \([0, \infty)\), and assume (1.1)–(1.3) hold, with \(\psi \notin L^1(\mathbb{R}^+)\), and that (1.27) holds. Then \(\mu\) satisfies (1.7).

The special case (1.8) has already been mentioned. We turn to the case (1.6), for which (1.1) leads to (1.7) with

\[
\int_0^R \psi(\lambda) \, d\lambda = \int_1^R \lambda^{\alpha-1}(\log \lambda) \, d\lambda = \frac{1}{\alpha} \int_1^R \left(\frac{d}{d\lambda} \lambda^\alpha\right)(\log \lambda) \, d\lambda \\
= \frac{1}{\alpha} R^\alpha(\log R) + O(R^\alpha).
\]

This leads to the following.
Corollary 1.3. Let μ be a positive measure on [0, ∞). Assume
\[ (1.31) \quad \int_0^\infty e^{-s\lambda} \, d\mu(\lambda) \sim A \left( \log \frac{1}{s} \right) s^{-\alpha}, \quad s \searrow 0, \]
with \( \alpha > 0 \). then
\[ (1.32) \quad \mu([0, R]) \sim \frac{A}{\Gamma(\alpha + 1)} R^\alpha (\log R), \quad R \nearrow +\infty. \]

We mention some other results of Karamata. To state them, let us set
\[ (1.33) \quad \Psi(R) = \int_0^R \psi(\lambda) \, d\lambda. \]

Here is Karamata’s Abelian theorem.

Proposition 1.4. Let \( \psi > 0 \). Assume that \( \Psi \), given by (1.33), has the form
\[ (1.34) \quad \Psi(R) = R^\alpha F(R), \quad \text{with} \quad \alpha > 0, \]
where \( F \) is slowly varying at \( \infty \), in the sense that
\[ (1.35) \quad \lim_{R \to \infty} \frac{F(tR)}{F(R)} = 1, \]
uniformly in \( t \) in compact subsets of \((0, \infty)\). Then
\[ (1.36) \quad \mathcal{L}\psi(s) \sim \Gamma(\alpha + 1) s^{-\alpha} F(s^{-1}) = \Gamma(\alpha + 1) \Psi(s^{-1}), \]
as \( s \searrow 0 \).

Note that (1.4) and (1.6) provide special cases of the functions \( \psi \), considered here, with \( F(R) = 1 \) and \( F(R) = (\log R)_+ \), respectively.

The following result is Karamata’s Tauberian theorem.

Proposition 1.5. Take \( \psi(\lambda) \) and \( \Psi(R) \) as in Proposition 1.4. In particular, assume that (1.34)–(1.35) hold. Let \( \mu \) be a positive measure on \([0, \infty)\), and assume
\[ (1.37) \quad \int_0^\infty e^{-s\lambda} \, d\mu(\lambda) \sim A\varphi(s), \]
as \( s \searrow 0 \), with
\[ (1.38) \quad \varphi(s) = \Gamma(\alpha + 1) \Psi(s^{-1}) = \Gamma(\alpha + 1) s^{-\alpha} F(s^{-1}). \]

Then
\[ (1.39) \quad \mu([0, R]) \sim A\Psi(R), \quad \text{as} \quad R \nearrow +\infty. \]
2. Another special case

We want to extend the treatment of (1.4) to $\alpha = 0$. Since $\psi(\lambda) = \lambda^{-1}$ is not integrable on $(0, 1]$, we instead take

\begin{equation}
\psi(\lambda) = \lambda^{-1}, \quad \text{for } \lambda \geq 1, \\
0, \quad \text{for } 0 < \lambda < 1.
\end{equation}

Then

\begin{equation}
\varphi(s) = \int_0^\infty e^{-s\lambda} \psi(\lambda) \, d\lambda = \int_s^\infty e^{-y} \frac{dy}{y} \\
= \int_s^1 e^{-y} \frac{dy}{y} + \int_1^\infty e^{-y} \frac{dy}{y} \\
= \log \frac{1}{s} - \int_s^1 (1 - e^{-y}) \frac{dy}{y} + c \\
\sim \log \frac{1}{s},
\end{equation}

as $s \searrow 0$.

Let $\mu$ be a positive measure on $[0, \infty)$, and assume (1.1) holds, with $\varphi(s)$ given by (2.1)–(2.2). We want to check that Proposition 1.2 holds. Certainly $\psi \notin L^1(\mathbb{R})$.

It remains to check (1.27). We look at

\begin{equation}
\max \left\{ \frac{1}{\lambda} : \left| \lambda - \frac{1}{s} \right| \leq \frac{\varepsilon}{s} \right\} = \max \left\{ \frac{1}{\lambda} : \lambda \geq \frac{1 - \varepsilon}{s} \right\} \\
= \frac{s}{1 - \varepsilon},
\end{equation}

while, for some $b > 0$,

\begin{equation}
s\varphi(s) \geq \frac{s}{2} \log \frac{1}{s}, \quad \text{for } 0 < s \leq b.
\end{equation}

Hence (1.27) holds, and we deduce from Proposition 1.2 the following.

**Corollary 2.1.** Let $\mu$ be a positive measure on $[0, \infty)$. Assume

\begin{equation}
\int_0^\infty e^{-s\lambda} \, d\mu(\lambda) \sim A \log \frac{1}{s}, \quad s \searrow 0.
\end{equation}

Then

\begin{equation}
\mu([0, R]) \sim A \int_1^R \frac{1}{\lambda} \, d\lambda = A \log R, \quad R \nearrow +\infty.
\end{equation}