

Observations on the Geodesic Flow And its Numerical Approximation

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Let $M \subset \mathbb{R}^k$ be a smooth n -dimensional surface. Given $x \in M$, $v \in T_x M$ (i.e., $(x, v) \in TM$) we define the geodesic flow $\mathcal{G}^t(x, v)$ to be $(\gamma(t), \gamma'(t))$, where $\gamma(0) = x$, $\gamma'(0) = v$, and γ is a constant-speed geodesic on M . We have $\mathcal{G}^t : TM \rightarrow TM$. The exponential map arises as

$$(1) \quad \text{Exp}_x : T_x M \rightarrow M, \quad \text{Exp}_x(v) = \Pi_M \mathcal{G}^1(x, v),$$

where $\Pi_M : TM \rightarrow M$ is the standard projection. We desire to numerically approximate $\mathcal{G}^t(x, v)$.

One can use geodesic equations in local coordinates on M , but this is typically not convenient. An alternative can be described as follows. For simplicity, we take $k = n + 1$, $u : \mathcal{O} \rightarrow \mathbb{R}$ smooth, \mathcal{O} a neighborhood of M in \mathbb{R}^k , and

$$(2) \quad M = \{x \in \mathbb{R}^k : u(x) = c\}, \quad \nabla u(x) \neq 0 \text{ on } M.$$

Then the geodesic equation can be written as

$$(3) \quad x''(t) = - \frac{\langle x'(t), D^2 u(x(t)) x'(t) \rangle}{\|\nabla u(x(t))\|^2} \nabla u(x(t)),$$

cf. [T], (6.1.37). We convert this to a first-order $2k \times 2k$ system, whose solution operator defines a flow

$$(4) \quad \mathcal{F}^t : T\mathcal{O} \longrightarrow T\mathcal{O}, \quad T\mathcal{O} = \mathcal{O} \times \mathbb{R}^k.$$

We have

$$(5) \quad \mathcal{F}^t : TM \rightarrow TM, \quad \mathcal{F}^t|_{TM} = \mathcal{G}^t.$$

But (and it's a big but), typically,

$$(6) \quad TM \text{ is an unstable invariant manifold for the flow } \mathcal{F}^t.$$

Thus a principal task is to stabilize a numerical approximation to the flow \mathcal{F}^t , so it reliably acts on TM .

To be specific, suppose we apply a 4th order accurate Runge-Kutta scheme to the first-order system arising from (3), with step size h . We start at a point $(x, v) \in TM$. At time h we obtain an approximation

$$(7) \quad (\tilde{x}(h), \tilde{v}(h)) \text{ to } \mathcal{F}^h(x, v).$$

In fact,

$$(8) \quad (\tilde{x}(h), \tilde{y}(h)) = \mathcal{F}^h(x, v) + O(h^5).$$

In particular,

$$(9) \quad \text{dist}((\tilde{x}(h), \tilde{v}(h)), TM) = O(h^5).$$

We can pick a “retraction” of a neighborhood of TM in $T\mathbb{R}^k$ onto TM , a “naive” one will do, and apply this to get an approximation

$$(10) \quad \begin{aligned} (\hat{x}(h), \hat{v}(h)) &= \mathcal{G}^t(x, v) + O(h^5), \\ (\hat{x}(h), \hat{v}(h)) &\in TM. \end{aligned}$$

This provides a useful modified Runge-Kutta approximation to the geodesic flow \mathcal{G}^t on TM . It remains 4th order accurate.

REMARK. We have been explicit about the case $k = n + 1$, but extensions to other codimensions work the same way.

Reference

[T] M. Taylor, Introduction to Analysis in Several Variables, Undergraduate Texts #46, AMS, Providence, RI, to appear, 2020.