WORKSHEETS for MATH 233H
Multivariable Calculus

Instructor: Michael Taylor

Contents

Introduction

These worksheets serve to guide the student through the text for Math 233H, *Multivariable Calculus*, by M. Taylor. They are designed so that each worksheet covers the material of one lecture. Each worksheet deals with material in a designated section of the text, and the idea is that a student can do the exercises in a worksheet in consultation with the text, and in that manner master the material in the text. There are also a handful of supplementary worksheets, to compensate for time lost due to the transition from in class to remote instruction.

These worksheets have been produced in response to the health crisis of 2020. They are dated to correspond to a class meeting twice a week.

Worksheet 1, Tuesday, 08/11

§§1.1–1.3, Basic one variable calculus (review)

1. Take $f : [a, b] \to \mathbb{R}$. Define what it means for f to be differentiable at $x \in (a, b)$, with derivative $f'(x)$.

2. State the product rule, and use it to show that

$$\frac{d}{dx} x^n = n x^{n-1}, \quad n \in \mathbb{N},$$

and also for $n \in \mathbb{Z}$, provided $x \neq 0$.

3. State the chain rule, and use it to show that

$$\frac{d}{dx} f(x)^n = n f(x)^{n-1} f'(x), \quad n \in \mathbb{N}.$$

4. State the Mean Value Theorem, and use it to show that, for $f : [a, b] \to \mathbb{R}$,

$$f'(x) \equiv 0 \Longrightarrow f \ \text{ constant}.$$

5. State the Inverse Function Theorem, and use it to show that

$$\frac{d}{dx} x^{1/n} = \frac{1}{n} x^{1/n-1}, \quad n \in \mathbb{N},\ x > 0.$$

6. Define the Riemann integral.

7. State the Fundamental Theorem of Calculus.

8. Show that

$$\int_0^x t^n \, dt = \frac{x^{n+1}}{n+1}, \quad n \in \mathbb{N},$$

and that

$$\int_1^x t^n \, dt = \frac{x^{n+1} - 1}{n+1}, \quad x > 0,\ n \in \mathbb{Z},\ n \neq -1.$$

9. Show that, for $|x| < 1$,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

10. Assume

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

is convergent for $|t| < R$. State Proposition 1.3.2, representing $f'(t)$ as a power series, for $t \in (-R, R)$.

Worksheet 2, Thursday, 08/13

§2.1, Euclidean spaces

1. Define the vector operations on $\mathbb{R}^n$.

2. Given $x, y \in \mathbb{R}^n$, define the dot product $x \cdot y$.

3. Given $x \in \mathbb{R}^n$, we define the norm $|x| \in [0, \infty)$ by

$$|x| = \sqrt{x \cdot x}.$$

Consult Proposition 2.1.1 and show that the triangle inequality

$$|x + y| \leq |x| + |y|$$

follows from Cauchy's inequality

$$|x \cdot y| \leq |x|\,|y|.$$

4. Consult Proposition 2.1.2 for the proof of Cauchy's inequality.

5. Given $p_j \in \mathbb{R}^n$, define what it means to say

$$\begin{aligned}&p_j \text{ converges to } p \text{ as } j \to \infty.\\&(p_j) \text{ is Cauchy.}\end{aligned}$$

6. Given $S \subset \mathbb{R}^n$, define what it means to say

$$\begin{aligned}&S \text{ is closed,}\\&S \text{ is open.}\end{aligned}$$

7. Given $x, y \in \mathbb{R}^n$, we say

$$x \perp y \Longleftrightarrow x \cdot y = 0.$$

Show that

$$|x + y|^2 = |x|^2 + |y|^2 \Longleftrightarrow x \perp y.$$

Worksheet 3, Tuesday, 08/18

§2.2, Vector spaces and linear transformations

1. Define the concept of a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Note that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. ($\mathbb{F}^n$ is a vector space over $\mathbb{F}$.)

2. Let $S = \{v_1, \dots, v_k\} \subset V$, a vector space. Define what it means to say

$$\begin{aligned} &S \text{ spans } V, \\ &S \text{ is linearly independent}, \\ &S \text{ is a basis of } V. \end{aligned}$$

3. Study Lemma 2.2.1 and Proposition 2.2.2, whose content is that
If V has a basis $\{v_1, \dots, v_k\}$ and if $\{w_1, \dots, w_\ell\} \subset V$ is linearly independent, then $\ell \le k$.
Show that this leads to Corollary 2.2.3:
If V is finite dimensional, then any two bases of V have the same number of elements.
In such a case, $\dim V$ denotes the number of elements in a basis of V.

4. State Propositions 2.2.4 and 2.2.5.

5. State Proposition 2.2.6, the Fundamental Theorem of Linear Algebra, and show how this follows from Propositions 2.2.4 and 2.2.5.

6. Deduce from the Fundamental Theorem of Linear Algebra that if V is finite dimensional and $A : V \to V$ is linear, then

$$A \text{ injective } \Leftrightarrow A \text{ surjective } \Leftrightarrow A \text{ isomorphism}.$$

7. State Proposition 2.2.9, characterizing when a matrix $A \in M(n, \mathbb{F})$ is invertible, in terms of the behavior of its columns.

Worksheet 4, Thursday, 08/20

§2.3, Determinants

1. Given
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{we set} \quad \det A = ad - bc.$$
Show that $A : \mathbb{F}^2 \to \mathbb{F}^2$ is invertible if and only if $\det A \neq 0$. If this holds,
$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

2. Consult Proposition 2.3.1 and define $\det A$ for $A \in M(n, \mathbb{F})$.

3. Show that the formula (2.3.30) for $\det A$ implies
$$\det A = \det A^t.$$

4. Read the proof of Proposition 2.3.3, that if $A, B \in M(n, \mathbb{F})$,
$$\det(AB) = (\det A)(\det B).$$
Show that this implies Corollary 2.3.4, i.e.,
$$A \text{ invertible} \implies \det A \neq 0.$$

5. Read the proof of Proposition 2.3.6, that if $A \in M(n, \mathbb{F})$,
$$A \text{ invertible} \iff \det A \neq 0.$$
See how this completes the result of Exercise 4.

6. Study Exercises 1–3 at the end of §2.3, treating the expansion of $\det A$ by minors down the kth column, given $A \in M(n, \mathbb{F})$.

7. Use an expansion by minors to evaluate $\det A$ for
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

Supplementary worksheet
§2.3, more on determinants

1. Verify the following method of computing 3×3 determinants. Given

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

form a 3×5 rectangular matrix by copying the first two columns of A to the right. The products in (2.3.16) with plus signs are the products of each of the three downward sloping diagonals marked in bold below.

$$\begin{pmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & a_{11} & a_{12} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{21}} & a_{22} \\ a_{31} & a_{32} & \mathbf{a_{33}} & \mathbf{a_{31}} & \mathbf{a_{32}} \end{pmatrix}.$$

The products in (2.3.16) with minus signs are the products of each of the three upward sloping diagonals marked in bold below.

$$\begin{pmatrix} a_{11} & a_{12} & \mathbf{a_{13}} & \mathbf{a_{11}} & \mathbf{a_{12}} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{21}} & a_{22} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} & a_{31} & a_{32} \end{pmatrix}.$$

2. Use the method described above to compute the determinants of

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

3. Given $A = (0\ 1\ 2)$, compute $\det A^t A$ and $\det AA^t$.

4. Compute the determinant of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Worksheet 5, Thursday, 08/27

§3.1, Curves and arclength

1. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1 curve, with

$$\text{velocity} \ \ \gamma'(t) \ \ \text{and speed} \ \ |\gamma'(t)|.$$

Study the statement and proof of Proposition 3.1.1, leading to the formula

$$\ell(\gamma) = \int_a^b |\gamma'(t)| \, dt$$

for the length of this curve.

2. State what it means for $\sigma : [\alpha, \beta] \to \mathbb{R}^n$ to be a reparametrization of γ, and study the identity

$$\ell(\sigma) = \ell(\gamma), \quad \text{in (3.1.17).}$$

3. Discuss the reparametrization by arclength of γ, given γ' nowhere vanishing.

4. Read about the parametrization of the unit circle S^1 by arclength, discussed in (3.1.22)–(3.1.34), leading to the definition of the trigonometric functions

$$\cos t, \quad \sin t.$$

5. Discuss the derivation in (3.1.34)–(3.1.39) of the identities

$$\frac{d}{dt} \cos t = -\sin t, \quad \frac{d}{dt} \sin t = \cos t.$$

6. Work out the arclength calculations proposed in Exercises 1–4 at the end of §3.1.

Worksheet 6, Tuesday, 09/01

§3.2, Exponential and trigonometric functions

1. Review Proposition 1.3.2, from Chapter 1, which states that, if you have a convergent power series

$$f(t)=\sum_{k=0}^{\infty} a_k t^k, \quad \text{for} \quad |t|<R,$$

with coefficients $a_k \in \mathbb{C}$, then f is differentiable on $(-R, R)$, and

$$f'(t)=\sum_{k=1}^{\infty} k a_k t^{k-1}.$$

2. Use the ratio test to show that

$$e^z=\sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

converges for each $z \in \mathbb{C}$. Then use the result of Exercise 1 to show that

$$\frac{d}{dt} e^{at}=a e^{at}, \quad \forall\, t \in \mathbb{R},\ a \in \mathbb{C}.$$

3. Show that

$$\frac{d}{dt} e^{at} e^{-at}=0,$$

and deduce that

$$e^{-at}=\frac{1}{e^{at}}, \quad \forall\, t \in \mathbb{R},\, a \in \mathbb{C}.$$

4. Show that

$$\frac{d}{dt} e^{(a+b)t} e^{-at} e^{-bt}=0,$$

and deduce that

$$e^{(a+b)t}=e^{at} e^{bt}, \quad \forall\, t \in \mathbb{R}, a, b \in \mathbb{C}.$$

5. Review (3.2.18)–(3.2.27), to the effect that $\operatorname{Exp}(t)=e^t$ satisfies

$$\operatorname{Exp}: \mathbb{R} \longrightarrow (0, \infty) \ \text{ is one-to-one and onto,}$$

with inverse

$$\log : (0, \infty) \longrightarrow \mathbb{R},$$

satisfying

$$\frac{d}{dx} \log x=\frac{1}{x}, \quad \text{hence} \quad \log x=\int_1^x \frac{dy}{y}.$$

6. Review (3.2.29)–(3.2.37), to the effect that

$$\gamma(t) = e^{it}, \quad t \in \mathbb{R}$$

is a unit-speed parametrization of the unit circle.

7. Recall definitions from basic trigonometry (see Exercise 4 of Worksheet 5) and deduce from Exercise 6 above that

$$e^{it} = \cot t + i \sin t, \quad t \in \mathbb{R}.$$

This is called Euler's identity.

8. Deduce from Exercise 7 that

$$\frac{d}{dt} e^{it} = i e^{it} \Longrightarrow \frac{d}{dt} \cos t = -\sin t, \quad \frac{d}{dt} \sin t = \cos t.$$

9. Deduce from Exercise 4 that

$$e^{i(s+t)} = e^{is} e^{it}.$$

Combine this with Exercise 7 to derive formulas for

$$\cos(s+t), \quad \sin(s+t).$$

Worksheet 7, Thursday, 09/03

§3.2, Exponential and trigonometric functions, II

1. Define

$$\tan t = \frac{\sin t}{\cos t}, \quad \sec t = \frac{1}{\cos t},$$

and check out Exercises 3–5 at the end of §3.2 of the text.

2. Check out Exercises 6–7 at the end of §3.2, leading to the formula

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

3. Look at Exercises 2 and 5 at the end of §3.2, and show that

$$\frac{\pi}{6} = \int_0^{\sqrt{3}/3} \frac{dx}{1+x^2}.$$

4. Set

$$\cosh t = \frac{1}{2}(e^t + e^{-t}), \quad \sinh t = \frac{1}{2}(t^t - e^{-t}),$$

and show that

$$\frac{d}{dt}\cosh t = \sinh t, \quad \frac{d}{dt}\sinh t = \cosh t,$$
$$\cosh^2 t - \sinh^2 t = 1.$$

5. Check out Exercise 14 at the end of §3.2. This involves evaluating

$$I(u) = \int_0^u \frac{dv}{\sqrt{1+v^2}}$$

in two ways:

$$v = \sinh y,$$
$$v = \tan t.$$

Show how this leads to the identity

$$\int_0^x \sec t \, dt = \sinh^{-1}(\tan x), \quad \text{for } |x| < \frac{\pi}{2}.$$

6. Check out the study of the functions x^r, defined by

$$x^r = e^{r \log x}, \quad x > 0, \quad r \in \mathbb{C},$$

in Exercises 18–22 at the end of §3.2. Show that

$$\frac{d}{dx} x^r = r x^{r-1}.$$

7. Consider the parabola

$$\gamma(t) = \left(t, \frac{t^2}{2}\right), \quad 0 \le t \le x.$$

Show that its length is

$$L(x) = \int_0^x \sqrt{1+t^2}\, dt.$$

8. Use the change of variable $t = \tan\theta$ to obtain

$$L(x) = \int_0^\psi \sec^3\theta\, d\theta, \quad x = \tan\psi.$$

9. On the other hand, use the change of variable $t = \sinh u$ to obtain

$$L(x) = \int_0^v \cosh^2 u\, du, \quad x = \sinh v.$$

10. Writing $\cosh u = (e^u + e^{-u})/2$, show that the last integral is equal to

$$\begin{aligned} \frac{1}{4}\int_0^v (e^u + e^{-u})^2\, du &= \frac{1}{4}\int_0^v (e^{2u} + 2 + e^{-2u})\, du \\ &= \frac{1}{4}\sinh 2v + \frac{v}{2} \\ &= \frac{1}{2}\sinh v\, \cosh v + \frac{v}{2}. \end{aligned}$$

11. Deduce from Exercise 9 above that

$$L(x) = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1} x.$$

Then deduce from Exercise 8 that, for $|\psi| < \pi/2$,

$$\int_0^\psi \sec^3\theta\, d\theta = \frac{1}{2}\sec\psi\, \tan\psi + \frac{1}{2}\sinh^{-1}(\tan\psi).$$

Supplementary worksheet

§3.2, Making a trig table

This worksheet is for students who have access to computer software allowing for numerical calculation (such as Matlab).

Follow exercises 31–39 at the end of §3.2, and make a table of

$$\cos \ell^\circ \quad \text{and} \quad \sin \ell^\circ, \tag{1}$$

for the integers ℓ between 0 and 45. Here

$$\ell^\circ = \frac{\pi \ell}{180} \text{ radians.} \tag{2}$$

A basis for the calculation of (1) is given by the identities

$$\begin{aligned} e^{\pi i/3} &= \frac{1}{2}(1 + i\sqrt{3}), \\ e^{\pi i/4} &= \frac{1}{\sqrt{2}}(1 + i), \\ e^{2\pi i/5} &= c_5 + i s_5, \end{aligned} \tag{3}$$

where

$$c_5 = \frac{1}{4}(\sqrt{5} - 1), \quad s_5 = \sqrt{1 - c_5^2}, \tag{4}$$

related to regular n-gons ($n = 6, 4, 5$), which are established in the text. Note that

$$\frac{\pi}{3} = 60^\circ, \quad \frac{\pi}{4} = 45^\circ, \quad \frac{2\pi}{5} = 72^\circ,$$

and using the identity

$$e^{(\alpha-\beta)i} = e^{\alpha i} e^{-\beta i}$$

repeatedly gives (1) when ℓ is an integral multiple of 3.

Exercise 39 presents a cube root construction that allows one to handle $\ell = 1$.

Exercise 33 discusses numerical evaluation of square roots, such as arise in (3)–(4).

Worksheet 8, Tuesday, 09/08

§3.3, Curvature of planar curves

1. Consult (3.3.1)–(3.3.8) to see that, if $\gamma : (a, b) \to \mathbb{R}^2$ is a unit speed planar curve, then its unit tangent vector $T(s)$ and normal $N(s)$ at $\gamma(s)$ are given by

$$\gamma'(s) = T(s), \quad N(s) = JT(s),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{is counterclockwise rotation by } 90^\circ.$$

Then the curvature $\kappa(s)$ at $\gamma(s)$ is given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \quad \text{i.e.,} \\ T'(s) &= \kappa(s)JT(s). \end{aligned}$$

2. Consider the treatment in (3.3.9)–(3.3.29) of the problem of constructing a unit speed curve $\gamma(s)$ with given curvature $\kappa(s)$.

Note the use of the matrix exponential, defined in (3.3.18) and used in (3.3.19) and in (3.3.29).

See Auxiliary worksheet C for material on the matrix exponential.

3. Now suppose we have a smooth curve $c : (\alpha, \beta) \to \mathbb{R}^2$, not necessarily unit speed, with velocity and acceleration

$$v(t) = c'(t), \quad a(t) = v'(t),$$

so

$$T(t) = \frac{v(t)}{\|v(t)\|}, \quad N(t) = JT(t).$$

Consider Exercises 1–2 at the end of §3.3, expressing the acceleration $a(t)$ as a linear combination of $T(t)$ and $N(t)$:

$$a(t) = \frac{d^2s}{dt^2}T(t) + \kappa(t)\Bigl(\frac{ds}{dt}\Bigr)^2 N(t),$$

and deriving the following formula for the curvature $\kappa(t)$ in this situation:

$$\kappa(t) = \frac{a(t) \cdot Jv(t)}{\|v(t)\|^3}.$$

4. Find the curvatures of the curves given in Exercises 3–6 at the end of §3.3.

Worksheet 9, Thursday, 09/10

§3.4, Curvature and torsion of curves in $\mathbb{R}^3$

1. Consult (3.4.1)–(3.4.7) to see that, if $c : (a, b) \to \mathbb{R}^3$ is a smooth curve (perhaps not of unit speed), with nowhere vanishing velocity $v(t) = c'(t)$, then the unit tangent $T(t)$ at $c(t)$ is given by

$$T(t) = \frac{v(t)}{\|v(t)\|}, \tag{1}$$

and, if $s(t)$ is the arclength parameter, given by (3.4.2), then the curvature $\kappa(s)$ and normal $N(s)$ are given by

$$\kappa(s) = \left\| \frac{dT}{ds} \right\|, \quad \frac{dT}{ds} = \kappa(s)N(s), \tag{2}$$

as long as $\kappa(s) \neq 0$. Furthermore, the binormal $B(s)$ is given by

$$B(s) = T(s) \times N(s). \tag{3}$$

Here we use the cross product in $\mathbb{R}^3$, treated in Chapter 2, §2.5.

The triple $T(s), N(s), B(s)$ is an orthonormal set in $\mathbb{R}^3$, for each s, called the Frenet frame at $\gamma(s)$, where $\gamma(s(t)) = c(t)$. Properties of the cross product given in §2.5 yield the following complements to (3):

$$T(s) = N(s) \times B(s), \quad N(s) = B(s) \times T(s). \tag{4}$$

2. Consult (3.4.8)–(3.4.9), to see that applying d/ds to (3) and using (2) yields

$$B'(s) \text{ parallel to } N(s). \tag{5}$$

We define the torsion at $\gamma(s)$ by

$$\frac{dB}{ds} = -\tau(s)N(s). \tag{6}$$

3. Consult (3.4.10)–(3.4.11) to see that applying d/ds to $N(s) = B(s) \times T(s)$ yields

$$\frac{dN}{ds} = -\kappa(s)T(s) + \tau(s)B(s). \tag{7}$$

Together, the equations (2), (6), and (7) are called the Frenet-Serret equations. See (3.4.19).

4. Consult (3.4.20)–(3.4.30) regarding the following problem:

Given smooth functions $\kappa(s)$ ad $\tau(s)$, find a unit speed curve $\gamma(s)$ for which the solution (T, N, B) to the Frenet-Serret equations is the Frenet frame.

Consult (3.4.31)–(3.4.44) for a treatment of the special case where κ and τ are constant. One again sees the matrix exponential, in (3.4.44). (Compare Exercise 2 in Worksheet 8.)

5. Consult Exercises 1–3 at the end of §3.4, expressing the acceleration $a(t) = v'(t)$ as a linear combination of $T(t)$ and $N(t)$,

$$a(t) = \frac{d^2 s}{dt^2} T(t) + \kappa(t) \Big(\frac{ds}{dt} \Big)^2 N(t)$$

(compare Exercise 3 of Worksheet 9), and deriving the formula

$$\kappa B = \frac{v \times a}{\|v\|^3},$$

for the curvature κ and binormal B, and furthermore deriving the formula

$$\tau = \frac{(v \times a) \cdot a'}{\|v \times a\|^2},$$

for the torsion.

6. Compute the Frenet frame and the curvature and torsion for the curve $c(t)$ given in Exercise 4 at the end of §3.4.

Supplementary worksheet

§C.4, The matrix exponential

1. As treated in Appendix C.4, the matrix exponential is defined by

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \quad t \in \mathbb{R},\ A \in M(n, \mathbb{C}).$$

Consult Exercise 1 at the end of §C.4 for a discussion of convergence issues.

2. Show that term by term differentiation of the power series given above for e^{tA} gives

$$\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} A.$$

3. Show that

$$\frac{d}{dt}(e^{tA} e^{-tA}) = 0,$$

and hence

$$(e^{tA})^{-1} = e^{-tA}.$$

4. Show that

$$\frac{d}{dt}(e^{(s+t)A} e^{-tA}) = 0,$$

and hence that

$$e^{(s+t)A} = e^{sA} e^{tA}, \quad \forall\, s, t \in \mathbb{R},\ A \in M(n, \mathbb{C}).$$

5. Show that, given $A, B \in M(n, \mathbb{C})$, if A and B commute, i.e., if

$$AB = BA,$$

then, for $t \in \mathbb{R}$,

$$e^{t(A+B)} = e^{tA} e^{tB}.$$

Hint. To start, show that commutativity yields

$$\frac{d}{dt}(e^{t(A+B)} e^{-tB} e^{-tA}) = 0.$$

6. Show that the chain rule plus Exercise 2 above give

$$\frac{d}{dt} e^{\varphi(t)A} = \varphi'(t) A e^{\varphi(t)A}.$$

when $\varphi : \mathbb{R} \to \mathbb{R}$ is a differentiable function.

Worksheet 10, Tuesday, 09/15

Review for test, Th 09/17

These review topics are tuned to Worksheets 1–9.

1. Review the material in §§1.1–1.3 on
The derivative:
  the product rule,
  the chain rule,
  the mean value theorem,
  the inverse function theorem.
The integral:
  the fundamental theorem of calculus.
Power series:
  particularly Proposition 1.3.2, which says that

$$f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad |t| < R \Longrightarrow f'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}.$$

2. Review the material in §2.1, on Euclidean spaces, with attention to
  the dot product on $\mathbb{R}^n$, the norm of $x \in \mathbb{R}^n$,
  the triangle inequality, and how it follows from Cauchy's inequality,
  convergent sequences, Cauchy sequences,
  closed set, open set.

3. Review material in §2.2 on vector spaces and linear transformations, with attention to
  spanning set, linear independence,
  basis, dimension,
  null space, range,
  Fundamental theorem of linear algebra,
  injective, surjective, invertible linear transformations.

4. Review material in §2.3 on the determinant of a square matrix. See Worksheet 4 and the supplementary worksheet that follows it. Particularly study the result that, if $A \in M(n, \mathbb{F})$, then

$$A \text{ is invertible } \Longleftrightarrow \det A \neq 0.$$

5. Review material in §3.1, on curves $\gamma : [a, b] \to \mathbb{R}^n$, with attention to
  velocity $\gamma'(t)$, speed $|\gamma'(t)|$,
  arc length,
  reparametrization of a curve, and comparison of arc lengths,
  reparametrization of a curve by arc length.

5A. Review how the trig functions $\cos t$ and $\sin t$ arise via taking the unit circle centered at the origin in $\mathbb{R}^2$ and parametrizing it by arc length, $C(t) = (\cos t, \sin t)$. Review how

$$\frac{d}{dt} C(t) \cdot C(t) \equiv 0 \Longrightarrow \frac{d}{dt} \cos t = -\sin t,\ \frac{d}{dt} \sin t = \cos t.$$

6. Review §3.2, on the exponential and trigonometric functions, with attention to
   power series formula for e^z,
   fact that $(d/dt)e^{at} = ae^{at}$, for $t \in \mathbb{R},\ a \in \mathbb{C}$,
   identity $e^{(a+b)t} = e^{at}e^{bt},\ t \in \mathbb{R},\ a, b \in \mathbb{C}$,
   $\log x$ as the inverse function to $x = e^t,\ t \in \mathbb{R},\ x \in (0, \infty)$.

Review the derivation of Euler's formula,

$$e^{it} = \cos t + i \sin t,$$

from the fact that $\gamma(t) = e^{it}$ is a unit-speed parametrization of the unit circle. See how $(d/dt)e^{it} = ie^{it}$ leads to formulas for the derivatives of $\cos t$ and $\sin t$, rederiving such formulas stated in Exercise 5A above.

7. Continue the review of §3.2, with attention to
   $\tan t,\ \sec t$, and their derivatives,
   π, integral formulas and numerical approximation of this number,
   $\cosh t,\ \sinh t$, and their derivatives,
   $x^r = e^{r \log x}$, and its derivative,
   two approaches to the evaluation of the length of a parabolic arc, i.e., of

$$\int_0^x \sqrt{1 + t^2}\, dt.$$

8. Review §3.3, on the curvature of a planar curve γ, with attention to
   unit tangent T and normal $N = JT$,
   curvature κ, given by $dT/ds = \kappa N$, where s is the arc length parameter,
   formula for κ when γ is not parametrized by arc length,
   use of the matrix exponential (see §C.4) in solving

$$\frac{dT}{ds} = \kappa(s) JT(s).$$

9. Review §3.4, on the curvature and torsion of a curve γ in $\mathbb{R}^3$, with attention to
   velocity $v(t) = \gamma'(t)$ and unit tangent $T(t) = v(t)/\|v(t)\|$,
   curvature $\kappa(s) = \|dT/ds\|$, and normal N, satisfying $dT/ds = \kappa(s)N(s)$,
   binormal $B(s) = T(s) \times N(s)$,
   Frenet-Serret formulas

$$\begin{aligned} \frac{dT}{ds} &= \kappa N, \\ \frac{dN}{ds} &= -\kappa T + \tau B, \\ \frac{dB}{ds} &= -\tau N, \end{aligned}$$

formulas for κ, B, and τ when γ is not parametrized by arc length.

10. **Computations.** Worksheets 1–9 point to various computational exercises in the text. Review these.

Worksheet 11, Tuesday, 09/22

§4.1, The derivative in several variables

As defined in §4.1, if $\mathcal{O} \subset \mathbb{R}^n$ is open, a function $F : \mathcal{O} \to \mathbb{R}^m$ is differentiable at $x \in \mathcal{O}$, with derivative $DF(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, if and only if

$$F(x+y) = F(x) + DF(x)y + R(x,y), \quad R(x,y) = o(\|y\|). \tag{1}$$

One compares this with the partial derivative,

$$\frac{\partial F}{\partial x_j}(x) = \lim_{h\to 0} \frac{1}{h}\big[F(x+he_j) - F(x)\big], \tag{2}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

1. Verify the identity (4.1.10) connecting (1) and (2), when F is differentiable.

2. Look at Exercise 1 at the end of §4.1, giving examples of functions to differentiate. Use the result of Proposition 4.1.1, as needed, and check out this result in the next worksheet.

3. Study the argument in (4.1.21)–(4.1.22) that, if

$$S : M(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad S(X) = X^2,$$

then

$$DS(X)Y = XY + YX.$$

Then look at Exercise 4, at the end of §4.1. Show that

$$P_3(X) = X^3 \Longrightarrow DP_3(X)Y = YX^2 + XYX + X^2Y.$$

4. Study the argument in (4.1.25)–(4.1.30) thet, if

$$\Phi : Gl(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad \Phi(X) = X^{-1},$$

then

$$D\Phi(I)Y = -Y.$$

Note the use of the evaluation of the infinite series

$$(I+Y)^{-1} = \sum_{k=0}^{\infty} (-1)^k Y^k, \quad \text{for} \quad \|Y\| < 1.$$

Going further, as indicated in Exercise 7 at the end of §4.1, show that

$$D\Phi(X)Y = -X^{-1}YX^{-1}.$$

Worksheet 12, Thursday, 09/24

§§4.1–4.2, The derivative in several variables II

1. Proposition 4.1.1 says that if F is of class C^1 (i.e., $\partial F/\partial x_j$ is continuous on $\mathcal{O}$ for each j) then F is differentiable at each $x \in \mathcal{O}$. Study its proof.

2. The chain rule is given in Proposition 4.1.2. If $F : \mathcal{O} \to U$ and $G : U \to \mathbb{R}^k$ are differentiable, then $G \circ F$ is differentiable, and

$$D(G \circ F)(x) = DG(F(x))DF(x).$$

Study its proof.

3. Look at Exercise 2 at the end of §4.1, dealing with a strengthening of Proposition 4.1.1.

4. Look at Exercise 16 at the end of §4.1, describing a function that is not differentiable at $(0, 0) \in \mathbb{R}^2$, despite the fact that both its partial derivatives exist (but they are not continuous).

5. Proposition 4.2.1 says that if $\mathcal{O} \subset \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}^m$ is of class C^2, then

$$\partial_j \partial_k F(x) = \partial_k \partial_j F(x), \quad x \in \mathcal{O},\ j, k \in \{1, \dots, n\}. \tag{1}$$

Study the proof. It involves difference quotients and the mean value theorem.

6. Check out Exercise 5, at the end of §4.2, regarding a function $g \in C^1(\mathbb{R}^2)$ for which $\partial_x \partial_y g$ and $\partial_y \partial_x g$ exist at each point of $\mathbb{R}^2$, but

$$\partial_x \partial_y g(0, 0) \neq \partial_y \partial_x g(0, 0),$$

in contrast to (1) above. (In this case, $g \notin C^2$.)

Worksheet 13, Tuesday, 09/29

§4.2, Higher derivatives and power series

1. Formulas (4.2.8)–(4.2.12) present two different multi-index notations for derivatives of order k:

$$
\begin{aligned}
&f^{(J)}(x) = \partial_{j_k} \cdots \partial_{j_1} f(x), \quad J = (j_1, \dots, j_k),\ |J| = k, \\
&f^{(\alpha)}(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f(x), \quad \alpha = (\alpha_1, \dots, \alpha_n),\ |\alpha| = \alpha_1 + \cdots + \alpha_n = k.
\end{aligned} \tag{1}
$$

Become familiar with these notations.

2. Review the treatment of power series for functions of one variable in Chapter 1, §1.3. The formula (1.3.35) can be written

$$
f(t) = \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} t^j + R_k(t), \quad t \in I = (-R, R), \tag{2}
$$

and two integral formulas are derived for the remainder $R_k(t)$, depending on whether $f \in C^{k+1}(I)$ or $f \in C^k(I)$. See (1.3.40) and (1.3.67).

Note also the Cauchy and Lagrange formulas for the remainder, given in (1.3.39) and (1.3.42).

3. Study formulas (4.2.13)–(4.2.20), leading to the power series formula

$$
F(x) = \sum_{|J| \le k} \frac{1}{|J|!} F^{(J)}(0) x^J + R_k(x), \tag{3}
$$

with the remainder $R_k(x)$ given by (4.2.20).

4. Study formulas (4.2.21)–(4.2.27), leading to the power series formula

$$
F(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_k(x), \tag{4}
$$

with $R_k(x)$ given by (4.2.27).

5. The results (4.2.20) and (4.2.27) require F to be of class C^{k+1}. Study Proposition 4.2.4, which establishes (4), with $R_k(x)$ given by (4.2.30), for F of class C^k.

Worksheet 14, Thursday, 10/01

§4.2, Higher derivatives II, critical points

1. Study formulas (4.2.32)–(4.2.37), in which it is established that, if $\mathcal{O} \subset \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}$ is C^2, then, for $y \in \mathcal{O}$,

$$F(x) = F(y) + DF(y)(x-y) + \frac{1}{2}(x-y) \cdot D^2F(y)(x-y) + R_2(x,y), \tag{1}$$

where $D^2F(y)$ is the $n \times n$ Hessian matrix, given by (4.2.33), and

$$R_2(x,y) = o(|x-y|^2). \tag{2}$$

2. In the setting of Exercise 1, we say $x_0 \in \mathcal{O}$ is a critical point of F if $DF(x_0) = 0$. Proposition 4.2.5 says that if $F : \mathcal{O} \to \mathbb{R}$ is C^2 and $x_0 \in \mathcal{O}$ is a critical point, then
(i) $D^2F(x_0)$ positive definite $\Rightarrow F$ has a local min at x_0,
(ii) $D^2F(x_0)$ negative definite $\Rightarrow F$ has a local max at x_0,
(iii) $D^2F(x_0)$ strongly indefinite $\Rightarrow x_0$ is a saddle point for F.
Show how this result follows from (1)–(2) above.

3. Study Proposition 4.2.6, characterizing when a matrix $A = A^t \in M(n, \mathbb{R})$ is positive definite, in terms of the behavior of the determinants of all the $\ell \times \ell$ upper left submatrices, $1 \le \ell \le n$.

4. Returning to the setting of Exercise 1 above, check out the remainder formula in (4.2.54).

5. Check out Proposition 4.2.9, regarding smoothness and derivatives of functions given by convergent power series on a domain $\widetilde{R} \subset \mathbb{R}^n$.

6. Check out the Leibnitz identity, for $\partial^\alpha(fg)$, discussed in Exercise 7 at the end of §4.2.

7. Do Exercise 9, at the end of §4.2.

Worksheet 15, Tuesday, 10/06

§4.3, Inverse function theorem

1. Review the inverse function theorem in one variable, given in Chapter 1, Theorem 1.1.3.

2. Theorem 4.3.1 says that if $\Omega \subset \mathbb{R}^n$ is open,

$$F : \Omega \to \mathbb{R}^n \ \text{ is } \ C^1,$$

$p_0 \in \Omega$, and $DF(p_0)$ is invertible, then F maps some open neighborhood U of p_0 one-one and onto a neighborhood V of $q_0 = F(p_0)$, and the inverse map

$$F^{-1} : V \longrightarrow U \ \text{ is } \ C^1.$$

Study its proof.

3. Proposition 4.3.2 gives a condition that guarantees that a C^1 map $f : \Omega \to \mathbb{R}^n$ be one-one. State it and write down its proof. Describe the role it plays in the proof of Theorem 4.3.1.

4. Theorem 4.3.3 is called the contraction mapping theorem. State it and write down its proof. Describe its role in the proof of Theorem 4.3.1.

5. The map $F : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ in (4.3.21) defines polar coordinates. Study how it illustrates the inverse function theorem.

6. Compare the iterative method (4.3.19) for solving $F(x) = y$ for x with that given in Exercise 1 at the end of §4.3 (Newton's method).

7. Do Exercise 2 at the end of §4.3.

Worksheet 16, Thursday, 10/08

§4.3, Implicit function theorem

1. Explain how

$$x^2 + y^2 = 1$$

defines y implicitly as a sommth function of x, in two ways, for $x \in (-1, 1)$.

2. Theorem 4.3.5 is the implicit function theorem. It says that if $x_0 \in U$, open in $\mathbb{R}^m$, $y_0 \in V$, open in $\mathbb{R}^\ell$, and

$$F : U \times V \longrightarrow \mathbb{R}^\ell \text{ is } C^k, \quad F(x_0, y_0) = u_0, \tag{1}$$

and if

$$D_y F(x_0, y_0) \text{ is invertible} \tag{2}$$

(as an $\ell \times \ell$ matrix), then the equation

$$F(x, y) = u_0 \tag{3}$$

defines

$$y = g(x, u_0), \tag{4}$$

for x near x_0 (satisfying $g(x_0, u_0) = y_0$), and g is a C^k map.

Study the proof given in (4.3.39)–(4.3.45), which brings in

$$H : U \times V \to \mathbb{R}^m \times \mathbb{R}^\ell, \quad H(x, y) = \begin{pmatrix} x \\ F(x, y) \end{pmatrix}, \quad H(x_0, y_0) = \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}. \tag{5}$$

See from

$$DH = \begin{pmatrix} I & 0 \\ D_x F & D_y F \end{pmatrix} \tag{6}$$

that the hypothesis (2) implies $DH(x_0, y_0)$ is invertible. See how the inverse function theorem (Theorem 4.3.1) yields a smooth inverse

$$G : \mathcal{O} \longrightarrow U \times V \tag{7}$$

to H, where $\mathcal{O}$ is a neighborhood of (x_0, u_0) in $\mathbb{R}^m \times \mathbb{R}^\ell$, and that $G(x, u)$ has the form

$$G(x, u) = \begin{pmatrix} x \\ g(x, u) \end{pmatrix}, \tag{8}$$

yielding the identity

$$F(x, g(x, u)) = u, \tag{9}$$

and hence satisfying (3)–(4).

3. Check out Proposition 4.3.6, which treats a C^k map

$$F : \Omega \longrightarrow \mathbb{R}, \quad F(x_0) = u_0, \quad \Omega \subset \mathbb{R}^n \text{ open},$$

under the hypothesis that

$$\nabla F(x_0) \neq 0.$$

It shows that if in particular

$$\partial_n F(x_0) \neq 0,$$

then you can solve $F(x) = u_0$ for

$$x_n = g(x_1, \dots, x_{n-1}),$$

with $(x_{10}, \dots, x_{n-1,0}, x_{n0}) = x_0$, for a C^k function g.

4. Check out the relevance of the material of #3 to the example introduced in #1.

5. Do Exercise 7, at the end of §4.3.

Worksheet 17, Tuesday, 10/13

§1.2, The Riemann integral in one variable

1. Let $I = [a, b]$ be a closed, bounded interval in $\mathbb{R}$. Read the definition of a *partition* $\mathcal{P}$ of I into intervals $J_k = [x_k, x_{k+1}]$, associated to a collection of points $\{x_j\}$, satisfying
$$a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b.$$
Write down the definitions of maxsize$(\mathcal{P})$, $\ell(J_k)$, and $\mathcal{Q} \succ \mathcal{P}$.

2. Let $f : I \to \mathbb{R}$ be a bounded function. Read (1.2.1)–(1.2.7), and write down formulas for
$$\overline{I}_{\mathcal{P}}(f), \quad \underline{I}_{\mathcal{P}}(f), \quad \overline{I}(f), \quad \underline{I}(f).$$

3. Take f as in #2. Note that
$$\underline{I}(f) \le \overline{I}(f).$$
We say that f is Riemann integrable, and write $f \in \mathcal{R}(I)$, provided
$$\underline{I}(f) = \overline{I}(f).$$
Then we write
$$\int_a^b f(x)\,dx = \int_I f\,dx = \overline{I}(f) = \underline{I}(f).$$

4. Proposition 1.2.1 says that $f, g \in \mathcal{R}(I) \Rightarrow f + g \in \mathcal{R}(I)$ and
$$\int_I (f+g)\,dx = \int_I f\,dx + \int_I g\,dx.$$
Read its proof.

5. Proposition 1.2.2 says that if $f : I \to \mathbb{R}$ is continuous (we write $f \in C(I)$), then $f \in \mathcal{R}(I)$. Read its proof.

6. Write down Darboux's theorem, Theorem 1.2.4. One implication is that, if $f \in \mathcal{R}(I)$, and if we have a sequence
$$\mathcal{P}_\nu = \{J_{\nu k} : 1 \le k \le \nu\}$$
of partitions of I, satisfying maxsize$(\mathcal{P}_\nu) \to 0$, then
$$\int_a^b f(x)\,dx = \lim_{\nu\to\infty} \sum_{k=1}^{\nu} f(\xi_{\nu k})\,\ell(J_{\nu k}),$$
where we take arbitrary $\xi_{\nu k} \in J_{\nu k}$. These sums are called Riemann sums.

7. State the Fundamental Theorem of Calculus, given in Theorems 1.2.6 and 1.2.7, and follow the proof. Note the use of the Mean Value Theorem in the proof of Theorem 1.2.7.

8. Take a set $S \subset I$. Write down the definitions of upper content and outer measure,

$$\mathrm{cont}^{+}(S), \quad \text{and} \quad m^{*}(S),$$

given in (1.2.21) and (1.2.25).

9. Write down the sufficient conditions for a bounded function $f : I \to \mathbb{R}$ to be Riemann integrable, given in Proposition 1.2.11 and in Proposition 1.2.12. These involve two different evaluations of the "size" of the set S of points in I at which f is not continuous, namely $\mathrm{cont}^{+}(S)$ and $m^{*}(S)$.

10. Note the example of a bounded function that is not Riemann integrable, given in (1.2.16). Note the examples of bounded, discontinuous functions that are Riemann integrable, given in (1.2.51) and in (1.2.53).

11. Do exercise 4, at the end of §1.2.

Worksheet 18, Thursday, 10/15

§5.1, The Riemann integral in n variables

1. Let $R = I_1 \times \cdots \times I_n$ be a cell in $\mathbb{R}^n$, where each $I_\nu = [a_\nu, b_\nu]$ is a closed, bounded interval in $\mathbb{R}$. Study the definition of a partition $\mathcal{P}$ of R, into cells R_α, $\alpha = (\alpha_1, \dots, \alpha_n)$, given in the beginning of §5.1. Write down the definitions of

$$\operatorname{maxsize}(\mathcal{P}), \quad V(R_\alpha), \quad \text{and} \quad \mathcal{Q} \succ \mathcal{P}.$$

2. Let $f : R \to \mathbb{R}$ be a bounded function. Read (5.1.1)–(5.1.8), and write down formulas for

$$\overline{I}_{\mathcal{P}}(f), \quad \underline{I}_{\mathcal{P}}(f), \quad \overline{I}(f), \quad \underline{I}(f).$$

3. Take f as in #2. Note that

$$\underline{I}(f) \le \overline{I}(f).$$

We say f is Riemann integrable, and write $f \in \mathcal{R}(R)$, provided

$$\underline{I}(f) = \overline{I}(f).$$

Then we write

$$\int\limits_R f(x)\, dV(x) = \overline{I}(f) = \underline{I}(f).$$

4. Proposition 5.1.1 is the multi-D Darboux theorem. One implication is that if $f \in \mathcal{R}(R)$, and if we have a sequence

$$\mathcal{P}_\nu = \{R_{\nu\alpha} : \alpha \in S_\nu\}$$

of partitions of R, satisfying $\operatorname{maxsize}(\mathcal{P}_\nu) \to 0$, then

$$\int\limits_R f(x)\, dV(x) = \lim_{\nu \to 0} \sum_{\alpha \in S_\nu} f(\xi_{\nu\alpha})\, V(R_{\nu\alpha}),$$

where we take arbitrary $\xi_{\nu\alpha} \in R_{\nu\alpha}$. These sums are called Riemann sums. Compare this result with Theorem 1.2.4. A key ingredient behind this result is that, whenever $g : R \to \mathbb{R}$ is bounded,

$$\overline{I}_{\mathcal{P}_\nu}(g) \to \overline{I}(g), \quad \underline{I}_{\mathcal{P}_\nu}(g) \to \underline{I}(g).$$

5. Proposition 5.1.2 says that $f_j \in \mathcal{R}(R),\ c_j \in \mathbb{R} \Rightarrow c_1 f_1 + c_2 f_2 \in \mathcal{R}(R)$ and

$$\int\limits_R (c_1 f_1 + c_2 f_2)\, dV = c_1 \int\limits_R f_1\, dV + c_2 \int\limits_R f_2\, dV.$$

Compare this result with Proposition 1.2.1.

6. Proposition 5.1.3 says that if $f : R \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}(R)$. Read its proof. Compare this with Proposition 1.2.2.

7. Take a set $S \subset R$, where R is a cell. Write down the definitions of upper content, lower content, and volume,

$$\operatorname{cont}^+(S), \quad \operatorname{cont}^-(S), \quad \text{and} \quad V(S),$$

given in (5.1.15)–(5.1.17).

8. A set $S \subset R$ is called *contented* provided $\operatorname{cont}^+(S) = \operatorname{cont}^-(S)$, in which case the common value is denoted $V(S)$. Proposition 5.1.4 says that a set $S \subset R$ is contented if and only if its boundary bS satisfies

$$\operatorname{cont}^+(bS) = 0.$$

Read its proof.

9. Proposition 5.1.6 says that if $f : R \to \mathbb{R}$ is bounded and if S is the set of its points of discontinuity, then

$$\operatorname{cont}^+(S) = 0 \Longrightarrow f \in \mathcal{R}(R).$$

Read its proof.
Note. A stronger result is established in §5.4, namely $m^*(S) = 0 \Rightarrow f \in \mathcal{R}(R)$.

10. Propositions 5.1.7–5.1.8 give sufficient conditions that a set $S \subset R$ have upper content 0 (we then say S is a nil set). Write down these conditions.

Remark. This brings us to the part of §5.1 that takes up iterated integrals. We will take this up after the Oct. 22 test.

Worksheet 19, Tuesday, 10/20

Review for test, Th 10/22

These review topics are tuned to Worksheets 11–18.

1. Review the material in §4.1 on the derivative in several variables:
   the definition of $DF(x)$ as a linear transformation,
   the partial derivatives $\partial F/\partial x_j$ and their relation to $DF(x)$
   examples of computing $DF(x)$, including maps $F : M(n, \mathbb{R}) \to M(n, \mathbb{R})$, such as

$$F(X) = X^2, \quad F(X) = X^{-1} \text{ (on } \ Gl(n, \mathbb{R})).$$

2. Continue the review of §4.1, with attention to:
   criterion on $\partial F/\partial x_j$ to guarantee that F is differentiable,
   definition of $F : \mathcal{O} \to \mathbb{R}^m$ being C^1,
   the chain rule.

3. Review the material of §4.2:

Higher derivatives
   definition of $F : \mathcal{O} \to \mathbb{R}^m$ being C^2, or C^k,
   equality of mixed partial derivatives,
   multi-index notations, involving $f^{(J)}$ and $f^{(\alpha)}$.

Power series
   Taylor formula with remainder for

$$f(t) = \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} t^j + R_k(t),$$

   defined on $I \subset \mathbb{R}$,
   Taylor formula with remainder

$$F(x) = \sum_{|J| \le k} \frac{1}{|J|!} F^{(J)}(0) x^J + R_k(x),$$
$$F(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} F^{(\alpha)}(0) x^\alpha + R_k(x).$$

4. Continue the review of §4.2, with attention to

Power series to second order
   $F(x) = F(y) + DF(y)(x-y) + \frac{1}{2}(x-y) \cdot D^2F(y)(x-y) + R_2(x, y)$,
   $D^2F(y)$ is the Hessian matrix,
   $R_2(x, y) = o(|x-y|^2)$ if F is of class C^2.

Critical points (where $DF(x_0) = 0$), for $F : \mathcal{O} \to \mathbb{R}$.
   local max
   local min

saddle

5. Review the material on §4.3

Inverse function theorem

one-to-one condition
contraction mapping theorem
polar coordinates

6. Continue the review of §4.3.

Implicit function theorem

statement of result
derivation from the inverse function theorem
implication of a C^k function $F : \Omega \to \mathbb{R}$ satisfying

$$\nabla F(x_0) \neq 0.$$

7. Review the material of §1.2 on the Riemann integral in one variable:
interval I, partition $\mathcal{P}$,
$\overline{I}_{\mathcal{P}}(f), \quad \underline{I}_{\mathcal{P}}(f), \quad \overline{I}(f), \quad \underline{I}(f).$
definition of $\mathcal{R}(I)$ and of the integral,
additivity of the integral,
integrability of each $f \in C(I)$,
Darboux theorem,
$\text{cont}^+(S), \quad \text{cont}^-(S).$
integrability criterion in Proposition 1.2.11.

8. Review the material in §5.1, on the Riemann integral in n variables:
cell R, partition $\mathcal{P}$,
$\overline{I}_{\mathcal{P}}(f), \quad \underline{I}_{\mathcal{P}}(f), \quad \overline{I}(f), \quad \underline{I}(f),$
definition of $\mathcal{R}(R)$ and of the integral,
integrability of each $f \in C(R)$,
Darboux theorem,
additivity of the integral,
$\text{cont}^+(S), \quad \text{cont}^-(S), \quad V(S),$
integrability condition in Proposition 5.1.6.

Worksheet 20, Tuesday, 10/27

§5.1, The Riemann integral in n variables II
Iterated integrals

1. Theorem 5.1.10 takes a closed, bounded, contented set $\Sigma \subset \mathbb{R}^{n-1}$

$$\Omega = \{(x,y) \in \mathbb{R}^n : x \in \Sigma,\ g_0(x) \le y \le g_1(x)\},$$

where g_j are continuous on Σ, $g_0 < g_1$. One is given $f : \Omega \to \mathbb{R}$, continuous, and

$$\varphi(x) = \int_{g_0(x)}^{g_1(x)} f(x,y)\, dy,$$

which is continuous on Σ. The conclusion is that

$$\int\limits_{\Omega} f\, dV_n = \int\limits_{\Sigma} \varphi\, dV_{n-1}.$$

Study its proof. Note the role of Proposition 5.1.9. Note Corollary 5.1.11, which says

$$V(\Omega) = \int\limits_{\Sigma} [g_1(x) - g_0(x)]\, dx.$$

2. Study the application of these results to

$$A(D) = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx = \pi,$$

in (5.1.57)–(5.1.59). Here, $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

3. Study the application of results of # 1 to

$$V(B^n) = 2\int\limits_{B^{n-1}} \sqrt{1-|x|^2}\, dx,$$

where $B^n = \{x \in \mathbb{R}^n : |x| \le 1\}$, hence to

$$\begin{aligned} V(B^3) &= 2\int\limits_{D} \sqrt{1-|x|^2}\, dx \\ &= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy\, dx \\ &= \frac{4}{3}\pi, \end{aligned}$$

in (5.1.63)–(5.1.68).

4. Proposition 5.1.12 extends Theorem 5.1.10. We take $\Sigma \subset \mathbb{R}^k$, closed, bounded, and contented, $g_j : \Sigma \to [0, \infty)$ continuous, $g_0 < g_1$, and

$$\Omega = \{(x, y) \in \mathbb{R}^n : x \in \Sigma,\ y \in \mathbb{R}^{n-k},\ g_0(x) \le |y| \le g_1(x)\}.$$

We take $f : \Omega \to \mathbb{R}$ continuous. The conclusion is that

$$\varphi(x) = \int\limits_{g_0(x) \le |y| \le g_1(x)} f(x, y)\, dy$$

is continuous on Σ and

$$\int\limits_{\Omega} f\, dV_n = \int\limits_{\Sigma} \varphi\, dV_k.$$

Consider how the proof of Theorem 5.1.10 adapts to this situation.

5. Consider the application of Proposition 5.1.12 to solids of revolution

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : a \le x \le b, \sqrt{y^2 + z^2} \le g(x)\}$$

in (5.1.72)–(5.1.75), including

$$V(\Omega) = \pi \int_a^b g(x)^2\, dx,$$

and the alternative derivation of the formula

$$V(B^3) = \pi \int_{-1}^{1} (1 - x^2)\, dx = \frac{4}{3}\pi,$$

in (5.1.76).

6. Study the application of Proposition 5.1.12 to the recursive formula

$$V(B^n) = \beta_n V(B^{n-1}), \quad \beta_n = \int_{-1}^{1} (1 - x^2)^{(n-1)/2}\, dx,$$

including

$$V(B^4) = \beta_4 V(B^3), \quad \beta_4 = 2 \int_0^{\pi/2} \cos^4 t\, dt,$$

given in (5.1.77)–(5.1.83).

7. State the general Fubini theorem, Theorem 5.1.15. See how it implies Theorem 5.1.10 and Proposition 5.1.12.

Worksheet 21, Thursday, 10/29

§5.1, The Riemann integral in n variables III
Change of variable formulas

The central result of this part of §5.1, encompassing Propositions 5.1.16–5.1.24, is Theorem 5.1.20, which says the following. Take

$$\mathcal{O}, \Omega \subset \mathbb{R}^n \text{ open}, \quad G : \mathcal{O} \to \Omega \ \text{ a } C^1 \text{ diffeomorphism.}$$

Assume $f : \Omega \to \mathbb{R}$ is supported on a compact subset of Ω and is Riemann integrable (we say $f \in \mathcal{R}_c(\Omega)$). Then $f \circ G \in \mathcal{R}_c(\mathcal{O})$, and

$$\int\limits_{\Omega} f(y)\, dV(y) = \int\limits_{\mathcal{O}} f(G(x))\, |\det DG(x)|\, dV(x). \tag{1}$$

This is established in stages.

1. Proposition 5.1.16 says that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and has compact support (we say $f \in C_c(\mathbb{R}^n)$), and if $A \in G\ell(n, \mathbb{R})$ (invertible matrix), then

$$\int f(x)\, dx = |\det A| \int f(Ax)\, dx. \tag{2}$$

Study its proof. Note the role of Proposition 5.1.9.

2. Proposition 5.1.13 establishes other characterizations of $\overline{I}(f)$ and $\underline{I}(f)$, given bounded $f : R \to \mathbb{R}$. In particular,

$$\begin{aligned} \overline{I}(f) &= \inf \Big\{ \int\limits_R g\, dV : g \in C(R), g \ge f \Big\}, \\ \underline{I}(f) &= \sup \Big\{ \int\limits_R g\, dV : g \in C(R), g \le f \Big\}. \end{aligned} \tag{3}$$

Study its proof.

3. See how (3) leads to the extension of (2) to all compactly supported Riemann integrable functions f on $\mathbb{R}^n$ (we say $f \in \mathcal{R}_c(\mathbb{R}^n)$), in Proposition 5.1.17.

Note also Corollary 5.1.18:

$$V(A(\Sigma)) = |\det A|\, V(\Sigma), \tag{4}$$

when $\Sigma \subset \mathbb{R}^n$ is compact and contented, and $A \in G\ell(n, \mathbb{R})$.

4. Proposition 5.1.19 establishes Theorem 5.1.20, under the additional hypothesis that f is continuous. Study its proof, following (5.1.111)–(5.1.119).

5. See how Theorem 5.1.20 is derived from Proposition 5.1.19, via (3).

6. Check out the use of polar coordinates to do double integrals, in (5.1.121)–(5.1.136), including the formula

$$\int_{D_\rho} f(x,y)\,dx\,dy = \int_0^\rho \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\,d\theta\,dr, \tag{5}$$

where $D_\rho = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \rho^2\}$. Note the use of polar coordinates in (5.1.125):

$$A(D_\rho) = \pi\rho^2,$$

and in the analysis of $V(B^3)$ in (5.129)–(5.130):

$$\begin{aligned} V(B^3) &= 2\int_D \sqrt{1-x^2-y^2}\,dx\,dy \\ &= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r\,dr\,d\theta \\ &= \frac{4}{3}\pi. \end{aligned} \tag{6}$$

Compare approaches described in #3 and #5 of Worksheet 20.

7. Study the use of spherical polar coordinates on $\mathbb{R}^3$ in (5.1.137)–(5.1.142), including, in formula (5.1.141),

$$\int_{B^3} f(x)\,dV(x) = \int_0^{2\pi}\int_0^\pi\int_0^1 f(G(\rho,\theta,\psi))\rho^2 \sin\theta\,d\rho\,d\theta\,d\psi. \tag{7}$$

See from this yet a fourth derivation in (5.1.142) of the formula (6).

8. Check out the end of §5.1, extending various results to integrals over all of $\mathbb{R}^n$. In particular, see (5.1.152)–(5.1.155), deriving the identity

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = I = \sqrt{\pi} \tag{8}$$

by representing I^2 as an integral over $\mathbb{R}^2$ and switching to polar coordinates. More generally, show that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = I^n = \pi^{n/2}. \tag{9}$$

9. Look at Exercises 5–7, 11–13, 23, and 24, at the end of §5.1.

Worksheet 22, Tuesday, 11/03

§6.1, Surfaces and surface integrals

1. A C^k smooth m-dimensional surface M in $\mathbb{R}^n$ is covered by coordinate charts. Each $p \in M$ has a neighborhood $U \subset M$ for which there is a C^k map

$$\varphi : \mathcal{O} \longrightarrow U, \quad \text{one-to-one and onto,} \tag{1}$$

with $\mathcal{O} \subset \mathbb{R}^m$ open, such that

$$D\varphi(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \ \text{ is injective, for all } \ x \in \mathcal{O}. \tag{2}$$

Given $p \in U$, we set

$$\begin{aligned} T_pM &= \text{ Range } D\varphi(x_0), \quad \varphi(x_0) = p, \\ N_pM &= \perp \text{ complement of } T_pM \text{ in } \ \mathbb{R}^n. \end{aligned} \tag{3}$$

If

$$\psi : \Omega \longrightarrow U, \quad \text{one-to-one and onto,} \tag{4}$$

is another C^k coordinate chart, we set

$$F = \psi^{-1} \circ \varphi : \mathcal{O} \longrightarrow \Omega. \tag{5}$$

Check out Lemma 6.1.1, saying F is a C^k diffeomorphism, and study its proof. Note the role of the Inverse Function Theorem. See that also

$$T_pM = \text{ Range } D\psi(x_1), \quad \text{if } \ \psi(x_1) = p. \tag{6}$$

2. If $M \subset \mathbb{R}^n$ is a smooth m-dimensional surface, as in #1, we associate to each coordinate chart φ, as in (1), an $m \times m$ matrix function of the form

$$G(x) = D\varphi(x)^t D\varphi(x), \tag{7}$$

called a metric tensor. See in (6.1.17) the connection with the inner product on T_pM $(p = \varphi(x))$, inherited from the dot product on $\mathbb{R}^n$. Note that

$$v \cdot G(x)w = D\varphi(x)w \cdot D\varphi(x)v, \tag{7A}$$

for $v, w \in \mathbb{R}^m$, and that

$$G(x) = \big(g_{jk}(x)\big), \quad g_{jk}(x) = \partial_j\varphi(x) \cdot \partial_k\varphi(x). \tag{7B}$$

Show via the chain rule that if we have another coordinate chart ψ (connected with φ via (5)), with metric tensor

$$H(y) = D\psi(y)^t D\psi(y), \tag{8}$$

then

$$G(x) = DF(x)^t \, H(y) \, DF(x), \quad \text{for} \;\; y = F(x). \tag{9}$$

3. If $f : M \to \mathbb{R}$ is a continuous function supported in a coordinate patch U (as in (1)), we define the surface integral of f as

$$\int_M f \, dS = \int_{\mathcal{O}} f(\varphi(x)) \sqrt{g(x)} \, dx, \quad g(x) = \det G(x), \tag{10}$$

with $G(x)$ given by (7). Check out (6.1.22)–(6.1.25) to see that if $\psi : \Omega \to U$ in (4) is another coordinate chart, then (10) is equal to

$$\int_\Omega f(\psi(y)) \sqrt{h(y)} \, dy, \quad h(y) = \det H(y), \tag{11}$$

with $H(y)$ given by (8). Note the role of the change of variable formula, addressed in Worksheet 21.

4. Check out (6.1.28)–(6.1.30), which says that if $M \subset \mathbb{R}^3$ is a 2D surface with coordinate chart $\varphi : \mathcal{O} \to U \subset M$, and f is supported on U, then

$$\int_M f \, dS = \int_{\mathcal{O}} f(\varphi(x)) \, |\partial_1 \varphi \times \partial_2 \varphi| \, dx_1 \, dx_2. \tag{12}$$

5. Check out (6.1.31)–(6.1.32), which says that if $\Omega \subset \mathbb{R}^{n-1}$ is open and $M \subset \mathbb{R}^n$ is the graph of $z = u(x)$, then

$$\varphi(x) = (x, u(x)) \tag{13}$$

provides a coordinate chart, in which the metric tensor formula (7B) becomes

$$g_{jk}(x) = \delta_{jk} + \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k}, \tag{14}$$

and in such a case

$$\sqrt{g} = (1 + |\nabla u|^2)^{1/2}. \tag{15}$$

Worksheet 23, Thursday, 11/05

§6.1, Surfaces and surface integrals II

1. Show that the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is a smooth surface. Show that solving

$$x_1^2 + \cdots + x_k^2 + \cdots + x_n^2 = 1$$

for x_k yields coordinate charts

$$\varphi_k^\pm : B^{n-1} \longrightarrow U_k^\pm \subset S^{n-1},$$

where

$$U_k^\pm = \{x \in S^{n-1} : \pm x \cdot e_k > 0\},$$

and $\{e_1, \dots, e_n\}$ denotes the standard orthonormal basis of $\mathbb{R}^n$.

2. Writing φ_n^+ in #1 as

$$\varphi_n^+(x) = (x, u(x)) = (x, \sqrt{1 - |x|^2}), \quad x \in B^{n-1},$$

show that

$$|\nabla u(x)|^2 = \frac{|x|^2}{1 - |x|^2},$$

and hence, from #5 of Worksheet 22,

$$\sqrt{g(x)} = (1 - |x|^2)^{-1/2},$$

in this coordinate system. Deduce that the area A_{n-1} of S^{n-1} is given by

$$A_{n-1} = 2 \int_{B^{n-1}} (1 - |x|^2)^{-1/2} \, dx.$$

3. Deduce from #2 that

$$\begin{aligned} A_2 &= 2 \int_D (1 - |x|^2)^{-1/2} \, dx \\ &= 2 \int_0^{2\pi} \int_0^1 (1 - r^2)^{-1/2} \, r \, dr \, d\theta \\ &= 4\pi. \end{aligned}$$

4. Check out the argument in (6.1.34)–(6.1.38), yielding

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_{S^{n-1}} \Big[\int_0^\infty f(r\omega) r^{n-1} \, dr \Big] \, dS(\omega), \tag{4}$$

for f integrable on $\mathbb{R}^n$. If f is radial, i.e., $f(x) = \varphi(|x|)$, deduce that

$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\,dx = A_{n-1} \int_0^\infty \varphi(r) r^{n-1}\,dr, \tag{5}$$

where A_{n-1} is the area of S^{n-1}.

5. Deduce from (5) that

$$V(B^n) = \frac{1}{n} A_{n-1}. \tag{6}$$

In particular, $V(B^3) = A_2/3$. Compare the computations of $V(B^3)$ in worksheets 20–21 and the computation of A_2 in #3 above.

6. Combine (2), with $n-1$ replaced by n, with (5) to show that

$$A_n = 2A_{n-1} \int_0^1 (1-r^2)^{-1/2} r^{n-1}\,dr. \tag{7}$$

Use this with $n = 2$ to relate A_2 to $A_1 = 2\pi$. Compare the calculation in #3 above.

7. See #8 of Worksheet 21 for a derivation of the identity

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx = I^n = \pi^{n/2}. \tag{8}$$

Take $\varphi(r) = e^{-r^2}$ in (5), and follow the arguments in (6.1.39)–(6.1.43) to see the formula

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{9}$$

where $\Gamma(z)$ is Euler's gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\,ds. \tag{10}$$

8. Study the treatment of $\Gamma(z)$ in (6.1.43)–(6.1.47), including Lemma 6.1.2, which uses integration by parts to establish

$$\Gamma(z+1) = z\Gamma(z), \quad z > 0, \tag{11}$$

and also the particular identities

$$\Gamma(1) = 1, \quad \Gamma\Big(\frac{1}{2}\Big) = \pi^{1/2}, \quad \Gamma(k) = (k-1)! \tag{12}$$

See also how these identities together with (9) yield the formulas

$$A_{2k-1} = \frac{2\pi^k}{(k-1)!}, \quad A_{2k} = \frac{2\pi^k}{(k-\frac{1}{2})\cdots(\frac{1}{2})}. \tag{13}$$

9. Study Proposition 6.15, and its special case, arising in Exercise 10 at the end of §6.1, which says that if $\Omega \subset \mathbb{R}^n$ is open, $c \in \mathbb{R}$, and

$$\begin{aligned} &u : \Omega \longrightarrow \mathbb{R} \ \text{ is } \ C^k, \\ &S = \{x \in \Omega : u(x) = c\}, \\ &S \neq \emptyset, \ \text{ and } \ x \in S \Rightarrow \nabla u(x) \neq 0, \end{aligned} \tag{14}$$

then S is a C^k smooth, $(n-1)$-dimensional surface. Also, for $x \in S$,

$$\begin{aligned} N_x S &= \text{ span of } \ \nabla u(x), \\ T_x S &= \perp \ \text{complement of } N_x S. \end{aligned} \tag{15}$$

Apply this to

$$S^{n-1} = \{x \in \mathbb{R}^n : u(x) = 1\}, \quad u(x) = |x|^2. \tag{16}$$

10. Apply #9 to

$$S = \{(x', f(x')) : x' \in \mathbb{R}^{n-1}\},$$

which takes the form (14) with

$$u(x) = x_n - f(x'), \quad x' = (x_1, \dots, x_{n-1}), \quad c = 0.$$

11. Look at Exercises 1, 3, 6–9, and 15, at the end of §6.1.

Supplementary worksheet

§6.6, Partitions of unity

1. Let X be a compact subset of $\mathbb{R}^n$, and $\{u_j : 1 \le j \le N\}$ an open cover of X . A continuous partition of unity subordinate to this cover is a family of continuous functions $\varphi_j : X \to \mathbb{R}$ such that

$$\varphi_j \ge 0, \quad \operatorname{supp} \varphi_j \subset U_j, \quad \sum_j \varphi_j = 1.$$

Follow the construction in (6.6.1)–(6.6.3) of such a partition of unity.

2. If M is a C^k smooth compact surface, covered by coordinate patches U_j, a smooth partition of unity on M, subordinate to this cover, is a partition of unity $\{\varphi_j\}$, as in #1, with the additional property that

$$\varphi_j \in C^k(M).$$

Follow the discussion in §6.6 of how to construct such a partition of unity.

Worksheet 24, Tuesday, 11/10

§6.3, Formulas of Gauss, Green, and Stokes

1. Gauss's formula, Theorem 6.3.1, also known as the Divergence Theorem, says that, if $\Omega \subset \mathbb{R}^n$ is a bounded open set, with C^1 boundary $\partial\Omega$, and if F is a C^1 smooth vector field on $\overline{\Omega}$, then

$$\int_{\Omega} (\operatorname{div} F)\, dx = \int_{\partial\Omega} N \cdot F\, dS, \tag{1}$$

where, for $x \in \partial\Omega$, $N(x)$ is the unit outward-pointing normal.

Follow the reduction of Theorem 6.3.1 to Proposition 6.3.2, which says that

$$\int_{\Omega} e \cdot \nabla f\, dx = \int_{\partial\Omega} (e \cdot N) f\, dS, \tag{2}$$

given a C^1 function $f : \overline{\Omega} \to \mathbb{R}$ and $e \in \mathbb{R}^n$.

2. Follow the proof of (2) given in (6.3.3)–(6.3.8). A key ingredient in the proof of (2) is to check it for f supported in a set U of the form (6.3.3), with $\partial\Omega$ the graph of a function u, and N given by

$$N = (1 + |\nabla u|^2)^{-1/2}(-\nabla u, 1),$$

while

$$dS = (1 + |\nabla u|^2)^{1/2}\, dx',$$

thus leading to (6.3.6). The path to (2) is then provided by (6.3.8).

3. In the setting of #1, assume $n = 2$, so $\overline{\Omega} \subset \mathbb{R}^2$. Follow the arguments in (6.3.9)–(6.3.11) to deduce from (1) that

$$\int_{\Omega} (\operatorname{div} JF)\, dx = -\int_{\partial\Omega} F \cdot T\, ds, \tag{3}$$

where J is the 2×2 matrix in (6.3.10), $T = JN$ is the unit tangent to $\partial\Omega$, and ds denotes arclength.

4. If $\gamma : [a, b] \to \mathbb{R}^n$ is a C^1 curve and $F = (f_1, \dots, f_n)$, we set

$$\int_{\gamma} f_1\, dx_1 + \cdots + f_n\, dx_n = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt, \tag{4}$$

Show that (4) is equal to $\int_\gamma F \cdot T\, ds$, and deduce that (3) can be written as (6.3.14), i.e.,

$$\int_{\Omega} \Big(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\Big)\, dx_1\, dx_2 = \int_{\partial\Omega} f_1\, dx_1 + f_2\, dx_2. \tag{5}$$

5. Switching notation, obtain the following common formulation of Green's theorem:

If $\Omega \subset \mathbb{R}^2$ is a C^1 smoothly bounded open set, and $f, g \in C^1(\overline{\Omega})$, then

$$\int_{\Omega} \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \Big)\, dx\, dy = \int_{\partial\Omega} f\, dx + g\, dy. \tag{6}$$

Worksheet 25, Thursday, 11/12

§6.3, Formulas of Gauss, Green, and Stokes II

1. Let $F = (f, g, h)$ be a C^1 vector field on an open set $\mathcal{O} \subset \mathbb{R}^3$. We define

$$\operatorname{curl} F = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{pmatrix}, \tag{1}$$

where $\{i, j, k\}$ denotes the standard basis of $\mathbb{R}^3$. Cf. (6.3.16). Follow the computation in (6.3.17)–(6.3.19) that, if $\mathcal{O}$ contains the planar domain

$$\overline{U} = \{(x, y, 0) : (x, y) \in \overline{\Omega}\}, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a smoothly bounded open set, then Green's formula, from Worksheet 24, can be written

$$\int_U (\operatorname{curl} F) \cdot k \, dA = \int_{\partial U} (F \cdot T) \, ds. \tag{3}$$

Hint. (1) gives $(\operatorname{curl} F) \cdot k = \partial_x g - \partial_y f$.

2. Stokes' formula (Proposition 6.3.4) says the following. Let $S \subset \mathbb{R}^3$ be a smooth surface, and let $\overline{M} \subset S$ be a smoothly bounded subset (see Figure 6.3.2). Assume there is a smooth unit normal field N on S. Let T denote the unit tangent to ∂M (satisfying (6.3.4)). If F is a C^1 vector field on a neighborhood $\mathcal{O}$ of $\overline{M}$ in $\mathbb{R}^3$, then

$$\int_M (\operatorname{curl} F) \cdot N \, dS = \int_{\partial M} (F \cdot T) \, ds. \tag{4}$$

Follow the proof of this, in (6.3.22)–(6.3.31).

Note that part of the strategy involves covering $\overline{M}$ with coordinate charts that are given as graphs of functions, either of the form (6.3.22) or (6.3.31), and using a partition of unity subordinate to this cover. Then Stokes' formula on each piece is established, in (6.3.22)–(6.3.30), by deriving it from the 2D Green formula.

Worksheet 26, Tuesday, 11/17

Review of course, Exam Wed., 11/18

I. The following review topics are tuned to worksheets 20–25.

1. Review material in §5.1, on iterated integrals:
   General identities of the form

$$\int\limits_{X\times Y} f\,dV = \int\limits_X \Bigl(\int\limits_Y f(x,y)\,dy\Bigr)\,dx,$$

   in Proposition 5.1.9, Theorem 5.1.10, Proposition 5.1.12, and Theorem 5.1.15.
   Applications to

$$A(D)=\pi, \quad V(B^3)=\frac{4}{3}\pi, \quad V(B^n)=\beta_n V(B^{n-1}).$$

2. Review material in §5.1, on change of variables in multiple integrals:
   Proposition 5.1.19: for a C^1 diffeomorphism $G:\mathcal{O}\to\Omega$,

$$\int\limits_\Omega f(y)\,dV(y) = \int\limits_{\mathcal{O}} f(G(x))|\det DG(x)|\,dV(x).$$

   Polar coordinates on $\mathbb{R}^2$, applications to

$$A(D)=\pi, \quad \int\limits_{\mathbb{R}^2} e^{-|x|^2}\,dx=\pi, \quad \int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx=\pi^{n/2}.$$

   Spherical polar coordinates on $\mathbb{R}^3$. Application to $V(B^3)=(4/3)\pi$.

3. Review material in §6.1, on surfaces and surface integrals:
   Coordinate charts $\varphi:\mathcal{O}\to U\subset M$, for surface $M\subset\mathbb{R}^n$.
   Tangent space T_pM.
   Metric tensor, $g_{jk}(x)=\partial_j\varphi(x)\cdot\partial_k\varphi(x)$, or

$$G(x)=\bigl(g_{jk}(x)\bigr)=D\varphi(x)^t D\varphi(x).$$

   Surface integral

$$\int\limits_M f\,dS = \int\limits_{\mathcal{O}} f(\varphi(x))\sqrt{g(x)}\,dx, \quad g(x)=\det G(x).$$

   Invariance under change of coordinate chart.

4. Review material in §6.1, more on surfaces and surface integrals:
Metric tensor for a graph of $z = u(x)$, yielding
$$\sqrt{g(x)} = (1 + |\nabla u(x)|^2)^{1/2}$$
Application to $A_{n-1} = A(S^{n-1})$:
$$A_{n-1} = 2 \int\limits_{B^{n-1}} (1 - |x|^2)^{-1/2}\, dx, \quad A(S^2) = 4\pi.$$
Spherical polar coordinates on $\mathbb{R}^n :\ x = r\omega$. Application to
$$\int\limits_{\mathbb{R}^n} f(x)\, dx = \int\limits_{S^{n-1}} \Bigl(\int_0^\infty f(r\omega) r^{n-1}\, dr \Bigr)\, dS(\omega),$$
hence to
$$\int\limits_{\mathbb{R}^n} \varphi(|x|)\, dx = A_{n-1} \int_0^\infty \varphi(r) r^{n-1}\, dr, \quad \text{so} \ \ V(B^n) = \frac{1}{n} A_{n-1}.$$
Apply to $\varphi(r) = e^{-r^2}$ to obtain
$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad \Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\, ds.$$
Euler gamma function $\Gamma(z),\ z > 0$, satisfies
$$\Gamma(1) = 1, \quad \Gamma(\tfrac{1}{2}) = \pi^{1/2}, \quad \Gamma(z+1) = z\Gamma(z).$$

5. Review material in §6.3, on Gauss and Green formulas:
Unit normal to $\{x \in \mathbb{R}^n : v(x) = c\}$ is
$$N(x) = \frac{\nabla v(x)}{|\nabla v(x)|}.$$
In case $v(x) = x_n - u(x'),\ c = 0$,
$$N(x) = (1 + |\nabla u|^2)^{-1/2}(-\nabla u, 1).$$
Gauss formula (divergence theorem) for vector field $F \in C^1(\overline{\Omega})$,
$$\int\limits_{\Omega} \operatorname{div} F(x)\, dx = \int\limits_{\partial\Omega} (N \cdot F)\, dS.$$
If $F = (f_1, \dots, f_n)$, $\operatorname{div} F = \partial_1 f_1 + \cdots + \partial_n f_n$.
N is the outward-pointing unit normal.
For $n = 2$, applying the divergence theorem with F replaced by JF gives Green's theorem:
$$\int\limits_{\Omega} \Bigl(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \Bigr)\, dx\, dy = \int\limits_{\partial\Omega} f\, dx + g\, dy.$$

6. Review material in §6.3, on the Stokes formula:

$$\int_M (\operatorname{curl} F) \cdot N \, dS = \int_{\partial M} (F \cdot T) \, ds.$$

Here $M \subset \mathbb{R}^3$ is a smoothly bounded 2D surface, assumed to have a smooth unit normal N, T is a unit tangent to the curve ∂M, F is a C^1 vector field on an open set $\mathcal{O} \supset M$, and $\operatorname{curl} F$ is a vector field, defined in Worksheet 25.

II. Having done these reviews, look back over Worksheets 10 and 19, reviewing previous course material.