WORKSHEETS for MATH 524
Differential Equations
Fall 2020

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Contents

Introduction

These worksheets serve to guide the student through the text for Math 524, *Introduction to Differential Equations*, by M. Taylor. They are designed so that each worksheet covers the material of one lecture. Each worksheet deals with material in a designated section of the text, and the idea is that a student can do the exercises in a worksheet in consultation with the text, and in that manner master the material in the text. There are also a handful of supplementary worksheets, to compensate for time lost due to the transition from in-class to remote instruction.

These worksheets have been produced in response to the health crisis of 2020. They are dated to correspond to a class meeting twice a week.

Worksheet 1, Tuesday, 08/11

Ch. 1, §§1–3, Single differential equations (review)

1. Suppose

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

is convergent for $|t| < R$. Give the power series for $f'(t)$, for $t \in (-R, R)$. Consult Appendix C of Chapter 1 (revised version) for a demonstration of this fact.

2. Give the definition of e^z, for $z \in \mathbb{C}$. Show that

$$\frac{d}{dt} e^{at} = a e^{at}, \quad \text{for} \ \ t \in \mathbb{R}, \ a \in \mathbb{C}.$$

3. Show that

$$\frac{d}{dt} e^{at} e^{-at} = 0,$$

and deduce that $e^{at} e^{-at} \equiv 1$.

4. Show that

$$\frac{d}{dt} e^{(a+b)t} e^{-at} e^{-bt} = 0,$$

and deduce that $e^{(a+b)t} = e^{at} e^{bt}$, for $t \in \mathbb{R}, \ a, b \in \mathbb{C}$.

5. Show that

$$\gamma(t) = e^{it}$$

is a unit speed curve on the unit circle in the complex plane, and relate this to the Euler identity

$$e^{it} = \cos t + i \sin t.$$

6. Consider the differential equation

$$\frac{dx}{dt} + a(t)x = b(t), \quad x(t_0) = x_0.$$

Use the identity

$$\frac{d}{dt}\Big(e^{A(t)} x\Big) = e^{A(t)} \Big(\frac{dx}{dt} + A'(t)x\Big)$$

to produce a solution.

7. Use separation of variables to solve

$$\frac{dx}{dt} = x^2, \quad x(0) = 1.$$

Worksheet 2, Thursday, 08/13

Ch. 1, §9, Second order linear equations (review)

1. Let a, b, c be constants, and consider the differential equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0.$$

Show that $x(t) = e^{rt}$ is a solution if and only if

$$ar^2 + br + c = 0.$$

This is called the characteristic equation.

2. Find the general solution to

$$\frac{d^2x}{dt^2} - x = 0.$$

3. Show that solutions to

$$\frac{d^2x}{dt^2} + x = 0$$

are linear combinations of e^{it} and e^{-it}. Use Euler's formula to show that such solutions are also linear combinations of $\cos t$ and $\sin t$.

4. Find the general solution to

$$x'' + x' + x = 0,$$

first in terms of complex exponentials, then, using Euler's formula, in real terms.

5. Find the general solution to

$$x'' + 2x' + x = 0.$$

Note that the characteristic equation here has a double root.

Preview of first order linear systems

6. Show that the 2×2 system

$$x_1' = ax_1 + bx_2,$$
$$x_2' = cx_1 + dx_2$$

is equivalent to the matrix ODE

$$\frac{dx}{dt} = Ax,$$

for

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

7. Consider the matrix ODE

$$\frac{dx}{dt} = Ax,$$

with $A \in M(n, \mathbb{C}),\ x(t) \in \mathbb{C}^n$. Show that this has a solution of the form

$$x(t) = e^{\lambda t} v, \quad \lambda \in \mathbb{C},\ v \in \mathbb{C}^n,\ v \neq 0,$$

if and only if

$$Av = \lambda v.$$

If this holds (and $v \neq 0$) we say λ is an eigenvalue of A and v is an eigenvector.

Thus a major problem is to analyze the eigenvalues and eigenvectors of a given matrix $A \in M(n, \mathbb{C})$.

Worksheet 3, Tuesday, 08/18

Ch. 2, §§1–3, Vector spaces and linear transformations

1. Define the concept of a vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Note that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. ($\mathbb{F}^n$ is a vector space over $\mathbb{F}$.)

2. Let $S = \{v_1, \dots, v_k\} \subset V$, a vector space. Define what it means to say

$$S \text{ spans } V,$$
$$S \text{ is linearly independent},$$
$$S \text{ is a basis of } V.$$

3. Study Lemma 3.1 and Proposition 3.2, whose content is:
If V has a basis $\{v_1, \dots, v_k\}$ and if $\{w_1, \dots, w_\ell\} \subset V$ is linearly independent, then $\ell \le k$.
Show that this leads to Corollary 3.3:
If V is finite dimensional, then any two bases of V have the same number of elements.
In such a case, $\dim V$ denotes the number of elements in a basis of V.

4. State Propositions 3.4 and 3.5.

5. State Proposition 3.6, the Fundamental Theorem of Linear Algebra, and show how it follows from Propositions 3.4 and 3.5.

6. Deduce from the Fundamental Theorem of Linear Algebra that if V is finite dimensional and $A : V \to V$ is linear, then

$$A \text{ injective } \Leftrightarrow A \text{ surjective } \Leftrightarrow A \text{ isomorphism}.$$

7. State Proposition 3.9, characterizing when a matrix $A \in M(n, \mathbb{F})$ is invertible in terms of the behavior of its columns.

Worksheet 4, Thursday, 08/20

Ch. 2, §5, Determinants and invertibility

1. Given

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{we set} \quad \det A = ad - bc.$$

Show that $A : \mathbb{F}^2 \to \mathbb{F}^2$ is invertible if and only if $\det A \neq 0$. If this holds,

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

2. Consult Proposition 5.1 of Chapter 2 and define $\det A$ for $A \in M(n, \mathbb{F})$.

3. Show that the formula (5.30) for $\det A$ implies

$$\det A = \det A^t.$$

4. Read the proof of Proposition 5.3, that if $A, B \in M(n, \mathbb{F})$,

$$\det(AB) = (\det A)(\det B).$$

Show that this implies Corollary 5.4, i.e.,

$$A \text{ invertible} \implies \det A \neq 0.$$

5. Read the proof of Proposition 5.6, that, if $A \in M(n, \mathbb{F})$,

$$A \text{ invertible} \iff \det A \neq 0.$$

See how this completes the result of Exercise 4.

6. Study Exercises 1–3 at the end of §5, treating the expansion of $\det A$ by minors down the kth column, given $A \in M(n, \mathbb{F})$.

7. Use an expansion by minors to evaluate $\det A$ for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

Supplementary worksheet
§5, more on determinants

1. Verify the following method of computing 3×3 determinants. Given

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

form a 3×5 rectangular matrix by copying the first two columns of A to the right. The products in (5.16) with plus signs are the products of each of the three downward sloping diagonals marked in bold below:

$$\begin{pmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} & a_{11} & a_{12} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{21}} & a_{22} \\ a_{31} & a_{32} & \mathbf{a_{33}} & \mathbf{a_{31}} & \mathbf{a_{32}} \end{pmatrix}.$$

The products in (5.16) with minus signs are the products of each of the three upward sloping diagonals marked in bold below:

$$\begin{pmatrix} a_{11} & a_{12} & \mathbf{a_{13}} & \mathbf{a_{11}} & \mathbf{a_{12}} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} & \mathbf{a_{21}} & a_{22} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} & a_{31} & a_{32} \end{pmatrix}.$$

2. Use the method described above to compute the determinants of

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}.$$

3. Given $A = (0\ 1\ 2)$, compute $\det A^t A$ and $\det AA^t$.

4. Compute the determinant of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

5. Consult the exercises at the end of §5 and write down a definition of

$$\det T, \quad \text{for} \quad T : V \to V,$$

when V is a finite dimensional vector space over $\mathbb{F}$.

Worksheet 5, Thursday, 08/27

Ch. 2, §6, Eigenvalues and eigenvectors

1. Let V be a finite dimensional vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), and suppose $T : V \to V$ is linear. For $\lambda \in \mathbb{F}$, define the eigenspace

$$\mathcal{E}(T,\lambda), \quad \text{and show that} \quad T : \mathcal{E}(T,\lambda) \to \mathcal{E}(T,\lambda).$$

2. We say λ is an eigenvalue of T ($\lambda \in \operatorname{Spec}(T)$) provided $\mathcal{E}(T,\lambda) \neq 0$. Then a nonzero element of $\mathcal{E}(T,\lambda)$ is called a λ-eigenvector of T. Show that, for $\lambda \in \mathbb{F}$,

$$\lambda \in \operatorname{Spec}(T) \Longleftrightarrow \det(\lambda I - T) = 0.$$

3. State the Fundamental Theorem of Algebra, and show how it is used in the proof of Proposition 6.1, which says that, if $\mathbb{F} = \mathbb{C}$, then each linear $T : V \to V$ has at least one eigenvector in V.

4. Proposition 6.2 says that if $\{v_1, \dots . v_k\}$ are eigenvectors of $T \in \mathcal{L}(V)$, $Tv_j = \lambda_j v_j$, and if $\{\lambda_1, \dots, \lambda_k\}$ are all distinct, then these eigenvectors are linearly independent.
Read its proof.

5. Find the eigenvalues and eigenvectors of the 2×2 matrices listed in Exercise 1 of §6.

6. Find the eigenvalues and eigenvectors of the 3×3 matrices listed in Exercise 2 of §6.

Worksheet 6, Tuesday, 09/01

Ch. 2, §§9–10, Vector and matrix norms

1. On $\mathbb{R}^n$ we have the dot product and norm, given by

$$v \cdot w = v_1 w_1 + \cdots + v_n w_n, \quad \|v\|^2 = v \cdot v = v_1^2 + \cdots + v_n^2,$$

and on $\mathbb{C}^n$ we have the inner product and norm, given by

$$(v, w) = v \cdot \overline{w} = v_1 \overline{w}_1 + \cdots + v_n \overline{w}_n, \quad \|v\|^2 = (v, v) = |v_1|^2 + \cdots + |v_n|^2.$$

Look at the definition of inner product spaces given in (9.5)–(9.8) of Chapter 2, and check that $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner product spaces (over $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, respectively).

2. Consider the definition of normed vector space, given in (9.12)–(9.14). Show that, if V is an inner product space, the definition

$$\|v\| = \sqrt{(v, v)}$$

makes it a normed space, via the triangle inequality

$$\|v + w\| \le \|v\| + \|w\|,$$

cf. (9.14). See the verification of the triangle inequality in (9.15)–(9.16), using the Cauchy inequality,

$$|(v, w)| \le \|v\| \, \|w\|,$$

established in Proposition 9.1.

3. If V is an inner product space, a set of vectors $\{u_j\} \subset V$ is said to be an orthonormal set provided

$$(u_j, u_k) = \delta_{jk},$$

where $\delta_{jk} = 1$ if $j = k$, 0 otherwise. The Gramm-Schmidt construction takes an arbitrary basis $\{v_1, \dots, v_n\}$ of V and produces an orthonormal basis, $\{u_1, \dots, u_n\}$. Read about this construction, in (9.25)–(9.31).

4. Let V be an n-dimensional inner product space and let $A : V \to V$ be linear ($A \in \mathcal{L}(V)$). We define the operator norm of A,

$$\|A\| = \max \{\|Av\| : v \in V, \|v\| \le 1\}.$$

Equivalently, $\|A\|$ is the smallest constant K for which the inequality

$$\|Av\| \le K\|v\|, \quad \forall\, v \in V$$

holds. Show that $\|A\|$ satisfies the triangle inequality:

$$\|A + B\| \le \|A\| + \|B\|, \quad \forall\, A, B \in \mathcal{L}(V).$$

5. In the setting of Exercise 4, show that

$$\|AB\| \le \|A\| \, \|B\|.$$

Then show that, for each $k \in \mathbb{N}$,

$$\|A^k\| \le \|A\|^k.$$

6. Read about the relation between the operator norm of $A \in \mathcal{L}(V)$ and its Hilbert-Schmidt norm, $\|A\|_{HS}$, discussed in (10.14)–(10.21).

Worksheet 7, Thursday, 09/03

Ch. 3, §1, The matrix exponential

1. Review Appendix C of Chapter 1, particularly Proposition C.4, to the effect that if $F(t)$ is given by a convergent power series,

$$F(t) = \sum_{k=0}^{\infty} A_k t^k, \quad |t| < R,$$

then F is differentiable on $(-R, R)$, and

$$F'(t) = \sum_{k=1}^{\infty} k A_k t^{k-1}, \quad \text{for } \ t \in (-R, R).$$

There the result was stated for $A_k \in \mathbb{C}$. See how it continues to work for $A_k \in M(n, \mathbb{C})$.

2. Use the estimates from Worksheet 6 to show that, given $A \in M(n, \mathbb{C})$, the function e^{tA}, defined by the series

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k,$$

is convergent for each $A \in M(n, \mathbb{C}),\ t \in \mathbb{R}$.

3. Deduce from Exercises 1 and 2 above that

$$\frac{d}{dt} e^{tA} = A e^{tA}, \quad \forall\, t \in \mathbb{R},\ A \in M(n, \mathbb{C}),$$

and also

$$\frac{d}{dt} e^{tA} = e^{tA} A.$$

Note that this generalizes the result of Exercise 2 in Worksheet 1.

4. Show that

$$\frac{d}{dt} e^{tA} e^{-tA} \equiv 0,$$

and deduce that $e^{tA} e^{-tA} \equiv I$, hence

$$e^{-tA} = \left(e^{tA}\right)^{-1}, \quad \forall\, t \in \mathbb{R},\ A \in M(n, \mathbb{C}).$$

5. Show that

$$\frac{d}{dt} e^{(s+t)A} e^{-tA} = 0,$$

hence $e^{(s+t)A}e^{-tA} \equiv e^{sA}$, and therefore

$$e^{(s+t)A} = e^{sA}e^{tA}, \quad \forall\, s,t \in \mathbb{R},\ A \in M(n,\mathbb{C}).$$

6. Read the proof of Proposition 1.2 in Chapter 3, §1, which establishes that, for $A, B \in M(n,\mathbb{C})$,

$$AB = BA \Longrightarrow e^{t(A+B)} = e^{tA}e^{tB}, \quad \forall\, t \in \mathbb{R}.$$

7. Show that if $A \in M(n,\mathbb{C}),\ \lambda \in \mathbb{C}$,

$$v \in \mathcal{E}(A,\lambda) \Longrightarrow e^{tA}v = e^{t\lambda}v.$$

Compare the result previewed in Exercise 7 of Worksheet 2.

8. Compute e^{tA} for the matrices A given in Exercise 1 of Ch. 3, §1.

Worksheet 8, Tuesday, 09/08

Ch. 3, §§1–3, Matrix exponentials, II

How to compute $e^{tA}v$.

1. Find a basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$, consisting of generalized eigenvectors of A. Here, if $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in M(n, \mathbb{C})$, the generalized eigenspace is

$$\mathcal{GE}(A, \lambda) = \{u \in \mathbb{C}^n : (A - \lambda)^k u = 0, \text{ for some } k \in \mathbb{N}\}.$$

Clearly $\mathcal{GE}(A, \lambda) \supset \mathcal{E}(A, \lambda)$. Nonzero elements of $\mathcal{GE}(A, \lambda)$ are called generalized eigenvectors of A. See Chapter 2, §7 for material on this subject, including a proof that $\mathbb{C}^n$ always has a basis of generalized eigenvectors of any $A \in M(n, \mathbb{C})$.

2. Find $c_1, \dots, c_n \in \mathbb{C}$ such that $v = c_1 v_1 + \cdots + c_n v_n$. Then

$$e^{tA} v = c_1 e^{tA} v_1 + \cdots + c_n e^{tA} v_n.$$

3. Here is how to compute $e^{tA} v_j$.

A. If v_j is an eigenvector, so $A v_j = \lambda_j v_j$, then

$$e^{tA} v_j = e^{t\lambda_j} v_j.$$

B. If v_j is a generalized eigenvector, satisfying $(A - \lambda_j I)^\ell v_j = 0$, then

$$e^{tA} v_j = e^{t\lambda_j} \sum_{k=0}^{\ell-1} \frac{t^k}{k!} (A - \lambda_j I)^k v_j.$$

How to compute the $n \times n$ matrix e^{tA}.

The jth column of e^{tA} is $e^{tA} e_j$, where e_j is the jth standard basis vector of $\mathbb{C}^n$. For example, if $n = 3$,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

4. Check out the treatment of e^{tA} for A given by (1.51) in Ch. 3, §1. Then compute e^{tA} for the matrices A given in Exercise 2, at the end of Ch. 3, §1.

5. Consider the treatment given in Ch. 3, §2, of

$$e^{tJ}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Relate the identity

$$e^{tJ} = (\cos t)I + (\sin t)J$$

to Euler's formula.

6. Consider the treatment of

$$\sin tA, \quad \cos tA, \quad A \in M(n, \mathbb{C}),$$

in Exercises 2–5 of Ch. 3, §2. In particular, show that

$$\frac{d}{dt} \sin tA = A \cos tA, \quad \frac{d}{dt} \cos tA = -A \sin tA.$$

7. Verify the formula

$$B = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix} \Longrightarrow e^{tB} = \begin{pmatrix} \cos tA & -\sin tA \\ \sin tA & \cos tA \end{pmatrix},$$

given $A \in M(n, \mathbb{C})$, discussed in Exercise 6 of Ch. 3, §2.
Hint. Denote the right side by $X(t)$. Compute $X'(t)$, using Exercise 6 above, and show that

$$\frac{dX}{dt} = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix} X(t), \quad X(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Worksheet 9, Thursday, 09/10

Ch. 3, §4, Nonhomogeneous equations and Duhamel's formula

1. The goal here is to treat the non-homogeneous equation

$$\frac{dx}{dt} - Ax = f(t), \quad x(0) = x_0 \in \mathbb{C}^n, \tag{1}$$

given $A \in M(n, \mathbb{C})$, $f(t) \in \mathbb{C}^n$, continuous in t. To this end, set

$$y(t) = e^{-tA} x(t),$$

and show that (1) yields

$$\frac{dy}{dt} = e^{-tA} f(t), \quad y(0) = x_0.$$

Deduce that

$$y(t) = x_0 + \int_0^t e^{-sA} f(s)\, ds,$$

hence

$$\begin{aligned} x(t) &= e^{tA} x_0 + e^{tA} \int_0^t e^{-sA} f(s)\, ds \\ &= e^{tA} x_0 + \int_0^t e^{(t-s)A} f(s)\, ds. \end{aligned} \tag{2}$$

This is called Duhamel's formula for the solution to (1).

2. Study (4.6)–(4.10) in Ch. 3 of the text, regarding how the solution to the second order equation

$$y'' + y = f(t), \quad y(0) = y_0, \ y'(0) = y_1$$

is given by

$$y(t) = (\cos t) y_0 + (\sin t) y_1 + \int_0^t \sin(t-s) f(s)\, ds.$$

3. Consider the inhomogeneous systems in Exercises 3–5 at the end of Ch. 3, §4.

4. Read Appendix B of Ch. 3, treating the matrix Laplace transform and connecting it to Duhamel's formula.

Worksheet 10, Tuesday, 09/15

Review for test, Th 09/17

These review topics are tuned to Worksheets 1–9.

1. Review material in Ch. 1, §§1–3 on
   exponential function e^{at},
   Euler identity, $e^{it} = \cos t + i \sin t$.
   first order linear differential equations $x' + a(t)x = b(t)$,
   separation of variables.

2. Review material in Ch. 1, §9 on second order equations

$$ax'' + ax' + cx = 0,$$

the associated characteristic equation $ar^2 + br + c = 0$, and the solutions arising from the roots of the characteristic equation.

2A. Review the matrix formulation of a first order system of differential equations, described in Exercises 6–7 of Worksheet 2.

3. Review material in Ch. 2, §§1–3 on vector spaces and linear transformations. Make note of the following concepts.
   spanning, linear independence, basis, dimension,
   range, null space,
   Fundamental theorem of linear algebra,
   injective, surjective, invertible linear transformations.

4. Review material on the determinant of a linear transformation, in Ch. 2, §5. See Worksheet 4 and the supplementary worksheet that follows it. Particularly study the result that, if $A \in M(n, \mathbb{F})$, then

$$A \text{ is invertible } \Longleftrightarrow \det A \neq 0.$$

5. Review the material on eigenvalues and eigenvectors, in Ch. 2, §6. Make note of the following concepts.
   eigenspace $\mathcal{E}(T, \lambda)$,
   characteristic polynomial $\det(\lambda I - T)$.

Particularly note the result that, if $T \in \mathcal{L}(V)$,

$$\lambda \text{ is an eigenvalue of } T \Longleftrightarrow \det(\lambda I - T) = 0.$$

6. Review material in §§9–10 of Ch. 2, on vector and matrix norms. Make note of the following concepts:
  inner product spaces, and the norm on such a space,
  triangle inequality, deduced from Cauchy's inequality,
  orthonormal set, and Gramm-Schmidt construction.

6A. Define the concept of the operator norm of a linear transformation in $\mathcal{L}(V)$, when V is an inner product space. Review the demonstrations that, if $A, B \in \mathcal{L}(V)$,

$$\|A+B\| \le \|A\| + \|B\|, \quad \|AB\| \le \|A\| \, \|B\|.$$

7. Review material in Ch. 3, §1, on the matrix exponential. Given $A \in M(n, \mathbb{C})$, define e^{tA}, as a power series, and review the demonstration that

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A.$$

Review the demonstration that $e^{(s+t)A} = e^{sA} e^{tA}$, and, going further, for $A, B \in M(n, \mathbb{C})$,

$$e^{t(A+B)} = e^{tA} e^{tB} \quad \forall \, t \in \mathbb{R} \Longleftrightarrow AB = BA.$$

8. Continue the review of material in §§1–3 in Ch. 3. Review how to compute $e^{tA} v$ when

$$v \in \mathcal{GE}(A, \lambda), \text{ i.e., } (A - \lambda I)^k v = 0, \quad \text{for some} \quad k \in \mathbb{N}.$$

9. Review material in Ch. 3, §4, on nonhomogeneous equations, of the form

$$\frac{dx}{dt} - Ax = f(t), \quad x(0) = x_0 \in \mathbb{C}^n,$$

given $A \in M(n, \mathbb{C})$, $f(t) \in \mathbb{C}^n$. See how the substitution $y(t) = e^{-tA} x(t)$ leads to Duhamel's formula for the solution to this equation.

10. **Computations.** Worksheets 1–9 point to various computational exercises in the text. Review these.

Worksheet 11, Tuesday, 09/22

Ch. 3, §8, Variable coefficient linear systems

Section 8 of Chapter 3 deals with the system

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0 \in \mathbb{C}^n, \tag{1}$$

given smooth $A : (a, b) \to M(n, \mathbb{C})$. A key ingredient is a function $M : (a, b) \to M(n, \mathbb{C})$ (for whose existence see Ch. 3, §10 and Ch. 4, §1), solving

$$\frac{dM}{dt} = A(t)M(t). \tag{2}$$

1. Study (8.7)–(8.10), regarding the Wronskian

$$W(t) = \det M(t), \quad \text{solving} \quad \frac{dW}{dt} = \bigl(\operatorname{Tr} A(t)\bigr)W(t). \tag{3}$$

See how the analysis of this Wronskian equation yields that

$$M(t_0) \text{ invertible} \Rightarrow M(t) \text{ invertible}, \ \forall\, t \in (a, b). \tag{4}$$

Consider Exercises 1–3, at the end of Ch. 3, §8, bearing on the derivation of (3).

2. Proposition 8.1 says that, if $M(t)$ is as above, and invertible, then the equation (1) is uniquely solved by

$$x(t) = M(t)M(t_0)^{-1}x_0.$$

Show how this follows from the change of variable

$$x(t) = M(t)y(t), \tag{5}$$

and consideration of dy/dt.

3. The variable coefficient version of Duhamel's formula, for the solution to

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(t_0) = x_0, \tag{6}$$

is given by (8.20):

$$x(t) = M(t)M(t_0)^{-1}x_0 + M(t)\int_{t_0}^{t} M(s)^{-1}f(s)\,ds. \tag{7}$$

Show how this also follows from the change of variable (5), introduced in Exercise 2 above.

4. Look at the exercises 9–10 at the end of Ch. 3, §8, dealing with (1) when $A(t)$ is periodic, satisfying $A(t+1) = A(t)$. Learn about the Floquet representation.

Worksheet 12, Thursday, 09/24

Ch. 3, §10, Power series expansions

This section deals with constructing solutions to

$$\frac{dx}{dt} = A(t)x + f(t), \quad x(0) = x_0, \tag{1}$$

where $A(t)$ and $f(t)$ are given by convergent power series

$$A(t) = \sum_{k=0}^{\infty} A_k t^k, \quad f(t) = \sum_{k=0}^{\infty} f_k t^k, \quad |t| < R_0, \tag{2}$$

with $A_k \in M(n, \mathbb{C}),\ f_k \in \mathbb{C}^n$. One seeks the solution $x(t)$ as a convergent power series:

$$x(t) = \sum_{k=0}^{\infty} x_k t^k, \quad x_k \in \mathbb{C}^n. \tag{3}$$

1. Follow the derivation in (10.4)–(10.6) of the iterative formula

$$(k+1)x_{k+1} = \sum_{j=0}^{k} A_{k-j} x_j + f_k, \tag{4}$$

for the coefficients x_k in (3).

2. Proposition 10.1 says that, under the hypothesis (2), the construction (4) leads to a power series (3) that converges for $|t| < R_0$ (same R_0 as in (2)). Study its proof.

3. Study (10.19)–(10.25), extending the scope of (1) to $x(t_0) = x_0$, and involving power series expansions about t_0.

4. Find the definition of a real analytic function on an interval $I = (a, b)$ in (10.30). Proposition 10.4 says that if $A(t)$ and $f(t)$ are real analytic on an interval (a, b), $t_0 \in (a, b)$, then the initial value problem (1) (with initial condition $x(t_0) = x_0$) has a unique solution $x(t)$, analytic on (a, b). Study its proof.

Note the role of Proposition 10.2, which implies that a function given by a convergent power series in $t - t_0$ on the interval $|t - t_0| < R$ is real analytic on this interval.

5. Check out Exercises 2–3, at the end of Section 10, dealing with the Airy equation,

$$y'' = ty, \quad y(0) = y_0,\ y'(0) = y_1,$$

which one converts to a 2×2 system.

Supplementary worksheet

Complex analytic functions

As stated in (10.34) of the text, if $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$, then f is said to be complex differentiable at $z_0 \in \Omega$, with derivative $f'(z_0)$, provided

$$\lim_{w\to 0} \frac{f(z_0+w)-f(z_0)}{w} = f'(z_0). \tag{1}$$

One also denotes the limit by df/dz. Other terms used for such functions are "complex analytic" and "holomorphic." Here we sketch some results that lead to lots of examples of holomorphic functions.

First, clearly $f_1(z) = z$ is holomorphic, with $f_1'(z) = 1$. To go from here, we have the following:

$$f, g : \Omega \to \text{holomorphic } \mathbb{C} \Longrightarrow fg \ \ \text{holomorphic.}, \tag{2}$$

where $fg(z) = f(z)g(z)$. In fact, one can write

$$\begin{aligned} &\frac{f(z_0+w)g(z_0+w)-f(z_0)g(z_0)}{w} \\ &= \frac{f(z_0+w)-f(z_0)}{w}\, g(z_0+w) + f(z_0)\, \frac{g(z_0+w)-g(z_0)}{w}, \end{aligned} \tag{3}$$

and take $w \to 0$ to deduce that

$$\frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z). \tag{4}$$

We can apply this to $f_2(z) = z^2 = z \cdot z$ to get $f_2'(z) = 2z$, and, inductively,

$$\frac{d}{dz} z^n = n z^{n-1}, \quad z \in \mathbb{C},\ n \in \mathbb{N}. \tag{5}$$

Next, we claim that $1/z$ is holomorphic on $\mathbb{C} \setminus 0$. Indeed, for $z_0 \neq 0$, $|w| < |z_0|$,

$$\frac{1}{w}\Bigl(\frac{1}{z_0+w} - \frac{1}{z_0}\Bigr) = -\frac{1}{z_0(z_0+w)}, \tag{6}$$

and taking $w \to 0$ yields

$$\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}, \quad z \in \mathbb{C} \setminus 0. \tag{7}$$

Again an inductive application of (4) yields that $1/z^n$ is holomorphic on $\mathbb{C} \setminus 0$, and

$$\frac{d}{dz}\frac{1}{z^n} = -\frac{n}{z^{n+1}}, \quad z \in \mathbb{C} \setminus 0,\ n \in \mathbb{N}. \tag{8}$$

We turn to the exponential function $\mathrm{Exp}(z) = e^z$, introduced in Chapter 1. We claim that this is holomorphic in $\mathbb{C}$, and

$$\frac{d}{dz} e^z = e^z, \quad z \in \mathbb{C}, \tag{9}$$

extending from $\mathbb{R}$ to $\mathbb{C}$ the formula for the derivative established there. To see this, use the identity $e^{z_0+w} = e^{z_0} e^w$, established in Chapter 1, to write

$$\begin{aligned} \frac{1}{w}(e^{z_0+w} - e^{z_0}) &= e^{z_0} \, \frac{e^w - 1}{w} \\ &= e^{z_0} \sum_{k=1}^{\infty} \frac{1}{k!} w^{k-1}, \end{aligned} \tag{10}$$

and note that the last sum is equal to

$$\sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)!} w^{\ell} = 1 + \frac{w}{2} + \cdots \longrightarrow 1, \quad \text{as} \quad w \to 0, \tag{11}$$

yielding (9).

We can get lots more holomorphic functions by combining the examples above with the following general result, known as the *chain rule* for holomorphic functions.

Proposition. Assume $\Omega, \mathcal{O} \subset \mathbb{C}$ are open and

$$f : \Omega \longrightarrow \mathcal{O}, \quad g : \mathcal{O} \longrightarrow \mathbb{C} \tag{12}$$

are holomorphic. Then $f = g \circ f : \Omega \to \mathbb{C}$, defined by

$$h(z) = g(f(z)), \tag{13}$$

is holomorphic, and

$$\frac{d}{dz} g \circ f(z) = g'(f(z)) f'(z), \quad z \in \Omega. \tag{14}$$

Proof. We can write the definition (1) as

$$f(z_0) = f(z_0) + f'(z_0) w + r(z_0, w), \quad z_0 \in \Omega, \tag{15}$$

where $r(z_0, w)/w \to 0$ as $w \to 0$ (we say $r(z_0, w) = o(w)$). Similarly for g. Then, for $z_0 \in \Omega$,

$$\begin{aligned} h(z_0 + w) &= g(f(z_0 + w)) \\ &= g(f(z_0) + f'(z_0) w + r) \\ &= g(f(z_0)) + g'(f(z_0))(f'(z_0) w + r) + r_1(z_0, w), \end{aligned} \tag{16}$$

with also $r_1(z_0, w) = o(w)$. Hence

$$h(z_0 + w) = h(z_0) + g'(f(z_0)) f'(z_0) + r_2(z_0, w), \tag{17}$$

with $r_2(z_0, w) = o(w)$, and we have (14).

Putting together these results yields such holomorphic functions as

$$\frac{1}{z^2+1}, \quad e^{z/(z^2+1)}, \quad z \neq \pm i, \tag{18}$$

and a host of others, which the reader can play around with.

Worksheet 13, Tuesday, 09/29

Ch. 4, §1, Nonlinear systems – Picard iteration

This section deals with local existence (and uniqueness) of solutions to an $n \times n$ system of ODEs, of the form

$$\frac{dx}{dt} = F(t,x), \quad x(t_0) = x_0, \tag{1}$$

given $x_0 \in \Omega$, open in $\mathbb{R}^n$, $t_0 \in (\alpha, \beta) \subset [\alpha, \beta] = I$. We make the following hypotheses:

$$F : I \times \Omega \to \mathbb{R}^n \text{ is continuous, } x_0 \in \Omega,\ t_0 \in I, \tag{2}$$

$$\|F(t,x)\| \le M, \quad \forall\, (t,x) \in I \times \Omega, \tag{3}$$

$$\|F(t,x) - F(t,y)\| \le L\|x-y\|, \quad \forall\, t \in I,\ x, y \in \Omega. \tag{4}$$

Here $M, L \in (0, \infty)$ are constants. Proposition 1.1 asserts that there exists $T_0 > 0$ and a unique C^1 solution to (1) for $|t - t_0| < T_0$.

1. Use the fundamental theorem of calculus to show that (1) is equivalent to the "integral equation"

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\, ds. \tag{5}$$

2. Study the Picard iteration method, defined in (1.5)–(1.9), giving an infinite sequence of approximate solutions $x_n(t)$ to (5). We start with $x_0(t) \equiv x_0$, and define the sequence iteratively by

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} F(x, x_n(s))\, ds. \tag{6}$$

We complement hypothesis (2) with the hypothesis

$$x_0 \in \overline{B_R(x_0)} \subset \Omega, \quad R > 0. \tag{2A}$$

Follow the argument involving (1.8) that, if $[t_0 - T_0, t_0 + T_0] \subset I$ and

$$x_n : [t_0 - T_0, t_0 + T_0] \longrightarrow \overline{B_R(x_0)},$$

then x_{n+1} also has this property, provided (2), (2A), and (3) hold, and T_0 satisfies

$$T_0 \le \frac{R}{M}. \tag{7}$$

Hence the sequence (x_n) is well defined for all n, if (7) holds.

3. Study (1.10)–(1.17) to see that, in the setting described above, if also

$$T_0 \le \frac{1}{2L}, \tag{8}$$

with L as in (4), then

$$\max_{|t-t_0|\le T_0} \|x_{n+1}(t) - x_n(t)\| \le 2^{-n} R, \tag{9}$$

and hence we have convergence,

$$x_n \longrightarrow x, \quad \text{uniformly on} \ \ |t - t_0| \le T_0, \tag{10}$$

to a limit $x \in C([t_0 - T_0, t_0 + T_0])$, satisfying (5). Note that the right side of (5) is then of class C^1.

4. To finish the proof of Proposition 1.1, study the uniqueness argument, involving (1.18).

5. Check out Exercises 1–4, at the end of §1 of Ch. 4.

6. Use separation of variables to solve

$$\frac{dx}{dt} = x^2, \quad x(0) = 1.$$

Verify that $x(t) \to +\infty$ as $t \nearrow 1$. This illustrates a failure of global existence.

Worksheet 14, Thursday, 10/01

Ch. 4, §1, Nonlinear systems II – global solvability

We continue to study the equation

$$\frac{dx}{dt} = F(t,x), \quad x(t_0) = x_0. \tag{1}$$

We relax the hypotheses (3)–(4) of Worksheet 13 to the following: for each compact (i.e., closed and bounded) $K \subset \Omega$, there exist L_K, $M_K < \infty$ such that

$$\begin{aligned} &\|F(t,x)\| \le M_K, \quad \forall\, t \in I,\ x \in K, \\ &\|F(t,x) - F(t,y)\| \le L_K \|x - y\|, \quad \forall\, t \in I,\ x, y \in K. \end{aligned} \tag{2}$$

1. Follow the arguments given in (1.19)–(1.21) to see the following:
Proposition. For each compact $K \subset \Omega$, there exist $R_K > 0$ and compact $\widetilde{K} \subset \Omega$, such that

$$\overline{B_{R_K}(x)} \subset \widetilde{K}, \quad \forall\, x \in K. \tag{3}$$

Then, for each $x_0 \in K$, the solution to (1) exists on the interval

$$\Big\{ t \in I : |t - t_0| \le \min\Big(\frac{R_K}{M_{\widetilde{K}}}, \frac{1}{2L_{\widetilde{K}}} \Big) \Big\}, \tag{4}$$

where $M_{\widetilde{K}}$ and $L_{\widetilde{K}}$ are as in (2), but with K replaced by $\widetilde{K}$.

2. Proposition 1.2 says the following. In the setting above, assume $[a,b] \subset (\alpha,\beta) \subset I$, and assume $x(t)$ solves (1) for $t \in (a,b)$. Assume further that there exists a compact $K \subset \Omega$ such that

$$x(t) \in K, \quad \forall\, t \in (a,b).$$

Then there exist $a_1 < a$ and $b_1 > b$ such that

$$x(t) \text{ solves (1) for } t \in (a_1, b_1).$$

Study the proof of Proposition 1.2, and see how it follows from the Proposition stated in Exercise 1 above, involving (4).

3. Follow the arguments (1.25)–(1.26), showing that the system

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = -y^3$$

has a global solution, for all $t \in \mathbb{R}$, for each initial condition $y(0) = y_0,\ v(0) = v_0$. See how Proposition 1.2 plays a role in demonstrating this.

4. Follow the arguments in (1.27)–(1.32), concerning global solvability of a linear system

$$\frac{dx}{dt} = A(t)x, \quad x(0) = x_0,$$

given continuous $A : I \to M(n, \mathbb{C})$. Note the use of Proposition 1.3, which says that

$$\|A(t)\| \le K \ \forall\, t \in I \Longrightarrow \|x(t)\| \le e^{K|t|} \|x_0\|.$$

5. Check out Exercise 6 at the end of Ch. 4, §1, dealing with the system

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = -y^3 - v,$$

asserted to be globally solvable for $t \ge 0$.

6. Check out Gronwall's inequality, Proposition 1.5, treated after Exercise 10, and its application to global existence results in Exercises 11–13.

Worksheet 15, Tuesday, 10/06

Ch. 4, §2, Dependence of solutions on initial data

In this section, we consider

$$\frac{dx}{dt} = F(x), \quad x(0) = y, \tag{1}$$

with solution denoted $x(t, y)$, and study how $x(t, y)$ depends on y.

1. In conjunction with material from Supplementary Worksheet C, identify the following objects and see how they fit into the program to analyze $x(t, y)$.

$$\begin{aligned}
W(t, x) &= D_y x(t, y), \\
\frac{dW}{dt} &= DF(x)W, \quad W(0, y) = I, \\
w(t, y) &= W(t, y)w_0, \\
\frac{dw}{dt} &= DF(x)w, \quad w(0) = w_0, \\
x(t) &= x(t, y), \quad x_1(t) = x(t, y + w_0), \quad z(t) = x_1(t) - x(t), \\
\frac{dz}{dt} &= F(x_1) - F(x), \quad z(0) = w_0, \\
\frac{dz}{dt} &= G(x_1, x)z, \quad z(0) = w_0, \\
G(x_1, x) &= \int_0^1 DF(\tau x_1 + (1 - \tau)x)\, d\tau, \\
\|DF(u)\| \le L,\ \forall\, u \in \Omega &\Rightarrow \|z(t)\| \le e^{|t|L}\|w_0\|, \\
\|x(t, y) - x(t, y + w_0)\| &\le e^{|t|L}\|w_0\|.
\end{aligned}$$

2. The last item of #1 implies that $x(t, y)$ is Lipschitz continuous in y. Now parse the following steps.

$$\begin{aligned}
G(x, x) &= DF(x), \\
\frac{dz}{dt} &= G(x + z, x)z = DF(x)z + R(x, z), \quad z(0) = w_0, \\
\|R(x, z)\| &= o(\|z\|) = o(\|w_0\|), \\
\frac{d}{dt}(z - w) &= DF(x)(z - w) + R(x, z), \quad (z - w)(0) = 0, \\
z(t) - w(t) &= \int_0^t S(t, s)R(x(s), z(s))\, ds, \\
\|S(t, s)\| &\le e^{|t-s|L}, \\
\|z(t) - w(t)\| &= o(\|w_0\|) \\
x(t, y + w_0) &= x(t, y) + z(t) = x(t, y) + w(t) + o(\|w_0\|).
\end{aligned}$$

3. See how these steps lead to
Proposition 2.1 If $F \in C^1(\Omega)$ and if solutions to (1) exist for $|t| < T_0$, then, for each such t, $x(t, y)$ is C^1 in y, and

$$D_y x(t, y) = W(t, y)$$

satisfies the "linearized equation"

$$\frac{dW}{dt} = DF(x)W, \quad W(0, y) = I.$$

Worksheet 16, Thursday, 10/08

Ch. 4, §2, Dependence of solutions on paramaters

1. If F is smooth of class C^2, we can couple the ODEs,

$$\begin{aligned} \frac{dx}{dt} &= F(x), \\ \frac{dW}{dt} &= DF(x)W, \end{aligned} \tag{1}$$

with initial data

$$x(0) = y, \quad W(0) = I,$$

and Proposition 2.1 applies to this system. Deduce from this that

$$F \in C^2(\Omega) \Longrightarrow x(t,y) \text{ is } C^2 \text{ in } y.$$

Proceed to check out Proposition 2.2.

2. Next, consider the family of ODEs,

$$\frac{dx}{dt} = F(\tau, x), \quad x(0) = y. \tag{2}$$

Assume F is C^k in (τ, x). Denote the solution by $x(t, \tau, y)$. Compare this with the system for (x, z):

$$\begin{aligned} \frac{dx}{dt} &= F(z, x), \\ \frac{dz}{dt} &= 0, \end{aligned} \tag{3}$$

with initial data

$$x(0) = y, \quad z(0) = \tau. \tag{4}$$

Deduce smoothness of $x(t, \tau, y)$ in (τ, y) from Propositions 2.1–2.2.

3. Consider the following special case of $F(\tau, x)$ in (2),

$$F(\tau, x) = \tau F(x). \tag{5}$$

Show that

$$x(t_0, \tau, y) = x(t_0\tau, y), \tag{6}$$

where $x(t, y) = x(t, 1, y)$ is the solution to

$$\frac{dx}{dt} = F(x), \quad x(0) = y. \tag{7}$$

Deduce from smoothness results for $x(t_0, \tau, y)$ in (τ, y) from #2 that

$$F \in C^k(\Omega) \Longrightarrow x(t,y) \text{ is } C^k \text{ jointly in } (t,y),$$
$$\text{and so is } \frac{dx}{dt}(t,y).$$

4. Suppose $\tau \in \mathbb{R}$ in #2. Show that $\xi = \partial x/\partial \tau$ satisfies the ODE

$$\frac{d\xi}{dt} = D_x F(\tau, x)\xi + \frac{\partial F}{\partial \tau}(\tau, x), \quad \xi(0) = 0. \tag{8}$$

Hint. Use the formulation (3)–(4) and apply Proposition 2.1.
Alternative. Granted sufficient smoothness of $x(t, \tau, y)$ in τ and t, apply $\partial/\partial\tau$ to (2). Use the chain rule.

5. Let $x = x_\tau(t),\ y = y_\tau(t)$ solve

$$\begin{aligned} \frac{dx}{dt} &= -y + \tau(x^2 + y^2), \\ \frac{dy}{dt} &= x, \end{aligned}$$

with initial data $x(0) = 1,\ y(0) = 0$. Find differential equations for the coefficients $X_j(t), Y_j(t)$ in power series expansions

$$\begin{aligned} x_\tau(t) &= X_0(t) + \tau X_1(t) + \tau^2 X_2(t) + \cdots, \\ y_\tau(t) &= Y_0(t) + \tau Y_1(t) + \tau^2 Y_2(t) + \cdots. \end{aligned}$$

Note that $X_0(t) = \cos t,\ Y_0(t) = \sin t$.
Hint.

$$x_\tau^2 = \sum_{j\ge 0} \tau^j X_j(t) \sum_{k \ge 0} \tau^k X_k(t) = \sum_{\ell \ge 0} \sum_{k=0}^{\ell} \tau^\ell X_{\ell - k}(t) X_k(t).$$

6. Suppose that y in (7) is a critical point of F, i.e., $F(y) = 0$. Show that the linearized equation becomes

$$\frac{dW}{dt} = LW, \quad W(0) = I, \quad L = DF(y).$$

Hence

$$F(y) = 0 \Longrightarrow D_y x(t,y) = e^{tL}.$$

Supplementary worksheet

Ch. 4, Appendix A, The derivative in several variables

1. Read Appendix A, at the end of Chapter 4, which discusses the derivative of a function of several variables. In particular, if $\Omega \subset \mathbb{R}^n$ is open and

$$F : \Omega \longrightarrow \mathbb{R}^m, \quad F(x) = (F_1(x), \dots, F_m(x)),$$

then, for $x \in \Omega$,

$$DF(x) = \Bigl(\frac{\partial F_j}{\partial x_k}(x)\Bigr)$$

is the $m \times n$ matrix of partial derivatives of F, and differentiability is expressed as

$$F(x + w) = F(x) + DF(x)w + r(x, w),$$

for $w \in \mathbb{R}^n$ small, where $r(x, w)/\|w\| \to 0$ as $w \to 0$ (we say $r(x, w) = o(\|w\|)$. We say

$$F \in C^1(\Omega)$$

provided $DF(x)$ exists for each $x \in \Omega$ and is a continuous function of x. Equivalently, each partial derivative $\partial F_j/\partial x_k$ exists and is continuous on Ω.

2. Read about the chain rule, regarding the derivative of a composition $G \circ F$, in the form

$$D(G \circ F)(x) = DG(F(x))DF(x),$$

and variants, given in (A.7)–(A.8).

Worksheet 17, Tuesday, 10/13

Ch. 4, §3, Vector fields, orbits, and flows

1. The system of ODEs

$$\frac{dx}{dt} = F(x), \quad x(0) = y.$$

with solution $x = x(t, y)$, studied in §4.2, produces a flow, Φ_F^t, defined by

$$\Phi_F^t(y) = x(t, y).$$

We call Φ_F^t the *flow* generated by the *vector field* F.

Study results (3.2)–(3.8) on Φ_F^t, including the identity

$$\Phi_f^{s+t} = \Phi_F^t \circ \Phi_F^s.$$

2. Given $v \in C_0^1(\Omega)$, (3.9) defines

$$v^t(x) = v(\Phi_F^t(x)).$$

Study the formulas (3.10)–(3.13), leading to the identity

$$\frac{d}{dt} \int_\Omega v(\Phi_F^t(x))\, dx = -\int_\Omega \operatorname{div} F(x)\, v(\Phi_F^t(x))\, dx,$$

where $\operatorname{div} F$ is the *divergence* of F, given by (3.14), i.e.,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Follow (3.15)–(3.17) to the conclusion given in Proposition 3.1, that

$$\frac{d}{dt} \operatorname{Vol}(\Phi_F^t(B)) = \int_{\Phi_F^t(B)} \operatorname{div} F(x)\, dx.$$

Appreciate the conclusion that the flow Φ_F^t expands volumes (as t increases) where $\operatorname{div} F > 0$ and shrinks volumes where $\operatorname{div} F < 0$.

3. Read the definition regarding when a flow Φ_F^t is complete, or is forward complete. Proposition 3.2 says that if F is a smooth vector field on $\Omega = \mathbb{R}^n$, and if there exist $R < \infty$ and $V \in C^1(\mathbb{R}^n)$ such that

$$V(x) \to +\infty \text{ as } |x| \to \infty, \text{ and}$$
$$|x| \geq R \Rightarrow \nabla V(x) \cdot F(x) \leq 0,$$

then Φ_F^t is forward complete. Read its proof.

4. Follow the discussion in (3.22)–(3.34) of the first-order system arising from the pendulum equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin\theta = 0,$$

yielding the vector field

$$F(\theta, \psi) = (\psi, -(g/\ell)\sin\theta).$$

(a) See that the orbits for the flow generated by F lie on level curves of the function $\mathcal{E}(\theta, \psi)$ in (3.25).

(b) See the discussion of the critical points of F, i.e., points where F vanishes. See the classification of its critical points as

centers and saddles.

5. See the discussion in (3.35)–(3.38) of a vector field F, with critical point at 0, which is not a center for F, though it is a center for the linearization of F at 0.

6. See the discussion of critical points of vector fields of the special (Hamiltonian) form

$$F(x) = -J\nabla\mathcal{E}(x),$$

given in (3.39)–(3.47), including Proposition 3.3.

Worksheet 18, Thursday, 10/15

Ch. 4, §3, Vector fields, orbits, and flows II

1. Check out the discussion in (3.48)–(3.54) of the first-order system arising from the damped pendulum equation,

$$\frac{d^2\theta}{dt^2} + \frac{\alpha}{m}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = 0,$$

yielding the vector field

$$F(\theta,\psi) = \Bigl(\psi, -\frac{\alpha}{m}\psi - \frac{g}{\ell}\sin\theta\Bigr).$$

Here, $g, \ell, \alpha, m > 0$.

(a) See that, if $\mathcal{E}(\theta,\psi)$ is given by (3.26), then, along an orbit for F,

$$\frac{d}{dt}\mathcal{E}(\theta,\psi) = -\frac{\alpha}{m}\psi^2 \le 0.$$

(b) See the discussion of the critical points of F. See the classification of these critical points as

sinks and saddles.

2. Check out Proposition 3.4, which says that if F is a smooth vector field on $\Omega \subset \mathbb{R}^n$, with a critical point at $x_0 \in \Omega$, and if

all eigenvalues of $DF(x_0)$ have negative real part,

the x_0 is a sink, i.e., (3.56) holds.
An ingredient in the proof is Lemma 3.5, whose proof occupies (3.65)–(3.67).

3. Write down the definitions that a critical point x_0 of a vector field F be a

source, sink, saddle, center.

4. Do Exercise 10, at the end of Section 3.

Worksheet 19, Tuesday, 10/20

Review for test, Th 10/22

These review topics are tuned to Worksheets 11–18.

1. Review material in Ch. 3, §8 on variable coefficient linear systems:
   matrix solution M to $M'(t) = A(t)M(t)$,
   Wronskian $W(t) = \det M(t)$,
   Abel's equation: $W'(t) = (\operatorname{Tr} A(t))W(t)$,
   Duhamel's formula for solution to $x'(t) = A(t)x + f(t)$,
   derived via substitution $x(t) = M(t)y(t)$,
   which yields $M(t)y'(t) = f(t)$.

2. Review material in Ch. 3, §10 on power series expansions:
   to solve $x'(t) = A(t)x + f(t)$, seek $x(t) = \sum x_k t^k$,
   recursive formula for coefficients x_k,
   $(k+1)x_{k+1} = \sum_{j=0}^{k} A_{k-j}x_j + f_k$,
   convergence result for power series given by recursive formula,
   analytic functions, given by convergent power series.

3. Review material in Ch. 4, §1, on nonlinear systems:
   $x'(t) = F(t,x),\ x(t_0) = x_0$.
   $x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\,ds$.
   Picard iteration,
   Lipschitz condition,
   short-time existence and uniqueness,
   lower bound for t interval of existence,
   examples of finite-time blowup.

4. Continuation of review of Ch. 4, §1:
   conditions for global solvability,
   global solvability in linear case,
   energy conservation and global solvability,
   energy dissipation and global forward-time solvability.

5. Review material in Ch. 4, §2, dependence of solutions on initial data:
   $dx/dt = F(x),\ x(0) = y$, solution $x = x(t,y)$,
   Linearized equation, $W = W(t,y)$:

$$\frac{dW}{dt} = DF(x(t,y))W, \quad W(0) = I.$$

   $F \in C^1(\Omega) \Rightarrow x(t,y)$ is C^1 in y, and

$$D_y x(t,y) = W(t,y).$$

6. Continuation of review of Ch. 4, §2.
   $F \in C^2(\Omega) \Rightarrow x(t, y)$ is C^2 in y,
   Proof: Couple the equations

$$\frac{dx}{dt} = F(x), \quad \frac{dW}{dt} = DF(x)W, \quad x(0) = y, \ W(0) = I.$$

   and use results from #5. Iterate this argument:
   $F \in C^k(\Omega) \Rightarrow x(t, y)$ is C^k in y,

Dependence of solutions on parameters:
   $x'(t) = F(\tau, x), \ x(0) = y, \quad x = x(t, \tau, y).$
   $F \in C^1$ in $(\tau, y) \Rightarrow x(t, \tau, y)$ is C^1 in (τ, y),
   Trick: form the $(n + 1) \times (n + 1)$ system for (x, z):

$$\frac{dx}{dt} = F(z, x), \ \frac{dz}{dt} = 0, \quad x(0) = y, \ z(0) = \tau,$$

   and use results from #5. Get differential equation for $\xi(t) = D_\tau x(t, \tau, y)$:

$$\frac{d\xi}{dt} = D_x F(\tau, x)\xi + D_\tau F(\tau, x), \quad \xi(0) = 0.$$

6A. Check out Ch. 4, Appendix A, the derivative in several variables:
   the derivative as a linear transformation,
   connection with partial derivatives,
   the chain rule.

7. Review material in Ch. 4, §3, vector fields, orbits, and flows:
   Solution to $dx/dt = F(x), \ x(0) = y$ defines a flow,

$$\Phi_F^t(y) = x(t, y).$$

   We have $\Phi_F^{s+t} = \Phi_F^t \circ \Phi_F^s$.
   Action of this flow on volumes:

$$\frac{d}{dt}\mathrm{Vol}(\Phi_F^t(B)) = \int_{\Phi_F^t(B)} \operatorname{div} F(x)\, dx,$$

   where $\operatorname{div} F = \partial_1 F_1 + \cdots + \partial_n F_n$ is the *divergence* of F.
   Review when a flow is complete, resp., forward complete,
   see criterion for forward completeness in Proposition 3.2; variant in Exercise 2.

8. Continuation of review of Ch. 4, §3:
   phase plane portraits of flows,
      example: 2×2 system arising from pendulum equation,
      example: 2×2 system arising from damped pendulum equation,
   energy conservation and energy dissipation,
   critical points of vector fields
      examples: saddles, centers, sources, sinks.
   Proposition 3.4 gives a criterion for a critical point to be a sink.
   (Appendix C of Ch. 4 gives a criterion for a critical point to be a saddle.)

Worksheet 20, Tuesday, 10/27

Ch. 4, §5, Newtonian equations

1. Study the system of m interacting particles, moving in n-dimensional space, givden by

$$m_k \frac{d^2 x_k}{dt^2} = \sum_{\{j:j\neq k\}} F_{jk}(x_k - x_j), \quad 1 \le k \le m, \tag{1}$$

for $x_k(t)$ taking values in $\mathbb{R}^n$, described in (5.1)–(5.6). Note the emphasis on forces satisfying

$$F_{jk}(x_k - x_j) = -F_{kj}(x_j - x_k) = f_{jk}(\|x_k - x_j\|)(x_k - x_j), \tag{2}$$

in which case

$$F_{jk}(x_k - x_j) = -\nabla V_{jk}(x_k - x_j), \tag{3}$$

with

$$V_{jk}(u) = V_{kj}(u) = v_{jk}(\|u\|), \quad v'_{jk}(r) = -r f_{jk}(r). \tag{4}$$

2. The total energy of such a system is given by

$$E = \frac{1}{2}\sum_k m_k \Big\| \frac{dx_k}{dt} \Big\|^2 + \frac{1}{2}\sum_{j\neq k} V_{jk}(x_k - x_j), \tag{5}$$

kinetic energy + potential energy. Follow (5.7)–(5.11) to see that when (1) holds, with forces satisfying (2)–(4), then we have conservation of energy:

$$\frac{dE}{dt} = 0.$$

3. One converts the 2nd order system (1) to a 1st order system by bringing in the *momenta*, $p_k = m_k dx_k/dt$. See from (5.12)–(5.18) that the energy can be written as

$$E(x,p) = \sum_k \frac{1}{2m_k}\|p_k\|^2 + \frac{1}{2}\sum_{j\neq k} V_{jk}(x_k - x_j), \tag{6}$$

and the resulting 1st order system takes the *Hamiltonian form*

$$\frac{dx_k}{dt} = \frac{\partial E}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial E}{\partial x_k}. \tag{7}$$

4. Follow (5.18)–(5.21) to see how, whenever $(x(t), p(t))$ satisfies a Hamiltonian system of the form (7), one has

$$\frac{d}{dt} E(x(t), p(t)) = 0. \tag{8}$$

5. Return to the setting of (1)–(2) and define the total momentum,

$$P = \sum_k p_k = \sum_k m_k \frac{dx_k}{dt}. \tag{9}$$

Follow (5.22)–(5.27) to verify conservation of total momentum:

$$\frac{dP}{dt} = 0. \tag{10}$$

Deduce that, if $x(t) = (x_1(t), \dots, x_m(t))$ solves (1), then there exist $a, b \in \mathbb{R}$ such that

$$\frac{1}{M} \sum_k m_k x_k(t) = a + bt, \quad M = \sum_k m_k. \tag{11}$$

The left side of (11) is called the *center of mass* of the system of interacting particles.

6. Follow (5.24)–(5.28) to see that, if you set

$$y_k(t) = x_k(t) - (a + bt),$$

then $y(t)$ solves (1), and

$$\sum_k m_k y_k(t) \equiv 0.$$

7. Follow (5.29)–(5.32) to see that, when $m = 2$, one can use the identity $y_2 = -(m_1/m_2) y_1$ to transform the system (1) to

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2 x}{dt^2} = F_{21}(x), \tag{12}$$

for

$$x = x_1 - x_2 = y_1 - y_2 = \Bigl(1 + \frac{m_1}{m_2}\Bigr) y_1.$$

As before, $x(t) \in \mathbb{R}^n$.

8. Alternatively, when $m = 2$, write (1) as

$$\frac{d^2 x_1}{dt^2} = \frac{1}{m_1} F_{21}(x_1 - x_2), \quad \frac{d^2 x_2}{dt^2} = \frac{1}{m_2} F_{12}(x_2 - x_1),$$

and subtract (using (2)) to get (12).

The system (12) (with F_{21} as in (2)) is called a *central force problem.*

Worksheet 21, Thursday, 10/29

Ch. 4, §6, Central force problems and planetary motion

1. A central force problem on $\mathbb{R}^n$ has the form

$$m\frac{d^2x}{dt^2} = F(x), \tag{1}$$

with $F \in C^1(\mathbb{R}^n \setminus 0)$ satisfying

$$F(x) = f(\|x\|)x, \tag{2}$$

hence

$$F(x) = -\nabla V(x), \quad V(x) = v(\|x\|), \quad v'(r) = -rf(r). \tag{3}$$

Follow the results of (6.1)–(6.5), giving conservation of energy

$$E(t) = \frac{m}{2}\left\|\frac{dx}{dt}\right\|^2 + V(x).$$

2. Proposition 6.1 says that if $x(0) \neq 0$ and $W \subset \mathbb{R}^n$ is the linear span of $x(0)$ and $x'(0)$, and if $x(t)$ solves (1) for $t \in I$, then $x(t) \in W$ for $t \in I$. Study its proof.

3. Take $n = 3$. Follow (6.9)–(6.11), regarding conservation of angular momentum $\alpha(t) = mx(t) \times x'(t)$, i.e.,

$$x(t) \times x'(t) \equiv L. \tag{4}$$

4. Now take $n = 2$, identify $\mathbb{R}^2$ with $\mathbb{C}$, and write

$$x(t) = r(t)e^{i\theta(t)}. \tag{5}$$

Compute $x'(t)$ and $x''(t)$ and verify that (1)–(2) yield

$$m\Big[r'' - r(\theta')^2 + i(2r'\theta' + r\theta'')\Big] = f(r)r,$$

i.e.,

$$\begin{aligned} r'' - r(\theta')^2 &= \frac{f(r)r}{m}, \\ 2r'\theta' + r\theta'' &= 0. \end{aligned} \tag{6}$$

5. Show that the second equation in (6) implies

$$\frac{d}{dt}(r^2\theta') = 0,$$

hence

$$r^2\theta' = L, \tag{7}$$

for some $L \in \mathbb{R}$. Show that this result is equivalent to (4).

6. See from (6.22)–(6.24) that the signed area $A(t)$ swept out by the ray from 0 to $x(s)$, as s runs from t_0 to t, satisfies

$$A'(t) = \frac{1}{2}r^2\theta' = \frac{L}{2}. \tag{8}$$

See the identification of this in (6.27) as Kepler's second law.

7. Plug $\theta' = L/r^2$ (from (7)) into the first equation in (6), obtaining

$$\frac{d^2r}{dt^2} = \frac{f(r)r}{m} + \frac{L^2}{r^3} = g(r). \tag{9}$$

Follow (6.25)–(6.30), using techniques developed in Ch. 1, §5, to obtain the separable equation

$$\frac{dr}{dt} = \pm\sqrt{2E - 2w(r)}, \quad w'(r) = -g(r). \tag{10}$$

Note that dividing (10) by (7) yields

$$\frac{dr}{d\theta} = \pm\frac{r^2}{L}\sqrt{2E - w(r)}, \tag{11}$$

which is also separable.

8. We move to the Kepler problem, in which

$$F(x) = -Km\frac{x}{\|x\|^3}. \tag{12}$$

See from (6.36)–(6.38) that (9) becomes

$$\frac{d^2r}{dt^2} = -\frac{K}{r^2} + \frac{L^2}{r^3}. \tag{13}$$

The miracle

9. In the setting of #8, set

$$u = \frac{1}{r}, \tag{14}$$

and follow (6.42)–(6.46) to obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{K}{L^2}, \tag{15}$$

which is a *linear equation*. See from (6.47)–(6.49) that the general solution to (15) yields

$$
\begin{aligned}
r\Bigl[1 + e\cos(\theta - \theta_0)\Bigr] &= p, \\
p = \frac{L^2}{K}, \quad e &= A\frac{L^2}{K}.
\end{aligned}
\tag{16}
$$

10. In the setting of #9, the following cases hold:

$$
\begin{aligned}
e = 0, &\quad \text{circle}, \\
0 < e < 1, &\quad \text{ellipse}, \\
e = 1, &\quad \text{parabola}, \\
e > 1, &\quad \text{hyperbola}.
\end{aligned}
$$

See Exercise 4 at the end of Ch. 4, §6, for a discussion of the ellipse case.

11. Check out the further discussion of Kepler's three laws, stated below (6.34), and pursued further between (6.49) and (6.52).

Worksheet 22, Tuesday, 11/03

Ch. 4, §7, Variational problems, stationary action principle

1. Given $L \in C^2(\Omega \times \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ open, say $L = L(x, v)$, and given a path $u : [a, b] \to \Omega$, from p to q, we consider the path function

$$I(u) = \int_a^b L(u(t), u'(t))\, dt, \tag{1}$$

and look for critical paths, i.e., paths satisfying

$$\frac{d}{ds} I(u_s)\big|_{s=0} = 0, \tag{2}$$

for all smooth families of paths u_s such that $u_0 = u,\ u_s(a) \equiv p,\ u_s(b) \equiv q$.

Follow (4.7.1)–(4.7.7) to obtain the result that a critical path solves the second-order $n \times n$ system of ODE, called Lagrange's equation,

$$\frac{d}{dt} L_{v_k}(u(t), u'(t)) = L_{x_k}(u(t), u'(t)), \quad k \in \{1, \dots, n\}. \tag{3}$$

Here $L_{x_k} = \partial L/\partial x_k,\ L_{v_k} = \partial L/\partial v_k$. Note the equivalent form (4.7.8), which puts the system in a form to which basic existence and uniqueness results of §4.1 apply, provided the $n \times n$ matrix

$$\Big(L_{v_k v_\ell}(x, v) \Big) \ \text{ is invertible.} \tag{4}$$

2. Follow (4.7.10)–(4.7.13), and see how if

$$L(x, v) = T - V = \frac{1}{2} m|v|^2 - V(x), \tag{5}$$

where T = kinetic energy, V = potential energy, then the Lagrange equation (3) becomes the standard Newtonian equation:

$$m \frac{d^2 u}{dt^2} = -\nabla V(u). \tag{6}$$

In this setup, $L = T - V$ is called the *action*, and the approach to the Lagrange equation (3) via (2) is called the stationary action principle.

3. Follow the treatment of the pendulum in (4.7.14)–(4.7.20), and see how the formulas

$$T = \frac{m\ell^2}{2} \theta'(t)^2, \quad V = -mg\ell \cos\theta,$$

for the kinetic energy and the potential energy arise, and how then the Lagrange equation, applied to $L(\theta, \theta') = T - V$, yields

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell} \sin\theta,$$

the same pendulum equation that arose, by different means, in Chapter 1, §6.

3A. (For the eager reader) Check out the treatment of the double pendulum, in §4.9.

4. Check out the treatment of *constrained variational problems* in (4.7.21)–(4.7.24). Here, we let $M \subset \Omega$ be a smooth $(n-1)$-dimensional surface, take $p, q \in M$, and look for critical paths for (1) subject to the constraint that

$$u(t) \in M, \quad \forall\, t \in [a,b]. \tag{7}$$

Follow how the Lagrange equation (3) is modified to the condition

$$\frac{d}{dt} L_v(u(t), u'(t)) - L_x(u(t), u'(t)) \ \text{ is parallel to } \ n(u(t)). \tag{8}$$

5. Follow the treatment in (4.7.26)–(4.7.33) of constrained variational problems when the Lagrangian is specialized to $L(x,v) = (1/2)|v|^2$, so $I(u)$ becomes the *energy integral*,

$$E(u) = \frac{1}{2} \int_a^b |u'(t)|^2 \, dt. \tag{9}$$

See how (8) becomes

$$u''(t) = a(t) n(u(t)), \quad a(t) = n(u(t)) \cdot u''(t), \tag{10}$$

and how (10) in turn yields the $n \times n$ system

$$u''(t) + u'(t) \cdot \Bigl(\frac{d}{dt} n(u(t))\Bigr) n(u(t)) = 0. \tag{11}$$

See also in (4.7.33) that, in this setting, critical paths all have constant speed.

5A. (For the eager reader) Check out Appendix 4.G, relating the material of #5 above to the study of geodesics on the surface M.

6. Returning to unconstrained variational problems, check out the treatment in (4.7.39)–(4.7.43) of the Lagrange equation arising from

$$L(x,v) = \frac{1}{2} v \cdot G(x) v, \tag{12}$$

where $G : \Omega \to M(n, \mathbb{R})$, and each $G(x)$ is symmetric and positive definite.

7. Look at Exercises 1–7, at the end of §4.7.

Worksheet 23, Thursday, 11/05

Ch. 4, §12, Limit sets and periodic orbits

1. Let F be a C^1 vector field on an open set $\mathcal{O} \subset \mathbb{R}^n$, generating the flow Φ^t. Take $x \in \mathcal{O}$. If $\Phi^t(x)$ is well defined (in $\mathcal{O}$) for all $t \geq 0$, the ω-limit set $L_\omega(x)$ consists of all points

$$y \in \mathcal{O} \text{ such that there exist } t_k \nearrow +\infty \text{ with } \Phi^{t_k}(x) \to y.$$

(There is a similar definition of $L_\alpha(x)$, with $t_k \searrow -\infty$.) See the illustrations of ω-limit sets in Figures 4.12.1 and 4.12.2.

2. Results on compactness treated in Appendix 4.B can be used to show that,

$$\text{if } \ \{\Phi^t(x) : t \geq 0\} = S \text{ and } \ \overline{S} = K \ \text{ is compact,}$$

then $L_\omega(x)$ is a nonempty compact subset of K, and

$$\Phi^t : L_\omega(x) \longrightarrow L_\omega(x), \quad \forall \, t \in \mathbb{R}.$$

Furthermore,

$$y \in L_\omega(x) \Longrightarrow L_\omega(y) \subset L_\omega(x).$$

3. The main result of §4.12 is the Poincaré-Bendixson theorem, which says the following.

Theorem. Let $\mathcal{O}$ be a planar domain, and let F generate a flow Φ^t on $\mathcal{O}$. Assume there is a set $K \subset \mathcal{O}$ that is a closed, bounded (i.e., compact) subset of $\mathbb{R}^2$ and satisfies

$$\Phi^t(K) \subset K, \quad \forall \, t > 0.$$

Take $x \in K$. If $L_\omega(x)$ contains no critical point of F, then it is a periodic orbit of Φ.

Study its proof, which occupies (4.12.1)–(4.12.12). Note how it makes use of the Jordan curve theorem.

4. Check out the discussion of the Van der Pol equation (4.12.13),

$$x'' - \mu(1 - x^2)x' + x = 0,$$

equivalent to the system (4.12.14), as an illustration of the Poincaré-Bendixson theorem. Note the use of a numerical calculation of an orbit through $(0, A)$, on the one hand, and an analysis of the critical point $(0, 0)$ in (4.12.15)–(4.12.17) on the other hand, to verify the hypotheses of Theorem 4.12.1. The resulting periodic orbit is illustrated in Figure 4.12.6.

5. See Exercises 6–8 at the end of §4.12 for a discussion of how ODEs like the Van der Pol equation arise in models of electronic circuits, in which Ohm's law $V = IR$ for the drop in voltage across a resistor, introduced in §1.13, is replaced by a more general formula

$$V = f(I),$$

such as $f(I) = \mu(\alpha I^3 - I)$. Transistors can behave as such circuit elements.

6. Another occurrence of a periodic orbit whose existence is forced by the Poincaré-Bendixson theorem will arise in a family of predator-prey equations. See Figure 4.13.6.

Worksheet 24, Tuesday, 11/10

Ch. 4, §13, Predator-prey equations

Section 4.13 studies 2×2 nonlinear systems for

$$\begin{aligned} x(t) &= \text{ population of predators,} \\ y(t) &= \text{ population of prey.} \end{aligned} \tag{1}$$

The systems also bring in

$$\zeta = \text{ rate at which each predator consumes prey,} \tag{2}$$

typically taking $\zeta = \zeta(y)$, with either

$$\zeta(y) = \kappa y, \tag{3}$$

or $\zeta(y)$ as pictured in Figure 4.13.2, given for example by

$$\zeta(y) = \frac{\kappa y}{1 + \gamma y}. \tag{4}$$

Two types of systems are considered, namely

$$\begin{aligned} \frac{dx}{dt} &= -ax + b\zeta x, \\ \frac{dy}{dt} &= ry - \zeta x, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{dx}{dt} &= -ax + b\zeta x, \\ \frac{dy}{dt} &= ry(1 - cy) - \zeta x, \end{aligned} \tag{6}$$

the difference being the first term on the right side of "$dy/dt =$", following either the exponential growth model or the logistic model for the prey species (in the absence of predators).

In all cases, one takes constants

$$a, b, c, r, \kappa, \gamma > 0,$$

and the quarter plane $\{x \geq 0,\ y \geq 0\}$ is invariant under the flow.

1. The Volterra-Lottka system, introduced in (4.13.11), is

$$
\begin{aligned}
\frac{dx}{dt} &= -ax + \sigma xy, \quad \sigma = b\kappa, \\
\frac{dy}{dt} &= ry - \kappa xy.
\end{aligned}
\tag{7}
$$

This has the form (5), with $\zeta(y)$ as in (3). Follow the analysis in (4.13.11)–(4.13.20), and verify the following.

See that the vector field $V(x, y)$ arising from the right side of (7) has two critical points,

$$
(0, 0), \quad \text{and} \quad (x_0, y_0) = \Bigl(\frac{r}{\kappa}, \frac{a}{\sigma}\Bigr).
\tag{8}
$$

We have

$$
DV(0, 0) = \begin{pmatrix} -a & 0 \\ 0 & r \end{pmatrix}, \quad DV(x_0, y_0) = \begin{pmatrix} 0 & \sigma x_0 \\ -\kappa y_0 & 0 \end{pmatrix},
\tag{9}
$$

so $(0, 0)$ is a saddle and the eigenvalues of $DV(x_0, y_0)$ are purely imaginary.

Follow the steps (4.13.15)–(4.13.20) to see that

$$
H(x, y) = \sigma y - a \log y + \kappa x - r \log x
\tag{10}
$$

is a smooth function on $\{x, y > 0\}$, and

$$
V(x, y) \cdot \nabla H(x, y) = 0,
\tag{11}
$$

hence the orbits in this quadrant of the flow generated by V lie on the level curves of H. See that H has a minimum at (x_0, y_0), which is hence a center. Study the phase portrait in Figure 4.13.3.

2. Follow the observation of Volterra, discussed in (4.13.21)–(4.13.23), regarding the effect of fishermen on the interaction of sharks and their prey, as modeled by (7).

3. The first modification of Volterra-Lotka, introduced in (4.13.24), is

$$\begin{aligned}\frac{dx}{dt} &= -ax + \sigma xy, \qquad \sigma = b\kappa, \\ \frac{dy}{dt} &= ry(1-cy) - \kappa xy.\end{aligned} \tag{12}$$

This has the form (6), again with $\zeta(y)$ given by (3). Follow the analysis in (4.13.24)–(4.13.37), and verify the following.

From an evaluation of $V(x,1/c)$, the region

$$\mathcal{R} = \Big\{(x,y) : x \geq 0,\ 0 \leq y \leq \frac{1}{c}\Big\} \tag{13}$$

is invariant under the flow generated by the vector field $V(x,y)$, arising from the right side of (12). This time, V has 3 critical points,

$$(0,0), \quad \Big(0, \frac{1}{c}\Big), \quad (x_0, y_0) = \Big(\frac{r}{\kappa}\Big(1 - \frac{ca}{\sigma}\Big), \frac{a}{\sigma}\Big). \tag{14}$$

In this situation $DV(0,0)$ is still given by (9), so $(0,0)$ is a saddle, and

$$DV\Big(0, \frac{1}{c}\Big), \ \ DV(x_0,y_0) \ \text{ are given by (4.13.29) and (4.13.32).} \tag{15}$$

Three cases are distinguished, according to whether

$$\frac{\sigma}{c} - a \ \text{ is } \ < 0, \quad > 0, \quad \text{or} \quad = 0. \tag{16}$$

Case I. $(0,1/c)$ is a sink, and the population of predators is driven to extinction.

Case III. $(0,1/c)$ is a degenerate critical point of V. Nevertheless, by (4.13.37), the population of predators is also driven to extinction. This leaves

Case II. $\sigma/c - a > 0$, or equivalently $x_0 > 0$. In this case,

$$(x_0, y_0) \in \mathcal{R}. \tag{17}$$

Also (4.13.29) implies $(0,1/c)$ is a saddle, and

$$\text{(4.13.32)} \Rightarrow (x_0,y_0) \text{ is a sink.} \tag{18}$$

Now follow (4.13.34)–(4.13.36) to deduce from Proposition 4.12.5 that

$$p \in \mathcal{R} \Longrightarrow \mathcal{L}_\omega(p) = (x_0, y_0). \tag{19}$$

See how these results are illustrated in the phase portrait, Figure 4.13.4.

4. The second modification of Volterra-Lotka, taken up in (4.13.38), is

$$
\begin{aligned}
\frac{dx}{dt} &= -ax + bx\zeta(y), \\
\frac{dy}{dt} &= ry(1-cy) - x\zeta(y).
\end{aligned}
\tag{20}
$$

This has the form (6). This time we take $\zeta(y)$ to be given by (4), or more generally to satisfy (a)–(e) of (4.13.39). Follow the analysis in (4.13.38)–(4.13.62), and verify the following.

Again from an evaluation of $V(x,1/c)$, the region $\mathcal{R}$ described in (13) is invariant under the flow Φ^t generated by the vector field $V(x,y)$, arising from the right side of (20).

As in (14), two of the critical points of V are $(0,0)$ and $(0,1/c)$. Again $DV(0,0)$ is given by (9), so $(0,0)$ is a saddle. This time,

$$
DV\Bigl(0,\frac{1}{c}\Bigr) \ \text{ is given by (4.13.42).} \tag{21}
$$

We also have

$$
V(x_0,y_0)=0, \ \ \text{provided} \ \ \zeta(y_0)=\frac{a}{b}, \ \ x_0=\frac{b}{a}ry_0(1-cy_0). \tag{22}
$$

Solving $\zeta(y_0)=a/b$ for y_0 can be done if and only if $a/b<\sup\zeta$. We assume a/b satisfies this condition.

Then there are 3 cases, distinguished by whether

$$
x_0<0, \quad x_0>0, \quad \text{or} \ \ x_0=0. \tag{23}
$$

In Case I, $(x_0,y_0)\notin\mathcal{R}$, and $(0,1/c)$ is a sink. In Case III, $(x_0,y_0)=(0,1/c)$. We concentrate on

Case II. Then $(x_0,y_0)\in\mathcal{R}$.

In this case, computation of (21) implies $(0,1/c)$ is a saddle. Meanwhile,

$$
DV(x_0,y_0) \ \text{ is given by (4.13.48).} \tag{24}
$$

A calculation in (4.13.49) gives

$$
\det DV(x_0,y_0)>0. \tag{25}
$$

We then divide Case II into 3 subcases, depending on whether

$$
\operatorname{Tr} DV(x_0,y_0) \ \text{ is } \ <0, \ \ >0, \ \ \text{or} \ \ =0, \tag{26}
$$

denoted, respectively, Cases IIA, IIB, and IIC.

Case IIA. Here (x_0,y_0) is a sink.

Case IIB. Here (x_0, y_0) is a source.

Follow the analysis in (4.13.56)–(4.13.62), to see that

$$\text{In Case IIB, } \mathcal{L}_\omega(p) \text{ is a periodic orbit,} \tag{27}$$

for all $p \neq (x_0, y_0)$ in the interior of $\mathcal{R}$. Note the role of the Poincaré-Bendixson theorem here.

For an illustration of (27), see Figure 4.13.6. This figure was produced using the parameters

$$a = 1,\ b = 2,\ \kappa = 1,\ \gamma = 1,\ r = 1,\ c = \frac{1}{4}.$$

Worksheet 25, Thursday, 11/12

Ch. 4, §15, Chaos in multidimensional systems

Section 4.15 discusses two 3D systems that exhibit chaotic behavior. This sheet will concentrate on one of them: the **Lorenz equations**

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= rx - y - xz, \\ z' &= xy - bz. \end{aligned} \tag{1}$$

The parameters σ, b, and r are all positive. Lorenz took $\sigma = 10$, $b = 8/3$, and considered various values of r, with emphasis on $r = 28$.

Denote the vector field defined by the right side of (1) by V.

1. Given

$$f(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2, \tag{2}$$

show that, if (x, y, z) solves (1), then

$$\frac{d}{dt} f(x, y, z) = -2\sigma\Big[rx^2 + y^2 + b(z - r)^2 - br^2\Big]. \tag{3}$$

Show that there exists $K \in (0, \infty)$ such that

$$B = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) \le K\} \tag{4}$$

is a compact subset of $\mathbb{R}^3$ and the right side of (3) is < 0 on $\mathbb{R}^3 \setminus B$. Deduce that

$$\Phi^t(B) \subset B, \quad \forall\, t > 0, \tag{5}$$

where Φ^t is the flow generated by V, and, for each $(x, y, z) \in \mathbb{R}^3$,

$$\Phi^t(x, y, z) \subset B, \quad \text{for all sufficiently large } t > 0. \tag{6}$$

Deduce that the flow Φ^t is forward complete on $\mathbb{R}^3$.

2. Set $B(t) = \Phi^t(B)$. Deduce from (5) that $B(t) \subset B(s)$ for $0 < s < t$. Set

$$\mathcal{B} = \bigcap_{t \in \mathbb{R}^+} B(t) = \bigcap_{k \in \mathbb{Z}^+} B(k). \tag{7}$$

Show that

$$\Phi^t(\mathcal{B}) = \mathcal{B}, \quad \forall\, t \ge 0. \tag{8}$$

The set $\mathcal{B}$ is called the *attractor* for (1).

3. Show that

$$\operatorname{div} V = -\sigma - 1 - b < 0, \tag{9}$$

and use results from §4.3 (worksheet 17) to deduce that

$$\operatorname{Vol}(\mathcal{B}) = 0. \tag{10}$$

Given σ and b as set in #1, the attractor $\mathcal{B}$ depends on the parameter r. Figure 4.15.1 illustrates how $\mathcal{B}$ becomes more complex as r varies from 10 to 28.

The next step in the analysis is to consider the critical points of V.

4. The origin is a critical point for each $\sigma, b, r \in (0, \infty)$. Follow the arguments in (4.15.16)–(4.15.18) to see that

$$\begin{aligned} DV(0) &\text{ has 3 negative eigenvalues, for } 0 < r < 1, \\ &\quad 2 \text{ negative and 1 positive eigenvalue, for } r > 1. \end{aligned} \tag{11}$$

Eigenvectors are specified in (4.15.17).

5. For $r > 1$, V has 2 additional critical points, given by (4.15.20), i.e.,

$$C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \tag{12}$$

Follow the description of $DV(C_\pm)$, given in (4.15.23), implying that

$$C_\pm \text{ are sinks for } 1 < r < 24.74\cdots, \tag{13}$$

while $DV(C_\pm)$ has

$$1 \text{ negative eigenvalue and 2 with positive real part, for } r > 24.74\cdots. \tag{14}$$

See how the transition from (13) to (14) is reflected in the phase portraits in Figure 4.15.1, as far as regards whether the pictured orbits are drawn in to the critical points $C_\pm$ or are pushed away.

6. We look further at the behavior of the orbit that leaves the origin in the direction of the eigenvector v_+ of $DV(0)$. This is the orbit illustrated in Figure 4.15.1. Follow the material between (4.15.23) and (4.15.24), making the following points.

(a) For $10 < r < r_0 \approx 14$, the orbit spirals into C_+.

(b) For $r_0 < r < r_c \approx 24.74\cdots$, the orbit crosses over into the half-space $\{x < 0\}$ and spirals into C_-.

(c) For $r > r_c$, the orbit also crosses over into the half-space $\{x < 0\}$, but C_- is no longer a sink. The orbit gets pushed away from C_-. From there, it is batted back and forth like a ping pong ball between paddles centered at C_+ and C_-, except more sporadically. This is chaos.

Worksheet 26, Tuesday, 11/17

Review of course, Exam Th., 11/19

I. The following review topics are tuned to worksheets 20–25.

1. Review material in Ch. 4, §5, on Newtonian equations:
ODE systems resulting from $F = ma$.
Energy and energy conservation.
Energy as a function of position and momentum. Hamiltonian form of ODE system.
Reduction of 2 body problem to central force problem.

2. Review material in Ch. 4, §6, on central force problems and planetary motion:
Basic equation

$$m\frac{d^2x}{dt^2} = f(|x|)x.$$

Energy and energy conservation.
Conservation of angular momentum.
Central force problems in polar coordinates.
Kepler problem. Kepler's laws.
Change of variable $u = 1/r$. Ellipse in polar coordinates.

3. Review material in Ch. 4, §7, on variational problems and the stationary action principle:
Critical paths for

$$I(u) = \int_a^b L(u(t), u'(t))\,dt, \quad L = L(x, v).$$

Lagrange equation

$$\frac{d}{dt}L_v(u, u') = L_x(u, u').$$

Newtonian equations captured by

$$L(x, v) = T - V = \frac{m}{2}|v|^2 - V(x).$$

Lagrangian treatment of the pendulum.
Constrained variational problems.
Geodesics on a surface as critical paths of energy integral. Geodesic equations.

4. Review material in Ch. 4, §12, on limit sets and periodic orbits:
   ω-limit set $L_\omega(x)$ of orbit $\Phi^t(x),\ t \ge 0$.
   Existence for $x \in K$, given K compact, Φ^t-invariant, for $t \ge 0$.
   Poincaré-Bendixson theorem.
   Role of Jordan curve theorem.
   Van der Pol equation.

5. Review material in Ch. 4, §13, on predator-prey equations:
   Two paradigmatic 2×2 systems, for $x(t)$ predators, $y(t)$ prey:

$$x' = -ax + b\zeta(y)x, \quad y' = ry - \zeta(y)x, \tag{1}$$

$$x' = -ax + b\zeta(y)x, \quad y' = ry(1 - cy) - \zeta(y)x. \tag{2}$$

   Two types of consumption rates:

$$\zeta(y) = \kappa y, \quad \zeta(y) = \kappa y/(1 + \gamma y).$$

   In all cases, $a, b, c, r, \kappa, \gamma > 0$.
   Volterra-Lotka system – type (1) with $\zeta(y) = \kappa y$.
   Critical point (x_0, y_0) a center.
   First modification – type (2), also with $\zeta(y) = \kappa y$.
   Critical point (x_0, y_0) a sink.
   Second modification – type (2) with $\zeta(y) = \kappa y/(1 + \gamma y)$.
   Critical point (x_0, y_0) a source. Poincaré-Bendixson type periodic orbit.

6. Review material in Ch. 4, §15, on chaos in multidimensional systems:
   Lorenz system

$$x' = \sigma(y - x), \quad y' = rx - y - xz, \quad z' = xy - bz.$$

   Parameters $\sigma = 10,\ b = 8/3,\ r > 1$.
   Compact invariant set $B = \{(x, y, z) : f(x, y, z) \le K\}$, f given by (4.15.6).
   Attractor $\mathcal{B} = \cap_{t \ge 0} \Phi^t(B)$. $\mathrm{Vol}(\mathcal{B}) = 0 \Leftarrow \operatorname{div} V < 0$.
   Critical points: $(0, 0, 0)$, $\quad C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$.
   $C_\pm$ are sinks for $r < r_c \approx 24.74\cdots$, hyperbolic for $r > r_c$.
   Leads to chaotic behavior for $r > r_c$.

II. Having done these reviews, look back over worksheets 10 and 19, reviewing previous course material.