Let $M$ be a compact, $n$-dimensional Riemannian manifold, with Laplace operator $\Delta$. We consider the following Dirichlet problem for the wave equation,

\begin{align}
(\partial_{tt} - \Delta) u &= 0, \quad 0 \leq t \leq T, \\
u(0, x) &= f(x), \quad u(T, x) = g(x),
\end{align}

and aim to establish the following.

**Proposition 1.** Take $s > n$. Then, for a.e. $T \in (0, \infty)$, (1) has a unique solution for each $f, g \in H^s(M)$, satisfying

\begin{align}
\|u(t, \cdot)\|_{L^2(M)} &\leq C\left(\|f\|_{H^s(M)} + \|g\|_{H^s(M)}\right).
\end{align}

To get this, we let $\{u_k : k \geq 0\}$ be an orthonormal basis of $L^2(M)$, consisting of eigenfunctions of $\Delta$,

\begin{align}
\Delta u_k &= -\lambda_k^2 u_k, \quad 0 \leq \lambda_0 < \lambda_1 < \cdots \to \infty.
\end{align}

(Assume $M$ is connected.) We seek a solution

\begin{align}
u(t, x) &= (a_0 + b_0 t)u_0 + \sum_{k \geq 1} (a_k e^{i t \lambda_k} + b_k e^{-i t \lambda_k})u_k.
\end{align}

We write

\begin{align}
f = \sum \hat{f}(k)u_k, \quad g = \sum \hat{g}(k)u_k,
\end{align}

with $\hat{f}(k) = (f, u_k)$. The boundary condition in (1) requires

\begin{align}
a_0 = \hat{f}(0), \quad a_0 + b_0 T = \hat{g}(0),
\end{align}

and, for $k \geq 1$,

\begin{align}
a_k + b_k &= \hat{f}(k), \\
e^{iT \lambda_k} a_k + e^{-i T \lambda_k} b_k &= \hat{g}(k).
\end{align}

From (7) we get

\begin{align}
b_k &= \frac{e^{iT \lambda_k} \hat{f}(k) - \hat{g}(k)}{2i \sin T \lambda_k}.
\end{align}
Thus our task is to show that, for a.e. \( T > 0 \), \(|\sin T \lambda_k|\) stays away from 0, in a quantifiable way, for \( k \geq 1 \) (hence for \( \lambda_k \geq \lambda_1 = b > 0 \)).

To this end, let us take \( A \geq 1 \) and \( \gamma \in (0, 1) \), and estimate

\[
(9) \quad \int_0^A |\sin t\lambda|^{-\gamma} \, dt, \quad \text{for } \lambda \geq b.
\]

A change of variable shows that (9) is equal to

\[
(10) \quad \frac{1}{\lambda} \int_0^{A\lambda} |\sin s|^{-\gamma} \, ds \leq C,
\]

where \( C = C_{\gamma,b,A} \). Hence, if we fix \( \sigma > 0 \) and set

\[
(11) \quad X_{k,N} = \left\{ t \in (0, A] : |\sin t\lambda_k|^{-\gamma} \geq N\lambda_k^\sigma \right\},
\]

we have, by Chebecheff’s inequality,

\[
(12) \quad m(X_{k,N}) \leq \frac{C}{N} \lambda_k^{-\sigma}.
\]

It follows that, for each \( N \),

\[
(13) \quad m\left( \bigcup_{k \geq 1} X_{k,N} \right) \leq \frac{C}{N} \sum_{k \geq 1} \lambda_k^{-\sigma} = \frac{C'}{N},
\]

provided

\[
(14) \quad C \sum_{k \geq 1} \lambda_k^{-\sigma} = C' < \infty.
\]

We then get

\[
(15) \quad m\left( [0, A] \setminus \bigcup_{k \geq 1} X_{k,N} \right) \geq A - \frac{C'}{N}.
\]

Now, for \( T \) in the set measured in (15),

\[
(16) \quad |\sin T \lambda_k|^{-1} \leq N^{1/\gamma} \lambda_k^{\sigma/\gamma}, \quad \forall k \geq 1.
\]

It follows that, for such \( T \),

\[
(17) \quad |b_k| \leq N^{1/\gamma} (|f(k)| + |g(k)|) \lambda_k^{\sigma/\gamma},
\]
with a similar estimate on $|a_k|$, so (2) holds as long as

\begin{equation}
\sum_{k \geq 1} \lambda_k^{-s} < \infty.
\end{equation}

In view of the Weyl asymptotic formula

\begin{equation}
\lambda_k \sim C(M)k^{1/n}, \quad k \nearrow \infty,
\end{equation}

this holds as long as $s > n$.

**Remark 1.** Once one has (17), we also have, for each $r \in \mathbb{R}$,

\begin{equation}
\|u(t, \cdot)\|_{H^r(M)} \leq C\left(\|f\|_{H^{r+s}(M)} + \|g\|_{H^{r+s}(M)}\right).
\end{equation}

In addition,

\begin{equation}
\|\partial_t u(t, \cdot)\|_{H^{r-1}(M)} \leq \text{RHS (20)}.
\end{equation}

Furthermore, the solution to (1) holds for all $t \in \mathbb{R}$, not just for $t \in [0, T]$.

**Remark 2.** Note that the assertion that (18) holds for all $s > n$ is equivalent to the assertion that

\begin{equation}
(-\Delta + 1)^{-s/2} \text{ is Hilbert-Schmidt on } L^2(M), \text{ for } s > n/2,
\end{equation}

hence a consequence of the fact that any continuous linear operator

\begin{equation}
T: L^2(M) \rightarrow H^s(M), \quad s > \frac{n}{2},
\end{equation}

is Hilbert-Schmidt on $L^2(M)$. This is more elementary than a derivation of eigenvalue asymptotics (19).