Calculating Natural Logarithms

Michael Taylor

The integral formula

$$\log(1+x) = \int_0^x \frac{dy}{1+y} \tag{1}$$

leads to the power series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \tag{2}$$

convenient for calculating $\log a$ when $|a-1|$ is small. However, we desire to compute $\log a$ when this quantity is not so small, for example, $a = 2$. Setting $x = 1$ in (2) yields

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots, \tag{3}$$

which actually does converge, but very slowly. One would not want to use (3) to compute $\log 2$ to 12 decimal places. A better formula arises from replacing x by $-x$ and using

$$\log \frac{1}{a} = -\log a. \tag{4}$$

We get

$$\lambda(x) := \log \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \tag{5}$$

(the first identity defining $\lambda(x)$), also convergent for $|x| < 1$, and

$$\log 2 = \lambda(\tfrac{1}{2}) = \sum_{k=1}^{\infty} \frac{1}{k}\Big(\frac{1}{2}\Big)^k. \tag{6}$$

This has better convergence properties; 40 terms yield 12 digits of accuracy. Similarly,

$$\log 5 = \lambda(\tfrac{4}{5}) = \sum_{k=1}^{\infty} \frac{1}{k}\Big(\frac{4}{5}\Big)^k, \tag{7}$$

but this converges a bit more slowly than (6).

Here we use the identity

$$\log ab = \log a + \log b \tag{8}$$

to show how we can get power series for $\log n$, for various values of n, that converge much faster than (6) (not to mention (7)). The key is to simultaneously produce power series for $\log n$ for several different values of n. For example,

$$\begin{aligned} \log 3 &= \log 2 + \log \tfrac{3}{2} = \log 2 + \lambda(\tfrac{1}{3}), \\ \log 4 &= \log 3 + \log \tfrac{4}{3} = \log 3 + \lambda(\tfrac{1}{4}), \end{aligned} \tag{9}$$

gives the pair of equations

$$\begin{aligned} \log 2 - \log 3 &= -\lambda(\tfrac{1}{3}), \\ 2\log 2 - \log 3 &= \lambda(\tfrac{1}{4}), \end{aligned} \tag{10}$$

hence we get

$$\begin{aligned} \log 2 &= \lambda(\tfrac{1}{3}) + \lambda(\tfrac{1}{4}), \\ \log 3 &= 2\lambda(\tfrac{1}{3}) + \lambda(\tfrac{1}{4}), \end{aligned} \tag{11}$$

and the power series $\lambda(1/3)$ and $\lambda(1/4)$ converge faster than (6), at least moderately so. We can do better, starting with

$$\begin{aligned} 2^3 = 8,\ 3^2 = 9 &\Longrightarrow 2\log 3 = 3\log 2 + \lambda(\tfrac{1}{9}), \\ 2^8 = 256,\ 3^5 = 243 &\Longrightarrow 8\log 2 - 5\log 3 = \lambda(\tfrac{13}{256}), \end{aligned} \tag{12}$$

leading to the pair of equations

$$\begin{aligned} 3\log 2 - 2\log 3 &= -\lambda(\tfrac{1}{9}), \\ 8\log 2 - 5\log 3 &= \lambda(\tfrac{13}{256}). \end{aligned} \tag{12A}$$

These can be solved, to yield

$$\begin{aligned} \log 2 &= 2\lambda(\tfrac{13}{256}) + 5\lambda(\tfrac{1}{9}), \\ \log 3 &= 3\lambda(\tfrac{13}{256}) + 8\lambda(\tfrac{1}{9}). \end{aligned} \tag{12B}$$

Alternatively, we can combine the first identity in (12) with

$$\begin{aligned} 2^4 = 16,\ 3\cdot 5 = 15 &\Longrightarrow 4\log 2 = \log 3 + \log 5 + \lambda(\tfrac{1}{16}), \\ 2\cdot 5 = 10,\ 3^2 = 9 &\Longrightarrow \log 2 + \log 5 = 2\log 3 + \lambda(\tfrac{1}{10}). \end{aligned} \tag{13}$$

Together, these identities provide 3 linear equations for $\log 2,\ \log 3$, and $\log 5$, which are readily solved to yield

$$\begin{aligned} \log 2 &= 3\lambda(\tfrac{1}{9}) + 2\lambda(\tfrac{1}{10}) + 2\lambda(\tfrac{1}{16}), \\ \log 3 &= 5\lambda(\tfrac{1}{9}) + 3\lambda(\tfrac{1}{10}) + 3\lambda(\tfrac{1}{16}), \\ \log 5 &= 2\log 3 - \log 2 + \lambda(\tfrac{1}{10}). \end{aligned} \tag{14}$$

Other identities involving the logs of 2, 3, and 5, arise as follows:

$$\begin{aligned} 5^2 = 25,\ 8\cdot 3 = 24 &\Longrightarrow 2\log 5 = 3\log 2 + \log 3 + \lambda(\tfrac{1}{25}), \\ 3^3 = 27,\ 5^2 = 25 &\Longrightarrow 3\log 3 = 2\log 5 + \lambda(\tfrac{2}{27}), \\ 3^4 = 81,\ 16\cdot 5 = 80 &\Longrightarrow 4\log 3 = 4\log 2 + \log 5 + \lambda(\tfrac{1}{81}), \\ 2^7 = 128,\ 5^3 = 125 &\Longrightarrow 7\log 2 = 3\log 5 + \lambda(\tfrac{3}{128}). \end{aligned} \tag{15}$$

For an improvement on the formulas in (14), we can use the first, third, and fourth identities in (15) to produce three linear equations for $\log 2,\ \log 3$, and $\log 5$, and solve this system, to get

$$\begin{aligned} \log 2 &= 12\lambda(\tfrac{1}{25}) + 3\lambda(\tfrac{1}{81}) + 7\lambda(\tfrac{3}{128}), \\ \log 5 &= \tfrac{7}{3}\log 2 - \tfrac{1}{3}\lambda(\tfrac{3}{128}), \\ \log 3 &= \tfrac{2}{3}\log 5 + \tfrac{1}{3}\big[\lambda(\tfrac{1}{25}) + \lambda(\tfrac{1}{81}) + \lambda(\tfrac{3}{128})\big]. \end{aligned} \tag{15A}$$

We move along to the computation of $\log 7$, using

$$3\cdot 7 = 21,\ 4\cdot 5 = 20 \Longrightarrow \log 3 + \log 7 = 2\log 2 + \log 5 + \lambda(\tfrac{1}{21}), \tag{16}$$

or alternatives, such as

$$\begin{aligned} 7^2 = 49,\ 16\cdot 3 = 48 &\Longrightarrow 2\log 7 = 4\log 2 + \log 3 + \lambda(\tfrac{1}{49}), \\ 2\cdot 5^2 = 50,\ 7^2 = 49 &\Longrightarrow 2\log 5 + \log 2 = 2\log 7 + \lambda(\tfrac{1}{50}). \end{aligned} \tag{17}$$

Next we can tackle $\log 11$, using

$$\log 11 = \log 2 + \log 5 + \lambda(\tfrac{1}{11}), \tag{18}$$

or, alternatively,

$$\begin{aligned} 2\cdot 11 = 22,\ 3\cdot 7 = 21 &\Longrightarrow \log 11 + \log 2 = \log 3 + \log 7 + \lambda(\tfrac{1}{22}), \\ 3\cdot 11 = 33,\ 2^5 = 32 &\Longrightarrow \log 3 + \log 11 = 5\log 2 + \lambda(\tfrac{1}{33}), \\ 11^2 = 121,\ 3\cdot 5\cdot 8 = 120 &\Longrightarrow 2\log 11 = 3\log 2 + \log 3 + \log 5 + \lambda(\tfrac{1}{121}). \end{aligned} \tag{19}$$

To calculate $\log 13$, we can use

$$\log 13 = 2\log 2 + \log 3 + \lambda(\tfrac{1}{13}), \tag{20}$$

or alternatively,

$$\begin{aligned} 3^3 = 27,\ 2\cdot 13 = 26 &\Longrightarrow 3\log 3 = \log 2 + \log 13 + \lambda(\tfrac{1}{27}),\\ 8\cdot 5 = 40,\ 3\cdot 13 = 39 &\Longrightarrow 3\log 2 + \log 5 = \log 3 + \log 13 + \lambda(\tfrac{1}{40}),\\ 5\cdot 13 = 65,\ 2^6 = 64 &\Longrightarrow \log 5 + \log 13 = 6\log 2 + \lambda(\tfrac{1}{65}). \end{aligned} \tag{21}$$

Moving to $\log 17$, we can use

$$\log 17 = 4\log 2 + \lambda(\tfrac{1}{17}), \tag{22}$$

or alternatively

$$\begin{aligned} 5\cdot 7 = 35,\ 2\cdot 17 = 34 &\Longrightarrow \log 5 + \log 7 = \log 2 + \log 17 + \lambda(\tfrac{1}{35}),\\ 3\cdot 17 = 51,\ 2\cdot 25 = 50 &\Longrightarrow \log 3 + \log 17 = \log 2 + 2\log 5 + \lambda(\tfrac{1}{51}),\\ 2\cdot 5\cdot 17 = 170,\ 13^2 = 169 &\Longrightarrow \log 2 + \log 5 + \log 17 = 2\log 13 + \lambda(\tfrac{1}{170}). \end{aligned} \tag{23}$$

We can proceed further, using

$$\begin{aligned} \log 19 &= \log 18 + \lambda(\tfrac{1}{19}) = 2\log 3 + \log 2 + \lambda(\tfrac{1}{19}),\\ \log 23 &= \log 24 - \lambda(\tfrac{1}{24}) = 3\log 2 + \log 3 - \lambda(\tfrac{1}{24}), \end{aligned} \tag{24}$$

and so forth.

Having logs of primes, we can evaluate $\log n$ when n is a product of primes, via (8). For example,

$$\begin{aligned} \log 10 &= \log 2 + \log 5\\ &= 2\log 3 + \lambda(\tfrac{1}{10})\\ &= 10\lambda(\tfrac{1}{9}) + 7\lambda(\tfrac{1}{10}) + 6\lambda(\tfrac{1}{16}), \end{aligned} \tag{25}$$

the latter two identities by (14). Alternatively, we can use the last identity in (15) to write

$$\log 10 = \tfrac{10}{3}\log 2 - \tfrac{1}{3}\lambda(\tfrac{3}{128}). \tag{26}$$

Then from (15A) we get

$$\log 10 = 40\lambda(\tfrac{1}{25}) + 10\lambda(\tfrac{1}{81}) + 23\lambda(\tfrac{3}{128}). \tag{27}$$

Log Table

Here is a table of $\log n$, for $n \in \{2, 3, 4, \ldots, 21\}$, to 15 digits after the decimal point.

$$
\begin{aligned}
\log 2 &= 0.693147180559945 & \log 3 &= 1.098612288668110 \\
\log 4 &= 1.386294361119891 & \log 5 &= 1.609437912428055 \\
\log 6 &= 1.791759469228055 & \log 7 &= 1.945910149055313 \\
\log 8 &= 2.079441541679836 & \log 9 &= 2.197224577336220 \\
\log 10 &= 2.302585092994046 & \log 11 &= 2.397895272798371 \\
\log 12 &= 2.484906649788000 & \log 13 &= 2.564949357461537 \\
\log 14 &= 2.639057329615258 & \log 15 &= 2.708050201102210 \\
\log 16 &= 2.772588722239781 & \log 17 &= 2.833213344056216 \\
\log 18 &= 2.890371757896165 & \log 19 &= 2.944438979166440 \\
\log 20 &= 2.995732273553991 & \log 21 &= 3.044522437723423
\end{aligned}
$$

An evaluation of $\lambda(y)$, at the values used in (15A) and (27), obtained by summing the first 11 terms in (5), yields

$$
\begin{aligned}
\lambda(\tfrac{1}{25}) &= 0.04082199452025513, \\
\lambda(\tfrac{1}{81}) &= 0.01242251999855715, \\
\lambda(\tfrac{3}{128}) &= 0.02371652661731604.
\end{aligned}
\tag{28}
$$

One can check that using these values in (15A) and (27) yields $\log 2$ and $\log 10$ to full accuracy.