Row Rank = Column Rank

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Let $A \in M(m \times n, \mathbb{F})$, defining a linear map $A : \mathbb{F}^n \to \mathbb{F}^m$. Let $\mathrm{Row}(A) \subset \mathbb{F}^n$ denote the linear span of the rows of A, $\mathrm{Col}(A) \subset \mathbb{F}^m$ the linear span of the columns of A. Then we set

$$\mathrm{Rowrank}(A) = \dim \mathrm{Row}(A), \quad \mathrm{Colrank}(S) = \dim \mathrm{Col}\, A. \tag{1}$$

An important result in linear algebra is that $\mathrm{Rowrank}(A)$ and $\mathrm{Colrank}(A)$ are equal. One proof, involving showing that $\dim \mathcal{R}(A) = \dim \mathcal{R}(A^t)$, is sketched in Exercise 4 at the end of §3.2 in [T]. Here we give another, based on row and column operations, but simpler than arguments based on reduced row echelon forms.

To start, as noted in (1.3.18) of [T],

$$\mathrm{Col}(A) = \mathcal{R}(A), \tag{2}$$

the range of A. As shown in Proposition 1.6.2 of [T], applying a sequence of column operations to A does not alter its range, hence does not alter its column rank. Also, by Proposition 1.6.1 of [T], applying a sequence of row operations to A does not alter its null space $\mathcal{N}(A)$; it might alter its range, but it does not alter the dimension of the range, since one always has (cf. Proposition 1.3.6 of [T])

$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n. \tag{3}$$

We summarize:

Proposition 1. *Applying a row operation or a column operation to A does not alter its column rank.*

Replacing A by its transpose yields:

Proposition 2. *Applying a row operation or a column operation to A does not alter its row rank.*

Having these results, we can prove the following.

Theorem 3. *Given $A \in M(m \times n, \mathbb{F})$,*

$$\mathrm{Rowrank}(A) = \mathrm{Colrank}(A). \tag{4}$$

Proof. If $A \neq 0$, there is a nonzero entry a_{ij}, and applying a row operation and a column operation moves it to the $(1,1)$ position. Then row operations set all the other entries in the first column equal to 0, and column operations set all the other entries in the first row equal to 0. Hence a sequence of row operations and column operations transform A into

$$\widetilde{A} = \begin{pmatrix} 1 & \\ & B \end{pmatrix}, \quad B \in M((m-1) \times (n-1), \mathbb{F}). \tag{5}$$

By Propositions 1–2, we have $\mathrm{Rowrank}(A) = \mathrm{Rowrank}(\widetilde{A})$ and $\mathrm{Colrank}(A) = \mathrm{Colrank}(\widetilde{A})$. It remains to verify (4) with A replaced by B. But this follows by induction on $\min(m, n)$, since the result is clear for $m = 1$ and for $n = 1$.

Reference

M. Taylor, Linear Algebra, Undergratuate Text #45, AMS, Providence RI, 2010.